A STUDY ON FUZZY SEQUENTIAL TOPOLOGICAL SPACES

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By

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August, 2016

DEDICATED TO MY PARENTS*......*

DECLARATION

The thesis entitled "**A STUDY ON FUZZY SEQUENTIAL TOPOLOGICAL SPACES**", is a presentation of my original research work which has been supervised by Prof. S. De Sarkar, Department of Mathematics, University of North Bengal. No part of this thesis has formed the basis for the award of any other degree or diploma.

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CERTIFICATE

This is to certify that the thesis entitled "A STUDY ON FUZZY SEQUENTIAL TOPOLOGICAL SPACES", is an authentic record of research carried out by Miss Nita Tamang under my supervision and guidance in the Department of Mathematics, University of North Bengal for the PhD degree and no part of it has previously formed the basis for the award of any other degree or diploma.

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Abstract

Fuzzy mathematics became known in the second half of the 20^{th} century after L.A. Zadeh published the 1st article on fuzzy sets [42] in 1965. The article generated lots of research and implicitly leads to the founding of a new branch in mathematics. Fuzzy mathematics may be regarded as a parallel to the classical mathematics or as a natural continuation of the latter, but in most cases, it is considered a new mathematics, very useful in solving problems expressed through a vague language. The study of fuzzy mathematics is organized on several sub domains. Among them, we can mention: the fuzzy logic, the fuzzy arithmetic, the fuzzy topology etc. Compared to other research domains, fuzzy topology gained interest after its introduction by C. L. Chang [7] in 1968, who studied a number of the basic concepts, including fuzzy continuous maps and compactness. Goguen [14] in 1973, presented the fundamental ideas of basis, sub basis and product in an investigation of compactness. After the introduction of compactness in fuzzy topological spaces by C.L. Chang, a series of different notions of fuzzy compactness have been presented by T.E. Gantner, R.C. Steinlage, R.H. Warren [12], R. Lowen [21, 22], Wang Guojun [40], Gunther Jager [18] etc. and they were compared in detail by R. Lowen [23]. In 1980, Pu and Liu introduced a new concept of so called Q-neighbourhood which could reflect the features of neighbourhood structures in

fuzzy topological spaces and by this new neighbourhood structure the Moore - Smith convergence was established [26]. In the same year, B. Hutton and I. Reilly presented separation axioms in fuzzy topological spaces [17]. Fuzzy closure operators and fuzzy closure systems have been studied by Mashour and Ghanim [24], G. Gerla [13], Bandler and Kohout [2], R. Belohlavek [3], whereas fuzzy interior operators and fuzzy interior systems have appeared in the studies of R. Belohlavek and T. Funiokova [4], Bandler and Kohout [2].

Kazimierz Kuratowski $(1896 - 1980)$, a Polish mathematician, introduced the concept of closure operators known in mathematical circles as the Kuratowski closure operators, which was fundamental for the development of topological space theory and irreducible continuum theory between two points. Kuratowski closure operator generates a topological space in which it coincides with the closure of a set. After the initiation of Kuratowski closure operators, many mathematicians studied various combinations of Kuratowski closure axioms. Dr. M. Singha and Prof. S. De Sarkar (2012), in their article "*On K*Ω *and Relative closure operators in* $P(X)^{\mathbb{N}}$"[30], defined KΩ-closure operators in $P(X)^{\mathbb{N}}$ which generates a topology like structure and is same as so called sequential topology introduced by Prof. M.K. Bose and Prof. I. Lahiri [5]. In [5], Prof. M.K. Bose and Prof. I. Lahiri introduced the concept of sequential topological spaces and they developed separation axioms in sequential topological spaces upto regularity. Encountering some problems in the definition of regularity in a sequential topological space while going through [5], N. Tamang, M. Singha and S. De Sarkar, presented an article entitled "*Separation Axioms in Sequential Topological Spaces in the Light of Reduced and Augmented Bases*" [32] in which they studied the separation axioms in a sequential topological space with some modified definitions so as to solve the problems encountered. Apart from this, the studies of sequential topology comprises of Semi open and Weakly Semi open sequential sets by S. Das, M. Singha and S. De Sarkar [9], monotonic sequential operators by M. Singha and S. De Sarkar [31] etc.

After going through the above mentioned topics on fuzzy topology and that on sequential topology, it is natural to ask whether the study of the concept of sequential topology is possible in case of fuzzy sets. In this thesis, we investigate various notions of a sequential topological space like separation axioms, operators, mappings, compactness etc. in the fuzzy setting.

The thesis comprises of seven chapters. **Chapter** 1 is introductory in nature and to provide a suitable background for the rest of the chapters, it consists of basic definitions and results from the theory of fuzzy topological spaces.

Chapter 2 is devoted to the introduction of fuzzy sequential sets and fuzzy sequential topology. Different notions of a general topological space, like basis, subbasis, closure and interior of a set, derived set, etc. are studied in the setting of fuzzy sequential topological spaces. The concept of Q-neighbourhoods is also introduced. Given a fuzzy topological space, one can always associate a fuzzy sequential topology with the given space as one of its components and given a fuzzy sequential topological space, it is possible to construct countably many fuzzy topological spaces, called components of the given space. Many pleasant properties of a fuzzy sequential topology are compared with that of its components and many properties of a fuzzy topological space are compared with that of the associated fuzzy sequential topology. A variant of Yang's result [19] in a general topological space, is also achieved in a fuzzy sequential topological space. The contents of this chapter is published in [Singha M., Tamang N. and De Sarkar S., *Fuzzy Sequential Topological Spaces*, International Journal of Computer and Mathematical Sciences, **3** (4) (June 2014), 2347-8527.]

In Chapter 3, we present separation axioms in a fuzzy sequential topological space, where fs-T_0, fs-T_1, fs-Hausdorff, fs-regular and fs-normal spaces are studied. A necessary and sufficient condition for a space to be each of these spaces is established. Further, the results relating the separation axioms in component fuzzy topological spaces with that of the corresponding fuzzy sequential topological space are also obtained. The results of this chapter is published in [Tamang N., Singha M. and De Sarkar S., "*Separation Axioms in Fuzzy Sequential Topological Spaces*", Journal of Advanced Studies In Topology, **4** (1) (2013), 83-97.]

Closure and interior operators on an ordinary set belong to the very fundamental mathematical structures with direct applications on many fields like topology, logic etc. The approach in

Chapter 4 is inspired by the importance of closure and interior operators. In this Chapter, the concepts of fs-closure and fs-interior operators on a set are presented. Other studied notions are the concepts of fs-closure and fs-interior systems. With an fs-closure system (fs-interior system), one can always associate an fs-closure operator (fs-interior operator) and vice-versa. An fs-closure and an fs-interior operator on a set may not induce the same fuzzy sequential topologies in general. So the question arises whether there is any condition under which, these operators induce the same fuzzy sequential topologies. The answer is in affirmative and a necessary and sufficient condition under which an fs-closure and an fs-interior operator on a set induce the same fuzzy sequential topologies is obtained.

In general set theory, composition of closure (interior) operators is again a closure (interior) operator and hence it also induces a topology. It is natural to ask whether the result is true in case of fs-closure and fs-interior operators and if the answer is in affirmative, is there any relation among the topologies induced by the composition and that induced by the participants to the composition? The answers to these questions are also given in **Chapter** 4 and a relation between such fuzzy sequential topologies is obtained. The relation between the collections of fs-closure and fs-interior operators is also investigated. Further, the Relative fs-closure operators and fs-connectors connecting two fuzzy topologies on a set are introduced and studied. The contents of this Chapter are published in [Tamang N., Singha M. and De

Sarkar S., *FS-closure operators and FS-interior operators*, Ann. Fuzzy Math. Inform., **6** (3) (November 2013), 589-603.] and [Tamang N., Singha M. and De Sarkar S., *Composition of Fuzzy Sequential Operators with Special Emphasis on FS-Connectors*, Palestine Journal of Mathematics, **4** (1) (2015), 37-43.]

Chapter 5 deals with the study of the concepts of continuity and compactness in fuzzy sequential topological spaces. Some characterizations of continuity are given. Other notions like open maps, closed maps and homeomorphisms are also studied. Two kinds of compactness, fs-compactness and Ωfs-compactness in a fuzzy sequential topological space are introduced, studied and the Chapter has been concluded by investigating the behavior of the product of these compact spaces.

One major area of research in general topology during the last few decades that mathematicians have been pursuing is to investigate different types of generalized open sets, generalized continuous functions and study their structural properties. **Chapter** 6 and **Chapter** 7 is devoted to the study of different types of generalized open sets, generalized continuous functions in the setting of a fuzzy sequential topological space. Their interrelations have been shown using a diagram in **Chapter** 7. Further, various properties of these sets and functions have been studied in both **Chapter** 6 and **Chapter** 7. Finally, **Chapter** 7 is concluded showing a decomposition of fs-continuity.

Preface

The main idea of this thesis is to study the concept of sequential topology in the fuzzy setting. During this work, a plenty of results were obtained and a good number of examples were constructed wherever necessary. This thesis has been completed hoping that it may be useful to the researchers for future research on similar topics or any other related field.

The whole work of this thesis would not have been possible without the support of many people to whom I owe heartfelt thanks. Foremost, I express my deep sense of gratitude and indebtedness to my supervisor Prof. S. De Sarkar, Department of Mathematics, University of North Bengal, who enlightened me with the first glimpse of research and guided me to carry out the entire work of this thesis.

I would like to thank all the faculty members of the Department of Mathematics, University of North Bengal who have always inspired me to work hard, especially Dr. M. Singha who has always supported and helped me. My sincere gratitude to the non teaching staffs of the Department of Mathematics, University of North Bengal, who always assisted me nicely in the official works. I must not forget to thank all the faculty members, non-teaching staffs of the Department of Mathematics, Jadavpur University as they have always provided an encouraging environment for me while I was serving there.

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List of Publications:

1. Singha M., Tamang N. and De Sarkar S., "*Fuzzy Sequential Topological Spaces*", International Journal of Computer and Mathematical Sciences, **3** (4) (June 2014), 2347-8527.

2. Tamang N., Singha M. and De Sarkar S., "*Separation Axioms in Fuzzy Sequential Topological Spaces*", Journal of Advanced Studies in Topology, **4** (1) (2013), 83-97.

3. Tamang N., Singha M. and De Sarkar S., "*FS-closure operators and FS-interior operators*", Ann. Fuzzy Math. Inform., **6** (3) (November 2013), 589-603.

4. Tamang N., Singha M. and De Sarkar S., "*Composition of Fuzzy Sequential Operators with Special Emphasis on FS-Connectors*", Palestine Journal of Mathematics, **4** (1) (2015), 37-43.

5. Tamang N., Singha M. and De Sarkar S., "*On the notions of continuity and compactness in fuzzy sequential topological*

spaces", communicated.

6. Tamang N. and De Sarkar S., "*Some Nearly Open Sets in a Fuzzy Sequential Topological Space*", International Journal of Mathematical Archive, **7** (2) (2016), 120-128.

7. Tamang N. and De Sarkar S., "*A Note on FS-preopen Sets and FS-precontinuity*", Communicated.

8. Tamang N. and De Sarkar S., "*Decomposition of FS-continuity*", communicated.

Symbols and Notations

ϕ	$\Leftrightarrow$	void set.
$\mathbb{R}$	$\Leftrightarrow$	Set of real numbers
$\mathbb{N}$	$\Leftrightarrow$	Set of natural numbers
$\mathbb{Z}$	$\Leftrightarrow$	Set of integers
$\mathbb{Q}$	$\Leftrightarrow$	Set of rational numbers
Y^X	$\Leftrightarrow$	collection of all functions from X to Y.
$x \in A$	$\Leftrightarrow$	x belongs to A
$x \notin A$	$\Leftrightarrow$	just negation of $x \in A$
$A \cup B$	$\Leftrightarrow$	Union of sets A and B
$A \cap B$	$\Leftrightarrow$	Intersection of sets A and B
$A - B$	$\Leftrightarrow$	complement of B in A
$A \subseteq B$	$\Leftrightarrow$	A is a subset of B
$A \subsetneqq B$	$\Leftrightarrow$	A is a proper subset of B

CHAPTER

1

Introduction

This Chapter deals with some of the basic concepts of the theory of fuzzy topological spaces. The essence of this Chapter is to throw some light on the literature of fuzzy topology, which will be quite helpful to study this thesis. Throughout the thesis, X will denote a non empty set and I denotes the unit closed interval $[0, 1]$. A fuzzy set in X, is a function with domain X and values in I, that is, a fuzzy set is an element of I^X. Let $A \in I^X$. The subset of X in which A assumes non zero values is known as the support of A. For every $x \in X$, $A(x)$ is called the grade of membership of x in A. If A takes only values 0 and 1, then A is called a crisp set. Let $r \in I$, then the constant fuzzy set which takes the value r at every $x \in X$, is denoted by $\overline{r}$.

Let $A, B \in I^X$. Then we have the following definitions:

1. A is contained in B, denoted by $A \leq B$, if $A(x) \leq B(x)$ for all $x \in X$.
2. The union of A and B, denoted by $A \vee B$, is a fuzzy set defined by $(A \vee B)(x) = \max\{A(x), B(x)\}$ for all $x \in X$.
3. The intersection of A and B, denoted by $A \wedge B$, is a fuzzy set defined by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for all $x \in X$.
4. The complement of A, denoted by A^c, is a fuzzy set defined by $A^c(x) = 1 - A(x)$ for all $x \in X$.
5. Let $f : X \to Y$ be a map, $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y, defined by

$$\begin{aligned} f(A)(y) &= sup\{A(x) : x \in f^{-1}(y)\} \text{ if } f^{-1}(y) \neq \phi \\ &= 0 \text{ if } f^{-1}(y) = \phi \end{aligned}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x))$ for all $x \in X$.

A family δ of fuzzy sets in X is called a fuzzy topology on X if it satisfies the following conditions:

$i)$ $\overline{0}, \overline{1} \in \delta$,

$ii)$ $A,\ B \in \delta \Rightarrow A \wedge B \in \delta$ and

$iii)$ for any family $\{A_j \in \delta;\ j \in J\}$, $\underset{j \in J}{\vee} A_j \in \delta$

and the ordered pair $(X,\ \delta)$ is called a fuzzy topological space (FTS). The members of δ are called fuzzy open set. Complement of a fuzzy open set is called a fuzzy closed set. In a fuzzy topological space, the closure $\overline{A}$ and interior $\overset{o}{A}$ of a fuzzy set A

are defined as $\overline{A} = \wedge\{C; A \leq C$ and $C^c \in \delta\}$ and $\overset{o}{A} = \vee\{O; O \leq A$ and $O \in \delta\}$.

A fuzzy set in X is called a fuzzy point if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$, the fuzzy point is denoted by p_x^λ. A fuzzy point p_x^λ is said to belong to a fuzzy set A if $\lambda < A(x)$ and is denoted by $p_x^\lambda \in A$. A fuzzy point p_x^λ is said to be quasi-coincident with A, denoted by $p_x^\lambda qA$, if $\lambda + A(x) > 1$. Two fuzzy sets A and B are said to be quasi-coincident with each other, denoted by AqB, if there exists $x \in X$ such that $A(x) + B(x) > 1$. The fact that A and B are not quasi-coincident, is denoted by $A\overline{q}B$. Two fuzzy sets A and B are said to be strongly quasi-discoincident if they are not quasi-coincident with each other and if $A(x) + B(x) = 1$ for some $x \in X$, then either $A(x) = 1$ or $B(x) = 1$.

In a fuzzy topological space, a fuzzy set A is said to be a neighbourhood (Q-neighbourhood) of a fuzzy set B if there exists a fuzzy open set O such that $B \leq O \leq A$ $(BqO \leq A)$.

Many mathematicians studied separation axioms in a fuzzy topological space. A fuzzy topological space is said to be fuzzy T_0 if for any two distinct fuzzy points p_x^λ and p_y^μ, there exists a Q-neighbourhood of one of them which is not quasi-coincident with the other. A fuzzy topological space, in which every fuzzy point is closed, is called a fuzzy T_1 space. A fuzzy topological space is said to be fuzzy T_2 if for any two distinct fuzzy points p_x^λ and p_y^μ with p_y^μ not contained in p_x^λ, there exist neighbourhoods

U, V of p_x^λ and $p_y^{1-\mu}$ respectively, which are not quasi-coincident with each other. A fuzzy topological space is said to be fuzzy regular if for any fuzzy point p_x^λ and its any open neighbourhood A, there exists a fuzzy set B such that $p_x^\lambda \in \overset{o}{B} \leq \overline{B} \leq A$. A fuzzy topological space is said to be fuzzy normal if for any closed fuzzy set C and its any open neighbourhood B, there exists a fuzzy set A such that $C \leq \overset{o}{A} \leq \overline{A} \leq B$.

An operator $\psi : I^X \to I^X$ is a fuzzy closure operator if

(i) $\psi(\overline{0}) = \overline{0}$.

(ii) $A \leq \psi(A)$ for all $A \in I^X$.

(iii) For all A, $B \in I^X$, $\psi(A \vee B) = \psi(A) \vee \psi(B)$.

(iv) $\psi(\psi(A)) = \psi(A)$ for all $A \in I^X$.

An operator $\varphi : I^X \to I^X$ is a fuzzy interior operator if

(i) $\varphi(\overline{1}) = \overline{1}$.

(ii) $\varphi(A) \leq A$ for all $A \in I^X$.

(iii) For all A, $B \in I^X$, $\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B)$.

(iv) $\varphi(\varphi(A)) = \varphi(A)$ for all $A \in I^X$.

It is easily seen that, in a fuzzy topological space, the closure and interior of a fuzzy set are respectively closure and interior operators. Thus, with a fuzzy topology, we can associate a fuzzy closure operator and a fuzzy interior operator as well. Also, given a fuzzy closure (respectively a fuzzy interior) operator, we can associate a fuzzy topology in the following way: If ψ be a fuzzy closure (respectively, φ a fuzzy interior) operator on X, then the associated fuzzy topology is given by $\delta(\psi) = \{A^c; \psi(A) = A\}$

(respectively, $\delta(\varphi) = \{A; \varphi(A) = A\}$). Further, these associations are reflexive in the sense that the associated fuzzy topology of closure (respectively interior) of a fuzzy set in some fuzzy topology δ is δ itself, and that the fuzzy closure (respectively fuzzy interior) operator associated with the fuzzy topology of some closure operator ψ (respectively fuzzy interior operator φ) is ψ (respectively φ) itself.

A mapping g from a fuzzy topological space (X, δ) to another fuzzy topological space (Y, η), is said to be fuzzy continuous if the inverse image of every fuzzy open set in Y, is fuzzy open in X. In a fuzzy topology, constant maps may not fuzzy continuous. The necessary and sufficient condition for a constant mapping $g : (X, \delta) \to (Y, \eta)$ to be fuzzy continuous is that δ contains all the constant fuzzy sets.

Compactness has also been studied in case of a fuzzy topological space. A family of fuzzy sets $\{A_i; i \in I\}$ is called a cover of a fuzzy set A if $A \leq \bigvee_{i\in I} A_i$ and a cover of a fuzzy set is called an open cover if its every member is open. A fuzzy topological space (X, δ) is said to be fuzzy compact if every open cover of $\overline{1}$ has a finite subcollection covering $\overline{1}$. Continuous image of a fuzzy compact space is fuzzy compact but unlike in a general topological space, an arbitrary product of fuzzy compact spaces may not be fuzzy compact.

Many more developments have taken place in the theory of fuzzy topological spaces, but as the things we stated so far on fuzzy topology is sufficient to provide a background to study this thesis, we conclude this introductory Chapter here.

CHAPTER

2

Basic Concepts of Fuzzy Sequential Topological Spaces

A sequence of fuzzy sets in X is called a fuzzy sequential set or an fs-set in X. We denote the fs-sets by the symbols $A_f(s)$, $B_f(s)$, $C_f(s)$ etc. For each $n \in \mathbb{N}$, A_f^n denotes the n^{th} term or component of an fs-set $A_f(s)$. Let $A_f(s)$ and $B_f(s)$ be fuzzy sequential sets or fs-sets in X, then we define

- $A_f(s) \ \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (union),
- $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (intersection),
- $A_f(s)$ is contained in $B_f(s)$, symbolically $A_f(s) \leq B_f(s)$, if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$,

- $A_f(s)$ is weakly contained in $B_f(s)$, symbolically $A_f(s) \leq_w B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$,
- $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$,
- $A_f(s)(x) = \{A_f^n(x)\}_n$, $x \in X$,
- Let $M \subseteq \mathbb{N}$. Then, $A_f(s)(x) \geq_M r$ if and only if $A_f^n(x) \geq r_n$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular, if $M = \mathbb{N}$, we write $A_f(s)(x) \geq r$,
- $X_f^l(s) = \{X_f^n\}_n$, where $l \in I$ and $X_f^n = \bar{l}$ for all $n \in \mathbb{N}$,
- $A_f^c(s) = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$,
- A fuzzy sequential set $P_f(s)$ is called a fuzzy sequential point or an fs-point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that for all $n \in \mathbb{N}$,

$$\begin{aligned} P_f^n(t) &= r_n, \text{ if } t = x, \\ &= 0, \text{ if } t \in X - \{x\}. \end{aligned}$$

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$\begin{aligned} P_f^n(x) &= r_n, \text{ whenever } n \in M, \\ &= 0, \text{ whenever } n \in \mathbb{N} - M. \end{aligned}$$

The point x is called the support, M is called the base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (p^M_{fx},\, r)$. Further, if $M = \{n\}$, then the fs-point is called a simple fs-point and it is denoted by $(p^n_{fx},\, r_n)$. An fs-point is called complete if its base is the set of natural numbers. An fs-point $P_f(s) = (p^M_{fx},\, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$, symbolically $P_f(s) \in_w A_f(s)$, if and only if there exists $n \in M$ such that $P^n_f(x) \leq A^n_f(x)$. If $R \subseteq M$ and s is the sequence in I same to r in R and vanishes outside R, then the fs-point $P_{rf}(s) = (p^R_{fx},\, s)$ is called a reduced fs-point of $P_f(s) = (p^M_{fx},\, r)$. A sequence $(x, L) = \{S_n\}_n$ of subsets of X, where $S_n = \{x\}$ for all $n \in L$ and $S_n = \phi$ for all $n \in \mathbb{N} - L$, is called a sequential point in X.

2.1 Fuzzy Sequential Topological Space

Definition 2.1.1 *A family $\delta(s)$ of fuzzy sequential sets on a set X satisfying the properties*

i) $X^r_f(s) \in \delta(s)$ for all $r \in \{0,\, 1\}$,

ii) $A_f(s),\, B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and

iii) for any family $\{A_{jf}(s) \in \delta(s),\, j \in J\}$, $\underset{j \in J}{\vee} A_{jf}(s) \in \delta(s)$,

is called a fuzzy sequential topology on X and the ordered pair $(X, \delta(s))$ is called a fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets or fs-open sets. Complement of an open fuzzy sequential set is called a closed fuzzy sequential set or an fs-closed set.

Definition 2.1.2 *If $\delta_1(s)$ and $\delta_2(s)$ be two fuzzy sequential topologies on X such that $\delta_1(s) \subseteq \delta_2(s)$, then we say that $\delta_2(s)$ is finer than $\delta_1(s)$ or $\delta_1(s)$ is coarser than $\delta_2(s)$.*

Proposition 2.1.1 *If δ be a fuzzy topology on X, then $\delta^{\mathbb{N}}$ forms a fuzzy sequential topology on X.*

Proof. Let δ be a fuzzy topology on X. Now,
(i) $X_f^0(s), X_f^1(s) \in \delta^{\mathbb{N}}$, since $\overline{0}, \overline{1} \in \delta$.
(ii) Let $A_f(s)$, $B_f(s) \in \delta^{\mathbb{N}}$. Then $A_f^n, B_f^n \in \delta$ for all $n \in \mathbb{N}$ and hence $A_f^n \wedge B_f^n \in \delta$ for all $n \in \mathbb{N}$. Thus $A_f(s) \wedge B_f(s) \in \delta^{\mathbb{N}}$.
(iii) Let $\{A_{\lambda f}(s), \lambda \in \Lambda\}$ be a family of fs-sets in $\delta^{\mathbb{N}}$. Then for each $n \in \mathbb{N}$, $A_{\lambda f}^n \in \delta$ for all $\lambda \in \Lambda$ and hence $\underset{\lambda\in\Lambda}{\vee} A_{\lambda f}^n \in \delta$ for all $n \in \mathbb{N}$. Thus $\underset{\lambda\in\Lambda}{\vee} A_{\lambda f}(s) \in \delta^{\mathbb{N}}$. ■

Proposition 2.1.2 *If $(X,\ \delta(s))$ be an FSTS, then $(X,\ \delta_n)$ is a fuzzy topological space, where $\delta_n = \{A_f^n;\ A_f(s) \in \delta(s)\}$, $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Then,
(i) $\overline{0}, \overline{1} \in \delta_n$, since $X_f^0(s)$, $X_f^1(s) \in \delta(s)$.
(ii) Let $A, B \in \delta_n$. Then, there exist fs-sets $A_f(s), B_f(s) \in \delta(s)$ such that $A_f^n = A$ and $B_f^n = B$. Since $A_f(s) \wedge B_f(s) \in \delta(s)$, $A \wedge B \in \delta_n$, .
(iii) Let $\{A_\lambda; \lambda \in \Lambda\}$ be a family of fuzzy open sets in (X, δ_n). Then for each $\lambda \in \Lambda$, there exists an fs-set $A_{\lambda f}(s) \in \delta(s)$ having n^{th} component A_λ. Hence $\underset{\lambda\in\Lambda}{\vee} A_\lambda \in \delta_n$, since $\underset{\lambda\in\Lambda}{\vee} A_{\lambda f}(s) \in \delta(s)$. ■

Definition 2.1.3 *$(X,\, \delta_n)$ in the Proposition 2.1.2, is called the n^{th} component fuzzy topological space of the FSTS $(X,\, \delta(s))$.*

We may construct different fuzzy sequential topologies on X from a given fuzzy topology δ on X, $\delta^{\mathbb{N}}$ is the finest of all these fuzzy sequential topologies. Not only that, any fuzzy topology δ on X can be considered as a component of some fuzzy sequential topology on X. There are at least countably many fuzzy sequential topologies on X weaker than $\delta^{\mathbb{N}}$ of which δ is a component. One of them is $\delta'(s) = \{A_f^n(s);\, A_f^n = A$ for all $n \in \mathbb{N}$ and $A \in \delta\}$.

From Definition 2.1.3, it is obvious that if $A_f(s)$ be an fs-open (fs-closed) set in an FSTS $(X,\, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in $(X,\, \delta_n)$ but the converse may not necessarily be true, as shown by the following Example.

Example 2.1.1 *Let us take an FSTS $(X,\, \delta(s))$, where $\delta(s) = \{X_f^r(s),\, r \in I\}$. Let $\{r_n\}_n$ be a strictly increasing sequence in I and $A_f(s)$ be an fs-set in X such that $A_f^n = \overline{r_n}$ for all $n \in \mathbb{N}$. Clearly, for each $n \in \mathbb{N}$, A_f^n is an open fuzzy set in $(X,\, \delta_n)$ but $A_f(s)$ is not an open fuzzy sequential set in $(X,\, \delta(s))$.*

Definition 2.1.4 *Fuzzy sequential sets $A_f(s)$ and $B_f(s)$ are called quasi-coincident, denoted by $A_f(s)qB_f(s)$, if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$ for all $n \in \mathbb{N}$. We write $A_f(s)\overline{q}B_f(s)$ to mean that $A_f(s)$ and $B_f(s)$ are not quasi-coincident.*

Definition 2.1.5 *Fuzzy sequential sets $A_f(s)$ and $B_f(s)$ are called weakly quasi-coincident, denoted by $A_f(s)q_wB_f(s)$, if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$ for some $n \in \mathbb{N}$. We write $A_f(s)\overline{q}_wB_f(s)$ to mean that $A_f(s)$ and $B_f(s)$ are not weakly quasi-coincident.*

Definition 2.1.6 *A fuzzy sequential point $P_f(s) = (p_{fx}^M,\ r)$ is called quasi-coincident with $A_f(s)$, denoted by $P_f(s)qA_f(s)$, if $P_f^n(x) > (A_f^n)^c(x)$ for all $n \in M$. If $P_f(s)$ is not quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q}A_f(s)$.*

Definition 2.1.7 *A fuzzy sequential point $P_f(s) = (p_{fx}^M,\ r)$ is called weakly quasi-coincident with $A_f(s)$, denoted by $P_f(s)q_wA_f(s)$, if $P_f^n(x) > (A_f^n)^c(x)$ for some $n \in M$. If $P_f(s)$ is not weakly quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q}_wA_f(s)$. If $P_f^n(x) > (A_f^n)^c(x)$ for some $n \in L \subseteq M$, then we say that $P_f(s)$ is weakly quasi-coincident with $A_f(s)$ at the sequential point $(x,\ L)$.*

From Definition 2.1.4, it is clear that, if the fuzzy sequential sets $A_f(s)$ and $B_f(s)$ are quasi-coincident, then each pair of non zero fuzzy sets A_f^n and B_f^n are also so but the converse need not be true, as shown by the following Example.

