Chapter 1

Preliminaries

1.1 Introduction

The study of nonlinear differential equations is of great interest in understanding the fundamental laws of nature. A basic fact of nature is that the laws governing the evolution of natural processes are generally described *locally* by differential equations. The local rate of variation of a relevant dynamical quantity in an infinitesimal neighbourhood of a point is related functionally to itself and other relevant dynamical variables so as to give an analytical (differential) representation of the law of variation of a given natural system. For a linear functional relationship involving the concerned dynamical variables, the natural law under study is linear and thus satisfies the principle of superposition; otherwise the concerned law and the associated differential equation is nonlinear. The theory of linear differential equations is fairly well understood in the sense that a linear differential equation with regular singularity is exactly solvable in the form of modified power series by the Frobenius method in the class of special functions at least locally in a neighbourhood of the singular point. In the case of an irregular singular point, on the other hand, a Frobenius series does not generally exists. One, however, is interested in estimating dominant asymptotic behaviour of the solution by well known asymptotic methods such as the method of dominance balance, WKB method, boundary layers method and so on [1, 2].

The difficulty with nonlinear DE (NDE) is that only a very few can admit exact solutions respecting general integrability criteria, thus exposing special symmetry properties of the underlying dynamical problem. As an example, let us recall that the pendulum equation

$$\ddot{x} + \mu^2 \sin x = 0 \tag{1.1}$$

does admit exact solution involving elliptic integrals which in turn tells that the symmetry group of the nonlinear pendulum is much larger involving both librations or small oscillations about the stable centre and large rotations joining both the centre and unstable saddle, while the corresponding linearized orbit simply corresponds to small oscillations. Although study of possible exact solutions and integrability of NDEs is a topic of great interest, the class of exactly integrable equations using special functions is rather very small, that is to say a class of measure zero. Majority of NDEs are generally known to be non-integrable and can not be solved explicitly using known functions. As a consequence the study of qualitative methods to determine important features of a NDE without solving it exactly, in one hand, and also developing new efficient asymptotic methods to estimate accurately the approximate nature of the exact solutions and other relevant dynamical parameters, on the other, are both of great current interests in the literature of applied mathematics [1, 3, 4].

The main objective of the thesis is to study some aspects of qualitative theory of nonlinear differential equations in the context of a class of Lienard equations. The thesis also reports on an application of a new improved asymptotic method in estimating the periodic orbit and the corresponding amplitude for a class of Rayleigh-Van der Pol equations. The thesis is therefore divided into two parts. In Part I we present some simple but nevertheless interesting extensions of the classical Lienard theorem on the existence of unique limit cycle. In chapter 2 and 3, we present some theorems on the existence of exactly 2 and N number of distinct limit cycles for a class of generalized Lienard equations with symmetric potentials. In Chapter 4, the above theorem is extended for at least n limit cycles for more general class of Lienard equations with non-symmetric potential. In Part II of the thesis, we present analytic formula for amplitude of the unique limit cycle for both Rayleigh and Van der Pol equations in Chapter 5 using the Homotopy

Analysis Method (HAM). Similar results are discussed in Chapter 6 by Renormalization Group Method (RGM). In Chapter 7, we discuss applications of an Improved Renormalization Group Method (IRGM) in obtaining approximate formulae both for the periodic orbit and the corresponding amplitude for Rayleigh Van der pol system and the results are compared with those obtained by Homotopy Analysis method.

1.2 Qualitative Theory of Nonlinear Ordinary Differential Equations

The basic reason for studying nonlinear ODE stems from the fact that nature is inherently nonlinear; majority of natural processes and laws governing them are nonlinear. In fact, linear superposition principle can at best be considered to hold approximately for a realistic natural system. As for example, the (linear) simple harmonic oscillation is an approximate small amplitude oscillation of the nonlinear pendulum equation (1.1). Historically, the qualitative theory of NDE was founded by J. Henri Poincare around 1880s when he initiated his pioneering investigations of three body problem in celestial mechanics [5]. His work in this field laid down the foundation of various key concepts such as the (Poincare) first return maps, sensitive dependence on initial conditions of the late time asymptotic behaviour of a solution, the phenomenon of recurrence and others, in reinterpreting a NDE as a deterministic dynamical system. Poincare's original contributions on the dynamical system approach to NDE remains as an isolated masterpiece over a considerable period of time until the thread was taken up slowly by a number of physicists and mathematicians in the second quarter of twentieth century such as Balthazar Van der Pol [6] on the existence of stable nonlinear (self-excited) oscillation in vacuum tube electric circuit in 1920s, Mary Cartwright, John Littlewood, and N Levinson [7, 8] on the proof of existence of sensitive dependence on initial condition and random-like motion in a forced Van der Pol oscillator during 1940-1950 and others. The modern enthusiasm and interest in the theory of dynamical system and NDE was triggered by the path breaking paper of Edward Lorenz in 1962 on a low ($n = 3$) dimensional NDE modeling of convective fluid flow for the long time predictability problem of weather system, discovering numerically the butter-fly effect and what is presently known as Chaotic or strange attractor for a deterministic system [9].

In the following paragraphs, we present a short review of the basic facts of phase space (plane) analysis of a deterministic system in the context of planar systems relevant for this thesis, viz., the Rayleigh, Van der Pol and the Lienard differential systems. As the present work is limited to applications to limit curves (cycles) of a planar autonomous system (rather than higher dimensional chaotic attractors), our review will be rather elementary. We assume the basic facts and concepts of phase plane analysis [2,3]. We begin our review, following Ref. [2,3], as preparation for the statement of the Poincare-Bendixson theorem that establishes the existence of an isolated closed curve as the limit set of a planar nonlinear system.

Let us consider a general planar autonomous system

$$\begin{aligned} \dot{x} &= X(x,y) \\ \dot{y} &= Y(x,y), \end{aligned} \tag{1.2}$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, the parameter t can be identified as time variable. We further assume that the system is regular [3] i.e., X and Y have continuous partial derivatives in the domain of definition which in general can be taken as entire phase plane.

Definition 1.2.1 (Half-Path) *Let Γ be a path of the system (1.2) and let $x = x(t)$ and $y = y(t)$ be a solution of (1.2) defining Γ. Then we shall call the set of all point of Γ for $t \geq t_0$, where t_0 is some value of t, a half-path of (1.2). In other words by a half-path of (1.2) we mean the set of all points with coordinates $(x(t), y(t))$ for $t_0 \leq t < \infty$. We denote such a half-path of (1.2) by Γ^+.*

It is well known that a linear planar autonomous system has a unique critical point i.e., the limit set of a linear system is a singleton set. The limit set of nonlinear system can, however, be more general; apart from the discrete set of critical points, a nonlinear system can also admit nontrivial limit set that is dense in $\mathbb{R}^2$ i.e. an isolated closed orbit. Such isolated closed limit sets are called *limit cycles*. Here we introduce the precise definitions.

Definition 1.2.2 (Limit Set) [2] *Let Γ^+ be a half-path of (1.2) defined by $x = x(t)$ and $y = y(t)$ for $t \geq t_0$. Let (x_1, y_1) be a point in xy plane. If there exists a sequence of real numbers $\{t_n\}$, $n = 1, 2, \ldots$, such that $t_n \to +\infty$ and $(x(t_n), y(t_n)) \to (x_1, y_1)$ as $n \to +\infty$, then we call (x_1, y_1) a limit point of Γ^+. The set of all limit points of a half-path Γ^+ will be called the limit set of Γ^+ and will be denoted by $L(\Gamma^+)$.*

Definition 1.2.3 (Limit Cycles) *An isolated closed path Γ of the system (1.2) which is approached spirally from either the inside or the outside by a nonclosed half-path Γ' of the same system either as $t \to \infty$ or as $t \to -\infty$ is called a limit cycle of (1.2). The word 'isolated' has been used in the sense that there is no other closed path in its sufficiently close neighbourhood.*

In a planar autonomous system the asymptotic behaviour of a phase path is guided by the following well known Poincare-Bendixson theorem.