Example 2.1.2 *Let $A_f(s)$ and $B_f(s)$ be fuzzy sequential sets on $\mathbb{R}$, where*

$$\begin{aligned} A_f^1(x) &= \frac{2}{3},\ x \in (-\infty,\ 0), \\ &= \frac{1}{3},\ x \in [0,\ \infty). \end{aligned}$$

$$\begin{aligned} A_f^2(x) &= \frac{1}{3},\ x \in (-\infty,\ 0), \\ &= \frac{2}{3},\ x \in [0,\ \infty). \end{aligned}$$

$$A_f^n(x) = \frac{3}{4} \textit{ for all } x \in \mathbb{R} \textit{ and } n \neq 1,\ 2.$$

$$\begin{aligned} B_f^1(x) &= \frac{1}{2},\ x \in (-\infty,\ 0), \\ &= \frac{2}{3},\ x \in [0,\ \infty). \end{aligned}$$

$$\begin{aligned} B_f^2(x) &= \frac{1}{4},\ x \in (-\infty,\ 0), \\ &= \frac{3}{7},\ x \in [0,\ \infty). \end{aligned}$$

$$B_f^n(x) = \frac{1}{2} \textit{ for all } x \in \mathbb{R} \textit{ and } n \neq 1,\ 2.$$

Clearly, $A_f^n q B_f^n$ *for all* $n \in \mathbb{N}$*. Since there does not exist any* $x \in X$ *such that* $A_f^n(x) + B_f^n(x) > 1$ *for all* $n \in \mathbb{N}$*, we have* $A_f(s)\overline{q}B_f(s)$*.*

Remark 2.1.1 *An fs-point* $P_f(s) = (p_{fx}^M,\ r)$ *is quasi-coincident with an fs-set* $A_f(s)$ *if and only if* P_f^n *and* A_f^n *are so for each* $n \in M$*.*

Definition 2.1.8 *An fs-set* $A_f(s)$ *in an FSTS* $(X,\ \delta(s))$ *is called a neighbourhood (in short nbd) of an fs-point* $P_f(s)$ *if there exists an fs-set* $B_f(s) \in \delta(s)$ *such that* $P_f(s) \in B_f(s) \leq A_f(s)$*.*

Definition 2.1.9 *An fs-set* $A_f(s)$ *in an FSTS* $(X,\ \delta(s))$ *is called a weak nbd of an fs-point* $P_f(s)$ *if there exists an fs-set* $B_f(s) \in \delta(s)$ *such that* $P_f(s) \in_w B_f(s) \leq A_f(s)$*.*

Definition 2.1.10 *An fs-set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a Q-nbd of an fs-point $P_f(s)$ if there exists an fs-set $B_f(s) \in \delta(s)$ such that $P_f(s)qB_f(s) \leq A_f(s)$.*

Definition 2.1.11 *An fs-set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a weak Q-nbd of an fs-point $P_f(s)$ if there exists an fs-set $B_f(s) \in \delta(s)$ such that $P_f(s)q_wB_f(s) \leq A_f(s)$.*

Proposition 2.1.3 *$A_f(s) \leq_w (\leq)B_f(s)$ if and only if $A_f(s)$ and $B^c_f(s)$ are not (weakly) quasi-coincident. In particular, $P_f(s) \in_w (\in)A_f(s)$ if and only if $P_f(s)$ is not (weakly) quasi-coincident with $A^c_f(s)$.*

Proof. We will only prove that $A_f(s) \leq_w B_f(s)$ if and only if $A_f(s)$ and $B^c_f(s)$ are not quasi-coincident.

$$\begin{aligned} & A_f(s) \leq_w B_f(s) \\ \Leftrightarrow\ & \exists\, n \in \mathbb{N} \text{ such that } A^n_f(x) \leq B^n_f(x) \text{ for all } x \in X \\ \Leftrightarrow\ & A_f(s)\overline{q}B^c_f(s) \end{aligned}$$

■

Proposition 2.1.4 *Let $\{A_{jf}(s),\ j \in J\}$ be a family of fuzzy sequential sets in X. Then, a fuzzy sequential point $P_f(s)\ q_w\ (\bigvee\limits_{j\in J} A_{jf}(s))$ if and only if $P_f(s)q_wA_{jf}(s)$ for some $j \in J$.*

Proof. Let $P_f(s) = (p^M_{fx}, r)$ and S^n_f denotes the n^{th} component of $\underset{j\in J}{\vee} A_{jf}(s)$. Suppose $P_f(s) q_w (\underset{j\in J}{\vee} A_{jf}(s))$. This implies,

$$P^k_f(x) + S^k_f(x) > 1$$

for some $k \in M$. Therefore,

$$S^k_f(x) = 1 - P^k_f(x) + \varepsilon_k, \text{ where } \varepsilon_k > 0 \tag{2.1.1}$$

Also,

$$S^k_f(x) - \varepsilon_k < A^k_{jf}(x) \text{ for some } j \in J \tag{2.1.2}$$

From 2.1.1 and 2.1.2 we have $p^k_f(x) + A^k_{jf}(x) > 1$, that is, $P_f(s)\ q_w\ A_{jf}(s)$ for some $j \in J$. Other implication is straightforward. ■

Corollary 2.1.1 *Let $\{A_{jf}(s),\ j \in J\}$ be a family of fuzzy sequential sets in X. If $P_f(s) q A_{jf}(s)$ for some $j \in J$, then $P_f(s) q$ $(\underset{j\in J}{\vee} A_{jf}(s))$ but not conversely.*

Proof. Proof of the first part is omitted. For second part, let $A_{jf}(s)$ $(j = 1, 2)$ be fuzzy sequential sets in $\mathbb{R}$, where

$$\begin{aligned} A^1_{1f}(x) &= 0 \text{ for all } x \in \mathbb{R} - (0, 1), \\ &= \frac{1}{4} \text{ for all } x \in (0, 1). \end{aligned}$$

$$\begin{aligned} A^2_{1f}(x) &= 0 \text{ for all } x \in \mathbb{R} - (\frac{1}{3}, \frac{2}{3}), \\ &= \frac{2}{3} \text{ for all } x \in (\frac{1}{3}, \frac{2}{3}). \end{aligned}$$

$$A^n_{1f}(x) = 0 = A^n_{f2}(x) \text{ for all } x \in \mathbb{R},\ n \neq 1,\ 2$$

$$\begin{aligned} A^1_{2f}(x) &= 0 \text{ for all } x \in \mathbb{R} - (-\frac{1}{2},\ 1), \\ &= \frac{1}{3} \text{ for all } x \in (-\frac{1}{2},\ 1). \end{aligned}$$

$$\begin{aligned} A^2_{2f}(x) &= 0 \text{ for all } x \in \mathbb{R} - (-\frac{1}{2},\ 2), \\ &= \frac{1}{5} \text{ for all } x \in (-\frac{1}{2},\ 2). \end{aligned}$$

Consider the fuzzy sequential point $P_f(s) = (p^M_{f0.5},\ r)$, where $M = \{1,\ 2\}$, $r = \{r_n\}_n$ with $r_1 = r_2 = \frac{7}{10}$ and $r_n = 0$ for all $n \neq 1, 2$. Then, $P_f(s)$ is quasi-coincident with $A_{1f}(s) \vee A_{2f}(s)$ but it is not so with any one of them. ■

Definition 2.1.12 *A subfamily β of a fuzzy sequential topology $\delta(s)$ on X is called a basis for $\delta(s)$ if to every $A_f(s) \in \delta(s)$, there exists a subfamily $\{B_{jf}(s),\ j \in J\}$ of β such that $A_f(s) = \underset{j\in J}{\vee} B_{jf}(s)$.*

Definition 2.1.13 *A subfamily $S = \{S_{\lambda f}(s);\ \lambda \in \Lambda\}$ of a fuzzy sequential topology $\delta(s)$ on X is called a sub-basis or subbase for $\delta(s)$ if the collection $\{\underset{j\in J}{\wedge} S_{jf}(s);\ J$ is a finite subset of $\Lambda\}$ forms a basis for $\delta(s)$.*

Theorem 2.1.1 *A subfamily β of a fuzzy sequential topology $\delta(s)$ on X is a basis for $\delta(s)$ if and only if for each fuzzy sequential point $P_f(s)$ in $(X,\ \delta(s))$ and for every open weak Q-nbd $A_f(s)$ of*

$P_f(s)$, there exists a member $B_f(s) \in \beta$ such that $P_f(s)q_w B_f(s) \leq A_f(s)$.

Proof. The necessary part is straightforward. To prove its sufficiency, if possible let β be not a basis for $\delta(s)$. Then, there exists a member $A_f(s) \in \delta(s) - \beta$ such that $O_f(s) = \vee\{B_f(s) \in \beta;\ B_f(s) \leq A_f(s)\} \neq A_f(s)$, and hence there is an $x \in X$ and an $M \subseteq \mathbb{N}$ such that $O_f^n(x) < A_f^n(x)$, for all $n \in M$. Let $r = \{r_n\}_n$, where $r_n = 1 - O_f^n(x) > 0$ whenever $n \in M$ and $r_n = 0$ whenever $n \in \mathbb{N} - M$. Then, $A_f^n(x) + r_n > O_f^n(x) + r_n = 1$ for all $n \in M$. Thus, the fuzzy sequential point $P_f(s) = (p_{fx}^M, r)q_w A_f(s)$. Therefore, $A_f(s)$ is an open weak Q-nbd of $P_f(s)$. Now, any $B_f(s) \in \beta$ and $B_f(s) \leq A_f(s)$ implies $B_f(s) \leq O_f(s)$, that is, $P_f(s)\overline{q_w}B_f(s)$, which is a contradiction. Hence the proof. ■

Proposition 2.1.5 *If β be a basis for a fuzzy sequential topology $\delta(s)$ on X, then $\beta_n = \{B_f^n;\ B_f(s) \in \beta\}$ will form a basis for the component fuzzy topology δ_n on X for each $n \in \mathbb{N}$ but not conversely.*

Proof. Proof of the first part is straightforward. For the converse part, we consider the FSTS $(\mathbb{R}, \delta^{\mathbb{N}})$, where $\delta = \{\overline{r};\ r \in [0, 1]\}$, which is a fuzzy topology on $\mathbb{R}$. For each $n \in \mathbb{N}$, $\beta_n = \{\overline{r};\ r \in (0, 1) \cap \mathbb{Q}\}$ is a basis for the component fuzzy topology $\delta_n^{\mathbb{N}} = \delta$ of $\delta^{\mathbb{N}}$, but, $\beta(s) = \{X_f^r(s);\ r \in (0, 1) \cap \mathbb{Q}\}$ is not a basis for the fuzzy sequential topology $\delta^{\mathbb{N}}$, as the fs-open set $A_f(s)$ of $(\mathbb{R}, \delta^{\mathbb{N}})$, with $A_f^n = \overline{(\frac{1}{n})}$ for all $n \in \mathbb{N}$, can not be written as a union of a subfamily of $\beta(s)$. ■

Definition 2.1.14 *Let $A_f(s)$ be any fuzzy sequential set in an FSTS $(X,\ \delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of an fs-set $A_f(s)$ are defined as*

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \leq C_f(s),\ C_f^c(s) \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \vee\{O_f(s);\ O_f(s) \leq A_f(s),\ O_f(s) \in \delta(s)\}.$$

Proposition 2.1.6 *Let $(X,\ \delta(s))$ be an FSTS. If for an fs-set $A_f(s)$ in X, $\overline{A_f(s)} = \{\overline{A_f^n}\}_n$, then $cl(A_f^n) \leq \overline{A_f^n}$ for each $n \in \mathbb{N}$, where $cl(A_f^n)$ is the closure of A_f^n in $(X,\ \delta_n)$.*

Proof. For each $n \in \mathbb{N}$, A_f^n is contained in the n^{th} component of $\overline{A_f(s)}$ and hence the result. ■

Here we cite an example, where the equality in the proposition 2.1.6 does not hold. Let $X = [0,\ 1]$ and $\delta(s) = \{X_f^r(s);\ r \in [0,\ 1]\}$. Let $A_f(s) = (p^{\mathbb{N}}_{f\frac{1}{3}},\ r)$, with $r = \{\frac{1}{2} - \frac{1}{3n}\}_n$. Then $\overline{A_f(s)} = X_f^{\frac{1}{2}}(s)$. Here $cl(A_f^n) = \overline{(\frac{1}{2} - \frac{1}{3n})}$, whereas $\overline{A_f^n} = \overline{(\frac{1}{2})}$.

Definition 2.1.15 *The dual of a fuzzy sequential point $P_f(s) = (p^M_{fx},\ r)$ is a fuzzy sequential point $P_{df}(s) = (p^M_{fx},\ t)$, where $r = \{r_n\}_n$, $t = \{t_n\}_n$ and*

$$\begin{aligned} t_n &= 1 - r_n \text{ for all } n \in M, \\ &= 0 \text{ for all } n \in \mathbb{N} - M. \end{aligned}$$

Theorem 2.1.2 *Every Q-nbd of a fuzzy sequential point $P_f(s)$ is weakly quasi-coincident with a fuzzy sequential set $A_f(s)$ implies $P_f(s) \in \overline{A_f(s)}$ implies every weak Q-nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident.*

Proof. Let $P_f(s) = (p_{fx}^M, r)$. Then, $P_f(s) \in \overline{A_f(s)}$ if for every closed fuzzy sequential set $C_f(s) \geq A_f(s)$, $P_f(s) \in C_f(s)$, that is, $P_f^n(x) \leq C_f^n(x)$ for all $n \in M \Longrightarrow P_f(s) \in \overline{A_f(s)}$ if for every open fuzzy sequential set $B_f(s) \leq A_f^c(s)$, $B_f^n(x) \leq 1 - P_f^n(x)$ for all $n \in M$; that is, $P_f(s) \in \overline{A_f(s)}$ if for every open fuzzy sequential set $B_f(s)$ satisfying $B_f^n(x) > 1 - P_f^n(x)$ for all $n \in M$, $B_f(s) \nleq A_f^c(s)$, which implies the first part. For the second part, let $P_f(s) \in \overline{A_f(s)}$. If possible, let there exists a weak Q-nbd $N_f(s)$ of $P_f(s)$ such that $N_f(s)\overline{q_w}A_f(s)$. Then, there exists an open fuzzy sequential set $B_f(s)$ such that $P_f(s)q_wB_f(s) \leq N_f(s)$. $N_f(s)\overline{q_w}A_f(s)$ and $B_f(s) \leq N_f(s)$ implies $B_f^n(x) + A_f^n(x) \leq 1$ for all $x \in X$ and $n \in \mathbb{N}$, that means, $A_f(s) \leq B_f^c(s)$ and hence $P_f(s) \in B_f^c(s)$. This contradicts the fact that $P_f(s)q_wB_f(s)$. Hence the result follows. ■

Corollary 2.1.2 *A fuzzy sequential point $P_f(s) \in \overline{A_f(s)}$ if and only if each nbd of its dual point $P_{df}(s)$ is weakly quasi-coincident with $A_f(s)$.*

Proof. Proof is straightforward, since Q-nbd of a fuzzy sequential point is exactly the nbd of its dual point. ■

Theorem 2.1.3 *A fuzzy sequential point* $P_f(s) \in \overset{o}{A}_f(s)$ *if and only if its dual point* $P_{df}(s) \notin \overline{A^c_f(s)}$.

Proof. Let $P_f(s) \in \overset{o}{A}_f(s)$. Then, there exists an open fuzzy sequential set $B_f(s)$ such that $P_f(s) \in B_f(s) \leq A_f(s)$, which implies $B_f(s)$ and $A^c_f(s)$ are not weakly quasi-coincident. Hence $P_{df}(s) \notin \overline{A^c_f(s)}$. Conversely, let $P_{df}(s) \notin \overline{A^c_f(s)}$. Then, there exists an open nbd $B_f(s)$ of $P_f(s)$ which is not weakly quasi-coincident with $A^c_f(s)$. This implies $P_f(s) \in B_f(s) \leq A_f(s)$ and hence $P_f(s) \in \overset{o}{A}_f(s)$. ■

Proposition 2.1.7 *In an FSTS* $(X,\delta(s))$, *the following hold:*

(*i*) $\overline{X^r_f(s)} = X^r_f(s)$, $r \in \{0, 1\}$,

(*ii*) $A_f(s)$ *is closed if and only if* $\overline{A_f(s)} = A_f(s)$,

(*iii*) $\overline{\overline{A_f(s)}} = \overline{A_f(s)}$,

(*iv*) $\overline{A_f(s) \vee B_f(s)} = \overline{A_f(s)} \vee \overline{B_f(s)}$,

(*v*) $\overline{A_f(s) \wedge B_f(s)} \leq \overline{A_f(s)} \wedge \overline{B_f(s)}$,

(*vi*) $(X^r_f(s))^o = X^r_f(s)$, $r \in \{0, 1\}$,

(*vii*) $A_f(s)$ *is open if and only if* $\overset{o}{A}_f(s) = A_f(s)$,

(*viii*) $(\overset{o}{A}_f(s))^o = \overset{o}{A}_f(s)$,

(*ix*) $(A_f(s) \wedge B_f(s))^o = \overset{o}{A}_f(s) \wedge \overset{o}{B}_f(s)$,

(*x*) $\overset{o}{A}_f(s) \vee \overset{o}{B}_f(s) \leq (A_f(s) \vee B_f(s))^o$,

(*xi*) $\overset{o}{A}_f(s) = (\overline{A^c_f(s)})^c$,

(*xii*) $\overline{A_f(s)} = ((A^c_f(s))^o)^c$,

(*xiii*) $(\overline{A_f(s)})^c = (A^c_f(s))^o$,

(*xiv*) $\overline{(A^c_f(s))} = (\overset{o}{A}_f(s))^c$.

Proof. Proof is omitted. ■

Definition 2.1.16 *A fuzzy sequential point $P_f(s)$ is called an adherence point of a fuzzy sequential set $A_f(s)$ if every weak Q-nbd of $P_f(s)$ is weakly quasi-coincident with $A_f(s)$.*

Definition 2.1.17 *A fuzzy sequential point $P_f(s)$ is called an accumulation point of a fuzzy sequential set $A_f(s)$ if $P_f(s)$ is an adherence point of $A_f(s)$ and every weak Q-nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident at some fuzzy sequential point having different base or support from that of $P_f(s)$ whenever $P_f(s) \in A_f(s)$.*

Proposition 2.1.8 *Any reduced fs-point of an accumulation point of a fuzzy sequential set is also an accumulation point of it.*

Proof. Let $P_f(s)$ be an accumulation point of an fs-set $A_f(s)$. Since every weak Q-nbd of a reduced fs-point of $P_f(s)$ is a weak Q-nbd of $P_f(s)$ itself, the result follows. ■

That the converse of proposition 2.1.8 may not true, is shown by the following Example.

Example 2.1.3 *Let $X = \{a,\ b\}$, $\delta(s) = \{X_f^0(s), X_f^1(s), G_f(s)\}$, where $G_f^n(a) = \frac{1}{2}$, $G_f^n(b) = 0$ for all $n \in \mathbb{N}$. Let $A_f(s)$ be an fs-set such that $A_f^n = \overline{(\frac{2}{3})}$ for $n = 1,\ 2$ and $A_f^n = \overline{0}$ otherwise. Then the fuzzy sequential point $P_f(s) = (p_{fa}^M,\ r)$, where $M = \{1, 2\}$*

and $r = \{r_n\}_n$ *with* $r_1 = r_2 = \frac{2}{3}$ *and* $r_n = 0$ *otherwise, is not an accumulation point of* $A_f(s)$ *although* $(p^1_{fa}, \frac{2}{3})$ *and* $(p^2_{fa}, \frac{2}{3})$ *both are accumulation points of* $A_f(s)$.

Definition 2.1.18 *The union of all accumulation points of a fuzzy sequential set* $A_f(s)$ *is called the fuzzy derived sequential set of* $A_f(s)$ *and it is denoted by* $\overset{d}{A}_f(s)$.

Theorem 2.1.4 *In an FSTS* $(X, \delta(s))$, $\overline{A_f(s)} = A_f(s) \vee \overset{d}{A}_f(s)$.

Proof. Let $P_f(s) \in \overline{A_f(s)}$. If $P_f(s) \notin A_f(s)$, then $P_f(s)$ is an accumulation point of $A_f(s)$ and hence belongs to $\overset{d}{A}_f(s)$. Thus ,

$$\overline{A_f(s)} \leq A_f(s) \vee \overset{d}{A}_f(s). \tag{2.1.3}$$

Again, $A_f(s) \leq \overline{A_f(s)}$ and since any accumulation point of $A_f(s)$ belongs to $\overline{A_f(s)}$. Therefore ,

$$A_f(s) \vee \overset{d}{A}_f(s) \leq \overline{A_f(s)}. \tag{2.1.4}$$

From 2.1.3 and 2.1.4, the result follows. ■

Corollary 2.1.3 *A fuzzy sequential set is closed in an FSTS if and only if it contains all its accumulation points.*

Proof. Proof is straightforward. ■

Remark 2.1.2 *The fuzzy derived sequential set of a fuzzy sequential set may not be closed, as shown by the following Example.*

Example 2.1.4 *Let* $X = \{a,\, b\}$ *and* $\delta(s)$ *be the fuzzy sequential topology having basis* $\beta = \{X_f^1(s), X_f^0(s)\} \cup \{P_f(s),\, G_f(s)\}$, *where* $G_f^n(b) = 1\ \forall\ n \in \mathbb{N}$, $G_f^3(a) = 0$, $G_f^n(a) = 1\ \forall\ n \neq 3$ *and* $P_f(s) = (p_{fa}^M,\, r)$, *where* $M = \{1,\, 2,\, 3\}$, $r = \{r_n\}_n$ *with* $r_1 = 0.5$, $r_2 = 1$, $r_3 = 0.3$, $r_n = 0\ \forall\ n \neq 1,\, 2,\, 3$. *Here the fuzzy derived sequential set of* $(p_{fa}^3,\, 0.3)$ *is* $(p_{fa}^3,\, 0.7)$, *which is not closed.*

Proposition 2.1.9 *The fuzzy derived sequential set of a fuzzy sequential point with a finite base, equals the union of the fuzzy derived sequential sets of all its simple reduced fuzzy sequential points.*

Proof. Let $P_f(s) = (p_{fx}^M, r)$ be a fuzzy sequential point, where M is finite. Let $D_f(s)$ be its fuzzy derived sequential set and $D_{nf}(s)$ be the fuzzy derived sequential set of the simple reduced fuzzy sequential point $(p_{fx}^n,\, r_n)$ for each $n \in M$. Since every accumulation point of $(p_{fx}^n,\, r_n)$ is an accumulation point of $P_f(s)$, we have

$$D_{nf}(s) \leq D_f(s) \text{ for all } n \in M$$

$$\Rightarrow \bigvee_{n \in M} D_{nf}(s) \leq D_f(s)$$

Again, suppose $Q_f(s)$ be an accumulation point of $P_f(s)$ and assume that $Q_f(s)$ is not an accumulation point of $(p_{fx}^n,\, r_n)$ for all $n \in M$. Then for each $n \in M$, there exists a weak Q-nbd $B_{nf}(s)$ of $Q_f(s)$ which is not weakly quasi coincident with $(p_{fx}^n,\, r_n)$. So,

$(p^n_{fx},\, r_n)$ is not weakly quasi coincident with $B_f(s) = \bigwedge_{n\in M} B_{nf}(s)$ for all $n \in M$ and hence $P_f(s)$ and $B_f(s)$ are not weakly quasi coincident. But $B_f(s)$ is a weak Q-nbd of $Q_f(s)$. Hence our assumption must be wrong. Thus,

$$D_f(s) = \bigvee_{n\in M} D_{nf}(s)$$

■

Proposition 2.1.10 *If the fuzzy derived sequential set of each of the simple reduced fuzzy sequential points of a fuzzy sequential point, having a finite base, is closed, then the fuzzy derived sequential set of the fuzzy sequential point is closed.*

Proof. Let $A_f(s) = (p^M_{fx},\, r)$ be a fuzzy sequential point, where M is finite. Let $D_f(s)$ be its fuzzy derived sequential set and Let $D_{nf}(s)$ be the fuzzy derived sequential set of $A_{nf}(s) = (p^n_{fx},\, r_n)$, $n \in M$. Suppose $D_{nf}(s)$ is closed for all $n \in M$. Let $P_f(s)$ be an accumulation point of $D_f(s)$.

Now, $P_f(s) \notin D_f(s)$

$\Rightarrow P_f(s)$ is not an accumulation point of $A_f(s)$

$\Rightarrow \exists$ a weak Q-nbd $B_f(s)$ of $P_f(s)$ which is not weakly quasi coincident with $A_f(s)$

$\Rightarrow B_f(s)$ is not weakly quasi coincident with $(p^n_{fx},\, r_n)\ \forall\ n \in M$.

$\Rightarrow P_f(s) \notin D_{nf}(s)\ \forall\ n \in M$

$\Rightarrow P_f(s)$ is not an accumulation point of $D_{nf}(s)\ \forall\ n \in M$ (since $D_{nf}(s)$ is closed $\forall\ n \in M$)

$\Rightarrow P_f(s)$ is not an accumulation point of $\underset{n\in M}{\vee} D_{nf}(s) = D_f(s)$, a contradiction. Hence proved. ■

Remark 2.1.3 *Converse of proposition* 2.1.10 *is not always true, as shown by the next Example.*

Example 2.1.5 *Let* $X = \{a,\ b\}$ *and* $\delta(s)$ *be the fuzzy sequential topology having basis* $\beta = \{X_f^1(s), X_f^0(s)\} \cup \{P_f(s),\ G_f(s)\}$, *where* $G_f^n(b) = 1\ \forall\ n \in \mathbb{N}$, $G_f^3(a) = 0$, $G_f^n(a) = 1\ \forall\ n \neq 3$ *and* $P_f(s) = (p_{fa}^M,\ r)$, *where* $M = \{1,\ 2,\ 3\}$, $r = \{r_n\}_n$ *with* $r_1 = 0.5$, $r_2 = 1$, $r_3 = 0.3$, $r_n = 0\ \forall\ n \neq 1,\ 2,\ 3$. *Here the fuzzy derived sequential set of* $P_f(s)$ *is* $X_f^1(s)$, *which is closed, whereas, the fuzzy derived sequential set of* $(p_{fa}^3,\ 0.3)$ *is* $(p_{fa}^3,\ 0.7)$, *which is not closed.*

Lemma 2.1.1 *Let* $A_f(s) = (p_{fx}^M,\ r)$ *be a fuzzy sequential point in an FSTS* $(X,\ \delta(s))$. *Then,*

(i) For $y \neq x$, $\overline{A_f(s)}(y) = \overset{d}{A}_f(s)(y)$.

(ii) If $\overline{A_f(s)}(x) >_P r$, $\overline{A_f(s)}(x) =_P \overset{d}{A}_f(s)(x)$, *where* $P \subseteq M$.

(iii) If $\overline{A_f(s)}(x) >_M r$, $\overline{A_f(s)}(x) = \overset{d}{A}_f(s)(x)$.

(iv) If $\overset{d}{A}_f(s)(x) = 0 =$ *sequence of real zeros, then* $\overline{A_f(s)}(x) = r$.

(v) If $A_f(s)$ *is simple, then converse of (iv) is true.*

Proof. (i), (ii), (iii) and (iv) follows from Theorem 2.1.4. To prove (v), let $A_f(s) = (p_{fx}^k,\ r)$, where $r = \{r_n\}_n$ with $r_n = 0$ for all $n \neq k$. We will show that no fuzzy sequential point having support x can be an accumulation point of $A_f(s)$. Let if possible, suppose $Q_f(s) = (p_{fx}^N, t)$ be an accumulation point of

$A_f(s)$. If $Q_f(s)$ is contained in $A_f(s)$, then it must be a simple fuzzy sequential point having base k and support x, but in that case, any weak Q-nbd of $Q_f(s)$ and $A_f(s)$ cannot be weakly quasi coincident at a fuzzy sequential point having different base or different support than that of $Q_f(s)$. Again, due to the fact that $\overline{A_f(s)}(x) = r$, if it is not contained in $A_f(s)$, then $Q_f(s) \notin \overline{A_f(s)}$, a contradiction. Hence $\overset{d}{A}_f(s)(x) = 0$. ■

Note 2.1.1 *Let $A_f(s)$ be a fuzzy sequential set in a set X, $x \in X$ and $r = \{r_n\}_n$ be a sequence in I. Then, $A_f(s)(x) = 1 - r$ means for each $n \in \mathbb{N}$, $A_f^n(x) = 1 - r_n$ and $A_f(s)(x) = 1$ means $A_f^n(x) = 1$ for all $n \in \mathbb{N}$.*

Lemma 2.1.2 *Let $A_f(s) = (p_{fx}^k, r_k)$ be a simple fuzzy sequential point in an FSTS $(X, \delta(s))$. Then,*
(i) If $\overset{d}{A}_f(s)(x)$ is a non zero sequence, then $\overline{A_f(s)} = \overset{d}{A}_f(s)$.
(ii) If $\overset{d}{A}_f(s)(x) = 0 =$ sequence of real zeros, then $\overset{d}{A}_f(s)$ is closed if and only if $\exists$ an fs-open set $B_f^{@}(s)$ such that $B_f^{@}(s)(x) = 1$ and for $y \neq x$, $B_f^{@}(s)(y) = (\overline{A_f(s)})^c(y) = (\overset{d}{A}_f(s))^c(y)$.
(iii) $A_f^d(s)(x) = 0 =$ sequence of real zeros if and only if $\exists$ an fs-open set $B_f(s)$ such that $B_f(s)(x) = 1 - r$, where $r = \{r_n\}_n$ with $r_n = 0$ if $n \neq k$ and $r_n = r_k$ if $n = k$.

Proof. (i) Follows from (i) and (v) of Lemma 2.1.1.
(ii) Suppose $\overset{d}{A}_f(s)$ is closed, then we take $B_f^{@}(s) = (\overset{d}{A}_f(s))^c$. Conversely, suppose the given condition holds, then $\overset{d}{A}_f(s) =$

$(B_f^{@}(s))^c$.

(iii) Let $\overset{d}{A}_f(s)(x) = 0$, then $\overline{A_f(s)}(x) = r$. Now, if we take $B_f(s) = (\overline{A_f(s)})^c$, the necessary part will be proved. Conversely, suppose there exists an fs-open set $B_f(s)$ satisfying the given condition. Then, $(B_f(s))^c$ is an fs-closed set containing $A_f(s)$ and hence contains $\overline{A_f(s)}$. Thus, $\overline{A_f(s)}(x) = r$ and therefore $A_f^d(s)(x) = 0$. ■

We conclude the chapter by proving a variant of Yang's Theorem [19] from general topology.