Theorem 1.2.1 (The Poincare-Bendixson Theorem) *Let R be a closed, bounded region that contains no critical point of a planar system (1.2) such that some positive half-path Γ of the system lies entirely within R. Then either Γ is itself a closed path, or it approaches asymptotically to a closed path as either $t \to \infty$ or $t \to -\infty$.*

It follows immediately that a planar nonlinear system can admit no limit set other than limit cycle or critical points. We consider the following example as an application of this theorem.

Example 1.2.1 *Consider the system*

$$\begin{aligned} \dot{x} &= x\left(1 - x^2 - y^2\right) + y \\ \dot{y} &= y\left(1 - x^2 - y^2\right) - x. \end{aligned}$$

The equilibrium points are given by

$$\dot{x} = 0, \ \dot{y} = 0 \Rightarrow x = 0, y = 0$$

so that $(0,0)$ *is the only equilibrium point of the given system. In order to identify limit cycles of the system we consider the polar transformation*

$$x = r\cos\theta,\ y = \dot{x} = r\sin\theta$$

so that

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}.$$

The given system then becomes

$$\dot{r} = r\left(1 - r^2\right),\ \dot{\theta} = -1.$$

One particular solution to the system is

$$r = 1,\ \theta = -t$$

which corresponds to the limit cycle

$$x = \cos t,\ y = -\sin t.$$

Also,

$$\dot{r} > 0 \text{ when } 0 < r < 1 \text{ and}$$
$$\dot{r} < 0 \text{ when } r > 1,$$

showing that the phase paths approach the limit cycle $r = 1$ *from inside and outside the cycle. The equation for* $\dot{\theta}$ *shows that the phase points move in a spiral clockwise direction around the limit cycle. The phase plane diagram of the system is shown in Figure 1.1.*

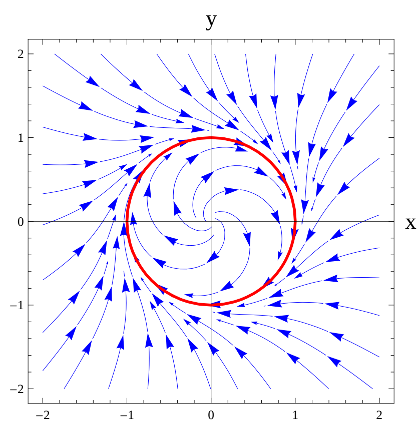

Figure 1.1: Stable Limit Cycle

The negatively directed system, on the other hand

$$\dot{x} = -x\left(1 - x^2 - y^2\right) - y$$
$$\dot{y} = -y\left(1 - x^2 - y^2\right) + x$$

has the phase plane diagram as shown in Figure 1.2.

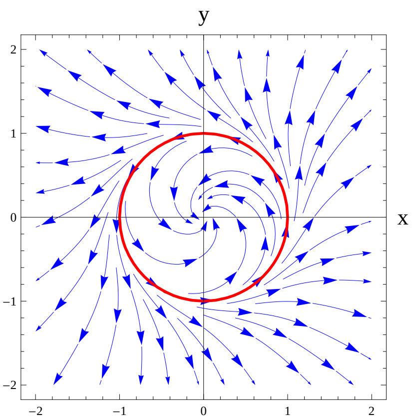

Figure 1.2: Unstable Limit Cycle

We observe that the positive half-path Γ is itself a closed path, or it approaches towards (or diverges from) a closed path, or it terminates at (or originates from) the equilibrium point $(0,0)$. Example 1.2.1 gives us the concept of stable and unstable limit cycle the formal definition of which are given below.

Definition 1.2.4 (Stable Limit Cycle) *A limit cycle is said to be stable if the neighbouring positive half-paths approach asymptotically towards the cycle.*

Definition 1.2.5 (Unstable Limit Cycle) *A limit cycle is said to be unstable if the neighbouring positive half-paths approach asymptotically away from the cycle.*

Figure 1.1 and 1.2 respectively represent stable and unstable limit cycles. In Part I of this thesis we shall study sufficient conditions for the existence of single and multiple limit cycles for the Lienard equation (1.3). It should be mentioned here that the problem is closely related to the famous Hilbert's 16-th problem which asks about the maximum number of limit cycles corresponding to the polynomial autonomous system (1.2) when X and Y are polynomials. We shall confine ourself to the Lienard equation introduced in the following subsection.

1.2.1 Lienard Equation

The Poincare-Bendixson theorem ensures the existence of at least one limit cycle in a bounded region. However it does not ensures the uniqueness of limit cycle for a given system. We shall restrict our study to generalized Lienard equation

$$\ddot{x} + f\left(x\right)\dot{x} + g\left(x\right) = 0 \tag{1.3}$$

and similar systems. This equation can be written as the autonomous system

$$\dot{x} = y - F\left(x\right), \qquad \dot{y} = -g\left(x\right), \tag{1.4}$$

known as Lienard system. This system has many applications in physical and engineering problems [6, 10, 11]. The Poincare-Bendixson theorem can ensure the existence of at least one limit cycle for the system (1.4). However, it does not always ensures the existence or the uniqueness of limit cycle for the system. One sufficient condition ensuring the existence and uniqueness of limit cycle for the Lienard system (1.4) is presented in the following theorem, which is well known as Lienard Theorem.

Theorem 1.2.2 (Lienard Theorem) *The equation (1.3) has a unique periodic solution if*

(*i*) f *and* g *are continuous;*
(*ii*) F *and* $g\left(x\right)$ *are odd functions with* $g\left(x\right) > 0$ *for* $x > 0$*;*
(*iii*) F *is zero only at* $x = 0$, $x = a$, $x = -a$ *for some* $a > 0$;
(*iv*) $F\left(x\right) \to \infty$ *as* $x \to \infty$ *monotonically for* $x > a$.

The detail of the theorem is reviewed in Chapter 2. The conditions are further generalized in remaining chapters of Part I of the thesis ensuring the existence of unique and multiple limit cycles for the system (1.4). The amplitude and shape of the limit cycles are also computed in Part II by applying Homotopy Analysis Method (HAM), Renormalization Group Method (RGM). Finally we complete this thesis by finding analytic approximation of the amplitude and the equation of the limit cycle for two special kind of Lienard systems, viz. Rayleigh Equation and Van der Pol Equation, in the context of a new improved RGM in Chapter 7.

1.2.2 Rayleigh and Van der Pol Equations

Before the introduction of HAM and RGM we present the Rayleigh equation

$$\ddot{y} + \varepsilon\left(\frac{1}{3}\dot{y}^3 - \dot{y}\right) + y = 0. \tag{1.5}$$

and the Van der Pol equation

$$\ddot{x} + \varepsilon\,\dot{x}\left(x^2 - 1\right) + x = 0 \tag{1.6}$$

where $\varepsilon > 0$ is a non-linearity parameter. Taking $x = \dot{y}$ one can generate (1.6) from (1.5). The equation (1.5) is used to model the dynamics of musical instruments such as the blown clarinet reed [4]. Rayleigh modeled the clarinet reed as a linear oscillator

$$\ddot{x} + kx = 0.$$

The effect of clarinetist is modeled by introducing a term $\alpha\dot{x} - \beta\dot{x}^3$, with $\alpha, \beta > 0$ on the right hand side, indicating negative damping for small $\dot{x}$ and positive damping for large $\dot{x}$. This gives rise to the model

$$\ddot{x} + kx = \alpha\dot{x} - \beta\dot{x}^3,$$

which is the dynamical model for the sustained oscillation of the blown clarinet reed. This system can be converted into the form (1.5) by the transformation

$$\tau = \sqrt{k}t,\ \beta = \frac{\varepsilon}{3\sqrt{k}},\ \alpha = \varepsilon\sqrt{k} \text{ with } \varepsilon > 0.$$