Theorem 2.1.5 *In an FSTS, the fuzzy derived sequential set of each fuzzy sequential set is closed if and only if the fuzzy derived sequential set of each simple fuzzy sequential point is closed.*

Proof. The necessary part is obvious. For the converse part, suppose $H_f(s)$ be a fuzzy sequential set. We will show that $\overset{d}{H}_f(s) = D_f(s)$ (say) is closed. Let $P_f(s) = (p_{fx}^k, r_k)$ be an accumulation point of $D_f(s)$. It is sufficient to show that $P_f(s) \in D_f(s)$. Let $r = \{r_n\}_n \in I$, where $\underline{r_n = r_k}$ for $n = k$ and $r_n = 0 \;\forall\; n \neq k$. Now $P_f(s) \in \overline{D_f(s)} = \overset{d}{H}_f(s) \leq \overline{\overline{H_f(s)}} = \overline{H_f(s)}$. Therefore $P_f(s)$ is an adherence point of $H_f(s)$. If $P_f(s) \notin H_f(s)$, then $P_f(s)$ is an accumulation point of $H_f(s)$, that is, $P_f(s) \in D_f(s)$ and we are done. Let us assume $P_f(s) \in H_f(s)$. Then,

$$\begin{aligned} & r \leq H_f(s)(x) = \rho \text{ (say)} \\ \Rightarrow\; & r_k \leq H_f^k(x) = \rho_k, \end{aligned}$$

where ρ_k is the k^{th} term of the sequence ρ. Now consider the simple fuzzy sequential point $A_f(s) = (p^k_{fx}, \rho_k)$. Let $\rho' = \{\rho'_n\}_n$, where $\rho'_k = \rho_k$ and $\rho'_n = 0 \; \forall \; n \neq k$. There are two possibilities concerning $\overset{d}{A}_f(s)$.

Case I. Suppose $A^d_f(s)(x) = \{\rho''_n\}_n$ is a non zero sequence. Now, $\overline{A_f(s)}(x) \geq A_f(s)(x) = \rho'$. By (i) of Lemma 2.1.2, $\overset{d}{A}_f(s) = \overline{A_f(s)}$ and by (v) of Lemma 2.1.1, $\overline{A_f(s)}(x) \neq \rho'$. Hence,

$$\Rightarrow \rho''_k > \rho_k = A^k_f(x) = H^k_f(x)$$

Hence, the simple fuzzy sequential point $Q_f(s) = (p^k_{fx}, \rho''_k) \notin H_f(s)$. But since $Q_f(s) \in \overset{d}{A}_f(s) \leq \overline{A_f(s)} \leq \overline{H_f(s)}$, $Q_f(s)$ is an accumulation point of $H_f(s)$, that is, $Q_f(s) \in D_f(s)$. Moreover,

$$\begin{aligned} & r_k \leq \rho_k < \rho''_k \\ \Rightarrow \; & r_k < \rho''_k \\ \Rightarrow \; & P_f(s) \in D_f(s) \end{aligned}$$

Case II. Suppose $\overset{d}{A}_f(s)(x) = 0$. Let $B_f(s)$ be an arbitrary weak Q-nbd of $P_f(s)$. In view of Lemma 2.1.2 (ii), $\exists$ an fs-open set $B^@_f(s)$ such that $B^@_f(s)(x) = 1$ and for $y \neq x$, $B^@_f(s)(y) = (\overline{A_f(s)})^c(y)$. Let $C_f(s) = B_f(s) \wedge B^@_f(s)$. Then, $C_f(s)(x) = B_f(s)(x)$ which implies $C^k_f(x) = B^k_f(x) > 1 - r_k$. Thus, $C_f(s)$ is a weak Q-nbd of $P_f(s)$. Hence, $C_f(s)$ and $D_f(s)$ are weakly quasi-coincident, that is, $\exists$ a point $z \in X$ and $n \in \mathbb{N}$ such that

$D_f^n(z) + C_f^n(z) > 1$. Owing to the fact that $D_f(s)$ is the union of all the accumulation points of $H_f(s)$, $\exists$ an accumulation point $P'_f(s) = (p^n_{fz}, \mu_n)$ of $H_f(s)$ such that $\mu_n + C_f^n(z) > 1$. Therefore, $C_f(s)$ is a weak Q-nbd of $P'_f(s)$. Again, by Lemma 2.1.2 (iii), there exists an fs-open set $B'_f(s)$ such that $B'_f(s)(x) = 1 - \rho'$. The proof will be carried out, according to the following subcases:

Subcase I. When $n = k$.

(a) If $z = x$ and $\mu_k \leq \rho'_k$, then $P'_f(s) \in H_f(s)$. Since $P'_f(s)$ is an accumulation point of $H_f(s)$, every weak Q-nbd of $P'_f(s)$ (and hence $B_f(s)$) and $H_f(s)$ are weakly quasi coincident at some fuzzy sequential point having different base or different support than that of $P_f(s)$.

(b) If $z = x$ and $\mu_k > \rho'_k$, then $P'_f(s) \notin H_f(s)$. Therefore, $G_f(s) = C_f(s) \wedge B'_f(s)$ is a weak Q-nbd of $P'_f(s)$ and hence is weakly quasi-coincident with $H_f(s)$. Since $G_f(s)(x) \leq B'_f(s)(x) = 1 - \rho'$, we have $G_f^k(x) \leq B'^k_f(x) = 1 - \rho_k$. Thus, $G_f(s)$ (and hence $B_f(s)$) and $H_f(s)$ are weakly quasi coincident at some fuzzy sequential point having different base or different support than that of $P_f(s)$.

(c) If $z \neq x$, we have $B_f^{@}(s)(z) = (\overline{A_f(s)})^c(z)$. Also, $(\overline{A_f(s)})^c = (A_f^c(s))^\circ$. Since $(A_f^c(s))^\circ(z) = B_f^{@}(s)(z) \geq C_f(s)(z)$, $\exists$ an fs-open set $B''_f(s) \leq A_f^c(s)$ such that $B''^k_f(z) \geq C_f^k(z) > 1 - \mu_k$. Therefore, $G'_f(s) = B_f(s) \wedge B''_f(s)$ is a weak Q-nbd of $P'_f(s)$ and hence is weakly quasi coincident with $H_f(s)$. Since $B''_f(s) \leq A_f^c(s)$, we have $B''_f(s)(x) \leq 1 - A_f(s)(x)$ and hence $B''^k_f(x) \leq 1 - A_f^k(x) = 1 - H_f^k(x)$. Thus, $G'_f(s)$ (and hence $B_f(s)$) is weakly quasi coincident with $H_f(s)$ at some fuzzy sequential point having different base or different support than that of $P_f(s)$.

Subcase II. When $n \neq k$.

(a) Suppose $z = x$. We have $B'_f(s)(x) = 1 - \rho'$ which implies $B'^n_f(x) = 1 > 1 - \mu_n$. Thus, $G_f(s) = C_f(s) \wedge B'_f(s)$ is a weak Q-nbd of $P'_f(s)$ and hence is weakly quasi coincident with $H_f(s)$. Now,

$$\begin{aligned} & G_f(s)(x) \leq B'_f(s)(x) = 1 - \rho' \\ \Rightarrow\ & G^k_f(x) \leq 1 - \rho_k = 1 - H^k_f(x). \end{aligned}$$

So, $G_f(s)$ (and hence $B_f(s)$) is weakly quasi-coincident with $H_f(s)$ at some fuzzy sequential point having different base or different support than that of $P_f(s)$.

(b) When $z \neq x$, the proof is similar to Subcase I (c). ■

CHAPTER

3

Separation Axioms

After the initiation of fuzzy topology by C. L. Chang in 1968, a number of works have been done in fuzzy topology. One of them is the study of separation axioms. This Chapter mainly deals with the separation axioms in our setting of fuzzy sequential topology.

3.1 Separation Axioms in a Fuzzy Sequential Topological Space

Definition 3.1.1 *Two fuzzy sequential points* $P_f(s) = (p^M_{fx},\ r)$ *and* $Q_f(s) = (p^N_{fy},\ t)$ *are said to be identical if* $x = y$, $M = N$ *and* $r = t$*; otherwise they are distinct.*

Definition 3.1.2 *A set* $M \subseteq \mathbb{N}$ *is said to be the base of a fuzzy sequential set* $U_f(s)$ *if* $U^n_f \neq \overline{0}\ \forall n \in M$ *and* $U^n_f = \overline{0}\ \forall n \in \mathbb{N} - M$.

Definition 3.1.3 *A fuzzy sequential set $B_f(s)$ (having base N) is said to be completely contained in a fuzzy sequential set $A_f(s)$ (having base M) if $M = N$ and $B_f^n \leq A_f^n$ for all $n \in N$.*

Definition 3.1.4 *A fuzzy sequential set $B_f(s)$ (having base N) is said to be totally reduced from the fuzzy sequential set $A_f(s)$ (having base M) if $N \subsetneqq M$ and $B_f^n \leq A_f^n$ $\forall n \in N$.*

Definition 3.1.5 *An FSTS $(X, \delta(s))$ is said to be an fs-T_0 space if for any two distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$, there exist a weak Q-nbd of one of $P_f(s)$ and $Q_f(s)$ which is not weakly quasi-coincident with the other.*

Theorem 3.1.1 *An FSTS $(X, \delta(s))$ is an fs-T_0 space if and only if for every pair of distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$, either $P_f(s)$ does not belong to the closure of $Q_f(s)$ or $Q_f(s)$ does not belong to the closure of $P_f(s)$.*

Proof. Suppose $(X, \delta(s))$ be fs-T_0. Then, $\exists$ a weak Q-nbd $U_f(s)$ of $P_f(s)$ (say) which is not weakly quasi-coincident with $Q_f(s)$. This implies that $P_f(s) \notin \overline{Q_f(s)}$. Conversely, suppose $P_f(s)$ and $Q_f(s)$ be any two distinct fuzzy sequential points such that $P_f(s) \notin \overline{Q_f(s)}$. This implies that $\exists$ a weak Q-nbd of $P_f(s)$ which is not weakly quasi-coincident with $Q_f(s)$. Hence $(X, \delta(s))$ is fs-T_0. ■

Corollary 3.1.1 *An FSTS $(X, \delta(s))$ is fs-T_0 space if and only if distinct fuzzy sequential points have distinct closures.*

Theorem 3.1.2 *A fuzzy topology $(X,\, \delta)$ is fuzzy T_0 if and only if the fuzzy sequential topology $(X,\, \delta^{\mathbb{N}})$ is fs-T_0.*

Proof. Suppose $(X,\, \delta)$ be fuzzy T_0. Let $P_f(s) = (p_{fx}^M,\, r)$ and $Q_f(s) = (p_{fy}^N,\, t)$ be any two distinct fuzzy sequential points where $r = \{r_n\}_n$ and $t = \{t_n\}_n$.

Case I. Suppose $x \neq y$. Then, for $p_x^{r_m} \neq p_y^{t_m}$ $(m \in M)$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.

Case II. Suppose $x = y$, $M \cap N = \phi$. Then for $p_x^{r_m} \neq p_x^{t_m}$ $(m \in M$ $)$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.

Case III. Suppose $x = y$, $N \subseteq M$. If $r_m \neq t_m$ for some $m \in N$, then for $p_x^{r_m} \neq p_x^{t_m}$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$. If $r_n = t_n$ $\forall$ $n \in N$, then for $p_x^{r_m} \neq p_x^{t_m}$ $(m \in M - N)$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.

Case IV. Suppose $x = y$ and neither $N \subseteq M$ nor $M \subseteq N$ nor $M \cap N = \phi$. Then, for $p_x^{r_m} \neq p_x^{t_m}$ $(m \in M,\, m \notin N$ $)$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.

In all the above cases, the fuzzy sequential set $U_f(s)$ where $U_m = U$ and $U_n = \overline{0}$ $\forall n \neq m$, is a weak Q-nbd of $P_f(s)$ which is not weakly quasi-coincident with $Q_f(s)$.

Conversely, suppose $(X,\, \delta^{\mathbb{N}})$ is fs-T_0. Let p_x^{λ} and p_y^{μ} be any two distinct fuzzy points in X. Then, for each $n \in \mathbb{N}$, fuzzy sequential points $(p_{fx}^n,\, \lambda)$ and $(p_{fy}^n,\, \mu)$ are distinct. So $\exists$ a weak Q-nbd $U_f(s)$ of one of $(p_{fx}^n,\, \lambda)$ and $(p_{fy}^n,\, \mu)$ which is not weakly quasi-coincident with the other. This implies, U_f^n is a Q-nbd of one of p_x^λ and p_y^μ which is not quasi-coincident with the other . ■

Theorem 3.1.3 *If an FSTS $(X,\ \delta(s))$ is fs-T_0, then the fuzzy topological space $(X,\ \delta_n)$ is fuzzy T_0 for each $n \in \mathbb{N}$, where $\delta_n = \{A_f^n;\ A_f(s) \in \delta(s)\}$.*

Proof. Suppose $(X,\ \delta(s))$ be fs-T_0. Let p_x^λ and p_y^μ be any two distinct fuzzy points in X. Then for each $n \in \mathbb{N}$, fuzzy sequential points $(p_{fx}^n,\ \lambda)$ and $(p_{fy}^n,\ \mu)$ are distinct. So $\exists$ a weak Q-nbd $U_f(s)$ of one of $(p_{fx}^n,\ \lambda)$ and $(p_{fy}^n,\ \mu)$ which is not weakly quasi-coincident with the other. This implies, U_f^n is a Q-nbd of one of p_x^λ and p_y^μ which is not quasi-coincident with the other . ■

Converse of Theorem 3.1.3 may not be true, as shown by the following Example.

Example 3.1.1 *Let $(X,\ \delta)$ be a fuzzy topological space. For any $A \in \delta$, let us consider the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$, where $B_{fA}^n = A$ for odd n, $B_{fA}^n = \overline{0}$ for even n, $C_{fA}^n = \overline{0}$ for odd n, $C_{fA}^n = A$ for even n and $D_{fA}^n = A$ $\forall n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$ for all $A \in \delta$, forms a fuzzy sequential topology on X. If $(X,\ \delta)$ is fuzzy T_0, then the components of $(X,\ \delta(s))$ are fuzzy T_0 but $(X,\ \delta(s))$ is not fs-T_0.*

Definition 3.1.6 *Suppose $U_f(s)$ and $V_f(s)$ be two fuzzy sequential sets. If there exists an $M \subseteq \mathbb{N}$ such that $U_f^n q V_f^n$ $\forall n \in$*

M, we say that $U_f(s)$ is M-quasi coincident with $V_f(s)$ and we write $U_f(s)q^M V_f(s)$. If $U_f^n q V_f^n$ for at least one $n \in M$, we say that $U_f(s)$ is weakly M-quasi coincident with $V_f(s)$ and we write $U_f(s)q_w^M V_f(s)$.

Definition 3.1.7 *An FSTS $(X, \delta(s))$ is said to be an fs-T_1 space if every fuzzy sequential point in X is closed.*

Remark 3.1.1 *An fs-T_1 space is fs-T_0.*

Theorem 3.1.4 *A fuzzy topological space (X, δ) is fuzzy T_1 if and only if the fuzzy sequential topological space $(X, \delta^{\mathbb{N}})$ is fs-T_1.*

Proof. Proof is omitted. ■

Theorem 3.1.5 *If an FSTS $(X, \delta(s))$ is fs-T_1, then the component fuzzy topological space (X, δ_n) is fuzzy T_1 for each $n \in \mathbb{N}$.*

Proof. Proof is omitted. ■

The next example shows that the converse of Theorem 3.1.5 may not true.

Example 3.1.2 *Let (X, δ) be a fuzzy topological space. For any $A \in \delta$, let us consider the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$, where $B_{fA}^n = A$ for odd n, $B_{fA}^n = \overline{0}$ for even n, $C_{fA}^n = \overline{0}$ for odd n, $C_{fA}^n = A$ for even n and $D_{fA}^n = A$ $\forall n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$*

and $D_{fA}(s)$ for all $A \in \delta$, forms a fuzzy sequential topology on X. If $(X,\, \delta)$ is fuzzy T_1 then the components of $(X,\, \delta(s))$ are fuzzy T_1 but $(X,\, \delta(s))$ is not fs-T_1.

Theorem 3.1.6 *An FSTS $(X,\, \delta(s))$ is fs-T_1 if and only if for each $x \in X$ and each sequence $r = \{r_n\}_n$ in I, $\exists$ $B_f(s) \in \delta(s)$ such that $B_f(s)(x) = 1 - r$ and $B_f(s)(y) = 1$ for $y \neq x$.*

Proof. Suppose $(X,\, \delta(s))$ be fs-T_1. If r is a zero sequence, then it is sufficient to take $B_f(s) = X_f^1(s)$. Suppose r is a non zero sequence. Let $M \subseteq \mathbb{N}$ such that $r_n \neq 0$ $\forall n \in M$ and $r_n = 0$ $\forall n \in \mathbb{N} - M$. If $P_f(s) = (p_{fx}^M,\, r)$, then $B_f(s) = X_f^1(s) - P_f(s)$ is the required open fuzzy sequential set.

Conversely, suppose $P_f(s) = (p_{fx}^M,\, r)$ be an arbitrary fuzzy sequential point in X. By the given condition, there exists an open fuzzy sequential set $B_f(s)$ in X such that $B_f(s)(x) = 1 - r$ and $B_f(s)(y) = 1$ for $y \neq x$. It follows that $P_f(s)$ is the complement of $B_f(s)$ and hence is closed. ∎

Theorem 3.1.7 *The fuzzy derived sequential set of every fuzzy sequential set on an fs-T_1 space is closed.*

Proof. The fuzzy derived sequential set of a fuzzy sequential point in an fs-T_1 space is a fuzzy sequential point and hence is closed. Thus the result follows from Theorem 2.1.5. ∎

Definition 3.1.8 *An FSTS $(X,\, \delta(s))$ is said to be an fs-Hausdorff or an fs-T_2 space if for any two distinct fuzzy sequential points*

$P_f(s) = (p^M_{fx},\ r)$ and $Q_f(s) = (p^N_{fy},\ t)$, none of which is completely contained in the other, $\exists$ open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s)q_w^{M-N}U_f(s),\ Q_f(s)q_wV_f(s),\ P_f(s)\ \overline{q}_w^{M-N}\ \overline{V_f(s)},$$
$$\text{and } Q_f(s)\ \overline{q}_w\ \overline{U_f(s)},$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$; otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s)q_wU_f(s),\ Q_f(s)q_wV_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)},\ Q_f(s)\ \overline{q}_w\ \overline{U_f(s)}.$$

Definition 3.1.9 *An FSTS $(X, \delta(s))$ is said to be a weak fs-Hausdorff space or (w) fs-Hausdorff space if for any two distinct fuzzy sequential points $P_f(s) = (p^M_{fx},\ r)$ and $Q_f(s) = (p^N_{fy},\ t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that*

$$P_f(s) \in_w^{M-N} U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_wV_f(s),$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$; otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) \in_w U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_wV_f(s).$$

Theorem 3.1.8 *An fs-Hausdorff space is a weak fs-Hausdorff space.*

Proof. Proof is omitted. ■

Example 3.1.3 shows that a weak fs-Hausdorff space may not be an fs-Hausdorff space.

Example 3.1.3 *Let* $X = \{x,\, y\}$, $\delta = \{\overline{r},\, r \in [0,\, 1]\}$. *Then* $(X,\, \delta^{\mathbb{N}})$ *is weak fs-Hausdorff but not fs-Hausdorff.*

An fs-T_2 space may not be an fs-T_1 space, as shown by the following Example.

Example 3.1.4 *Let* $X = \{x,\, y\}$, $\delta = \{\overline{1},\, \overline{0},\, p_x^1,\, p_y^1\}$. *Fuzzy sequential topological space* $(X,\, \delta^{\mathbb{N}})$ *is fs-*T_2 *but not fs-*T_1.

Definition 3.1.10 *An FSTS* $(X,\, \delta(s))$ *is said to be a weak fs-*T_2 *or* (w) *fs-*T_2 *space if it is* (w) *fs-Hausdorff and fs-*T_1.

Remark 3.1.2 *An fs-*T_2 *space is weak fs-*T_2.

Theorem 3.1.9 *An FSTS* $(X,\, \delta(s))$ *is fs-Hausdorff if and only if for any two distinct fuzzy sequential points* $P_f(s) = (p_{fx}^M,\, r)$ *and* $Q_f(s) = (p_{fy}^N,\, t)$, *none of which is completely contained in the other, there exist open fuzzy sequential sets* $G_f(s)$, $H_f(s)$, $D_f(s)$ *and* $E_f(s)$ *in* $(X,\, \delta(s))$ *such that*

$$P_f(s) \in^{M-N} G_f(s),\ Q_f(s) q_w H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s) q_w^{M-N} D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s),$$

whenever $Q_f(s)$ *is a totally reduced fuzzy sequential point from* $P_f(s)$; *otherwise there exist open fuzzy sequential sets* $G_f(s)$, $H_f(s)$,

$D_f(s)$ and $E_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) \in G_f(s),\ Q_f(s) q_w H_f(s),\ G_f(s)\overline{q_w}H_f(s),$$
$$P_f(s) q_w D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s)\overline{q_w}D_f(s).$$

Proof. Let $P_f(s) = (p^M_{fx},\ r)$ and $Q_f(s) = (p^N_{fy},\ t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other. Suppose $(X,\ \delta(s))$ is fs-Hausdorff.

Case I. Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. Then, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) q_w U_f(s),\ Q_f(s) q_w V_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)},\ Q_f(s)\ \overline{q}_w\ \overline{U_f(s)}.$$

If we take $G_f(s) = X^1_f(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X^1_f(s) - \overline{U_f(s)}$, we are done.

Case II. Suppose one of $P_f(s)$ and $Q_f(s)$, say $Q_f(s)$ is totally reduced from $P_f(s)$. Then, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) q_w^{M-N} U_f(s),\ Q_f(s)\ q_w\ V_f(s),\ P_f(s)\ \overline{q}_w^{M-N}\ \overline{V_f(s)},$$
$$\text{and } Q_f(s)\ \overline{q}_w\ \overline{U_f(s)}.$$

Now if we take $G_f(s) = X^1_f(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X^1_f(s) - \overline{U_f(s)}$, we are done.

Conversely, suppose the given conditions are true. In both the cases, if we take $U_f(s) = D_f(s)$, $V_f(s) = H_f(s)$ and use the Definition 3.1.8, we are done. ■

Theorem 3.1.10 *An FSTS* $(X,\, \delta(s))$ *is fs-Hausdorff if and only if for any fuzzy sequential point* $P_f(s)$ *in* X,

$$P_f(s) = \wedge\{\overline{N_f(s)};\ N_f(s) \textit{ is a nbd of } P_f(s)\} \quad (3.1.1)$$

Proof. Suppose $(X,\, \delta(s))$ be fs-Hausdorff. Let $P_f(s) = (p^M_{fx},\, r)$ be any fuzzy sequential point in X and $Q_f(s) = (p^N_{fx},\, t)$ be another fuzzy sequential point distinct from $P_f(s)$ and $Q_f(s) \notin P_f(s)$.

If $P_f(s)$ is totally reduced from $Q_f(s)$, there exist an open fuzzy sequential set $V_f(s)$ in $(X,\, \delta(s))$ such that

$$Q_f(s)\ q_w^{N-M}\ V_f(s),\ P_f(s)\ \overline{q}_w \overline{V_f(s)}\ ;$$

otherwise there exist an open fuzzy sequential set $V_f(s)$ in $(X,\, \delta(s))$ such that

$$Q_f(s)\ q_w\ V_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)}.$$

In both the cases, if we take $U_f(s) = X^1_f(s) - \overline{V_f(s)}$, then $P_f(s) \in U_f(s)$ and $Q_f(s) \notin \overline{U_f(s)}$. Hence 3.1.1 is true.

Conversely, suppose 3.1.1 holds. Let $P_f(s) = (p^M_{fx},\, r)$ and $Q_f(s) = (p^N_{fy},\, t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other.

Case I. Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. By 3.1.1, there exist nbds $S_f(s)$ and $T_f(s)$ of $P_f(s)$ and $Q_f(s)$ respectively such that

$$P_f(s) \notin \overline{T_f(s)} \text{ and } Q_f(s) \notin \overline{S_f(s)}.$$

Case II. Suppose one of $P_f(s)$ and $Q_f(s)$, $Q_f(s)$ (say) is totally reduced from $P_f(s)$. Then, there exist nbds $S_f(s)$ and $T_f(s)$ of $P'_f(s)$ and $Q_f(s)$ respectively such that $P_f(s) \notin^{M-N} \overline{T_f(s)}$ and $Q_f(s) \notin \overline{S_f(s)}$, where $P'_f(s)$ is a reduced fuzzy sequential point of $P_f(s)$ with base $M - N$.
In both of the above two cases, if we take $U_f(s) = X^1_f(s) - \overline{T_f(s)}$, $V_f(s) = X^1_f(s) - \overline{S_f(s)}$ and use the Definition 3.1.8, we are done. ■

Theorem 3.1.11 *If a fuzzy topological space (X, δ) is fuzzy T_2, then the FSTS $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff.*

Proof. Proof is omitted. ■

That the converse of Theorem 3.1.11 is not true, is shown by the following Example.

Example 3.1.5 *Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p^1_x, p^1_y\}$. Then, the fuzzy sequential topological space $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff although the fuzzy topological space (X, δ) is not fuzzy T_2.*

Example 3.1.6 shows that even if (X, δ) is fuzzy T_2, the FSTS $(X, \delta^{\mathbb{N}})$ may not be fs-Hausdorff.

Example 3.1.6 *Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$. Then, the fuzzy topological space (X, δ) is fuzzy T_2 but the FSTS $(X, \delta^{\mathbb{N}})$ is not fs-Hausdorff.*

Example 3.1.7 shows that if an FSTS $(X,\ \delta(s))$ is fs-T_2, then its component fuzzy topological space $(X,\ \delta_n)$ may not be fuzzy T_2 for each $n \in \mathbb{N}$.

Example 3.1.7 *Let* $X = \{x,\ y\}$, $\delta = \{\overline{1},\ \overline{0},\ p_x^1,\ p_y^1\}$. *Let* $\delta(s) = \delta^{\mathbb{N}}$, *then* $(X,\ \delta(s))$ *is fs-*T_2 *but* $(X,\ \delta_n) = (X,\ \delta)$ $(\forall n \in \mathbb{N})$ *is not fuzzy* T_2.

Example 3.1.8 shows that even if all the component fuzzy topological spaces of an FSTS are fuzzy T_2, the FSTS may not be fs-T_2.

Example 3.1.8 *Let* $(X,\ \delta)$ *be a fuzzy topological space. For any* $G \in \delta$, *consider the fs-sets* $A_{fG}(s)$, $B_{fG}(s)$, $C_{fG}(s)$ *where* $A_{fG}^n = G$ *for odd* n, $A_{fG}^n = \overline{0}$ *for even* n; $B_{fG}^n = \overline{0}$ *for odd* n, $B_{fG}^n = G$ *for even* n; $C_{fG}^n = G\ \forall n \in \mathbb{N}$. *Then the collection* $\delta(s)$ *of all the fs-sets* $A_{fG}(s)$, $B_{fG}(s)$, $C_{fG}(s)$ $\forall G \in \delta$, *forms a fuzzy sequential topology on* X. *If* $(X,\ \delta)$ *is fuzzy* T_2, *then the components of* $(X,\ \delta(s))$ *are fuzzy* T_2 *but* $(X,\ \delta(s))$ *itself is not fs-*T_2.

Definition 3.1.11 *An FSTS* $(X,\ \delta(s))$ *is said to be fs-regular if for any fuzzy sequential point* $P_f(s) = (p_{fx}^M,\ r)$ *and a non zero closed fuzzy sequential set* $A_f(s)$ *such that* $P_f(s) \notin A_f(s)$ *and* $A_f(s)$ *is not completely contained in* $P_f(s)$, *there exist open fuzzy sequential sets* $U_f(s)$ *and* $V_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$P_f(s)\ q_w^{M-N}\ U_f(s),\ \ A_f(s)\ q_w\ V_f(s),\ \ P_f(s)\ \overline{q}_w^{M-N}\ \overline{V_f(s)}$$
$$\text{and } A_f(s) \leq X_f^1(s) - \overline{U_f(s)},$$

whenever $A_f(s)$ *is totally reduced from* $P_f(s)$ *and has base* N*; otherwise there exist open fuzzy sequential sets* $U_f(s)$ *and* $V_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$P_f(s)\ q_w\ U_f(s),\ A_f(s)\ q_w\ V_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)},$$
$$\text{and } A_f(s) \leq X_f^1(s) - \overline{U_f(s)}.$$

Definition 3.1.12 *An FSTS* $(X,\ \delta(s))$ *is said to be weak fs-regular if for any fuzzy sequential point* $P_f(s) = (p_{fx}^M,\ r)$ *and its any open weak nbd* $A_f(s)$*, there exists a fuzzy sequential set* $B_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$P_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s),$$

whenever $A_f^c(s)$ *is totally reduced from* $P_f(s)$ *and has base* N*; otherwise there exists a fuzzy sequential set* $B_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$P_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s).$$

Example 3.1.9 shows that an fs-regular space may not be weak fs-regular.

Example 3.1.9 *Let* $X = \{a\}$, $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.4}\}$*. Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-regular but not weak fs-regular.*

A weak fs-regular space may not be fs-regular and it has been shown in the following example.

Example 3.1.10 *Let* $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,\ 1]\}$, *then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is weak fs-regular but not fs-regular.*

Example 3.1.11 shows that an fs-regular space may not be fs-T_1.

Example 3.1.11 *Let* $X = \{x\}$ *and let* $\delta = \{\overline{0},\ \overline{1},\ p_x^{0.5}\}$. *Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-regular but not fs-*T_1.

Definition 3.1.13 *An FSTS* $(X,\ \delta(s))$ *is said to be fs-*T_3 *if it is fs-regular and fs-*T_1.

Remark 3.1.3 *An fs-*T_3 *space is fs-*T_2.