The Van der Pol oscillator is also a non-conservative oscillator in which energy is dissipated at high amplitudes and generated at low amplitudes. Balthazar Van der Pol discovered this oscillator while he was working in the field of radio and telecommunication at Phillips in course of building electronic circuit models of human heart [6].

The Van der Pol circuit is composed of one inductor, one capacitor and one resistor arranged in a loop. A voltage is applied to the circuit and removed. The problem is to determine the resulting current and voltage behaviour. The diagram of the circuit is given below. The

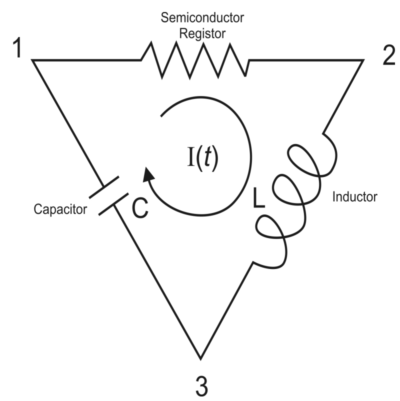

Figure 1.3: Van der Pol Circuit

problem is formulated under the following guiding laws:

Kirchhoff's first law on electrical circuit: It says that the current $I(t)$ through each of registor, inductor and capacitor are same at any time t.

Kirchhoff's second law on electrical circuit: It says that the sum of all voltage drops or voltage differences in a closed loop must vanish i.e.,

$$V_{12} + V_{23} + V_{31} = 0 \tag{1.7}$$

where V_{ij} is the voltage drop between nodes i and j; $i, j = 1, 2, 3$ and $i \neq j$.

We consider a semi-conductor as a registor so that a small voltage difference in one direction of the semi-conductor generates large current flow and in opposite direction it generates almost no current flow even if there is fairly large voltage difference between its ends 1 and 2. We take such a semi-conductor that the relation between the voltage difference V_{12} between the ends 1 and 2 and the current $I(t)$ through the registor be

$$V_{12} = I^3 - \mu I \tag{1.8}$$

where μ is a parameter which determines the sign of voltage difference V_{12} as I increases through the value $\sqrt{\mu}$. If $I < \sqrt{\mu}$ then $V_{12} > 0$ and if

$I > \sqrt{\mu}$ then $V_{12} < 0$.

The law stating the relationship between the inductance L of the inductor, the current flow $I(t)$ and the voltage difference V_{23} between the ends 2 and 3 is given by

$$V_{23} = L\frac{dI}{dt} = L\dot{I}(t). \tag{1.9}$$

The charge $q(t)$ at time t, the capacitance C of the capacitor and the voltage difference V_{31} between the ends 3 and 1 of the capacitor is related by the **Coulomb's law** as

$$C = V_{31}q(t)$$
$$\text{i.e., } V_{31} = \frac{q(t)}{C}. \tag{1.10}$$

We know if $I(t)$ is the current flowing through the capacitor at time t, then

$$\dot{q}(t) = I(t). \tag{1.11}$$

We now construct a differential equation from the above relations. We suppose that

$$-V_{31} = V_{13} = V \tag{1.12}$$

so that (1.10) gives

$$V = V_{13} = -V_{31} = -\frac{q(t)}{C}.$$

Here C is constant. So, differentiating the relation with respect to time t and using (1.11) we get

$$\dot{V}(t) = -\frac{\dot{q}(t)}{C} = -\frac{I(t)}{C}. \tag{1.13}$$

Using $(1.8), (1.9), (1.10)$ and (1.12) in (1.7) we get,

$$\left(I^3 - \mu I\right) + L\dot{I} - V = 0.$$

Differentiating again with respect to t we obtain,

$$\left(3I^2\dot{I} - \mu\dot{I}\right) + L\ddot{I} - \dot{V} = 0.$$

Using (1.13) we have,

$$\left(3I^2\dot{I} - \mu\dot{I}\right) + L\ddot{I} + \frac{I}{C} = 0$$
$$\Rightarrow \ddot{I} + \frac{\mu}{L}\left(\frac{3}{\mu}I^2 - 1\right)\dot{I} + \frac{I}{LC} = 0. \tag{1.14}$$

Applying the transformation

$$t = \sqrt{LC}\tau$$

so that

$$\dot{I} = \frac{dI}{d\tau}\frac{d\tau}{dt} = I'\frac{1}{\sqrt{LC}}$$
$$\text{and } \ddot{I} = \frac{d\dot{I}}{d\tau}\frac{d\tau}{dt} = \frac{d}{d\tau}\left(I'\frac{1}{\sqrt{LC}}\right)\frac{d\tau}{dt} = \frac{d}{d\tau}\left(I'\frac{1}{\sqrt{LC}}\right)\frac{1}{\sqrt{LC}} = I''\frac{1}{LC}$$

(1.14) reduces to

$$I''\frac{1}{LC} + \frac{\mu}{L}\left(\frac{3}{\mu}I^2 - 1\right)I'\frac{1}{\sqrt{LC}} + \frac{I}{LC} = 0$$
$$\Rightarrow I'' + \mu\sqrt{\frac{C}{L}}\left(\frac{3}{\mu}I^2 - 1\right)I' + I = 0.$$

Now taking further

$$I(\tau) = \sqrt{\frac{\mu}{3}}x(\tau)$$

the differential equation reduces to

$$\sqrt{\frac{\mu}{3}}x'' + \mu\sqrt{\frac{C}{L}}\left(x^2 - 1\right)\sqrt{\frac{\mu}{3}}x' + \sqrt{\frac{\mu}{3}}x = 0.$$

Writing

$$\varepsilon = \mu\sqrt{\frac{C}{L}}$$

we finally get

$$x'' + \varepsilon\left(x^2 - 1\right)x' + x = 0$$

which is the well known Van der Pol equation.

An important feature of this oscillator is that it has unique oscillation around a state at which energy generation and dissipation balance each other and as a result we get a unique limit cycle. The existence of such limit cycle is confirmed by Lienard theorem. This unique characteristic of the oscillator makes the Van der Pol equation (1.6) a benchmark in the study of nonlinear oscillation and limit cycle. This equation has become a standard model for nonlinear oscillatory processes in physics, biology, sociology, economics and many more fields. For instance, electrical potential across the cell membranes of neurons in the gastric mill circuit of lobsters is modeled by Van der Pol equation [12, 13]. It is also used to model spike generation in giant squid axons [14, 15]. Due to its wide applications detailed understanding of the Van der Pol equation is still of considerable interest.

In the solution of a non linear ordinary differential equation (ODE) the study of asymptotic behaviour and the development of a compatible calculation technique is a topic of key interest in literature. HAM and RGM are two such techniques which are discussed in the following subsections.