Theorem 3.1.12 *An FSTS* $(X,\ \delta(s))$ *is fs-regular if and only if for any fuzzy sequential point* $P_f(s) = (p_{fx}^M,\ r)$ *and a non zero closed fuzzy sequential set* $A_f(s)$ *such that* $P_f(s) \notin A_f(s)$ *and* $A_f(s)$ *is not completely contained in* $P_f(s)$, *there exist open fuzzy sequential sets* $G_f(s)$, $H_f(s)$, $D_f(s)$ *and* $E_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$P_f(s) \in^{M-N} G_f(s),\ A_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s)\ q_w^{M-N} D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s),$$

whenever $A_f(s)$ *is totally reduced from* $P_f(s)$ *and has base* N*; otherwise there exist open fuzzy sequential sets* $G_f(s)$, $H_f(s)$, $D_f(s)$ *and* $E_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$P_f(s) \in G_f(s),\ A_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s)\ q_w\ D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s).$$

Proof. Proof is omitted. ■

Theorem 3.1.13 *An FSTS $(X,\ \delta(s))$ is fs-regular if and only if for any fuzzy sequential point $P_f(s) = (p^M_{fx},\ r)$ and an open fuzzy sequential set $C_f(s)$ such that $P_f(s)q_wC_f(s)$ (where $X^1_f(s) - C_f(s)$ is not completely contained in $P_f(s)$), there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\ \delta(s))$ such that*

$$P_f(s) \in^{M-N} O_f(s),\ \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s)q_w^{M-N}B_f(s),\ \overline{B_f(s)} \leq C_f(s),$$

whenever $X^1_f(s) - C_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) \in O_f(s),\ \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s)\ q_w\ B_f(s),\ \overline{B_f(s)} \leq C_f(s).$$

Proof. Suppose $(X,\ \delta(s))$ be fs-regular. Let $P_f(s) = (p^M_{fx},\ r)$ be any fuzzy sequential point and $C_f(s)$ be an open fuzzy sequential set such that $P_f(s)q_wC(s)$ i.e $P_f(s) \notin X^1_f(s) - C_f(s) = A_f(s)$ (say). Then, there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$, $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in^{M-N} G_f(s),\ A_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s)q_w^{M-N}D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s),$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$,

$D_f(s)$ and $E_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in G_f(s),\ A_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s)\ q_w\ D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s).$$

If we take $O_f(s) = G_f(s)$ and $B_f(s) = D_f(s)$, we are done.

Conversely, suppose the given conditions hold. Let $P_f(s) = (p^M_{fx},\, r)$ be any fuzzy sequential point and $A_f(s)$ be any closed fuzzy sequential set such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, that is, $P_f(s) q_w (X^1_f(s) - A_f(s)) = C_f(s)$ (say). Then, there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in^{M-N} O_f(s),\ \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s) q^{M-N}_w B_f(s),\ \overline{B_f(s)} \leq C_f(s),$$

whenever $X^1_f(s) - C_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in O_f(s),\ \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s)\ q_w\ B_f(s),\ \overline{B_f(s)} \leq C_f(s).$$

If we take $G_f(s) = O_f(s)$, $H_f(s) = X^1_f(s) - \overline{O_f(s)}$, $D_f(s) = B_f(s)$, $E_f(s) = X^1_f(s) - \overline{B_f(s)}$ and use Theorem 3.1.12, then we are done.

■

Theorem 3.1.14 *If $(X,\, \delta(s))$ is fs-regular, then for any closed fuzzy sequential set $A_f(s)$ which is not a fuzzy sequential point,*

$$A_f(s) = \wedge\{N_f(s);\ N_f(s) \text{ is a closed nbd of } A_f(s)\} \quad (3.1.2)$$

Proof. Suppose $(X,\ \delta(s))$ be fs-regular and $A_f(s)$ be any closed fuzzy sequential set which is not a fuzzy sequential point. If $A_f(s) = X_f^0(s)$, then 3.1.2 is true. Suppose $A_f(s) \neq X_f^0(s)$. Let $P_f(s)$ be any fuzzy sequential point such that $P_f(s) \notin A_f(s)$. Let M and N be the bases of $P_f(s)$ and $A_f(s)$ respectively. Then, $P_f(s) q_w (X_f^1(s) - A_f(s)) = G_f(s)$ (say) and hence there exists an open fuzzy sequential set $B_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) q_w^{M-N} B_f(s),\ \overline{B_f(s)} \leq G_f(s)$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$; otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s)\ q_w\ B_f(s),\ \overline{B_f(s)} \leq G_f(s).$$

This implies, $A_f(s) \leq X_f^1(s) - \overline{B_f(s)} = H_f(s)$ (say). Again, $P_f(s) \notin X_f^1(s) - B_f(s) \Longrightarrow P_f(s) \notin \overline{H_f(s)}$. Thus 3.1.2 holds. ■

Example 3.1.12 shows that converse of Theorem 3.1.14 may not be true.

Example 3.1.12 *Consider a set X and $\delta = \{\overline{r},\ r \in [0,\ 1]\}$. Then the FSTS $(X,\ \delta^{\mathbb{N}})$ is not fs-regular although for any closed fuzzy sequential set $A_f(s)$ in $(X,\ \delta^{\mathbb{N}})$, $A_f(s) = \wedge\{N_f(s);\ N_f(s)$ is a closed nbd of $A_f(s)\}$.*

Example 3.1.13 shows that for a fuzzy sequential point, Theorem 3.1.14 may not hold.

Example 3.1.13 *Let* $X = \{x\}$ *and* $\delta = \{\overline{1},\ \overline{0},\ p_x^{0.2}\}$. *Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-regular but for the closed fuzzy sequential point* $A_f(s) = (p_{fx}^{\{1,\ 2\}},\ 0.8)$, $A_f(s) \neq \wedge\{N_f(s);\ N_f(s)$ *is a closed nbd of* $A_f(s)\}$.

Theorem 3.1.15 *A fuzzy topological space* $(X,\ \delta)$ *is fuzzy regular if and only if* $(X,\ \delta^{\mathbb{N}})$ *is weak fs-regular.*

Proof. Proof is omitted. ■

Even if the FSTS $(X,\ \delta^{\mathbb{N}})$ is fs-regular, $(X,\ \delta)$ may not be fuzzy regular, as shown by Example 3.1.14.

Example 3.1.14 *Let* $X = \{a\}$, $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$. *Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-regular but* $(X,\ \delta)$ *is not fuzzy regular.*

If $(X,\ \delta)$ be a fuzzy regular space, then it may not imply $(X,\ \delta^{\mathbb{N}})$ is fs-regular, as shown by Example 3.1.15.

Example 3.1.15 *Let* $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,\ 1]\}$, *then the fuzzy topological space* $(X,\ \delta)$ *is fuzzy regular but* $(X,\ \delta^{\mathbb{N}})$ *is not fs-regular.*

The component fuzzy topological spaces $(X,\ \delta_n)$, $n \in \mathbb{N}$, of an fs-regular space $(X,\ \delta(s))$ may not be fuzzy regular. This is shown by Example 3.1.16.

Example 3.1.16 *Let* $X = \{a\}$, $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$. *Let* $\delta(s) = \delta^{\mathbb{N}}$, *then* $(X,\ \delta(s))$ *is fs-regular but* $(X,\ \delta_n) = (X,\ \delta)\ \forall n \in \mathbb{N}$, *is not fuzzy regular.*

An FSTS $(X,\ \delta(s))$ may not be fs-regular even if its each component fuzzy topological space $(X,\ \delta_n)$, $n \in \mathbb{N}$, is fuzzy regular, as shown by Example 3.1.17.

Example 3.1.17 *Let* $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,\ 1]\}$. *Let* $\delta(s) = \delta^{\mathbb{N}}$. *Then* $(X,\ \delta(s))$ *is not be fs-regular but each of its component fuzzy topological spaces* $(X,\ \delta_n)$, $n \in \mathbb{N}$, *is fuzzy regular.*

Definition 3.1.14 *Fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ *are said to be strong quasi-discoincident if* A_f^n *and* B_f^n *are strong quasi-discoincident* $\forall n \in \mathbb{N}$.

Definition 3.1.15 *Fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ *are said to be partially quasi-discoincident if* A_f^n *and* B_f^n *are strong quasi-discoincident for some* $n \in \mathbb{N}$.

Definition 3.1.16 *An FSTS* $(X,\ \delta(s))$ *is said to be fs-normal if for any two partially quasi-discoincident non zero closed fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ (*having the respective bases* M *and* N *and none of which is completely contained in the other*), *there exist open fuzzy sequential sets* $U_f(s)$ *and* $V_f(s)$ *in* $(X,\ \delta(s))$

such that

$$A_f(s) q_w^{M-N} U_f(s),\ B_f(s) q_w V_f(s),\ A_f(s) \leq^{M-N} X_f^1(s) - \overline{V_f(s)},$$
$$\text{and } B_f(s) \leq X_f^1(s) - \overline{U_f(s)},$$

whenever $B_f(s)$ *is totally reduced from* $A_f(s)$*; otherwise there exist open fuzzy sequential sets* $U_f(s)$ *and* $V_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$A_f(s)\ q_w\ U_f(s),\ B_f(s)\ q_w\ V_f(s),\ A_f(s) \leq X_f^1(s) - \overline{V_f(s)},$$
$$\text{and } B_f(s) \leq X_f^1(s) - \overline{U_f(s)}.$$

Definition 3.1.17 *An FSTS* $(X,\ \delta(s))$ *is said to be weak fs-normal if for any non zero closed fuzzy sequential set* $C_f(s)$ *and its any open weak nbd* $A_f(s)$*, there exists a fuzzy sequential set* $B_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$C_f(s) \in_w^{M-N} \overset{o}{B_f}(s) \leq \overline{B_f(s)} \leq A_f(s),$$

whenever $A_f^c(s)$ *is totally reduced from* $C_f(s)$*; otherwise there exists a fuzzy sequential set* $B_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$C_f(s) \in_w \overset{o}{B_f}(s) \leq \overline{B_f(s)} \leq A_f(s)$$

(M *and* N *being the respective bases of* $C_f(s)$ *and* $A_f^c(s)$).

An fs-normal space may not be weak fs-normal, which is shown by Example 3.1.18.

Example 3.1.18 *Let* $X = \{a\}$ *and* $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$*. Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-normal but not weak fs-normal.*

Example 3.1.19 shows that a weak fs-normal space may not be fs-normal.

Example 3.1.19 *Let* $X = \{x, y\}$ *and* $\beta = \{p_x^r;\, r \in [0, 1]\} \cup \{p_y^r;\, r \in [0, 1]\}$ *be a basis for some fuzzy topology* δ *on* X. *Then the FSTS* $(X, \delta^{\mathbb{N}})$ *is weak fs-normal but not fs-normal.*

Definition 3.1.18 *An FSTS* $(X, \delta(s))$ *is said to be an fs-*T_4 *space if it is fs-normal and fs-*T_1.

An fs-normal space may not be fs-T_1, as shown by Example 3.1.20.

Example 3.1.20 *Let* $X = \{a, b\}$ *and* $\delta(s) = \{X_f^0(s),\, X_f^1(s),\, A_f(s),\, B_f(s)\}$, *where* $\forall n \in \mathbb{N}$, $A_f^n(a) = 1$, $A_f^n(b) = 0$, $B_f^n(a) = 0$, $B_f^n(b) = 1$. *Then the FSTS* $(X, \delta(s))$ *is fs-normal but not fs-*T_1.

An fs-normal space may not be fs-regular as shown by Example 3.1.21.

Example 3.1.21 *Let* $X = \{x, y\}$ *and* $\delta = \{\overline{r},\, r \in [0, 1]\}$. *Then the FSTS* $(X, \delta^{\mathbb{N}})$ *is fs-normal but not fs-regular.*

Theorem 3.1.16 *An FSTS* $(X, \delta(s))$ *is fs-normal if and only if for any two partially quasi-discoincident non zero closed fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ *(none of which is completely contained in the other) there exist open fuzzy sequential sets* $G_f(s)$,

$H_f(s)$, $D_f(s)$ and $E_f(s)$ in $(X,\, \delta(s))$ such that

$$A_f(s) \in^{M-N} G_f(s),\ B_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$A_f(s) q_w^{M-N} D_f(s),\ B_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s),$$

whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$); otherwise there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$, $D_f(s)$ and $E_f(s)$in $(X,\, \delta(s))$ such that

$$A_f(s) \in G_f(s),\ B_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$A_f(s)\ q_w\ D_f(s),\ B_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s).$$

Proof. Proof is omitted. ∎

Theorem 3.1.17 *If an FSTS $(X,\, \delta(s))$ is weak fs-normal, then for any two non-zero closed partially quasi-discoincident fs-sets $A_f(s)$ and $B_f(s)$ (none of which is completely contained in the other), there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\, \delta(s))$ such that*

$$A_f(s) \in_w^{M-N} U_f(s),\ B_f(s) \in_w V_f(s),\ U_f(s)\ \overline{q_w}\ V_f(s),$$

whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$); otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\, \delta(s))$ such that

$$A_f(s) \in_w G_f(s),\ B_f(s) \in_w V_f(s),\ U_f(s)\ \overline{q_w}\ V_f(s).$$

Proof. The proof is omitted. ■

Remark 3.1.4 *For an FSTS to be weak fs-normal, the condition given in Theorem* 3.1.17*, is only necessary but not sufficient, as shown by Example* 3.1.22*.*

Example 3.1.22 *Let* $X = \{a\}$ *and* $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$*. Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is not weak fs-normal but the condition in Theorem* 3.1.17 *is satisfied.*

Theorem 3.1.18 *A weak fs-regular space is weak fs-normal, when* X *is finite.*

Proof. Let $(X,\ \delta(s))$ be a weak fs-regular space, where X is finite. Let $C_f(s)$ be any non zero closed fuzzy sequential set in $(X,\ \delta(s))$ and $A_f(s)$ be its any open weak nbd. Let M and N be respectively the bases of $C_f(s)$ and $A_f^c(s)$. We choose $m \in M - N$ when $A_f^c(s)$ is totally reduced from $C_f(s)$ and we take $m \in M$ otherwise. Let $x \in X$ such that $C_f^m(x) \neq 0$ and let $C_f^m(x) = r_m$. Then, for the fuzzy sequential point $P_{xf}(s) = (p_{fx}^m,\ r_m)$, $A_f(s)$ is an open weak nbd. Hence, there exists an open fuzzy sequential set $B_{xf}(s)$ in $(X,\ \delta(s))$ such that

$$P_{xf}(s) \in_w^{M-N} B_{xf}(s) \leq \overline{B_{xf}(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is a totally reduced fuzzy sequential set from $C_f(s)$; otherwise there exists an open fuzzy sequential set $B_{xf}(s)$ in $(X,\ \delta(s))$ such that

$$P_{xf}(s) \in_w B_{xf}(s) \leq \overline{B_{xf}(s)} \leq A_f(s)$$

Corresponding to each $x \in X$ for which $C_f^m(x) \neq 0$, we get such fs-open set $B_{xf}(s)$. Taking $X = \{x_1, x_2, ..., x_k\}$, we get fs-open sets $B_{x_1 f}(s)$, $B_{x_2 f}(s)$,..........., $B_{x_k f}(s)$ such that
$P_{x_n f}(s) \in_w^{M-N} B_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s),\ \forall n = 1, 2, .., k,$
whenever $A_f^c(s)$ is totally reduced from $C_f(s)$; otherwise

$$P_{x_n f}(s) \in_w B_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s).$$

Now, let $B_f(s) = \bigvee_{n=1}^{k} B_{x_n f}(s)$. Then

$$C_f(s) \in_w^{M-N} B_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is totally reduced from $C_f(s)$; otherwise

$$C_f(s) \in_w B_f(s) \leq \overline{B_f(s)} \leq A_f(s).$$

Hence $(X,\, \delta(s))$ is weak fs-normal.

■

Theorem 3.1.19 *A fuzzy topological space $(X,\, \delta)$ is fuzzy normal if and only if $(X,\, \delta^{\mathbb{N}})$ is weak fs-normal.*

Proof. Proof is omitted. ■

Fuzzy topological space $(X,\, \delta)$ may not be fuzzy normal even if $(X,\, \delta^{\mathbb{N}})$ is fs-normal, as shown by Example 3.1.23.

Example 3.1.23 *Let $X = \{a\}$ and $\delta = \{\overline{0},\, \overline{1},\, p_a^{0.6}\}$. Then the FSTS $(X,\, \delta^{\mathbb{N}})$ is fs-normal but $(X,\, \delta)$ is not fuzzy normal.*

Example 3.1.24 shows that if $(X,\ \delta)$ is fuzzy normal, it may not imply $(X,\ \delta^{\mathbb{N}})$ is fs-normal.

Example 3.1.24 *Let* $X = \{x,\ y\}$ *and* $\beta = \{p_x^r;\ r \in [0,\ 1]\} \cup \{p_y^r;\ r \in [0,\ 1]\}$ *be a basis for some fuzzy topology* δ *on* X*. Then the fuzzy topological space* $(X,\ \delta)$ *is fuzzy normal but* $(X,\ \delta^{\mathbb{N}})$ *is not fs-normal.*

If an FSTS $(X,\ \delta(s))$ is fs-normal, then it may not imply $(X,\ \delta_n)$ is fuzzy normal for each $n \in \mathbb{N}$ and it is shown by Example 3.1.25.

Example 3.1.25 *Let* $X = \{a\}$ *and* $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$*. Let* $\delta(s) = \delta^{\mathbb{N}}$*, then the FSTS* $(X,\ \delta(s))$ *is fs-normal but* $(X,\ \delta_n) = (X,\ \delta)$ $(\forall n \in \mathbb{N})$*, is not fuzzy normal.*

An FSTS $(X,\ \delta(s))$ may not be fs-normal even if its each component fuzzy topological space $(X,\ \delta_n)$ is fuzzy normal, as shown by Example 3.1.26 .

Example 3.1.26 *Let* $X = \{x,\ y\}$, $\beta = \{p_x^r;\ r \in [0,\ 1]\} \cup \{p_y^r;\ r \in [0,\ 1]\}$ *be a basis for some fuzzy topology* δ *on* X *and let* $\delta(s) = \delta^{\mathbb{N}}$*. Then the FSTS* $(X,\ \delta(s))$ *is not fs-normal but each of the component fuzzy topological spaces* $(X,\ \delta_n)$, $n \in \mathbb{N}$, *is fuzzy normal.*

CHAPTER

4

FS-closure operators and FS-interior operators

Closure and interior operators on an ordinary set belong to the very fundamental mathematical structures with direct applications on many fields like topology, logic etc. Being motivated by the importance of closure and interior operators, we introduce the concepts of FS-closure and FS-interior operators on a set. Books ([8], [10] [19], [27]) and the articles ([2], [3], [4], [13], [24], [28], [30]) may provide a suitable background as some basic ideas have been derived from these sources.

4.1 FS-closure Operator

Definition 4.1.1 *An operator* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *is said to be an fs-closure operator on* X *if it satisfies the following conditions*:
$(FSC1)$ $\boldsymbol{Cl}(X_f^0(s)) = X_f^0(s)$.
$(FSC2)$ $A_f(s) \leq \boldsymbol{Cl}(A_f(s))$ *for all* $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC3)$ $\boldsymbol{Cl}(\boldsymbol{Cl}(A_f(s))) = \boldsymbol{Cl}(A_f(s))$ *for all* $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC4)$ $\boldsymbol{Cl}(A_f(s) \vee B_f(s)) = \boldsymbol{Cl}(A_f(s)) \vee \boldsymbol{Cl}(B_f(s))$ *for all* $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 4.1.1 *In an FSTS* $(X, \delta(s))$, *closure of a fuzzy sequential set is an fs-closure operator on* X.

Example 4.1.2 *The operator* $\boldsymbol{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by* $\boldsymbol{C}(A_f(s)) = A_f(s) \vee D_f(s)$ *whenever* $A_f(s) \neq X_f^0(s)$ *and* $\boldsymbol{C}(X_f^0(s)) = X_f^0(s)$, *where* $D_f(s)$ *is a fixed fuzzy sequential set in* X, *is an fs-closure operator on* X.

Theorem 4.1.1 *If* $\boldsymbol{Cl}$ *be an fs-closure operator on* X, *then*
(i) $\boldsymbol{Cl}$ *is monotonic increasing.*
(ii) *For all* $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$,

$$A_f(s) \leq \boldsymbol{Cl}(B_f(s)) \;\Rightarrow\; \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(B_f(s)).$$

Proof. Proof is omitted. ■

Theorem 4.1.2 *Let* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an operator satisfying* $(FSC1)$, $(FSC2)$ *and* $(FSC4)$, *then*

a) The collection $\delta'(s) = \{(A_f(s))^c; A_f(s) \in (I^X)^{\mathbb{N}}$ *and* $\boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ *forms a fuzzy sequential topology on* X*.*
b) If $\boldsymbol{Cl}$ *also satisfies* $(FSC3)$*, then for all* $A_f(s) \in (I^X)^{\mathbf{N}}$*, we have* $\overline{A_f(s)} = \boldsymbol{Cl}(A_f(s))$*, where* $\overline{A_f(s)}$ *is the closure of* $A_f(s)$ *in* $\delta'(s)$*.*

Proof. Proof is omitted. ■

Remark 4.1.1 *From Theorem* 4.1.2*, it follows that, if* $\boldsymbol{Cl}$ *be an fs-closure operator on* X*, then* $\delta'(s) = \{(A_f(s))^c; A_f(s) \in (I^X)^{\mathbb{N}}$ *and* $\boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ *forms a fuzzy sequential topology on* X*. Also,* $\overline{A_f(s)} = \boldsymbol{Cl}(A_f(s))$ *for all* $A_f(s) \in (I^X)^{\mathbf{N}}$*, where* $\overline{A_f(s)}$ *is the closure of* $A_f(s)$ *in* $\delta'(s)$*. This fuzzy sequential topology* $\delta'(s)$ *is called the fuzzy sequential topology induced by the fs-closure operator* $\boldsymbol{Cl}$ *and we denote it by* $\delta_{\boldsymbol{Cl}}(s)$*.*

Example 4.1.3 shows that, if an operator $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a set X, satisfies $(FSC1)$, $(FSC2)$ and $(FSC4)$ but does not satisfy $(FSC3)$, then $\delta_{\boldsymbol{Cl}}(s)$ forms a fuzzy sequential topology on X but for any $A_f(s) \in (I^X)^{\mathbb{N}}$, $\overline{A_f(s)}$ may not be equal to $\boldsymbol{Cl}(A_f(s))$.

Example 4.1.3 *Let* $X = \{a\}$*. Let* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an operator defined by*

$$\boldsymbol{Cl}(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty} \text{ for all } A_f(s) \in (I^X)^{\mathbb{N}}.$$

Then $\boldsymbol{Cl}$ *satisfies* $(FSC1)$*,* $(FSC2)$ *and* $(FSC4)$ *and hence* $(X, \delta_{\boldsymbol{Cl}}(s))$ *forms a fuzzy sequential topological space. Further,* $\boldsymbol{Cl}$

does not satisfy $(FSC3)$ *and in* $(X, \delta_{\boldsymbol{Cl}}(s))$, $\boldsymbol{Cl}(B_f(s)) \neq \overline{B_f(s)}$, *where* $B_f(s) \in (I^X)^{\mathbb{N}}$ *with* $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}$ $\forall\ n \neq 1, 2, 3$.

Definition 4.1.2 *Let* $\boldsymbol{Cl}$ *be an fs-closure operator on* X. *For* $n \in \mathbb{N}$, *if* ${}_{nA}X_f^0(s)$ *denotes a fuzzy sequential set whose* n^{th} *term is the fuzzy set* A *and others are* $\overline{0}$, *then an operator* $(\boldsymbol{Cl})_f^n$: $I^X \to I^X$ *defined by*

$$(\boldsymbol{Cl})_f^n(A) = n^{th} \text{ term of } \boldsymbol{Cl}({}_{nA}X_f^0(s)),$$

is called the n^{th} *component of* $\boldsymbol{Cl}$.

Theorem 4.1.3 *If* $\boldsymbol{Cl}$ *be an fs-closure operator on* X, *then each component* $(\boldsymbol{Cl})_f^n$ *is a fuzzy closure operator and* $(\delta_{\boldsymbol{Cl}})_n = \delta_{(\boldsymbol{Cl})_f^n}$, *where* $(\delta_{\boldsymbol{Cl}})_n$ *is the* n^{th} *component fuzzy topology of fuzzy sequential topology* $\delta_{\boldsymbol{Cl}}(s)$ *and* $\delta_{(\boldsymbol{Cl})_f^n}$ *is the fuzzy topology induced by the component* $(\boldsymbol{Cl})_f^n$ *of* $\boldsymbol{Cl}$.

Proof. $(\boldsymbol{Cl})_f^n(\overline{0}) = \overline{0}$ by definition. Let $A \in I^X$, then

$${}_{nA}X_f^0(s) \leq \boldsymbol{Cl}({}_{nA}X_f^0(s)) \ \Rightarrow\ A \leq (\boldsymbol{Cl})_f^n(A).$$

Hence,

$$(\boldsymbol{Cl})_f^n(A) \leq (\boldsymbol{Cl})_f^n((\boldsymbol{Cl})_f^n(A)).$$

Also,

$$\begin{aligned}
& \boldsymbol{Cl}(\boldsymbol{Cl}({}_{nA}X_f^0(s))) = \boldsymbol{Cl}({}_{nA}X_f^0(s)) \\
\Rightarrow\ & \boldsymbol{Cl}({}_{n(\boldsymbol{Cl})_f^n(A)}X_f^0(s)) \leq \boldsymbol{Cl}({}_{nA}X_f^0(s)) \\
\Rightarrow\ & (\boldsymbol{Cl})_f^n((\boldsymbol{Cl})_f^n(A)) \leq (\boldsymbol{Cl})_f^n(A)
\end{aligned}$$

Hence $(\boldsymbol{Cl})^n_f((\boldsymbol{Cl})^n_f(A)) = (\boldsymbol{Cl})^n_f(A)$.

Again let $A, B \in I^X$, then

$$\begin{aligned} & \boldsymbol{Cl}({}_{nA}X^0_f(s) \vee {}_{nB}X^0_f(s)) = \boldsymbol{Cl}({}_{nA}X^0_f(s)) \vee \boldsymbol{Cl}({}_{nB}X^0_f(s)) \\ \Rightarrow\ & \boldsymbol{Cl}({}_{n(A\vee B)}X^0_f(s)) = \boldsymbol{Cl}({}_{nA}X^0_f(s)) \vee \boldsymbol{Cl}({}_{nB}X^0_f(s)) \\ \Rightarrow\ & (\boldsymbol{Cl})^n_f(A \vee B) = (\boldsymbol{Cl})^n_f(A) \vee (\boldsymbol{Cl})^n_f(B) \end{aligned}$$

Thus, $(\boldsymbol{Cl})^n_f$ is a fuzzy closure operator.

For the next part, Let $A \in (\delta_{\boldsymbol{Cl}})_n$, then $\overline{1} - A$ is a closed fuzzy set in $(X, (\delta_{\boldsymbol{Cl}})_n)$. Let $B_f(s)$ be a closed fuzzy sequential set in $(X, \delta_{\boldsymbol{Cl}}(s))$ such that $B^n_f = \overline{1} - A$. Now,

$$\begin{aligned} & {}_{n(\overline{1}-A)}X^0_f(s) \le B_f(s) \\ \Rightarrow\ & \boldsymbol{Cl}({}_{n(\overline{1}-A)}X^0_f(s)) \le \boldsymbol{Cl}(B_f(s)) \\ \Rightarrow\ & (\boldsymbol{Cl})^n_f(\overline{1} - A) \le B^n_f = \overline{1} - A \\ \Rightarrow\ & (\boldsymbol{Cl})^n_f(\overline{1} - A) = \overline{1} - A \\ \Rightarrow\ & A \in \delta_{(\boldsymbol{Cl})^n_f} \end{aligned}$$

Also, $A \in \delta_{(\boldsymbol{Cl})^n_f}$ implies $(\boldsymbol{Cl})^n_f(\overline{1} - A) = \overline{1} - A$. Let $B_f(s) = \boldsymbol{Cl}({}_{n(\overline{1}-A)}X^o_f(s))$, then $B_f(s)$ is a closed fuzzy sequential set in $(X, \delta_{\boldsymbol{Cl}}(s))$ and its n^{th} component is $\overline{1} - A$. Therefore $A \in (\delta_{\boldsymbol{Cl}})_n$. Hence the theorem. ■

Theorem 4.1.4 *Let* $\boldsymbol{Cl}$ *be an fs-closure operator on* X *and* $Y \subseteq X$*. If* $Char(Y)$ *denotes the characteristic function of* Y*, then* $\boldsymbol{Cl}_Y \colon (I^Y)^{\mathbb{N}} \to (I^Y)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{Cl}_Y(B_f(s)) = \{Char(Y)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(B_f(s)) \ \forall B_f(s) \in (I^Y)^{\mathbb{N}},$$

is an fs-closure operator on Y*. Moreover,* $(\boldsymbol{Cl}_Y)^n_f(B) = Char(Y)$ $\wedge(\boldsymbol{Cl})^n_f(B)$ *for all* $B \in I^Y$*.*

Proof. Let $B_f(s) \in (I^Y)^{\mathbb{N}}$. Now,

$$\begin{aligned} &\boldsymbol{Cl}_Y(\boldsymbol{Cl}_Y(B_f(s))) = \boldsymbol{Cl}_Y(\{Char(Y)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(B_f(s))) \\ = \;& \{Char(Y)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(\{Char(Y)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(B_f(s))) \\ \leq \;& \{Char(Y)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(\{Char(Y)\}_{n=1}^{\infty}) \wedge \boldsymbol{Cl}(\boldsymbol{Cl}(B_f(s)))) \\ \leq \;& \{Char(Y)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(B_f(s))) \\ = \;& \boldsymbol{Cl}_Y(B_f(s)) \end{aligned}$$

All the other conditions being straightforward, we can conclude that $\boldsymbol{Cl}_Y$ is an fs-closure operator on Y. Also for $B \in I^Y$, $(\boldsymbol{Cl}_Y)^n_f(B) = n^{th}$ component of $\boldsymbol{Cl}_Y({}_{nB}X^0_f(s))= n^{th}$ component of $\{Char(Y)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}({}_{nB}X^0_f(s)) = Char(Y) \wedge n^{th}$ component of $\boldsymbol{Cl}({}_{nB}X^0_f(s)) = Char(Y) \wedge (\boldsymbol{Cl})^n_f(B)$. ■

Theorem 4.1.5 *Let* $\{\boldsymbol{Cl}_\lambda : (I^{X_\lambda})^{\mathbb{N}} \to (I^{X_\lambda})^{\mathbb{N}}; \lambda \in \Lambda\}$ *be a family of fs-closure operators, where* $X_\lambda \cap X_\mu = \phi$ *for all distinct* $\lambda, \mu \in \Lambda$*. If* $X = \underset{\lambda\in\Lambda}{\vee} X_\lambda$ *and* $Char(X_\lambda)$ *denotes the characteristic function of* X_λ*, then* $\boldsymbol{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{C}(A_f(s)) = \bigvee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)),$$

is an fs-closure operator on X*.*

Proof. For $A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned}
&\boldsymbol{C}(\boldsymbol{C}(A_f(s))) = \boldsymbol{C}(\bigvee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\
= &\bigvee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge (\bigvee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge \\
&A_f(s)))) \\
= &\bigvee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\
= &\bigvee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\
= &\bigvee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)) \\
= &\boldsymbol{C}(A_f(s))
\end{aligned}$$

Other conditions being straightforward, it follows that $\boldsymbol{C}$ is an fs-closure operator on X. ■

Definition 4.1.3 *A collection* $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}};\ \lambda \in \Lambda\}$ *is called an fs-closure system if* $\bigwedge_{\lambda\in\Lambda,\ A_f(s)\le A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s)$ *for each* $A_f(s) \in (I^X)^{\mathbb{N}}$.

Theorem 4.1.6 $\zeta(s)$ *is an fs-closure system if and only if* $\zeta(s)$ *is closed under arbitrary intersection.*

Proof. Suppose $\zeta(s)$ is closed under arbitrary intersection. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \subseteq \zeta(s)$ such that $A_f(s) \le A_{\lambda f}(s)\ \forall \lambda \in \Lambda$. Then $\bigwedge_{\lambda\in\Lambda} A_{\lambda f}(s) \in \zeta(s)$.

Conversely, suppose $\zeta(s)$ is an fs-closure system. Let $\{A_{\lambda f}(s);\ \lambda \in \Lambda\} \subseteq \zeta(s)$ and let $A_f(s) = \bigwedge_{\lambda\in\Lambda} A_{\lambda f}(s)$. Then

$$\begin{aligned} & A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\ \Rightarrow\ & \bigwedge_{\lambda\in\Lambda} A_{\lambda f}(s) = \bigwedge_{\lambda\in\Lambda,\ A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s) \end{aligned}$$

Hence $\zeta(s)$ is closed under arbitrary intersection. ∎

Lemma 4.1.1 *Let $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}}; \lambda \in \Lambda\}$ be an fs-closure system containing $X_f^0(s)$. Then an operator $\boldsymbol{Cl}_{\zeta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s)) = \bigwedge_{\lambda\in\Lambda,\ A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s), \quad \forall A_f(s) \in (I^X)^{\mathbb{N}}$$

and which commutes with finite union, is an fs-closure operator. Moreover, for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \zeta(s)$ if and only if $A_f(s) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s))$.

Proof. Since $\boldsymbol{Cl}_{\zeta(s)}(A_f(s)) \in \zeta(s)$ for $A_f(s) \in (I^X)^{\mathbb{N}}$, we have

$$\begin{aligned} \boldsymbol{Cl}_{\zeta(s)}(\boldsymbol{Cl}_{\zeta(s)}(A_f(s))) &= \bigwedge_{\lambda\in\Lambda, \boldsymbol{Cl}_{\zeta(s)}(A_f(s))\leq A_{\lambda f}(s)} A_{\lambda f}(s) \\ &\leq \boldsymbol{Cl}_{\zeta(s)}(A_f(s)) \end{aligned}$$

Hence, $\boldsymbol{Cl}_{\zeta(s)}$ is an fs-closure operator.

Now, if $A_f(s) \in \zeta(s)$, then $A_f(s) = A_{\lambda f}(s)$ for some $\lambda \in \Lambda$ and

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s)) = \bigwedge_{i\in\Lambda,\ A_f(s)\leq A_{if}(s)} A_{if}(s) \leq A_{\lambda f}(s) = A_f(s)$$

Also, $A_f(s) \leq \boldsymbol{Cl}_{\zeta(s)}((A_f(s))$. Hence $\boldsymbol{Cl}_{\zeta(s)}(A_f(s)) = A_f(s)$. Converse part follows from the definition of $\boldsymbol{Cl}_{\zeta(s)}$. ∎

Lemma 4.1.2 *Let* $\boldsymbol{Cl}$ *be an fs-closure operator on* X*. Then*

$$\zeta_{\boldsymbol{Cl}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \boldsymbol{Cl}(A_f(s))\}$$

is an fs-closure system.

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \subseteq \zeta_{Cl}(s)$ such that $B_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda$. Let $D_f(s) = \bigwedge_{\lambda \in \Lambda} A_{\lambda f}(s)$. Therefore,

$$\begin{aligned} \boldsymbol{Cl}(D_f(s)) &\leq \boldsymbol{Cl}(A_{\lambda f}(s))\ \forall \lambda \in \Lambda \\ \Rightarrow \boldsymbol{Cl}(D_f(s)) &\leq \bigwedge_{\lambda \in \Lambda} \boldsymbol{Cl}(A_{\lambda f}(s)) \\ &= \bigwedge_{\lambda \in \Lambda} A_{\lambda f}(s) \\ &= D_f(s) \end{aligned}$$

Thus, $D_f(s) = \boldsymbol{Cl}(D_f(s))$ and so $D_f(s) \in \zeta_{\boldsymbol{Cl}}(s)$. Hence $\zeta_{\mathbf{Cl}}(s)$ is an fs-closure system. ■

Note 4.1.1 *The fs-closure operator* $\boldsymbol{Cl}_{\zeta(s)}$*, defined in Lemma* 4.1.1*, is called an fs-closure operator generated by the fs-closure system* $\zeta(s)$ *and the fs-closure system* $\zeta_{\boldsymbol{Cl}}(s)$*, defined in Lemma* 4.1.2*, is called an fs-closure system generated by the fs-closure operator* $\boldsymbol{Cl}$*.*

Theorem 4.1.7 *Let* $\boldsymbol{Cl}$ *be an fs-closure operator and* $\zeta(s)$ *be an fs-closure system on* X *containing* $X_f^0(s)$*, then* $\zeta_{\boldsymbol{Cl}}(s)$ *and* $\boldsymbol{Cl}_{\zeta(s)}$ *are respectively fs-closure system and fs-closure operator on* X*. Also,* $\boldsymbol{Cl} = \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) = \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$*, that is, the mappings* $\boldsymbol{Cl} \to \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) \to \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$ *are mutually inverse.*

Proof. The first part follows from Lemma 4.1.1 and Lemma 4.1.2. For the second part, let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \subseteq \zeta_{\boldsymbol{Cl}}(s)$ such that $A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda$. Then, $\boldsymbol{Cl}_{\zeta_{Cl}(s)}(A_f(s)) = \bigwedge\limits_{\lambda\in\Lambda} A_{\lambda f}(s)$. Now,

$$\begin{aligned} & A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\ \Rightarrow\ & \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(A_{\lambda f}(s))\ \forall \lambda \in \Lambda \\ \Rightarrow\ & \boldsymbol{Cl}(A_f(s)) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\ \Rightarrow\ & \boldsymbol{Cl}(A_f(s)) \leq \bigwedge_{\lambda\in\Lambda} A_{\lambda f}(s) = \boldsymbol{Cl}_{\zeta_{Cl}(s)}(A_f(s)) \end{aligned}$$

Again,

$$\begin{aligned} & A_f(s) \leq \boldsymbol{Cl}(A_f(s)) \text{ and } \boldsymbol{Cl}(A_f(s)) \in \zeta_{\boldsymbol{Cl}}(s) \\ \Rightarrow\ & \boldsymbol{Cl}_{\zeta_{Cl}(s)}(A_f(s)) = \bigwedge_{\lambda\in\Lambda} A_{\lambda f}(s) \leq \boldsymbol{Cl}(A_f(s)) \end{aligned}$$

Hence $\boldsymbol{Cl} = \boldsymbol{Cl}_{\zeta_{Cl}(s)}$.
Also,

$$\begin{aligned} & A_f(s) \in \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s) \\ \Leftrightarrow\ & A_f(s) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s)) \\ \Leftrightarrow\ & A_f(s) \in \zeta(s) \end{aligned}$$

Thus $\zeta(s) = \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$. ■

4.2 FS-interior Operator

Definition 4.2.1 *An operator $\boldsymbol{I} \colon (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an fs-interior operator on X, if it satisfies the following conditions:*

$(FSI1)$ $\boldsymbol{I}(X_f^1(s)) = X_f^1(s)$.

$(FSI2)$ $\boldsymbol{I}(A_f(s)) \leq A_f(s)$ *for all* $A_f(s) \in (I^X)^{\mathbb{N}}$.

$(FSI3)$ $\boldsymbol{I}(\boldsymbol{I}(A_f(s))) = \boldsymbol{I}(A_f(s)))$ *for all* $A_f(s)) \in (I^X)^{\mathbb{N}}$.

$(FSI4)$ $\boldsymbol{I}(A_f(s) \wedge B_f(s)) = \boldsymbol{I}(A_f(s)) \wedge \boldsymbol{I}(B_f(s))$ *for all* $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 4.2.1 *For any FSTS* $(X, \delta(s))$*, interior of an fs-set is an fs-interior operator on* X*.*

Example 4.2.2 *The operator* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by* $\boldsymbol{I}(A_f(s)) = A_f(s) \wedge D_f(s)$ *whenever* $A_f(s) \neq X_f^1(s)$ *and* $\boldsymbol{I}(X_f^1(s)) = X_f^1(s)$*, where* $D_f(s)$ *is a fixed fuzzy sequential set in* X*, is an fs-interior operator on* X*.*

Theorem 4.2.1 *If* $\boldsymbol{I}$ *be an fs-interior operator on* X*, then*
(i) $\boldsymbol{I}$ *is monotonic increasing.*
(ii) For all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$,

$$\boldsymbol{I}(A_f(s)) \leq B_f(s) \;\Rightarrow\; \boldsymbol{I}(A_f(s)) \leq \boldsymbol{I}(B_f(s))$$

Proof. Proof is omitted. ■

Theorem 4.2.2 *Let* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an operator satisfying* $(FSI1)$, $(FSI2)$ *and* $(FSI4)$*, then*
a) the collection $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms a fuzzy sequential topology on* X*.*
b) if $\boldsymbol{I}$ *also satisfies* $(FSI3)$*, then for all* $A_f(s) \in (I^X)^{\mathbf{N}}$*, we have* $\overset{o}{A}_f(s) = \boldsymbol{I}(A_f(s))$*, where* $\overset{o}{A}_f(s)$ *is the interior of* $A_f(s)$ *in* $\delta(s)$*.*

Proof. Proof is omitted. ■

Remark 4.2.1 *From Theorem* 4.2.2, *it follows that, if* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an fs-interior operator on* X, *then* $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}} ; \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms a fuzzy sequential topology on* X. *Also* $\overset{o}{A}_f(s) = \boldsymbol{I}(A_f(s))$ *for all* $A_f(s) \in (I^X)^{\mathbf{N}}$, *where* $\overset{o}{A}_f(s)$ *is the interior of* $A_f(s)$ *in* $\delta(s)$. *This fuzzy sequential topology* $\delta(s)$ *is called the fuzzy sequential topology induced by the fs-interior operator* $\boldsymbol{I}$ *and we denote it by* $\delta_{\boldsymbol{I}}(s)$.

Example 4.2.3 shows that, if an operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a set X, satisfies $(FSI1)$, $(FSI2)$ and $(FSI4)$ but does not satisfy $(FSI3)$, then $\delta_{\mathbf{I}}(s)$ forms a fuzzy sequential topology on X but $\overset{o}{A}_f(s)$ may not be equal to $\mathbf{I}(A_f(s))$, where $A_f(s) \in (I^X)^{\mathbb{N}}$.

Example 4.2.3 *Let* $X = \{a\}$. *Let* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an operator defined by* $\boldsymbol{I}(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s) \in (I^X)^{\mathbb{N}}$. *Then,* $\boldsymbol{I}$ *satisfies* $(FSI1)$, $(FSI2)$ *and* $(FSI4)$ *and hence* $(X, \delta_{\boldsymbol{I}}(s))$ *forms an FSTS. Further,* $\boldsymbol{I}$ *does not satisfy* $(FSI3)$ *and in* $(X, \delta_{\boldsymbol{I}}(s))$, $\boldsymbol{I}(B_f(s)) \neq \overset{0}{B}_f(s)$, *where* $B_f(s)$ *is an fs-set with* $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}\ \forall\ n \neq 1, 2, 3$.

Definition 4.2.2 *Let* $\boldsymbol{I}$ *be an fs-interior operator on* X. *For* $n \in \mathbb{N}$, *if* ${}_{nA}X_f^1(s)$ *denotes a fuzzy sequential set whose* n^{th} *term is* A *and others are* $\overline{1}$, *then an operator* $(\boldsymbol{I})_f^n : I^X \to I^X$ *defined by*

$$(\boldsymbol{I})_f^n(A) = n^{th} \text{ term of } \boldsymbol{I}({}_{nA}X_f^1(s)),$$

is called the n^{th} *component of* $\boldsymbol{I}$.

Theorem 4.2.3 *If $\boldsymbol{I}$ be an fs-interior operator on X, then each component $(\boldsymbol{I})^n_f$ is a fuzzy interior operator. Also $(\delta_{\boldsymbol{I}})_n = \delta_{(\boldsymbol{I})^n_f}$, where $(\delta_{\boldsymbol{I}})_n$ is the n^{th} component fuzzy topology of fuzzy sequential topology $\delta_{\boldsymbol{I}}(s)$ and $\delta_{(\boldsymbol{I})^n_f}$ is the fuzzy topology induced by the component $(\boldsymbol{I})^n_f$ of $\boldsymbol{I}$.*

Proof. :$(\boldsymbol{I})^n_f(\overline{1}) = \overline{1}$ by definition. Let $A \in I^X$, then

$$\boldsymbol{I}({}_{nA}X^1_f(s)) \leq {}_{nA}X^1_f(s) \;\Rightarrow\; (\boldsymbol{I})^n_f(A) \leq A.$$

Hence

$$(\boldsymbol{I})^n_f((\boldsymbol{I})^n_f(A)) \leq (\boldsymbol{I})^n_f(A).$$

Also,

$$\begin{aligned} & \boldsymbol{I}(\boldsymbol{I}({}_{nA}X^1_f(s))) = \boldsymbol{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\; & \boldsymbol{I}({}_{nA}X^1_f(s)) \leq \boldsymbol{I}({}_{n(\boldsymbol{I})^n_f(A)}X^1_f(s)) \\ \Rightarrow\; & (\boldsymbol{I})^n_f(A) \leq (\boldsymbol{I})^n_f((\boldsymbol{I})^n_f(A)) \end{aligned}$$

Hence, $(\boldsymbol{I})^n_f((\boldsymbol{I})^n_f(A)) = (\boldsymbol{I})^n_f(A)$.
Again let A, B $\in I^X$, then

$$\begin{aligned} & \boldsymbol{I}({}_{nA}X^1_f(s) \wedge {}_{nB}X^1_f(s)) = \boldsymbol{I}({}_{nA}X^1_f(s)) \wedge \boldsymbol{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\; & \boldsymbol{I}({}_{n(A\wedge B)}X^1_f(s)) = \boldsymbol{I}({}_{nA}X^1_f(s)) \wedge \boldsymbol{I}({}_{nB}X^1_f(s)) \\ \Rightarrow\; & (\boldsymbol{I})^n_f(A \wedge B) = (\boldsymbol{I})^n_f(A) \wedge (\boldsymbol{I})^n_f(B) \end{aligned}$$

Thus, $(\boldsymbol{I})^n_f$ is a fuzzy interior operator.

For the next part, Let A$\in (\delta_{\boldsymbol{I}})_n$. Let $B_f(s)$ be an fs-open set in $(X, \delta_{\boldsymbol{I}}(s))$ such that $B_f^n = A$. Now,

$$\begin{aligned}
& B_f(s) \leq {}_{nA}X_f^1(s) \\
\Rightarrow\ & \boldsymbol{I}(B_f(s)) \leq \boldsymbol{I}({}_{nA}X_f^1(s)) \\
\Rightarrow\ & B_f(s) \leq \boldsymbol{I}({}_{nA}X_f^1(s)) \\
\Rightarrow\ & A \leq (\boldsymbol{I})_f^n(A) \\
\Rightarrow\ & (\boldsymbol{I})_f^n(A) = A \\
\Rightarrow\ & A \in \delta_{(\boldsymbol{I})_f^n}
\end{aligned}$$

Also, $A \in \delta_{(\boldsymbol{I})_f^n}$ implies $(\boldsymbol{I})_f^n(A) = A$. Let $B_f(s) = \boldsymbol{I}({}_{nA}X_f^1(s))$, then $B_f(s)$ is an fs-open set in $(X, (\delta_{\boldsymbol{I}}(s))$ and its n^{th} component is A. Therefore $A \in (\delta_{\boldsymbol{I}})_n$. Hence the theorem. ∎

Theorem 4.2.4 *Let $\boldsymbol{I}$ be an fs-interior operator on a set X and $Y \subseteq X$. If $Char(Y)$ denotes the characteristic function of Y, then $\boldsymbol{I}_Y$: $(I^Y)^{\mathbb{N}} \to (I^Y)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I}_Y(B_f(s)) \;=\; \{Char(Y)\}_{n=1}^{\infty} \vee \boldsymbol{I}(B_f(s)) \;\forall B_f(s) \in (I^Y)^{\mathbb{N}},$$

is an fs-interior operator on Y and $(\boldsymbol{I}_Y)_f^n(B) = Char(Y) \vee (\boldsymbol{I})_f^n(B)$ for all $B \in I^Y$.

Proof. Let $B_f(s) \in (I^Y)^{\mathbb{N}}$. Now

$$\begin{aligned} \boldsymbol{I}_Y(B_f(s))) &= \{Char(Y)\}_{n=1}^{\infty} \vee \boldsymbol{I}(B_f(s)) \\ &= \{Char(Y)\}_{n=1}^{\infty} \vee \boldsymbol{I}(\{Char(Y)\}_{n=1}^{\infty}) \vee \boldsymbol{I}(\boldsymbol{I}(B_f(s))) \\ &\leq \{Char(Y)\}_{n=1}^{\infty} \vee \boldsymbol{I}(\{Char(Y)\}_{n=1}^{\infty} \vee \boldsymbol{I}(B_f(s))) \\ &= \boldsymbol{I}_Y(\{Char(Y)\}_{n=1}^{\infty} \vee \boldsymbol{I}(B_f(s))) \\ &= \boldsymbol{I}_Y(\boldsymbol{I}_Y(B_f(s))) \end{aligned}$$

All the other conditions being straightforward, we can conclude that $\boldsymbol{I}_Y$ is an fs-interior operator on X. Also,

$$\begin{aligned} (\boldsymbol{I}_Y)_f^n(B) &= n^{th} \text{ component of } \boldsymbol{I}_Y({}_{nB}X_f^1(s)) \\ &= n^{th} \text{ component of } \{Char(Y)\}_{n=1}^{\infty} \vee \boldsymbol{I}({}_{nB}X_f^1(s)) \\ &= Char(Y) \vee n^{th} \text{ component of } \boldsymbol{I}({}_{nB}X_f^1(s)) \\ &= Char(Y) \vee (\boldsymbol{I})_f^n(B). \end{aligned}$$

■

Definition 4.2.3 *A collection* $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ *is called an fs-interior system if* $\bigvee_{j\in J,\ A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s)$ *for each* $A_f(s) \in (I^X)^{\mathbb{N}}$.

Theorem 4.2.5 $\eta(s)$ *is an fs-interior system if and only if* $\eta(s)$ *is closed under arbitrary union.*

Proof. Suppose $\eta(s)$ is closed under arbitrary union. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Let $\{A_{jf}(s); j \in J\} \subseteq \eta(s)$ such that $A_{jf}(s) \leq A_f(s)$

$\forall j \in J$. Then,

$$\bigvee_{j\in J} A_{jf}(s) \in \eta(s)$$

Conversely, suppose $\eta(s)$ is an fs-interior system. Let $\{A_{jf}(s); j \in J\} \subseteq \eta(s)$ and let $A_f(s) = \bigvee_{j\in J} A_{jf}(s)$. Then,

$$\begin{aligned} & A_{jf}(s) \leq A_f(s)\ \forall j \in J \\ \Rightarrow & \bigvee_{j\in J} A_{jf}(s) = \bigvee_{j\in J,\ A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s) \end{aligned}$$

Hence, $\eta(s)$ is closed under arbitrary union. ■

Lemma 4.2.1 *Let $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ be an fs-interior system containing $X_f^1(s)$. Then, an operator $\boldsymbol{I}_{\eta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I}_{\eta(s)}(A_f(s)) = \bigvee_{j\in J,\ A_{jf}(s)\leq A_f(s)} A_{jf}(s)\ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

and which commutes with finite intersection, is an fs-interior operator on X. Moreover, for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \eta(s)$ if and only if $A_f(s) = \boldsymbol{I}_{\eta(s)}(A_f(s))$.

Proof. Proof of the first part is straightforward. Now, if $A_f(s) \in \eta(s)$, then $A_f(s) = A_{jf}(s)$ for some $j \in J$ and

$$\boldsymbol{I}_{\eta(s)}(A_f(s)) = \bigvee_{i\in J,\ A_{if}(s)\leq A_f(s)} A_{if}(s) = A_f(s)$$

Converse part follows from the definition of $\boldsymbol{I}_{\eta(s)}$. ■

Lemma 4.2.2 *Let $\boldsymbol{I}$ be an fs-interior operator on X. Then, $\eta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ A_f(s) = \boldsymbol{I}(A_f(s))\}$ is an fs-interior system.*

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$ and $\{A_{jf}(s); j \in J\} \subseteq \eta_I(s)$ such that $A_{jf}(s) \leq B_f(s)$ for all $j \in J$. Let $D_f(s) = \bigvee_{j\in J} A_{jf}(s)$. Then,

$$\boldsymbol{I}(A_{jf}(s)) \leq \boldsymbol{I}(D_f(s))\ \forall j \in J$$

Therefore,

$$\begin{aligned} \bigvee_{j\in J} \boldsymbol{I}(A_{jf}(s)) &\leq \boldsymbol{I}(D_f(s)) \\ \Rightarrow D_f(s) = \bigvee_{j\in J} A_{jf}(s) &\leq \boldsymbol{I}(D_f(s)) \end{aligned}$$

Thus, $D_f(s) = \boldsymbol{I}(D_f(s))$ and so $D_f(s) \in \eta_{\boldsymbol{I}}(s)$. Hence, $\eta_{\boldsymbol{I}}(s)$ is an fs-interior system. ■

Note 4.2.1 *The fs-interior operator $\boldsymbol{I}_{\eta(s)}$, defined in Lemma 4.2.1, is called an fs-interior operator generated by the fs-interior system $\eta(s)$ and the fs-interior system $\eta_{\boldsymbol{I}}(s)$, defined in Lemma 4.2.2, is called an fs-interior system generated by the fs-interior operator $\boldsymbol{I}$.*

Theorem 4.2.6 *Let $\boldsymbol{I}$ be an fs-interior operator and $\eta(s)$ be an fs-interior system on X containing $X_f^1(s)$, then $\eta_{\boldsymbol{I}}(s)$ and $\boldsymbol{I}_{\eta(s)}$ are respectively fs-interior system and fs-interior operator on X. Also, $\boldsymbol{I} = \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) = \eta_{\boldsymbol{I}_{\eta(s)}}(s)$, that is, the mappings $\boldsymbol{I} \rightarrow \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) \rightarrow \eta_{\boldsymbol{I}_{\eta(s)}}(s)$ are mutually inverse.*

Proof. The first part follows from Lemma 4.2.1 and Lemma 4.2.2. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{jf}(s); j \in J\} \subseteq \eta_{\boldsymbol{I}}(s)$ such that $A_{jf}(s) \leq A_f(s)\ \forall j \in J$. Then,

$$\boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}(A_f(s)) = \bigvee_{j \in J} A_{jf}(s).$$

Now,

$$\begin{aligned}
& A_{jf}(s) \leq A_f(s)\ \forall j \in J \\
\Rightarrow\ & \boldsymbol{I}(A_{jf}(s)) \leq \boldsymbol{I}(A_f(s))\ \forall j \in J \\
\Rightarrow\ & \bigvee_{j \in J} \boldsymbol{I}(A_{jf}(s)) \leq \boldsymbol{I}(A_f(s)) \\
\Rightarrow\ & \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}(A_f(s)) = \bigvee_{j \in J} A_{jf}(s) \leq \mathbf{I}(A_f(s))
\end{aligned}$$

Again,

$$\begin{aligned}
& \boldsymbol{I}(A_f(s)) \leq A_f(s) \text{ and } \boldsymbol{I}(A_f(s)) \in \eta_{\boldsymbol{I}}(s) \\
\Rightarrow\ & \boldsymbol{I}(A_f(s)) \leq \bigvee_{j \in J} A_{jf}(s) = \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}(A_f(s)).
\end{aligned}$$

Hence $\boldsymbol{I} = \boldsymbol{I}_{\eta_{\mathbf{I}}(s)}$.

Also,

$$\begin{aligned}
& A_f(s) \in \eta_{\boldsymbol{I}_{\eta(s)}}(s) \\
\Leftrightarrow\ & A_f(s) = \boldsymbol{I}_{\eta(s)}(A_f(s)) \\
\Leftrightarrow\ & A_f(s) \in \eta(s).
\end{aligned}$$

Thus $\eta(s) = \eta_{\boldsymbol{I}_{\eta(s)}}(s)$. ■

Definition 4.2.4 *If $\boldsymbol{I}$ be an fs-interior operator on X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ forms an fs-*

closure system on X and we call it to be an fs-closure system generated by the fs-interior operator $\boldsymbol{I}$.

Definition 4.2.5 *If $\boldsymbol{Cl}$ be an fs-closure operator on X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}};\ \boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ forms an fs-interior system on X and we call it to be an fs-interior system generated by the fs-closure operator $\boldsymbol{Cl}$.*

Theorem 4.2.7 *Let $\boldsymbol{I}$ be an fs-interior operator on X, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ forms a fuzzy sequential topology on X.
(ii) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ forms an fs-interior system on X.
(iii) $\{A_f(s); (A_f(s))^c \in \delta_{\boldsymbol{I}}(s)\}$ forms an fs-closure system on X.

Proof. Proof is omitted. ■

Theorem 4.2.8 *Let $\boldsymbol{Cl}$ be an fs-closure operator on X, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ forms a fuzzy sequential topology on X.
(ii) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ forms an fs-interior system on X.
(iii) $\{A_f(s); (A_f(s))^c \in \delta_{\boldsymbol{Cl}}(s)\}$ forms an fs-closure system on X.

Proof. Proof is omitted. ■

Theorem 4.2.9 *If $\boldsymbol{Cl}$ be an fs-closure operator on X, then the operator $\boldsymbol{I_{Cl}} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I_{Cl}}(A_f(s)) = X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an fs-interior operator on X. Again, if $\boldsymbol{I}$ be an fs-interior operator on X, then the operator $\boldsymbol{Cl_I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$\boldsymbol{Cl_I}(A_f(s)) = X_f^1(s) - \boldsymbol{I}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an fs-closure operator on X.

Proof. Proof is omitted. ∎

It follows from Theorem 4.2.9, that, given an fs-closure operator on a set, we can define an fs-interior operator and given an fs-interior operator, we can define an fs-closure operator. In fact, there is a one to one correspondence between the collections of all fs-closure and fs-interior operators on a set (see Theorem 4.2.10). We denote the collection of all fs-closure operators and the collection of all fs-interior operators on X by $\mathcal{C}_X$ and $\mathcal{I}_X$ respectively.

Theorem 4.2.10 *For any set X, there exists a one to one correspondence between $\mathcal{C}_X$ and $\mathcal{I}_X$.*

Proof. Define $t : \mathcal{C}_X \to \mathcal{I}_X$ by

$$t(\boldsymbol{Cl}) = \boldsymbol{I}_{\boldsymbol{Cl}} \ \forall \boldsymbol{Cl} \in \mathcal{C}_X.$$

Then t is a well defined map. Now, for $\boldsymbol{Cl}_1, \boldsymbol{Cl}_2 \in \mathcal{C}_X$ such that $t(\boldsymbol{Cl}_1) = t(\boldsymbol{Cl}_2)$, we have $\boldsymbol{I}_{\boldsymbol{Cl}_1} = \boldsymbol{I}_{\boldsymbol{Cl}_2}$. Hence $\forall A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} \boldsymbol{I}_{\boldsymbol{Cl}_1}((A_f(s))^c) &= \boldsymbol{I}_{\boldsymbol{Cl}_2}((A_f(s))^c) \\ X_f^1(s) - \boldsymbol{Cl}_1(A_f(s)) &= X_f^1(s) - \boldsymbol{Cl}_2(A_f(s)) \\ \boldsymbol{Cl}_1(A_f(s)) &= \boldsymbol{Cl}_2(A_f(s)) \end{aligned}$$

Thus t is injective. Again for $\boldsymbol{I} \in \mathcal{I}_X$, there is $\boldsymbol{Cl}_{\boldsymbol{I}} \in \mathcal{C}_X$ such that $\forall A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\boldsymbol{Cl}_{\boldsymbol{I}}((A_f(s))^c) = X_f^1(s) - \boldsymbol{I}(A_f(s))$$

Now, $\forall A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} \boldsymbol{I}_{\boldsymbol{Cl}_{\boldsymbol{I}}}((A_f(s)) &= X_f^1(s) - \boldsymbol{Cl}_{\boldsymbol{I}}((A_f(s))^c) \\ &= X_f^1(s) - (X_f^1(s) - \boldsymbol{I}(A_f(s))) \\ &= \boldsymbol{I}(A_f(s)) \end{aligned}$$

Therefore, t is surjective and this completes the theorem. ■

If $\boldsymbol{I}$ is the t-image of $\boldsymbol{Cl}$ under the bijection t defined in Theorem 4.2.10, then $\boldsymbol{I}$ and $\boldsymbol{Cl}$ are called t-associated to each other.