1.2.3 Homotopy Analysis Method

In recent years there have been a lot of interests in the homotopy analysis method developed by S. Liao in his Ph.D. thesis [16]. The salient idea behind HAM is an extension of the topological concept of homotopy of paths into the function (solution) space of a given nonlinear differential equation, when the nonlinear differential operator ($\mathcal{N}$) itself is supposed to be a homotopic deformation of a simpler (linear) differential equation with well known solution set as the deformation parameter q is assumed to vary from the value 0 (corresponding to linear operator $\mathcal{L}$) to 1 (nonlinear operator $\mathcal{N}$). A known analytic solution of a simple (linear) problem is then continuously *deformed* into a solution of a more difficult (nonlinear) problem. Such deformation is called *homotopy*. A primary advantage of this approach is that the solution of the latter remains valid for all values of the small or large nonlinearity parameter that is present in the system. Moreover, HAM involves a free control parameter that makes the HAM generated solutions to converge generally to the exact solution of the nonlinear problem. A simple example of homotopy is $\mathcal{H}(x, q) = (1 - q) f(x) + q\, g(x)$ between two functions $f(x)$ and $g(x)$. If $f(x)$ represents the solution of some simple problem and $g(x)$ represents that of a complicated problem then $\mathcal{H}(x,0) = f(x)$ undergoes through a continuous deformation or change and becomes $\mathcal{H}(x,1) = g(x)$ as q changes continuously from zero to 1. The parameter $q \in [0,1]$ is called homotopy parameter.

As pointed out by Liao [17], the homotopy analysis essentially depends on the implicit function theorem, which is a basic principle behind continuation and bifurcation analysis. Since HAM makes use of a rather simple but nevertheless attractive concept of topological homotopy of paths, the method gains much attentions in recent decades in the literature of differential equations in deriving correct analytic approximation of the solution of a large class of nonlinear differential equations. Although we shall apply this method to ODE only it can be applied to partial differential equation (PDE) as well [17]. Lopez et al. [18] computed an analytic approximation to the amplitude of the limit cycle for Van der Pol equation

$$\ddot{x} + \varepsilon\, \dot{x}\left(x^2 - 1\right) + x = 0 \tag{1.15}$$

which is uniformly valid for the nonlinearity parameter $\varepsilon > 0$. Although the HAM may fail for some typical systems as pointed out recently by Meijer [19], this method has many applications [20–25] which establish HAM as a convenient method to find analytic approximations of the solution and different parameters involved in the system, provided, of course, this method is at all applicable. Liao has pointed out some limitations of HAM in [17] such as lack of rigorous theories to choose initial approximations, auxiliary linear operators, auxiliary functions, and auxiliary parameter etc. involved in the computation scheme. However, we do not go into the study of mathematical rigor behind this method.

In the present thesis, we report approximate analytic formulas of the amplitude a of the limit cycle for both Van der Pol and the Rayleigh Equations

$$\ddot{y} + \varepsilon\left(\frac{1}{3}\dot{y}^3 - \dot{y}\right) + y = 0 \tag{1.16}$$

as a function of the nonlinearity parameter ε in the nonperturbative regime $0 < \varepsilon \leq 50$ which can be extended uniformly for all $\varepsilon > 0$.

The analytic formula of the amplitude for this system is computed in literature for $\varepsilon \to 0$. However, to the authors' best knowledge, this is the first analytic formula in literature for non-perturbative ε giving a good approximation with the exact values of the amplitude for Rayleigh system. The detail discussion on HAM is given in Chapter 5.

1.2.4 Renormalization Group Method

Theory of renormalization group (RG) has a hallowed history; originally invented as an efficient technique in eliminating undesired divergences in quantum electrodynamics, in particular, and quantum field theory (QFT), in general, by factoring out generic divergent terms in a perturbative expansion of the relevant physical (dynamical) quantity, for instance, the electron self-energy, order by order, leading finally to a finite well defined (renormalized) theory [26]. The original algorithmic formulation of RG theory was later put into a more rigorous framework by Wilson in the context of continuous phase transition that enjoys some sort of scale invariance and has the property of large scale cooperative behaviour [27]. Over the past few decades the theory of RG has been accepted as an efficient method of extracting finite measurable (observable) results from a basically nonlinear problem, for which the standard methods, such as perturbation theory or some of its variants, normally fail i.e. yield meaningless/divergent values or are of limited validity. It is reasonable that the RG method has got wide applications in a variety of nonlinear problems starting from quantum field theory [26], phase transitions in statistical mechanics [28], theory of fractals [29], turbulence and chaotic attractors in dynamical systems [30] and finally to nonlinear ordinary and partial differential equations. Chen et al. initiated the first successful application of the RG method in nonlinear differential equations [31].