Theorem 4.2.11 *The fuzzy sequential topologies on a set X, induced by $\boldsymbol{Cl}$ and $\boldsymbol{I}_{\boldsymbol{Cl}}$ are identical and the fuzzy sequential topologies induced by $\boldsymbol{I}$ and $\boldsymbol{Cl}_{\boldsymbol{I}}$ are identical.*

Proof. Proof is omitted. ■

If we define an fs-interior and an fs-closure operator, separately, on a set, they will induce two fuzzy sequential topologies, which may not be identical in general. In view of Theorem 4.2.10 and Theorem 4.2.11, we give a necessary and sufficient condition that the two fuzzy sequential topologies induced by an fs-interior operator and an fs-closure operator on a set, are identical.

Theorem 4.2.12 *If $\boldsymbol{Cl} \in \mathcal{C}_X$ and $\boldsymbol{I} \in \mathcal{I}_X$, then $\delta_{\boldsymbol{Cl}}(s)$ and $\delta_{\boldsymbol{I}}(s)$ are identical if and only if $\boldsymbol{Cl}$ and $\boldsymbol{I}$ are t-associated to each other.*

Proof. Suppose $\boldsymbol{Cl}$ and $\boldsymbol{I}$ are t-associated to each other. Then $t(\boldsymbol{Cl}) = \boldsymbol{I}_{\boldsymbol{Cl}} = \boldsymbol{I}$. Now,
$A_f(s) \in \delta_{\boldsymbol{I}}(s)$
$\Leftrightarrow \boldsymbol{I}(A_f(s)) = A_f(s)$
$\Leftrightarrow \boldsymbol{I}_{\boldsymbol{Cl}}(A_f(s)) = A_f(s)$
$\Leftrightarrow X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) = A_f(s)$
$\Leftrightarrow \boldsymbol{Cl}((A_f(s))^c) = (A_f(s))^c$
$\Leftrightarrow A_f(s) \in \delta_{\boldsymbol{Cl}}(s)$.

Thus, $\delta_{\boldsymbol{I}}(s)$ and $\delta_{\boldsymbol{Cl}}(s)$ are identical.

Conversely, suppose $\delta_{\mathbf{I}}(s)$ and $\delta_{\boldsymbol{Cl}}(s)$ are identical. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Then

$$\begin{aligned} & (\boldsymbol{Cl}((A_f(s))^c))^c \in \delta_{\boldsymbol{I}}(s) \\ \Rightarrow \quad & \boldsymbol{I}((\boldsymbol{Cl}((A_f(s))^c))^c) = X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \end{aligned}$$

Now,

$$\begin{aligned} & (A_f(s))^c \leq \boldsymbol{Cl}((A_f(s))^c) \\ \Rightarrow\ & (\boldsymbol{Cl}((A_f(s))^c))^c \leq A_f(s) \\ \Rightarrow\ & \boldsymbol{I}((\boldsymbol{Cl}((A_f(s))^c))^c) \leq \boldsymbol{I}(A_f(s)) \\ \Rightarrow\ & X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \leq \boldsymbol{I}(A_f(s)). \end{aligned}$$

Again,

$$\begin{aligned} & \boldsymbol{I}(A_f(s)) \in \delta_{\boldsymbol{Cl}}(s) \\ \Rightarrow\ & \boldsymbol{Cl}((\boldsymbol{I}(A_f(s)))^c) = (\boldsymbol{I}(A_f(s)))^c = X_f^1(s) - \boldsymbol{I}(A_f(s)). \end{aligned}$$

Also,

$$\begin{aligned} & \boldsymbol{I}(A_f(s)) \leq A_f(s) \\ \Rightarrow\ & (A_f(s))^c \leq (\boldsymbol{I}(A_f(s)))^c \\ \Rightarrow\ & \boldsymbol{Cl}((A_f(s))^c) \leq \boldsymbol{Cl}((\boldsymbol{I}(A_f(s)))^c) = X_f^1(s) - \boldsymbol{I}(A_f(s)) \\ \Rightarrow\ & \boldsymbol{I}(A_f(s)) \leq X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c). \end{aligned}$$

Thus, $\boldsymbol{I}(A_f(s)) = X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) = \boldsymbol{I}_{\boldsymbol{Cl}}(A_f(s))\ \forall\ A_f(s) \in (I^X)^{\mathbb{N}}$. Hence $\boldsymbol{I} = \boldsymbol{I}_{\boldsymbol{Cl}} = t(\boldsymbol{Cl})$. ■

Theorem 4.2.13 *If $\boldsymbol{Cl} \in \mathcal{C}_X$, $\boldsymbol{I} \in \mathcal{I}_X$, then the following conditions are equivalent:*

(i) $\boldsymbol{I}$ and $\boldsymbol{Cl}$ are t-associated to each other.

(ii) The fuzzy sequential topologies $\delta_{\boldsymbol{I}}(s)$ and $\delta_{\boldsymbol{Cl}}(s)$ are identical.

(iii) fs-closure systems generated by $\boldsymbol{Cl}$ and $\boldsymbol{I}$ are identical.

(iv) fs-interior systems generated by $\boldsymbol{Cl}$ and $\boldsymbol{I}$ are identical.

Proof. Proof is omitted. ■

Theorem 4.2.13 gives two more necessary and sufficient conditions, ((*iii*) and (*iv*)), under which the fuzzy sequential topologies induced by an fs-interior operator and an fs-closure operator on a set X, are identical.

4.3 Composition of FS-closure and FS-interior operators

In this section, we study the composition of FS-closure and that of FS-interior operators.

Definition 4.3.1 *If $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be two fs-closure operators on X, then the mapping $\boldsymbol{C}_2 \circ \boldsymbol{C}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(A_f(s)) = \boldsymbol{C}_2(\boldsymbol{C}_1(A_f(s))) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is called the composition of the fs-closure operators $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$.

Theorem 4.3.1 *Composition of fs-closure operators is associative.*

Proof. Let $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$ be three fs-closure operators on X. Then $\forall A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} ((\boldsymbol{C}_1 \circ \boldsymbol{C}_2) \circ \boldsymbol{C}_3)(A_f(s)) &= (\boldsymbol{C}_1 \circ \boldsymbol{C}_2)(\boldsymbol{C}_3(A_f(s))) \\ &= \boldsymbol{C}_1(\boldsymbol{C}_2(\boldsymbol{C}_3(A_f(s)))) \\ &= \boldsymbol{C}_1((\boldsymbol{C}_2 \circ \boldsymbol{C}_3)(A_f(s))) \end{aligned}$$

$$= (\boldsymbol{C}_1 \circ (\boldsymbol{C}_2 \circ \boldsymbol{C}_3))(A_f(s))$$

Hence $(\boldsymbol{C}_1 \circ \boldsymbol{C}_2) \circ \boldsymbol{C}_3 = \boldsymbol{C}_1 \circ (\boldsymbol{C}_2 \circ \boldsymbol{C}_3)$. ■

Composition of two fs-closure operators may not be commutative, as shown by Example 4.3.1.

Example 4.3.1 *Let $\boldsymbol{C}_1$ be an fs-closure operator on X, defined by $\boldsymbol{C}_1(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\boldsymbol{C}_1(X_f^0(s)) = X_f^0(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also, let $\boldsymbol{C}_2$ be an fs-closure operator on X, defined by $\boldsymbol{C}_2(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty}$ $\forall A_f(s) \in (I^X)^{\mathbb{N}}$. Then $\boldsymbol{C}_2 \circ \boldsymbol{C}_1 \neq \boldsymbol{C}_1 \circ \boldsymbol{C}_2$.*

Theorem 4.3.2 *If $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be two fs-closure operators on X, then*
(i) $(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(X_f^0(s)) = X_f^0(s)$
(ii) $A_f(s) \leq (\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(A_f(s))$ *for all* $A_f(s) \in (I^X)^{\mathbb{N}}$.
(iii) $(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(A_f(s) \vee B_f(s)) = (\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(A_f(s)) \vee (\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(B_f(s))$ *for all* $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$.

Proof. Proof is omitted. ■

For two fs-closure operators $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ on X, $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ may not be idempotent, as shown by Example 4.3.2.

Example 4.3.2 *Consider the fs-closure operator $\boldsymbol{C}_1$ on X, defined by $\boldsymbol{C}_1(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\boldsymbol{C}_1(X_f^0(s)) = X_f^0(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also, consider the fs-closure operator $\boldsymbol{C}_2$ on X, defined by $\boldsymbol{C}_2(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty}$ $\forall A_f(s) \in (I^X)^{\mathbb{N}}$. Then $(\boldsymbol{C}_2 \circ \boldsymbol{C}_1) \circ (\boldsymbol{C}_2 \circ \boldsymbol{C}_1) \neq (\boldsymbol{C}_2 \circ \boldsymbol{C}_1)$.*

Theorem 4.3.3 *Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be two fs-closure operators on X. Under commutative composition, $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ is an fs-closure operator on X.*

Proof. By virtue of Theorem 4.3.2, we need only to show that $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ is idempotent. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Then,

$$\begin{aligned}
(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(A_f(s))) &= (\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(\boldsymbol{C}_2(\boldsymbol{C}_1(A_f(s)))) \\
&= \boldsymbol{C}_2(\boldsymbol{C}_1(\boldsymbol{C}_2(\boldsymbol{C}_1(A_f(s))))) \\
&= \boldsymbol{C}_2(\boldsymbol{C}_2(\boldsymbol{C}_1(\boldsymbol{C}_1(A_f(s))))) \\
&= \boldsymbol{C}_2(\boldsymbol{C}_1(A_f(s))) \\
&= (\boldsymbol{C}_2 \circ \boldsymbol{C}_1)(A_f(s))
\end{aligned}$$

Hence the theorem. ■

Theorem 4.3.4 *Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be two fs-closure operators on X. Under commutative composition, $\delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s) = \delta_{\boldsymbol{C}_2}(s) \cap \delta_{\boldsymbol{C}_1}(s)$, where $\delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$, $\delta_{\boldsymbol{C}_2}(s)$ and $\delta_{\boldsymbol{C}_1}(s)$ respectively denote the fuzzy sequential topologies on X, induced by $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_1$.*

Proof. Let $A_f(s) \in \delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$, then

$$(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c) = (A_f(s))^c$$

Now,

$$\begin{aligned}
\boldsymbol{C}_1((A_f(s))^c) &= \boldsymbol{C}_1((\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c)) \\
&= \boldsymbol{C}_1((\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c)) \\
&= \boldsymbol{C}_1(\boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c))) \\
&= \boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c)) \\
&= (\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c) \\
&= (A_f(s))^c.
\end{aligned}$$

Similarly, $\boldsymbol{C}_2((A_f(s))^c) = (A_f(s))^c$. Hence $A_f(s) \in \delta_{\boldsymbol{C}_2}(s) \cap \delta_{\boldsymbol{C}_1}(s)$.

Again, let $A_f(s) \in \delta_{\boldsymbol{C}_2}(s) \cap \delta_{\boldsymbol{C}_1}(s)$, then

$$\boldsymbol{C}_1((A_f(s))^c) = (A_f(s))^c \text{ and } \boldsymbol{C}_2((A_f(s))^c) = (A_f(s))^c$$

Now,

$$\begin{aligned}
(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c) &= \boldsymbol{C}_2(\boldsymbol{C}_1((A_f(s))^c)) \\
&= \boldsymbol{C}_2((A_f(s))^c) \\
&= (A_f(s))^c
\end{aligned}$$

Thus, $A_f(s) \in \delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$ and hence the theorem. ■

Theorem 4.3.5 *Under commutative composition, $(\mathcal{C}_X, \circ)$ forms a semigroup with identity.*

Proof. Proof is omitted. ■

Definition 4.3.2 *If $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be two fs-interior operators on X, then the mapping $\boldsymbol{I}_2 \circ \boldsymbol{I}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$(\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s)) = \boldsymbol{I}_2(\boldsymbol{I}_1(A_f(s))) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is called the composition of the fs-interior operators $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$.

Theorem 4.3.6 *Composition of fs-interior operators is associative.*

Proof. Let $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$ be three fs-interior operators on X. Then $\forall A_f(s) \in (I^X)^{\mathbb{N}}$

$$\begin{aligned} ((\boldsymbol{I}_1 \circ \boldsymbol{I}_2) \circ \boldsymbol{I}_3)(A_f(s)) &= (\boldsymbol{I}_1 \circ \boldsymbol{I}_2)(\boldsymbol{I}_3(A_f(s))) \\ &= \boldsymbol{I}_1(\boldsymbol{I}_2(\boldsymbol{I}_3(A_f(s)))) \\ &= (\boldsymbol{I}_1 \circ (\boldsymbol{I}_2 \circ \boldsymbol{I}_3))(A_f(s)) \end{aligned}$$

Hence $(\boldsymbol{I}_1 \circ \boldsymbol{I}_2) \circ \boldsymbol{I}_3 = \boldsymbol{I}_1 \circ (\boldsymbol{I}_2 \circ \boldsymbol{I}_3)$. ∎

Composition of two fs-interior operators may not be commutative, as shown by Example 4.3.3.

Example 4.3.3 *Consider the fs-interior operator $\boldsymbol{I}_1$ on X, defined by $\boldsymbol{I}_1(A_f(s)) = A_f(s) \wedge D_f(s)$ whenever $A_f(s) \neq X_f^1(s)$ and $\boldsymbol{I}_1(X_f^1(s)) = X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also, consider the fs-interior operator $\boldsymbol{I}_2$ on X, defined by $\boldsymbol{I}_2(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty} \ \forall A_f(s) \in (I^X)^{\mathbb{N}}$. Then $\boldsymbol{I}_2 \circ \boldsymbol{I}_1 \neq \boldsymbol{I}_1 \circ \boldsymbol{I}_2$.*

Theorem 4.3.7 *If $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be two fs-interior operators on X, then*
(i) $(\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(X_f^1(s)) = X_f^1(s)$
(ii) $(\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s)) \leq A_f(s)$ *for all* $A_f(s) \in (I^X)^{\mathbb{N}}$.
(iii) $(\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s) \wedge B_f(s)) = (\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s)) \wedge (\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(B_f(s))$ *for all* $A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}}$.

Proof. Proof is omitted. ■

For two fs-interior operators $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ on X, $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$ may not be idempotent, as shown by Example 4.3.4.

Example 4.3.4 *Consider the fs-interior operator $\boldsymbol{I}_1$ on X, defined by $\boldsymbol{I}_1(A_f(s)) = A_f(s) \wedge D_f(s)$ whenever $A_f(s) \neq X_f^1(s)$ and $\boldsymbol{I}_1(X_f^1(s)) = X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also, consider the fs-interior operator $\boldsymbol{I}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\boldsymbol{I}_2(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s) \in (I^X)^{\mathbb{N}}$. Then $(\boldsymbol{I}_2 \circ \boldsymbol{I}_1) \circ (\boldsymbol{I}_2 \circ \boldsymbol{I}_1) \neq (\boldsymbol{I}_2 \circ \boldsymbol{I}_1)$.*

Theorem 4.3.8 *Let $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be two fs-interior operators on X. If the composition is commutative, then $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$ forms an fs-interior operator on X.*

Proof. By virtue of Theorem 4.3.7, we need only to show that, $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$ is idempotent. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Then,

$$\begin{aligned}
(\boldsymbol{I}_2 \circ \boldsymbol{I}_1)((\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s))) &= (\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(\boldsymbol{I}_2(\boldsymbol{I}_1(A_f(s)))) \\
&= \boldsymbol{I}_2(\boldsymbol{I}_1(\boldsymbol{I}_2(\boldsymbol{I}_1(A_f(s))))) \\
&= \boldsymbol{I}_2(\boldsymbol{I}_2(\boldsymbol{I}_1(\boldsymbol{I}_1(A_f(s))))) \\
&= \boldsymbol{I}_2(\boldsymbol{I}_1(A_f(s))) \\
&= (\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s))
\end{aligned}$$

Hence the theorem. ■

Theorem 4.3.9 *Let $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ be two fs-interior operators on X. Under commutative composition, $\delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s) = \delta_{\boldsymbol{I}_2}(s) \cap \delta_{\boldsymbol{I}_1}(s)$, where $\delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s)$, $\delta_{\boldsymbol{I}_2}(s)$ and $\delta_{\boldsymbol{I}_1}(s)$ respectively denote the fuzzy sequential topologies on X, induced by $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$, $\boldsymbol{I}_2$ and $\boldsymbol{I}_1$.*

Proof. Let $A_f(s) \in \delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s)$, then

$$(\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s)) = A_f(s)$$

Now,

$$\begin{aligned}
\boldsymbol{I}_1(A_f(s)) &= \boldsymbol{I}_1((\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s))) \\
&= \boldsymbol{I}_1((\boldsymbol{I}_1 \circ \boldsymbol{I}_2)(A_f(s))) \\
&= \boldsymbol{I}_1(\boldsymbol{I}_1(\boldsymbol{I}_2(A_f(s)))) \\
&= \boldsymbol{I}_1(\boldsymbol{I}_2(A_f(s))) \\
&= (\boldsymbol{I}_1 \circ \boldsymbol{I}_2)(A_f(s)) \\
&= A_f(s).
\end{aligned}$$

Similarly, $\boldsymbol{I}_2(A_f(s)) = A_f(s)$. Hence $A_f(s) \in \delta_{\boldsymbol{I}_2}(s) \cap \delta_{\boldsymbol{I}_1}(s)$. Again, let $A_f(s) \in \delta_{\boldsymbol{I}_2}(s) \cap \delta_{\boldsymbol{I}_1}(s)$, then

$$\boldsymbol{I}_1(A_f(s)) = A_f(s) \text{ and } \boldsymbol{I}_2(A_f(s)) = A_f(s)$$

Now,

$$\begin{aligned}(\boldsymbol{I}_2 \circ \boldsymbol{I}_1)(A_f(s)) &= \boldsymbol{I}_2(\boldsymbol{I}_1(A_f(s))) \\ &= \boldsymbol{I}_2(A_f(s)) \\ &= A_f(s)\end{aligned}$$

Thus, $A_f(s) \in \delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s)$ and hence the theorem. ■

Theorem 4.3.10 *Under commutative composition, $(\mathcal{I}_X, \circ)$ forms a semigroup with identity.*

Proof. Proof is omitted. ■

Theorem 4.3.11 *If $\mathcal{I}_X$ and $\mathcal{C}_X$ are equipped with commutative composition, then there exists a semigroup isomorphism between them.*

Proof. Define $t : \mathcal{C}_X \to \mathcal{I}_X$ by

$$t(\boldsymbol{Cl}) = \boldsymbol{I}_{\boldsymbol{Cl}} \; \forall \boldsymbol{Cl} \in \mathcal{C}_X.$$

From Theorem 4.2.10, t is a bijection. Also for $\boldsymbol{C}_1,\ \boldsymbol{C}_2 \in \mathcal{C}_X$ and $A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned}(\boldsymbol{I}_{\boldsymbol{C}_1} \circ \boldsymbol{I}_{\boldsymbol{C}_2})(A_f(s)) &= \boldsymbol{I}_{\boldsymbol{C}_1}(X_f^1(s) - \boldsymbol{C}_2((A_f(s))^c)) \\ &= X_f^1(s) - \boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c)) \\ &= X_f^1(s) - (\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c) \\ &= \boldsymbol{I}_{\boldsymbol{C}_1 \circ \boldsymbol{C}_2}(A_f(s)).\end{aligned}$$

Therefore,

$$\begin{aligned} t(\boldsymbol{C}_1 \circ \boldsymbol{C}_2) &= \boldsymbol{I}_{\boldsymbol{C}_1 \circ \boldsymbol{C}_2} \\ &= \boldsymbol{I}_{\boldsymbol{C}_1} \circ \boldsymbol{I}_{\boldsymbol{C}_2} \\ &= t(\boldsymbol{C}_1) \circ t(\boldsymbol{C}_2) \end{aligned}$$

Hence t is an isomorphism. ■

4.4 Relative FS-closure Operators and FS-connectors

Relative fs-closure operators and FS-connectors connecting two fuzzy topologies on a set, have been studied in this section.

Definition 4.4.1 *Let $A_f(s)$ be an fs-set in X and $\boldsymbol{Cl}$ be an fs-closure operator on X. A function $(\boldsymbol{Cl})^n_{A_f(s)} : I^X \to I^X$ defined by*

$$(\boldsymbol{Cl})^n_{A_f(s)}(B) = n^{th} \text{ term of } \boldsymbol{Cl}({}_{nB}A_f(s)),$$

where ${}_{nB}A_f(s)$ is the fs-set in X obtained from $A_f(s)$ by replacing n^{th} term of it by B, is called the n^{th} relative fs-closure operator of $\boldsymbol{Cl}$ with respect to $A_f(s)$.

If $\boldsymbol{Cl}$ be an fs-closure operator on X, then it is obvious that $(\boldsymbol{Cl})^n_{X^0_f(s)} = (\boldsymbol{Cl})^n_f$ and consequently $\delta_{(\boldsymbol{Cl})^n_{X^0_f(s)}} = \delta_{(\boldsymbol{Cl})^n_f}$, where $\delta_{(\boldsymbol{Cl})^n_{X^0_f(s)}}$ and $\delta_{(\boldsymbol{Cl})^n_f}$ are the fuzzy topologies induced by $(\boldsymbol{Cl})^n_{X^0_f(s)}$ and $(\boldsymbol{Cl})^n_f$ respectively.

Theorem 4.4.1 *If* $\boldsymbol{Cl}$ *be an fs-closure operator on* X *and* $(\boldsymbol{Cl})^n_{A_f(s)}$ *be the* n^{th} *relative fs-closure operator of* $\boldsymbol{Cl}$*, with respect to an fs-set* $A_f(s)$*. Then,* $(\boldsymbol{Cl})^n_{A_f(s)}$ *satisfies*
(i) $B \leq (\boldsymbol{Cl})^n_{A_f(s)}(B)$ *for all* $B \in I^X$
(ii) $(\boldsymbol{Cl})^n_{A_f(s)}((\boldsymbol{Cl})^n_{A_f(s)}(B)) = (\boldsymbol{Cl})^n_{A_f(s)}(B)$ *for all* $B \in I^X$
(iii) $(\boldsymbol{Cl})^n_{A_f(s)}(B_1 \vee B_2) = (\boldsymbol{Cl})^n_{A_f(s)}(B_1) \vee (\boldsymbol{Cl})^n_{A_f(s)}(B_2)$ *for all* B_1*,* $B_2 \in I^X$*.*

Proof. (i) For $B \in I^X$,

$$_{nB}A_f(s) \leq \boldsymbol{Cl}(_{nB}A_f(s)) \Rightarrow B \leq (\boldsymbol{Cl})^n_{A_f(s)}(B).$$

(ii) For $B \in I^X$,

$$\begin{aligned} B \leq (\boldsymbol{Cl})^n_{A_f(s)}(B) &\Rightarrow\ _{nB}A_f(s) \leq\ _{n(\boldsymbol{Cl})^n_{A_f(s)}(B)}A_f(s) \\ &\Rightarrow\ \boldsymbol{Cl}(_{nB}A_f(s)) \leq \boldsymbol{Cl}(_{n(\boldsymbol{Cl})^n_{A_f(s)}(B)}A_f(s)) \\ &\Rightarrow\ (\boldsymbol{Cl})^n_{A_f(s)}(B) \leq (\boldsymbol{Cl})^n_{A_f(s)}((\boldsymbol{Cl})^n_{A_f(s)}(B)). \end{aligned}$$

Also,

$$\begin{aligned} & \boldsymbol{Cl}(\boldsymbol{Cl}(_{nB}A_f(s))) = \boldsymbol{Cl}(_{nB}A_f(s)) \\ \Rightarrow\ & \boldsymbol{Cl}(_{n(\boldsymbol{Cl})^n_{A_f(s)}(B)}A_f(s)) \leq \boldsymbol{Cl}(_{nB}A_f(s)) \\ \Rightarrow\ & (\boldsymbol{Cl})^n_{A_f(s)}((\boldsymbol{Cl})^n_{A_f(s)}(B)) \leq (\boldsymbol{Cl})^n_{A_f(s)}(B). \end{aligned}$$

Thus, $(\boldsymbol{Cl})^n_{A_f(s)}(B) = (\boldsymbol{Cl})^n_{A_f(s)}((\boldsymbol{Cl})^n_{A_f(s)}(B))$
(iii) Let $B_1, B_2 \in I^X$. Then,

$$\begin{aligned} & \boldsymbol{Cl}(_{nB_1}A_f(s) \vee\ _{nB_2}A_f(s)) = \boldsymbol{Cl}(_{nB_1}A_f(s)) \vee \boldsymbol{Cl}(_{nB_2}A_f(s)) \\ \Rightarrow\ & \boldsymbol{Cl}(_{n(B_1 \vee B_2)}A_f(s)) = \boldsymbol{Cl}(_{nB_1}A_f(s)) \vee \boldsymbol{Cl}(_{nB_2}A_f(s)) \\ \Rightarrow\ & (\boldsymbol{Cl})^n_{A_f(s)}(B_1 \vee B_2) = (\boldsymbol{Cl})^n_{A_f(s)}(B_1) \vee (\boldsymbol{Cl})^n_{A_f(s)}(B_2). \end{aligned}$$

■

The n^{th} relative fs-closure operator $(\boldsymbol{Cl})^n_{A_f(s)}$ of an fs-closure operator $\boldsymbol{Cl}$, with respect to an fs-set $A_f(s)$, may not satisfy $(\boldsymbol{Cl})^n_{A_f(s)}(\overline{0}) = \overline{0}$, as shown by Example 4.4.1. Hence $(\boldsymbol{Cl})^n_{A_f(s)}$ may not be a fuzzy operator.

Example 4.4.1 *Define a function* $\boldsymbol{Cl}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *by*

$$\begin{aligned} \boldsymbol{Cl}(B_f(s)) &= X^1_f(s) \text{ if } B_f(s) \neq X^0_f(s), \\ &= X^0_f(s) \text{ if } B_f(s) = X^0_f(s). \end{aligned}$$

Then for any fs-set $A_f(s) \neq X^0_f(s)$ *in* X, $(\boldsymbol{Cl})^n_{A_f(s)}(\overline{0}) = \overline{1}$ *for all* $n \in \mathbb{N}$.

Theorem 4.4.2 *Let* $(\boldsymbol{Cl})^n_{A_f(s)}$ *be the* n^{th} *relative fs-closure operator of an fs-closure operator* $\boldsymbol{Cl}$ *on* X, *with respect to an fs-set* $A_f(s)$. *Then,* $\delta_{(\boldsymbol{Cl})^n_{A_f(s)}} = \{\overline{1}\} \cup \{B \in I^X; (\boldsymbol{Cl})^n_{A_f(s)}(B^c) = B^c\}$ *forms a fuzzy topology on* X. *Further, the closure in the fuzzy topological space* $(X, \delta_{(\boldsymbol{Cl})^n_{A_f(s)}})$ *and* $(\boldsymbol{Cl})^n_{A_f(s)}$ *are identical on* $I^X - \{\overline{0}\}$.

Proof. Proof is omitted. ■

Definition 4.4.2 *The fuzzy topology* $\delta_{(\boldsymbol{Cl})^n_{A_f(s)}} = \{\overline{1}\} \cup \{B \in I^X;$ $(\boldsymbol{Cl})^n_{A_f(s)}(B^c) = B^c\}$, *induced by the* n^{th} *relative fs-closure oper-*

ator $(\boldsymbol{Cl})^n_{A_f(s)}$, *is called the* n^{th} *relative fuzzy topology induced by the fs-closure operator* $\boldsymbol{Cl}$, *with respect to the fs-set* $A_f(s)$.

Theorem 4.4.3 *Let* $A_f(s)$ *be an fs-set in a set* X *and* $\boldsymbol{Cl}$ *be an fs-closure operator on* X. *Let* $(\boldsymbol{Cl})^n_f$ *be the* n^{th} *component of* $\boldsymbol{Cl}$, $n \in \mathbb{N}$. *Then,*
(1) $\boldsymbol{Cl}(A_f(s)) \geq \{(\boldsymbol{Cl})^n_f(A^n_f)\}$ *and the equality holds if* $A_f(s)$ *is a closed fs-set in* $(X, \delta_{\boldsymbol{Cl}}(s))$.
(2) *If* $\boldsymbol{Cl}(A_f(s)) = \{(\boldsymbol{Cl})^n_f(A^n_f)\}$ *and* A_n *is closed in* $(X, \delta_{(\boldsymbol{Cl})^n_f})$ *for each* $n \in \mathbb{N}$, *then* $A_f(s)$ *is closed in* $(X, \delta_{\boldsymbol{Cl}}(s))$.
(3) $\boldsymbol{Cl}(A_f(s)) = \{(\boldsymbol{Cl})^n_{A_f(s)}(A^n_f)\}$.

Proof. Proof is omitted. ■

In an FSTS $(X, \delta(s))$, if $A_f(s)$ is closed, then A^n_f is closed in (X, δ_n) for each $n \in \mathbb{N}$ but the converse is not true by virtue of Example 2.1.1. Corollary 4.4.1 provides a pair of if and only if conditions for an fs-set $A_f(s)$, to be closed in an FSTS.

Corollary 4.4.1 *In an FSTS* $(X, \delta(s))$, *an fs-set* $A_f(s)$ *is closed:*
(1) *if and only if* $\overline{A_f(s)} = \{B^n_f\}$ *and* A^n_f *is closed in* (X, δ_n) *for each* $n \in \mathbb{N}$, *where* $B^n_f = n^{th}$ *component of* $\overline{{}_{nA^n_f}X^0_f(s)}$.
(2) *if and only if* A^n_f *is closed in* $(X, \delta_{R^n_{A_f(s)}})$ *for each* $n \in \mathbb{N}$, *where* $R^n_{A_f(s)}$ *is the* n^{th} *relative fs-closure operator of the closure operator in* $(X, \delta(s))$, *with respect to* $A_f(s)$.

Theorem 4.4.4 *If* $\{A_{\lambda f}(s);\ \lambda \in \Lambda\}$ *be a chain of fs-sets in*

$((I^X)^{\mathbb{N}}, \leq)$, *then* $\{\delta_{(\boldsymbol{Cl})^n_{A_{\lambda f}(s)}}; \lambda \in \Lambda\}$ *is a chain of fuzzy topologies on* X *for each* $n \in \mathbb{N}$, *where* $\boldsymbol{Cl}$ *is an fs-closure operator on* X.

Proof. Let $A_{\lambda f}(s) \leq A_{\mu f}(s)$, λ, $\mu \in \Lambda$. It suffices to show that $\delta_{(\boldsymbol{Cl})^n_{A_{\mu f}(s)}} \subseteq \delta_{(\boldsymbol{Cl})^n_{A_{\lambda f}(s)}}$. Let $B \in \delta_{(\mathbf{Cl})^n_{A_{\mu f}(s)}}$. Then

$$\begin{aligned} & (\boldsymbol{Cl})^n_{A_{\mu f}(s)}(\overline{1} - B) = \overline{1} - B \\ \Rightarrow\ & n^{th} \text{ term of } \boldsymbol{Cl}(_{n(\overline{1}-B)}A_{\mu f}(s)) = \overline{1} - B \\ \Rightarrow\ & n^{th} \text{ term of } \boldsymbol{Cl}(_{n(\overline{1}-B)}A_{\lambda f}(s)) \leq \overline{1} - B \\ \Rightarrow\ & (\boldsymbol{Cl})^n_{A_{\lambda f}(s)}(\overline{1} - B) \leq \overline{1} - B. \end{aligned}$$

Hence $B \in \delta_{(\boldsymbol{Cl})^n_{A_{\lambda f}(s)}}$. ■

Definition 4.4.3 *Each member of* $\delta_{(\boldsymbol{Cl})^n_{A_f(s)}}$, *except possibly* $\overline{1}$, *is contained in* $\overline{1} - (\boldsymbol{Cl})^n_{A_f(s)}(\overline{0})$ *and hence* $\delta_{(\boldsymbol{Cl})^n_{A_f(s)}}$ *is called* $(\overline{1} - (\boldsymbol{Cl})^n_{A_f(s)}(\overline{0}))$*-cut of* $\delta_{(\boldsymbol{Cl})^n_f}$.

Theorem 4.4.5 *Let* $\{C_n\}$ *be a sequence of fuzzy closure operators on* X. *Then the operator* $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by* $C(A_f(s)) = \{C_n(A^n_f)\}$ *for all* $A_f(s) \in (I^X)^{\mathbb{N}}$, *is an fs-closure operator on* X.

Proof. The proof is omitted. ■

Definition 4.4.4 *The operator* $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by* $C(A_f(s)) = \{C_n(A^n_f)\}$ *for all* $A_f(s) \in (I^X)^{\mathbb{N}}$, *is called an fs-closure operator induced by a sequence* $\{C_n\}$ *of fuzzy closure operators on* X.

Definition 4.4.5 *Let δ and δ' be two fuzzy topologies on a set X. A subset K_f of δ'^{δ} is called an fs-connector of δ to δ' if it satisfies the following conditions:*
(1) $A_\lambda \in \delta$ and $f_\lambda \in K_f$, $\lambda \in \Lambda \Rightarrow$ there exist $f \in K_f$ so that $f(\underset{\lambda\in\Lambda}{\vee} A_\lambda) = \underset{\lambda\in\Lambda}{\vee} f_\lambda(A_\lambda)$,
(2) $A_i \in \delta$ and $f_i \in K_f$, $i = 1, 2, ..., n \Rightarrow$ there exist $f \in K_f$ so that $f(\overset{n}{\underset{i=1}{\wedge}} A_i) = \overset{n}{\underset{i=1}{\wedge}} f_i(A_i)$ and
(3) $\delta' = \underset{f\in K_f}{\vee} f(\delta)$.

Example 4.4.2 *Let δ and δ' be two fuzzy topologies on a set X. A function $f : \delta \to \delta'$ defined by $f(A) = O$ for all $A \in \delta$, where O is a fixed member of δ', is called a constant function from δ into δ'. Now, let K_f be the collection of all such constant functions from δ into δ'. If $A_\lambda \in \delta$ and $f_\lambda \in K_f$, $\lambda \in \Lambda$, then $f_\lambda(A_\lambda) \in \delta'$ for all $\lambda \in \Lambda \Rightarrow \underset{\lambda\in\Lambda}{\vee} f_\lambda(A_\lambda) \in \delta'$. Consider the function $f \in K_f$ which maps each element of δ to $\underset{\lambda\in\Lambda}{\vee} f_\lambda(A_\lambda)$. Since $\underset{\lambda\in\Lambda}{\vee} A_\lambda \in \delta$, we have $f(\underset{\lambda\in\Lambda}{\vee} A_\lambda) = \underset{\lambda\in\Lambda}{\vee} f_\lambda(A_\lambda)$. Again, if $A_i \in \delta$ and $f_i \in K_f$ $(i = 1, 2, ..., n)$, then $f \in K_f$ defined by $f(A) = \overset{n}{\underset{i=1}{\wedge}} f_i(A_i)$ for all $A \in \delta$, is the desired function to satisfy the second condition to be an fs-connector. Also for each $O \in \delta'$, define a function $f_O : \delta \to \delta'$ by $f_O(A) = O$ for all $A \in \delta$. Then $K_f = \{f_O : \delta \to \delta'$, $O \in \delta'\}$ and $f_O(\delta) = \{O\} \Rightarrow \underset{O\in\delta'}{\vee} f_O(\delta) = \delta' \Rightarrow \underset{f\in K_f}{\vee} f(\delta) = \delta'$. Thus, K_f is an fs-connector from δ to δ'.*

Definition 4.4.6 *Let δ and δ' be two fuzzy topologies on a set X. Then the collection of all constant functions from δ into δ', forms* *an fs-connector of δ to δ' and is called the discrete fs-connector of δ to δ'.*

If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X, then any sequence $\{K_n\}$ of fs-connectors such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, provides a unique fuzzy sequential topology on X (See Theorem 4.4.6), which is denoted by $\delta(s) < \{\delta_n\}, \{K_n\} >$ such that the n^{th} components $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$ and it is called the fuzzy sequential topology generated by $\{\delta_n\}$ and $\{K_n\}$. Further, if each K_n is the discrete fs-connector of δ_n to δ_{n+1}, then the fuzzy sequential topology is said to be generated by $\{\delta_n\}$ and is denoted by $\delta(s) < \{\delta_n\} >$.

Theorem 4.4.6 *Let $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X. Then, for any sequence $\{K_n\}$ of fs-connectors, such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, there is a unique fuzzy sequential topology $\delta(s) < \{\delta_n\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$, $n \in \mathbb{N}$. Also, for any FSTS $(X, \delta(s))$, there is a sequence $\{K_n\}$ of fs-connectors, such that K_n connects δ_n to δ_{n+1} and $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$.*

Proof. Let $K = \prod_{n=1}^{\infty} K_n$, $g = \{g_n\} \in K$ and $A \in \delta_1$. Define $H_1 = A$ and $H_n = g_{n-1}g_{n-2}.....g_2g_1A$, $n > 1$. Let $H_A^g(s) = \{H_n\} \in (I^X)^{\mathbb{N}}$ and consider $\delta(s) < \{\delta_n\}, \{K_n\} >= \{X_f^1(s), X_f^0(s)\} \cup \{H_A^g(s); g \in K \text{ and } A \in \delta_1\}$. Consider

$$H_\lambda(s) = H_{A_\lambda}^{g_\lambda}(s) \in \delta(s),\ \lambda \in \Lambda,$$

where Λ is an index set and

$$A = \bigvee_{\lambda\in\Lambda} A_\lambda \in \delta_1.$$

For $g_{\lambda 1} \in K_1$ and $A \in \delta_1$, there exists $g_1 \in K_1$ such that

$$g_1 A = \bigvee_{\lambda\in\Lambda} g_{\lambda 1} A_\lambda, \text{ where } \ g_{\lambda n} \in K_n$$

and for $g_{n-1}g_{n-2}......g_2g_1A \in \delta_n$, there exists $g_n \in K_n$ such that

$$g_n g_{n-1}......g_2 g_1 A = \bigvee_{\lambda\in\Lambda} g_{\lambda n} g_{\lambda(n-1)}....g_{\lambda 2} g_{\lambda 1} A_\lambda.$$

Obviously,

$$\bigvee_{\lambda\in\Lambda} H_\lambda(s) = \bigvee_{\lambda\in\Lambda} H^{g_\lambda}_{A_\lambda}(s) = H^g_A(s) \in \delta(s) < \{\delta_n\}, \{K_n\} >,$$

where $g = g_n$. Arguing in the same way, it can be shown that $\delta(s) < \{\delta_n\}, \{K_n\} >$ is closed under finite intersection. Therefore, $(X, \delta(s) < \{\delta_n\}, \{K_n\} >)$ is a fuzzy sequential topological space. The third condition to be an fs-connector ensures that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$. For the next part, for each $n \in \mathbb{N}$, define a relation $R^{n,n+1}$ on $\delta(s)$ by

$$A_f(s) R^{n,n+1} B_f(s) \text{ if and only if } A^n_f = B^n_f.$$

Then, $R^{n,n+1}$ defines a partition of $\delta(s)$, say

$$\{Cls(A_f(s));\ A_f(s) \in \delta^{n,n+1}(s) \subseteq \delta(s)\},$$

where $\delta^{n,n+1}(s)$ is a family of fs-open sets taking exactly one from each class of the partition of $\delta(s)$ by $R^{n,n+1}$ and $Cls(A_f(s))$ represents the class of $A_f(s)$. Let

$$K^{n,n+1} = \prod_{A_f(s)\in\delta^{n,n+1}(s)} Cls(A_f(s))$$

Then, each $t \in K^{n,n+1}$ defines a function $g_t : \delta_n \to \delta_{n+1}$ and $K_n = \{g_t;\ t \in K^{n,n+1}\}$ is an fs-connector connecting δ_n to δ_{n+1} and properties of fs-connectors ensures that $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$. ■

Corollary 4.4.2 *If* ***Cl*** *be an fs-closure operator on X, then for any sequence $\{K_n\}$ of fs-connectors such that K_n connects $\delta_{(Cl)_f^n}$ to $\delta_{(Cl)_f^{n+1}}$ for all $n \in \mathbb{N}$, there is a unique fuzzy sequential topology $\delta(s) < \{\delta_{(Cl)_f^n}\}, \{K_n\} >$ on X, such that $(\delta(s) < \{\delta_{(Cl)_f^n}\}, \{K_n\} >)_n = \delta_{(Cl)_f^n}$ and the components of the closure operator on $(X, \delta(s) < \{\delta_{(Cl)_f^n}\}, \{K_n\} >)$ are $(Cl)_f^n$, $n \in \mathbb{N}$. Also, for any FSTS $(X,\delta(s))$, there is a sequence $\{K_n\}$ of fs-connectors, such that K_n connects $\delta_{(Cl)_f^n}$ to $\delta_{(Cl)_f^{n+1}}$ and $\delta(s) = \delta(s) < \{\delta_{(Cl)_f^n}\}, \{K_n\} >$.*

Corollary 4.4.3 *If* ***I*** *be an fs-interior operator on X, then for any sequence $\{K_n\}$ of fs-connectors, such that K_n connects $\delta_{(I)_f^n}$ to $\delta_{(I)_f^{n+1}}$ for all $n \in \mathbb{N}$, there is a unique fuzzy sequential topology $\delta(s) < \{\delta_{(I)_f^n}\}, \{K_n\} >$ on X, such that $(\delta(s) < \{\delta_{(I)_f^n}\}, \{K_n\} >)_n = \delta_{(I)_f^n}$ and the components of the interior operator on $(X, \delta(s) < \{\delta_{(I)_f^n}\}, \{K_n\} >)$ are $(I)_f^n$, $n \in \mathbb{N}$. Also, for any FSTS $(X,\delta(s))$, there is a sequence $\{K_n\}$ of fs-connectors, such that K_n connects $\delta_{(I)_f^n}$ to $\delta_{(I)_f^{n+1}}$ and $\delta(s) = \delta(s) < \{\delta_{(I)_f^n}\}, \{K_n\} >$.*

Corollary 4.4.4 *If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X such that $\delta_n = \delta$ for all $n \in \mathbb{N}$, then $\delta(s) < \{\delta_n\} >= \delta^{\mathbb{N}}$.*

Corollary 4.4.5 *If $\{C_n\}$ be a sequence of fuzzy closure operators on a set X and $\boldsymbol{C}$ be an fs-closure operator induced by $\{C_n\}$, then $\delta_{\boldsymbol{C}}(s) = \delta(s) < \{\delta_n\} >$, where δ_n is the fuzzy topology on X induced by C_n, $n \in \mathbb{N}$.*

CHAPTER

5

Compactness

Following the introduction of fuzzy sets by L. A. Zadeh [42], several authors studied various notions of fuzzy topological spaces. Among them, one of the most studied topics is compactness. Fuzzy compact spaces were first studied by C. L. Chang [7] in 1968 and then different kinds of fuzzy compactness were studied by various authors like J.A. Goguen [14], R. Lowen [21, 22], T.E. Gantner, R.C. Steinlage and R.H. Warren [12], Wang Guojun [15], Gunther Jager [18] etc. and they were compared in detail by R. Lowen [23].

Here we present a development of fuzzy sequential topology, which includes the introduction and study of the concepts of continuous functions and compact spaces. Section 5.1 deals with the study of continuous functions, where both nbds and Q-nbds have been used to characterize it and Section 5.2 deals with the notion of compactness, where two types of compactness have been discussed.

5.1 FS-continuity

Let $g : X \to Y$ be a map. For $A_f(s) \in (I^X)^{\mathbb{N}}$ and $B_f(s) \in (I^Y)^{\mathbb{N}}$, $g(A_f(s))$ is an fs-set in Y defined by

$$\begin{aligned} g(A_f(s))(y) &= \{ \sup_{x \in g^{-1}(y)} A_f^n(x)\}_{n=1}^{\infty} \text{ if } g^{-1}(y) \neq \phi, \\ &= X_f^0(s)(y) \text{ if } g^{-1}(y) = \phi, \end{aligned}$$

where $y \in Y$ and $g^{-1}(B_f(s))$ is an fs-set in X defined by

$$g^{-1}(B_f(s))(x) = B_f(s)(g(x)) \; \forall x \in X.$$

If $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$ and $C_f(s), D_f(s) \in (I^Y)^{\mathbb{N}}$, then it is seen that,

(*i*) $g(A_f(s)) = \{g(A_f^n)\}_{n=1}^{\infty}$.

(*ii*) $g^{-1}(C_f(s)) = \{g^{-1}(C_f^n)\}_{n=1}^{\infty}$.

(*iii*) $A_f(s) q_w B_f(s)$ if and only if $g(A_f(s)) q_w g(B_f(s))$.

(*iv*) $C_f(s) q_w D_f(s)$ at some point $y \in Y$ such that $g^{-1}(y) \neq \phi$ if and only if $g^{-1}(C_f(s)) q_w g^{-1}(D_f(s))$.

Theorem 5.1.1 *Let $g : X \to Y$ be a map. For $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$ and $C_f(s)$, $D_f(s) \in (I^Y)^{\mathbb{N}}$,*

(*i*) $g^{-1}((C_f(s))^c) = (g^{-1}(C_f(s)))^c$.

(ii) $(g(A_f(s)))^c(y) \leq g((A_f(s))^c)(y)$ $\forall y \in Y$ *such that* $g^{-1}(y) \neq \phi$ *and* $(g(A_f(s)))^c = g((A_f(s))^c)$ *if* g *is bijective.*
(iii) $A_f(s) \leq B_f(s) \Rightarrow g(A_f(s)) \leq g(B_f(s))$.
(iv) $C_f(s) \leq D_f(s) \Rightarrow g^{-1}(C_f(s)) \leq g^{-1}(D_f(s))$.
(v) $g(g^{-1}(C_f(s))) \leq C_f(s)$ *and the equality holds if* g *is onto.*
(vi) $A_f(s) \leq g^{-1}(g(A_f(s)))$ *and the equality holds if* g *is one-one.*
(vii) *If* $h : Y \to Z$ *be another map. Then,* $(h \circ g)^{-1}(G_f(s)) = g^{-1}(h^{-1}(G_f(s)))$ *for any fs-set* $G_f(s)$ *in* Z*, where* $h \circ g$ *is the composition of* h *and* g.

Proof. Proof is omitted. ■

Definition 5.1.1 *A map* g *from an FSTS* $(X, \delta(s))$ *to an FSTS* $(Y, \eta(s))$ *is called fs-continuous if* $g^{-1}(B_f(s))$ *is an fs-open set in* $(X, \delta(s))$ *for every fs-open set* $B_f(s)$ *in* $(Y, \eta(s))$.

Definition 5.1.2 *Fuzzy sequential sets* $X^l_f(s)$ $(l \in I)$*, in a set* X*, are called constant fs-sets.*

Definition 5.1.3 *An fs-set is called a component constant fs-set if its each component is a constant fuzzy set.*

Clearly, each constant fs-set is component constant.

Remark 5.1.1 *A constant function from an FSTS to another FSTS, may not be fs-continuous, as shown by Example* 5.1.1.

Example 5.1.1 *Let $(X, \delta(s))$ and $(Y, \gamma(s))$ be two fuzzy sequential topological spaces, where X be any set, $\delta(s) = \{X_f^0(s), X_f^1(s)\}$, $Y = [0,1]$, $\gamma(s) = \{X_f^0(s), X_f^1(s), \{id_{[0,1]}\}_{n=1}^{\infty}\}$. Define $g : X \to Y$ by*

$$g(x) = \frac{1}{2} \text{ for all } x \in X.$$

Here, g is a constant function but not fs-continuous.

Theorem 5.1.2 *If every constant function from an FSTS $(X, \delta(s))$ to any other FSTS is fs-continuous, then $\delta(s)$ must contain all the constant fs-sets.*

Proof. Proof is simple and hence omitted. ∎

Remark 5.1.2 *Unlike in case of fuzzy topological spaces, the converse of Theorem* 5.1.2 *may not true, as shown by Example* 5.1.2.

Example 5.1.2 *Let $(X, \delta(s))$ and $(Y, \gamma(s))$ be two fuzzy sequential topological spaces, where X be any set, $\delta(s) = \{X_f^r(s); r \in [0,1]\}$, $Y = [0,1]$, $\gamma(s) = \{X_f^0(s), X_f^1(s), G_f(s)\}$ with $G_f^n = \overline{\frac{1}{3}}$ for n odd and $G_f^n = \overline{\frac{1}{4}}$ for n even. Define $g : X \to Y$ by*

$$g(x) = \frac{1}{2} \text{ for all } x \in X.$$

Though $\delta(s)$ contains all the constant fs-sets, the constant function g is not fs-continuous.

Theorem 5.1.3 *Every constant function from an FSTS $(X, \delta(s))$ to any other FSTS is fs-continuous if and only if $\delta(s)$ contains all the component constant fs-sets.*

Proof. Proof is discarded. ■

Theorem 5.1.4 *If $g : X \to Y$ and $h : Y \to Z$ be fs-continuous functions, then $h \circ g$ is an fs-continuous function from X to Z.*

Proof. The proof is straightforward. ■

Theorem 5.1.5 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:*
(i) g is fs-continuous.
(ii) For each fs-set $A_f(s)$ in X, $g(\overline{A_f(s)}) \leq \overline{g(A_f(s))}$.
(iii) The inverse image of every fs-closed set under g is fs-closed.
(iv) For each fs-set $A_f(s)$ in X, the g inverse of every nbd of $g(A_f(s))$ is a nbd of $A_f(s)$.
(v) For each fs-set $A_f(s)$ in X and each nbd $V_f(s)$ of $g(A_f(s))$, there exists a nbd $W_f(s)$ of $A_f(s)$ such that $g(W_f(s)) \leq V_f(s)$.
(vi) For each fs-set $A_f(s)$ in X, the g inverse of every weak Q-nbd of $g(A_f(s))$ is a weak Q-nbd of $A_f(s)$.
(vii) $\overline{g^{-1}(A_f(s))} \leq g^{-1}(\overline{A_f(s)})$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be an fs-set in X and $P_f(s) \in \overline{A_f(s)}$ be a fuzzy sequential point. So $g(P_f(s)) \in g(\overline{A_f(s)})$. Let $V_f(s)$ be a weak Q-nbd of $g(P_f(s))$. So $g^{-1}(V_f(s))$ is a weak Q-nbd of $P_f(s)$ and hence $g^{-1}(V_f(s)) q_w A_f(s)$. This implies, $V_f(s) q_w g(A_f(s))$ and the result follows.
$(ii) \Rightarrow (iii)$ Let $B_f(s)$ be an fs-closed set in Y and let $A_f(s) = g^{-1}(B_f(s))$. Consider a fuzzy sequential point $P_f(s) \in \overline{A_f(s)}$.

Then, $g(P_f(s)) \in g(\overline{A_f(s)}) \leq \overline{g(A_f(s))} \leq \overline{B_f(s)} = B_f(s)$ so that $P_f(s) \in g^{-1}(B_f(s))$. Hence the result.

$(iii) \Rightarrow (i)$ is straightforward.

$(i) \Rightarrow (iv)$ Let $V_f(s)$ be a nbd of $g(A_f(s))$. So there exists an fs-open set $W_f(s)$ in $(Y, \eta(s))$ such that

$$g(A_f(s)) \leq W_f(s) \leq V_f(s).$$

Then, $g^{-1}(W_f(s))$ is an fs-open set in $(X, \delta(s))$ such that

$$A_f(s) \leq g^{-1}(W_f(s)) \leq g^{-1}(V_f(s)).$$

Hence, $g^{-1}(V_f(s))$ is a nbd of $A_f(s)$.

$(iv) \Rightarrow (i)$ Let $B_f(s)$ be an fs-open set in Y and let a fuzzy sequential point $P_f(s) \in g^{-1}(B_f(s))$. Then $g(P_f(s)) \in B_f(s)$. By (iv), $g^{-1}(B_f(s))$ is a nbd of $P_f(s)$ and thus there exists an fs-open set $O_f(s)$ in X such that $P_f(s) \in O_f(s) \leq g^{-1}(B_f(s))$. Hence the result.

$(iv) \Rightarrow (v)$ and $(v) \Rightarrow (iv)$ are easy to check.

$(i) \Rightarrow (vi)$ Let $V_f(s)$ be a weak Q-nbd of $g(A_f(s))$. So there exists an fs-open set $O_f(s)$ in $(Y, \eta(s))$ such that

$$g(A_f(s)) q_w O_f(s) \leq V_f(s).$$

Then, $g^{-1}(O_f(s))$ is an fs-open set in $(X, \delta(s))$ such that

$$A_f(s) q_w g^{-1}(O_f(s)) \leq g^{-1}(V_f(s)).$$

Hence, $g^{-1}(V_f(s))$ is a weak Q-nbd of $A_f(s)$.

$(vi) \Rightarrow (ii)$, $(i) \Rightarrow (vii)$ and $(vii) \Rightarrow (iii)$ are straightforward. ■

Definition 5.1.4 *A mapping g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, is called an fs-open map if the image of an fs-open set in $(X, \delta(s))$ is an fs-open set in $(Y, \eta(s))$.*

Definition 5.1.5 *A mappping g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, is called an fs-closed map if the image of an fs-closed set in $(X, \delta(s))$ is an fs-closed set in $(Y, \eta(s))$.*

Definition 5.1.6 *A map g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, is called an fs-homeomorphism if g is bijective, g and g^{-1} are both fs-continuous. Further, two fuzzy sequential topological spaces are said to be fs-homeomorphic if there exists an fs-homeomorphism between them.*

Theorem 5.1.6 *A bijective map g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$ is fs-open if and only if it is fs-closed.*

Proof. Proof is obvious. ■

Theorem 5.1.7 *(i) Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-closed map. Given any $B_f(s) \in (I^Y)^{\mathbb{N}}$ and any fs-open set $U_f(s)$ in $(X, \delta(s))$ containing $g^{-1}(B_f(s))$, there exists an fs-open set $V_f(s)$ in $(Y, \eta(s))$ containing $B_f(s)$ such that $g^{-1}(V_f(s)) \leq U_f(s)$.*
(ii) Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-open map. Given any $B_f(s) \in (I^Y)^{\mathbb{N}}$ and any fs-closed set $U_f(s)$ in $(X, \delta(s))$ containing $g^{-1}(B_f(s))$, there exists an fs-closed set $V_f(s)$ in $(Y, \eta(s))$ containing $B_f(s)$ such that $g^{-1}(V_f(s)) \leq U_f(s)$.

Proof. In both (i) and (ii), if we take $V_f(s) = Y_f^1(s) - g(X_f^1(s) - U_f(s))$, we are done. ■

Now, we characterize fs-open maps, fs-closed maps and fs-homeomorphisms, stating some theorems (Theorem 5.1.8 to Theorem 5.1.12), without proofs as the proofs are simple and straightforward.

Theorem 5.1.8 *Let g be a function from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then the following conditions are equivalent:*
(i) *g is fs-open.*
(ii) *$g((A_f(s))^\circ) \leq (g(A_f(s)))^\circ$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.*
(iii) *For each fs-set $A_f(s)$ in X, the g image of every nbd of $A_f(s)$ is a nbd of $g(A_f(s))$.*

Theorem 5.1.9 *Let g be a function from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then the following conditions are equivalent:*
(i) *g is fs-closed.*
(ii) *$\overline{g(A_f(s))} \leq g(\overline{A_f(s)})$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.*

Theorem 5.1.10 *If $g : (X, \delta(s)) \to (Y, \eta(s))$ be a bijective map, then the following conditions are equivalent:*
(i) *g is an fs-homeomorphism.*
(ii) *g is fs-continuous and fs-open.*
(iii) *g is fs-continuous and fs-closed*
(iv) *For each fs-set $A_f(s)$ in X, $g(\overline{A_f(s)}) = \overline{g(A_f(s))}$.*

Theorem 5.1.11 *Two fuzzy topological spaces (X, δ) and (Y, η) are homeomorphic to each other if and only if $(X, \delta^{\mathbb{N}})$ and $(Y, \eta^{\mathbb{N}})$ are fs-homeomorphic.*

Theorem 5.1.12 *If an FSTS $(X, \delta(s))$ is fs-homeomorphic to an FSTS $(Y, \eta(s))$, then the component fuzzy topologies of $(X, \delta(s))$ are homeomorphic to the corresponding component fuzzy topologies of $(Y, \eta(s))$.*

Converse of Theorem 5.1.12 may not be true, which is shown by Example 5.2.1 in our next section.

Theorem 5.1.13 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-closed bijection. If $(X, \delta(s))$ is fs-Hausdorff, so is $(Y, \eta(s))$.*

Proof. Let $P_f(s) = (p^M_{fx}, r)$ and $Q_f(s) = (p^N_{fy}, t)$ be any two distinct fs-points in Y, none of which is completely contained in the other. Then, $g^{-1}(P_f(s)) = P'_f(s)$ (say) and $g^{-1}(Q_f(s)) = Q'_f(s)$ (say) are distinct fs-points in X, with the respective bases M and N and none of which is completely contained in the other and thus there exist $U_f(s), V_f(s) \in \delta(s)$ such that

$$P'_f(s) q_w^{M-N} U_f(s),\ Q'_f(s) q_w V_f(s),\ P'_f(s) \overline{q}_w^{M-N} \overline{V_f(s)},\ Q'_f(s) \overline{q}_w \overline{U_f(s)},$$

whenever $Q'_f(s)$ is a totally reduced fuzzy sequential point from $P'_f(s)$; otherwise $\exists\ U_f(s), V_f(s) \in \delta(s)$ such that

$$P'_f(s) q_w U_f(s),\ Q'_f(s) q_w V_f(s),\ P'_f(s) \overline{q}_w \overline{V_f(s)},\ Q'_f(s) \overline{q}_w \overline{U_f(s)}.$$

If we take $U'_f(s) = g(U_f(s))$ and $V'_f(s) = g(V_f(s))$, then $U'_f(s), V'_f(s) \in \eta(s)$ such that

$$P_f(s)q_w^{M-N}U'_f(s),\ Q_f(s)q_wV'_f(s),\ P_f(s)\overline{q}_w^{M-N}\overline{V'_f(s)},\ Q_f(s)\overline{q}_w\overline{U'_f(s)},$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$; otherwise

$$P_f(s)q_wU'_f(s),\ Q_f(s)q_wV'_f(s),\ P_f(s)\overline{q}_w\overline{V'_f(s)},\ Q_f(s)\overline{q}_w\overline{U'_f(s)}$$

Hence the theorem. ■

Theorem 5.1.14 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-closed and an fs-continuous bijection. If $(X, \delta(s))$ is fs-normal, so is $(Y, \eta(s))$.*

Proof. Let $A_f(s)$ and $B_f(s)$ be any two partially quasi discoincident non zero fs-closed sets in Y, with the respective bases M and N and none of which is completely contained in the other. Then, $g^{-1}(A_f(s)) = A'_f(s)$ (say) and $g^{-1}(B_f(s)) = B'_f(s)$ (say) are partially quasi discoincident non zero fs-closed sets in X, with the respective bases M and N and none of which is completely contained in the other. Thus, there exist $U_f(s), V_f(s) \in \delta(s)$ such that

$$A'_f(s)q_w^{M-N}U_f(s),\ B'_f(s)q_wV_f(s),\ A'_f(s) \leq^{M-N} (\overline{V_f(s)})^c$$
$$\text{and } B'_f(s) \leq (\overline{U_f(s)})^c,$$

whenever $B'_f(s)$ is totally reduced from $A'_f(s)$; otherwise there exist $U_f(s), V_f(s) \in \delta(s)$ such that

$$A'_f(s)q_wU_f(s),\ B'_f(s)q_wV_f(s),\ A'_f(s) \leq (\overline{V_f(s)})^c,\ B'_f(s) \leq (\overline{U_f(s)})^c.$$

Now if we take $U'_f(s) = g(U_f(s))$ and $V'_f(s) = g(V_f(s))$, then $U'_f(s), V'_f(s) \in \eta(s)$ such that

$$A_f(s) q_w^{M-N} U'_f(s),\ B_f(s) q_w V'_f(s),\ A_f(s) \leq^{M-N} (\overline{V'_f(s)})^c$$
$$\text{and } B_f(s) \leq (\overline{U'_f(s)})^c,$$

whenever $B_f(s)$ is totally reduced from $A_f(s)$; otherwise

$$A_f(s) q_w U'_f(s),\ B_f(s) q_w V'_f(s),\ A_f(s) \leq (\overline{V'_f(s)})^c,\ B_f(s) \leq (\overline{U'_f(s)})^c.$$

Hence the theorem. ■

5.2 FS-compactness and ΩFS-compactness

In this section, fs-compact spaces are introduced and studied. It has also been proved that an arbitrary product of fs-compact spaces may not be fs-compact. For this, we introduce a modified version of fs-compactness so called Ωfs-compactness, where the said problem is solved.