In a renormalization theory one usually develops a scheme of viewing a system at different energy or distance scales. If we look at a large distance or time scale phenomena by smaller microscopic scale theory, such as in phase transitions or in nonlinear differential equations or if we want to explain low energy theory by high energy theory, as in QFT then we require to switch our view from one scale to another. Such mathematical transformations involving switching of views is known as renormalization-group (RG) transformation [32]. If the theory remains unaltered, as in the case of fractals, then the underlying problem is scale invariant, otherwise special techniques are required to understand the precise law behind predicted variations in the proposed theory. RG theory provides one such approach which is generally known to answer such questions by means of RG flow equations which describe the exact patterns of variations of relevant dynamical quantities in a nonlinear problem. We shall discuss this problem in the context of nonlinear ODE.

In the study of nonlinear ODE solving an ODE in closed form is always a challenging problem. No general technique is available to obtain the solution of non linear ODEs in closed or finite form. The reason behind it is the class of standard functions (e.g., polynomials, exponential, logarithmic or trigonometric functions etc.) that we have in our hand is insufficient to accommodate the variety of differential equations those arise in practice. We therefore, investigate the qualitative characteristics of the solution, such as its existence, periodicity, regularity etc. and their behaviour in the form of long time asymptotics. Naive perturbation method in small nonlinear coupling parameter $|\varepsilon| << 1$ fails to give uniformly valid results for sufficiently large time, requiring for new approaches to obtain uniformly valid finite results. Another more severe problem is to obtain meaningful results even in sufficiently large coupling $\varepsilon \sim O(1)$ nonperturbative regime. Different singular perturbation technique such as method of multiple scales, boundary layers, averaging, WKB methods, central manifold theory are used to deal with this problem. However, each of these methods has its own drawback and does not produce uniformly valid solution for all values of the non linearity parameter involved in the ODE.

As remarked already, the renormalization group was initially devised in QFT [26] and it is widely applied in solid state physics, fluid mechanics, cosmology, quantum field theory and even nanotechnology to manage divergences that arise in the solution of nonlinear problems. The RG method (RGM) gives us an algorithmic approach to derive asymptotic expansion of the solution for a large class of singularly perturbed ODEs. A first significant application of the RGM in the solution of nonlinear ODEs is given by Chen, Goldenfeld, Oono [33,34]. A brief introduction of RGM is given in Chapter 6 which is simplified latter by De Ville et al. [35] giving an algorithmic approach for achieving the solution. They studied interesting application of RGM for autonomous as well as nonautonomous systems. RGM is a technique which has many applications on differential equation in the recent past [35–39]. A major advantage of RGM is that it starts from naive perturbation expansion of a problem and is expected to yield automatically the gauge functions such as fractional powers of ε and logarithmic terms in ε in the renormalized expansion. One does not require to have any prior knowledge to prescribe these unexpected gauge functions in an ad hoc manner.

Although the RGM is formulated to give an analytic solution to a NDE and is based on naive perturbation expansion of a problem (which is formally possible for any system) and is expected to give an approximate solution for all values of the nonlinearity parameters involved in the problem, it is found in our work that the RG generated solution does not generally give good approximation uniformly for non-perturbative regime of the nonlinearity parameters.

To circumvent this undesired limitation of RGM, we present in the Part II of this these, a reformulation of RG method, called Improved RGM (IRGM), in the framework of a novel scale invariant extension of the ordinary analysis equipped with the so-called duality structure [40–42] and apply this scheme to calculate uniformly valid approximations for amplitudes for Rayleigh and Van der Pol equations for all values of the nonlinearity parameter. We have also shown the efficacy of the method by presenting efficient estimations of the limit cycle orbits of both these nonlinear oscillators.