Definition 5.2.1 *A family $\mathfrak{B}$ of fs-sets is said to be a cover of an fs-set $A_f(s)$ if $A_f(s) \leq \bigvee\{B_f(s);\ B_f(s) \in \mathfrak{B}\}$. If each member of $\mathfrak{B}$ is open, then it is called an open cover of $A_f(s)$. A subcover of $A_f(s)$ is a subfamily of $\mathfrak{B}$ which is also a cover of $A_f(s)$.*

Definition 5.2.2 *An fs-set $A_f(s)$ is said to be compact if its every open cover has a finite subcover.*

Definition 5.2.3 *An FSTS $(X, \delta(s))$ is called fs-compact if $X^1_f(s)$ is compact.*

Definition 5.2.4 *A family $\mathfrak{B}$ of fs-sets is said to have finite intersection property (FIP) if intersection of the members of each finite subfamily of $\mathfrak{B}$ is non zero.*

Theorem 5.2.1 *An FSTS $(X, \delta(s))$ is fs-compact if and only if each family of fs-closed sets which has the finite intersection property, has a non zero intersection.*

Proof. Suppose $(X, \delta(s))$ is fs-compact. Let $\mathfrak{B}$ be a family of fs-closed sets having the finite intersection property. Suppose further that,

$$\bigwedge\{B_f(s);\ B_f(s) \in \mathfrak{B}\} = X_f^0(s).$$

This implies, $\{(B_f(s))^c;\ B_f(s) \in \mathfrak{B}\}$ is an open cover of $X_f^1(s)$ and hence there exists $\{B_{1f}(s),\ B_{2f}(s), \ldots\ldots,\ B_{kf}(s)\} \subseteq \mathfrak{B}$ such that $\bigvee\limits_{i=1}^{k} (B_{if}(s))^c = X_f^1(s)$ - a contradiction.

Conversely, let $\mathfrak{B}$ be an open cover of $X_f^1(s)$ having no finite subcover. Then $\{(B_f(s))^c;\ B_f(s) \in \mathfrak{B}\}$ is a family of fs-closed sets having the FIP but zero intersection. Hence the result. ∎

Theorem 5.2.2 *An fs-continuous image of an fs-compact space is fs-compact.*

Proof. Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-continuous onto map, where $(X, \delta(s))$ is fs-compact. Let $\mathfrak{B}$ be an open cover of $Y_f^1(s)$, that is, $Y_f^1(s) = \bigvee\limits_{B_f(s) \in \mathfrak{B}} B_f(s)$. Then, $\{g^{-1}(B_f(s)); B_f(s) \in$

$\mathfrak{B}\}$ is an open cover of $X_f^1(s)$ and hence there exist a finite number of fs-sets, say $B_{1f}(s)$, $B_{2f}(s)$,....., $B_{kf}(s) \in \mathfrak{B}$ such that

$$X_f^1(s) = \bigvee_{i=1}^{k} g^{-1}(B_{if}(s)) = g^{-1}(\bigvee_{i=1}^{k} B_{if}(s)).$$

Since g is onto, we have

$$Y_f^1(s) = g(X_f^1(s)) = g(g^{-1}(\bigvee_{i=1}^{k} B_{if}(s))) = \bigvee_{i=1}^{k} B_{if}(s)$$

Hence, $\{B_{if}(s); i = 1, 2, ..., k\}$ is a finite subfamily of $\mathfrak{B}$ covering $Y_f^1(s)$. ■

Corollary 5.2.1 *An fs-homeomorphic image of an fs-compact space is fs-compact.*

Example 5.2.1 *Let (X, δ) be a fuzzy topological space, where $\delta = \{\overline{0}, \overline{1}\}$. Consider the FSTS's $(X, \delta^{\mathbb{N}})$ and $(X, \delta(s))$, where $\delta(s) = \{X_f^0(s), X_f^1(s)\}$. Both the FSTS's have each component fuzzy topologies δ. Again, $(X, \delta(s))$ is fs-compact but $(X, \delta^{\mathbb{N}})$ is not. Thus, $(X, \delta^{\mathbb{N}})$ and $(X, \delta(s))$ are not fs-homeomorphic although their component fuzzy topologies are homeomorphic.*

Theorem 5.2.3 *If an FSTS $(X, \delta(s))$ is fs-compact, then the component fuzzy topological space (X, δ_n) is fuzzy compact for each $n \in \mathbb{N}$.*

Proof. Proof is omitted. ■

Theorem 5.2.4 *If $(X, \delta^{\mathbb{N}})$ if fs-compact, then (X, δ) is fuzzy compact.*

Proof. Let $\mathbb{A}$ be an open cover of $\overline{1}$. For each $A \in \mathbb{A}$, consider the fs-sets $B_{Af}(s) = \{B^n_{Af}\}_{n=1}^{\infty}$, where $B^n_{Af} = A$ for all $n \in \mathbb{N}$. Then, $\{B_{Af}(s); A \in \mathbb{A}\}$ forms an open cover of $X^1_f(s)$ in $(X, \delta^{\mathbb{N}})$ and hence there exist A_1, A_2,..., $A_k \in \mathbb{A}$ such that $X^1_f(s) = \bigvee_{i=1}^{k} B_{A_if}(s)$. Hence, $\{A_1, A_2, ..., A_k\}$ is a finite subfamily of $\mathbb{A}$ covering $\overline{1}$. ∎

Remark 5.2.1 *Converse of Theorem* 5.2.3 *and Theorem* 5.2.4 *may not be true, as shown by Example* 5.2.2.

Example 5.2.2 *Let* (X, δ) *be a fuzzy topological space, where* $\delta = \{\overline{1}, \overline{0}\}$. *Then,* (X, δ) *is fuzzy compact but* $(X, \delta^{\mathbb{N}})$ *is not fs-compact, since the family* $\{A_{kf}(s), k \in \mathbb{N}\}$, *where*

$$\begin{aligned} A^n_{kf} &= \overline{1} \textit{ if } n = k \\ &= \overline{0} \textit{ otherwise}, \end{aligned}$$

is an open cover of $X^1_f(s)$ *in* $(X, \delta^{\mathbb{N}})$, *having no finite subfamily covering* $X^1_f(s)$.

Definition 5.2.5 *For each* $i \in J$, *let* $(X_i, \delta_i(s))$ *be an FSTS. A product fuzzy sequential topology* $\delta(s)$ *on the product* $X = \prod_{i \in J} X_i$ *is the coarsest fuzzy sequential topology on* X, *making all the projection mappings* $\pi_i : X \to X_i$ *fs-continuous. If* $\delta(s)$ *is the product fuzzy sequential topology on* $X = \prod_{i \in J} X_i$, *then* $(X, \delta(s))$ *is called product fuzzy sequential topological space.*

Theorem 5.2.5 *For each* $i \in J$, *let* $(X_i, \delta_i(s))$ *be an FSTS. A subbase for the product fuzzy sequential topology* $\delta(s)$ *on* $X =$

$\prod_{i\in J} X_i$ *is given by* $\mathbb{S} = \{\pi_i^{-1}(O_{if}(s)); O_{if}(s) \in \delta_i(s), i \in J\}$*, so that a basis for* $\delta(s)$ *can be taken to be* $\mathfrak{B} = \{\bigwedge_{j=1}^{n} \pi_{i_j}^{-1}(O_{i_jf}(s)); O_{i_jf}(s) \in \delta_{i_j}(s), i_j \in J, n \in \mathbb{N}\}$.

Proof. Proof is omitted. ■

Lemma 5.2.1 *If* $\mathbb{S}$ *be a subbase for a fuzzy sequential topology* $\delta(s)$ *on* X*, then* $(X, \delta(s))$ *is fs-compact if and only if every open cover of* $X_f^1(s)$ *by the members of* $\mathbb{S}$*, has a finite subcover.*

Proof. Proof is omitted. ■

Definition 5.2.6 *A collection of fs-sets in an FSTS is said to have the finite union property (FUP) if none of its finite sub-collection covers* $X_f^1(s)$.

Theorem 5.2.6 *Let* n *be a positive integer. If* $(X_i, \delta_i(s))$ *be fs-compact spaces for each* $i = 1, 2, \ldots n$ *and* $\delta(s)$ *be the product fuzzy sequential topology on* $X = \prod_{i\in J} X_i$*, then* $(X, \delta(s))$ *is fs-compact.*

Proof. We know that $\mathbb{S} = \{\pi_i^{-1}(O_{if}(s)); O_{if}(s) \in \delta_i(s), i = 1, 2, ..., n\}$ is a subbase for $\delta(s)$. By Lemma 5.2.1, it suffices to show that no sub-collection of $\mathbb{S}$ with FUP covers $X_f^1(s)$. Let $\mathbb{D}$ be a sub-collection of $\mathbb{S}$ with FUP. For each $i = 1, 2, \ldots, n$, let $\mathbb{D}_i = \{O_f(s) \in \delta_i(s) : \pi_i^{-1}(O_f(s)) \in \mathbb{D}\}$. Then $\mathbb{D}_i$ is a collection of fs-open sets in $(X_i, \delta_i(s))$ with FUP. By fs-compactness of

$(X_i, \delta_i(s))$, $\mathbb{D}_i$ cannot cover $X^1_{if}(s)$. So, there exists $x_i \in X_i$ and $m \in \mathbb{N}$ such that

$$\text{the } m^{th} \text{ component of } (\bigvee_{O_f(s)\in\mathbb{D}_i} O_f(s))(x_i) = a_i(\text{ say }) < 1$$

Now, if we consider the point $x = (x_1, x_2, ..., x_n) \in X$ and the collection $\mathbb{D}'_i = \{\pi_i^{-1}(O_f(s)); O_f(s) \in \delta_i(s)\} \cap \mathbb{D}$, then it follows that

$$\begin{aligned} &(\bigvee_{O'_f(s)\in\mathbb{D}'_i} O'_f(s))(x) \\ =\ & \bigvee\{\pi_i^{-1}(O_f(s))(x); O_f(s) \in \delta_i(s) \text{ and } \pi_i^{-1}(O_f(s)) \in \mathbb{D}\} \\ =\ & \bigvee\{O_f(s)(x_i); O_f(s) \in \delta_i(s) \text{ and } \pi_i^{-1}(O_f(s)) \in \mathbb{D}\} \end{aligned}$$

$$= (\bigvee_{O_f(s)\in\mathbb{D}_i} O_f(s))(x_i)$$

Further, noting that $\mathbb{D} = \bigcup_{i=1}^{n} \mathbb{D}'_i$, we obtain

$$\begin{aligned} (\bigvee_{O'_f(s)\in\mathbb{D}} O'_f(s))(x) &= \bigvee_{i=1}^{n}(\bigvee_{O'_f(s)\in\mathbb{D}'_i} O'_f(s))(x) \\ &= \bigvee_{i=1}^{n}(\bigvee_{O_f(s)\in\mathbb{D}_i} O_f(s))(x_i) \end{aligned}$$

Therefore, the m^{th} term of $(\bigvee_{O'_f(s)\in\mathbb{D}} O'_f(s))(x)$ is $\bigvee_{i=1}^{n} a_i$ which is less than 1 and hence the theorem. ■

Remark 5.2.2 *An arbitrary product of fs-compact spaces may not be fs-compact, as shown by Example* 5.2.3.

Example 5.2.3 *For each $i \in \mathbb{N}$, let $X_i = \mathbb{N}$. Let $A_{if}(s)$ be an fs-set in X_i such that $A^n_{if}(x_i) = \frac{i-1}{i}$ $\forall x_i \in X_i$ and $\forall n \in \mathbb{N}$. Let $\delta_i(s) = \{X^0_f(s), X^1_f(s), A_{if}(s)\} \cup \{A_{if}(s) \times \chi_{\{1,2,....,n\}}(s); n \in \mathbb{N}\}$, where $\chi_{\{1,2,....,n\}}(s)$ is an fs-set whose each component is the characteristic function of the set $\{1, 2, ..., n\}$. Then $(X_i, \delta_i(s))$ is an FSTS. Further, if $\{O_{\lambda f}(s); \lambda \in \Lambda\}$ be an open cover of $X^1_{if}(s)$ in $(X_i, \delta_i(s))$, then $O_{\lambda f}(s) = X^1_{if}(s)$ for some $\lambda \in \Lambda$. This implies that $(X_i, \delta_i(s))$ is fs-compact for all $i \in \mathbb{N}$.*

Now, let $\delta(s)$ be the product fuzzy sequential topology on $X = \prod_{i\in\mathbb{N}} X_i$. For $(i, n) \in \mathbb{N} \times \mathbb{N}$,

$$\begin{aligned} \pi_i^{-1}(A_{if}(s) \times \chi_{\{1,2,....,n\}}(s)) &= A_{if}(s) \times \chi_{\{1,2,....,n\}}(s) \circ \pi_i \\ &= B_{if}(s) \ (say) \end{aligned}$$

is a member of $\delta(s)$. Let $x = (x_i)_{i\in\mathbb{N}} \in X$. Then

$$B_{if}(s)(x) = A_{if}(s) \times \chi_{\{1,2,....,n\}}(s)(x_i)$$

which implies, $\forall\ n \in \mathbb{N}$,

$$B^n_{if}(x) = (A^n_{if} \times \chi_{\{1,2,....,n\}})(x_i) = \begin{cases} \frac{i-1}{i} \ if\ x_i \le n \\ \\ 0 \ if\ x_i > n \end{cases}$$

Given $\epsilon > 0$, we can find i with $1 - \epsilon < \frac{i-1}{i}$, which gives $B^n_{if}(x) > 1 - \epsilon$ $\forall\ n \ge x_i$. So $\bigvee_{(i,n)\in\mathbb{N}\times\mathbb{N}} B^n_{if}(x) = 1$ for all $n \in \mathbb{N}$, that is, $\bigvee_{(i,n)\in\mathbb{N}\times\mathbb{N}} B_{if}(s) = X^1_f(s)$. If L be a finite subset of $\mathbb{N} \times \mathbb{N}$, then we can find $N \in \mathbb{N}$ such that whenever $(i, n) \in L$, $n < N$. It follows that, for $x = (N, N, N, ...)$, we have $B^n_{if}(x) = 0$ for all

$(i, n) \in L$ and certainly $\bigvee_{(i,n)\in L} B^n_{if}(x) = 0$. Thus, $\bigvee_{(i,n)\in L} B_{if}(s) \neq X^1_f(s)$ and hence $(X, \delta(s))$ is not fs-compact.

Definition 5.2.7 *A fuzzy sequential topology is called Ω fuzzy sequential topology if it contains all the component constant fs-sets.*

Definition 5.2.8 *An fs-set $A_f(s)$ in an FSTS $(X, \delta(s))$ is said to be Ω-compact if for any open cover $\{B_{if}(s); i \in J\}$ of $A_f(s)$ and for any positive sequence $\epsilon = \{\epsilon_n\}$ of real numbers, there exist finitely many $B_{if}(s)$'s say $B_{i_1f}(s)$, $B_{i_2f}(s)$,......,$B_{i_kf}(s)$, such that*

$$\bigvee_{j=1}^{k} B_{i_jf}(s)(x) \geq A_f(s)(x) - \epsilon \text{ for all } x \in X.$$

Definition 5.2.9 *An Ω fuzzy sequential topological space or $\Omega-FSTS$ is called Ωfs-compact if every component constant fs-set is Ω-compact.*

Theorem 5.2.7 *An fs-continuous image of an Ωfs-compact space is Ωfs-compact.*

Proof. Proof is omitted. ■

Lemma 5.2.2 *Let $\mathbb{S}$ be a subbase for an Ω fuzzy sequential topology $\delta(s)$ on X. Then $(X, \delta(s))$ is Ωfs-compact if and only if for any component constant fs-set $\alpha_f(s)$ with $\bigvee_{i\in J} O_{if}(s) \geq \alpha_f(s)$, where $O_{if}(s) \in \delta(s)$ for all $i \in J$, and for any positive sequence* *of real numbers* $\epsilon = \{\epsilon_n\}$*, there are finitely many indices say* i_1, i_2,....,$i_k \in J$ *such that*

$$\bigvee_{j=1}^{k} O_{i_j f}(s)(x) \;\geq\; \alpha_f(s)(x) - \epsilon \text{ for all } x \in X.$$

Proof. Proof is omitted. ■

Definition 5.2.10 *For an fs-set* $\alpha_f(s) = \{\overline{\alpha}_n\}_{n=1}^{\infty}$ *and for a positive real sequence* $\epsilon = \{\epsilon_n\}$ *with* $\epsilon_n < \alpha_n$ *for each* $n \in \mathbb{N}$*, we say that a collection of fs-sets has* ϵ*-FUP for* $\alpha_f(s)$ *if none of its finite sub-collection covers* $\alpha_f(s) - \epsilon$*.*

Theorem 5.2.8 *Let* $(X_i, \delta_i(s))$ *be* Ω*fs-compact spaces for all* $i \in J$ *(*J *being an index set). Then the product fuzzy sequential topological space* $(X, \delta(s))$*, where* $X = \prod_{i \in J} X_i$*, is* Ω*fs-compact.*

Proof. Let $\alpha_f(s) = \{\overline{\alpha}_n\}_{n=1}^{\infty}$ be a component constant fs-set in X. We wish to show that $\alpha_f(s)$ is Ω-compact. A subbase for $\delta(s)$ is $\mathbb{S} = \{\pi_i^{-1}(O_{if}(s)); O_{if}(s) \in \delta_i(s), i \in J\}$. Let $\epsilon = \{\epsilon_n\}$ be any positive sequence of real numbers with $\epsilon_n < \alpha_n$ for all $n \in \mathbb{N}$ and let $\mathbb{D}$ be a sub-collection of $\mathbb{S}$ with ϵ-FUP for $\alpha_f(s)$. By Lemma 5.2.2, it is sufficient to show that $\mathbb{D}$ does not cover $\alpha_f(s)$.

For each $i \in J$, set $\mathbb{D}_i = \{O_f(s) \in \delta_i(s) : \pi_i^{-1}(O_f(s)) \in \mathbb{D}\}$. Let $O_{i_1 f}(s)$, $O_{i_2 f}(s)$,...,$O_{i_k f}(s) \in \mathbb{D}_i$. Then $\{\pi_i^{-1}(O_{i_j f}(s))\ ; j = 1, 2, ..., k\}$ is a finite sub-collection of $\mathbb{D}$, whence there exists a point $x = (x_i)_{i \in J} \in X$ and $r \in \mathbb{N}$ such that

$$\text{the } r^{th} \text{ component of } \bigvee_{j=1}^{k} \pi_i^{-1}(O_{i_j f}(s))(x) \;<\; \alpha_r - \epsilon_r.$$

It then follows that

$$\begin{aligned}
& \text{the } r^{th} \text{ component of } \bigvee_{j=1}^{k} O_{i_j f}(s)(x_i) \\
= \ & \text{the } r^{th} \text{ component of } \bigvee_{j=1}^{k} O_{i_j f}(s)(\pi_i(x)) \\
= \ & \text{the } r^{th} \text{ component of } \bigvee_{j=1}^{k} \pi_i^{-1}(O_{i_j f}(s))(x) \\
< \ & \alpha_r - \epsilon_r \\
= \ & (\alpha_r - \epsilon_r/2) - \epsilon_r/2
\end{aligned}$$

This implies, $\mathbb{D}_i$ is a collection of fs-open sets in $(X_i, \delta_i(s))$ with $\{\epsilon_n/2\}$-FUP for $\alpha_f(s) - \{\epsilon_n/2\}$.

By Ωfs-compactness of $(X_i, \delta_i(s))$, $\mathbb{D}_i$ cannot cover $\alpha_f(s) - \{\epsilon_n/2\}$. So, there exists $y_i \in X_i$ and $m \in \mathbb{N}$ such that

$$\text{the } m^{th} \text{ component of } (\bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s))(y_i) \ < \ \alpha_m - \epsilon_m/2$$

Having done this for each $i \in J$, set $y = (y_i)_{i \in J}$. If we set $\mathbb{D}'_i = \{\pi_i^{-1}(O_f(s)); O_f(s) \in \delta_i(s)\} \cap \mathbb{D}$, then as in Theorem 5.2.6, $\mathbb{D} = \bigcup_{i \in J} \mathbb{D}'_i$ and

$$\bigvee_{O'_f(s) \in \mathbb{D}'_i} O'_f(s)(y) \ = \ \bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s)(y_i)$$

so that

$$\begin{aligned}
(\bigvee_{O'_f(s) \in \mathbb{D}} O'_f(s))(y) \ &= \ \bigvee_{i \in J}(\bigvee_{O'_f(s) \in \mathbb{D}'_i} O'_f(s))(y) \\
&= \ \bigvee_{i \in J}(\bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s))(y_i)
\end{aligned}$$

Therefore, the m^{th} component of $(\bigvee_{O'_f(s)\in\mathbb{D}} O'_f(s))(y) \leq \alpha_m - \epsilon_m/2$, which is less than α_m. Thus $\mathbb{D}$ cannot cover $\alpha_f(s)$. ■

CHAPTER

6

Some Nearly Open Sets

In this Chapter, some nearly open sets like fs-semiopen, fs-preopen, fs-regular open sets have been studied and interrelations among the continuities associated with them have been investigated.

6.1 FS-semiopen sets and FS-semicontinuity

This section is devoted to the study of fs-semiopen sets and fs-semicontinuity.

Definition 6.1.1 *An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiopen set if $A_f(s) \leq \overline{\overset{o}{A}_f(s)}$. An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiclosed set if its complement is fs-semiopen.*

Fundamental properties of fs-semiopen (fs-semiclosed) sets are:

- Any union (intersection) of fs-semiopen (fs-semiclosed) sets is fs-semiopen (fs-semiclosed).
- Every fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).
- Closure (interior) of an fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).

Example 6.1.1 shows that an fs-semiopen (fs-semiclosed) set may not be fs-open (fs-closed), the intersection (union) of any two fs-semiopen (fs-semiclosed) sets need not be an fs-semiopen (fs-semiclosed) set. Unlike in a general topological space, the intersection of an fs-semiopen set with an fs-open set may fail to be an fs-semiopen set.

Example 6.1.1 *Consider the fs-sets $A_f(s)$, $B_f(s)$ and $C_f(s)$ in $X = [0,1]$, defined as follows:*

$$\begin{aligned} A_f^1(x) &= 0, \text{ if } 0 \le x \le \frac{1}{2} \\ &= \frac{1}{4}, \text{ if } \frac{1}{2} < x \le 1 \\ \text{and } A_f^n &= \overline{1} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} B_f^1(x) &= \frac{1}{2}, \text{ if } 0 \le x \le \frac{1}{2} \\ &= 0, \text{ if } \frac{1}{2} < x \le 1 \\ \text{and } B_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

$$C_f^1 = \overline{\frac{3}{8}} \text{ and } C_f^n = \overline{1} \text{ for all } n \neq 1.$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s),\ X_f^1(s)\}$. *Then* $(X, \delta(s))$ *is an FSTS. Now,*

(i) $B_f(s)$ *and* $C_f(s)$ *are fs-semiopen sets but their intersection is not fs-semiopen.*

(ii) $C_f(s)$ *is fs-semiopen but is not fs-open.*

Theorem 6.1.1 *Let* $(X, \delta(s))$ *be an FSTS. An fs-set* $A_f(s)$ *is fs-semiopen if and only if there exists an fs-open set* $O_f(s)$ *in* X *such that* $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$.

Proof. Straightforward. ■

Theorem 6.1.2 *Let* $(X, \delta(s))$ *be an FSTS. An fs-set* $A_f(s)$ *is fs-semiclosed if and only if there exists an fs-closed set* $C_f(s)$ *in* X *such that* $\overset{o}{C}_f(s) \leq A_f(s) \leq C_f(s)$.

Proof. Straightforward. ■

We will denote the set of all fs-semiopen sets in X by $FSSO(X)$.

Theorem 6.1.3 *In an FSTS* $(X, \delta(s))$, *(i)* $\delta(s) \subseteq FSSO(X)$, *(ii) If* $A_f(s) \in FSSO(X)$ *and* $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, *then* $B_f(s) \in FSSO(X)$.

Proof. (i) Follows from definition.

(ii) Let $A_f(s) \in FSSO(X)$. Then, there exists an fs-open set $O_f(s)$ such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. So,

$$\begin{aligned} & O_f(s) \leq A_f(s) \leq B_f(s) \leq \overline{A_f(s)} \leq \overline{O_f(s)} \\ \Rightarrow\ & O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \end{aligned}$$

Since $O_f(s)$ is fs-open, $B_f(s)$ is fs-semiopen. ■

Theorem 6.1.4 *If* $\overset{o}{C}_f(s) \leq B_f(s) \leq C_f(s)$ *holds in an FSTS* $(X, \delta(s))$*, where* $C_f(s)$ *is fs-semiclosed and* $B_f(s)$ *is any fs-set, then* $B_f(s)$ *is also fs-semiclosed.*

Proof. Omitted. ■

Theorem 6.1.5 *Let* $\mathfrak{A} = \{A_{\alpha f}(s); \alpha \in \Lambda\}$ *be a collection of fs-sets in an FSTS* $(X, \delta(s))$ *such that* (i) $\delta(s) \subseteq \mathfrak{A}$ *and* (ii) *if* $A_f(s) \in \mathfrak{A}$ *and* $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$*, then* $B_f(s) \in \mathfrak{A}$*. Then,* $FSSO(X) \subseteq \mathfrak{A}$*, that is,* $FSSO(X)$ *is the smallest class of fs-sets in* X *satisfying* (i) *and* (ii)*.*

Proof. Let $A_f(s) \in FSSO(X)$. Then, $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$ for some $O_f(s) \in \delta(s)$. By (i), $O_f(s) \in \mathfrak{A}$ and thus $A_f(s) \in \mathfrak{A}$ by (ii). ■

If $\mathfrak{A} = \{A_{\lambda f}(s); \lambda \in \Lambda\}$ be a collection of fs-sets in X, then $Int(\mathfrak{A})$ denotes the set $\{\overset{o}{A}_{\lambda f}(s); \lambda \in \Lambda\}$.

Theorem 6.1.6 *If* $(X, \delta(s))$ *be a fuzzy sequential topological space, then* $\delta(s) = Int(FSSO(X))$*.*

Proof. It is clear that $\delta(s) \subseteq Int(FSSO(X))$. Conversely, let $O_f(s) \in Int(FSSO(X))$. Then, $O_f(s) = \overset{o}{A}_f(s)$ for some $A_f(s) \in FSSO(X)$ and hence $O_f(s) \in \delta(s)$. ■

Definition 6.1.2 *Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. We define fs-semiclosure ${}_scl(A_f(s))$ and fs-semiinterior ${}_sint(A_f(s))$ of $A_f(s)$ by*

$$\begin{aligned} {}_scl(A_f(s)) &= \wedge\{B_f(s); A_f(s) \leq B_f(s) \text{ and } B_f^c(s) \in FSSO(X)\} \\ {}_sint(A_f(s)) &= \vee\{C_f(s); C_f(s) \leq A_f(s) \text{ and } C_f(s) \in FSSO(X)\} \end{aligned}$$

Obviously, ${}_scl(A_f(s))$ is the smallest fs-semiclosed set containing $A_f(s)$ and ${}_sint(A_f(s))$ is the largest fs-semiopen set contained in $A_f(s)$. Further,
(i) $A_f(s) \leq {}_scl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_sint(A_f(s)) \leq A_f(s)$.
(ii) $A_f(s)$ is fs-semiopen if and only if $A_f(s) = {}_sint(A_f(s))$
(iii) $A_f(s)$ is fs-semiclosed if and only if $A_f(s) = {}_scl(A_f(s))$
(iv) $A_f(s) \leq B_f(s) \Rightarrow {}_sint(A_f(s)) \leq {}_sint(B_f(s))$ and ${}_scl(A_f(s)) \leq_s cl(B_f(s))$.

Definition 6.1.3 *A mapping $g : (X, \delta(s)) \to (Y, \delta'(s))$ is said to be*
(i) fs-semicontinuous if $g^{-1}(B_f(s))$ is fs-semiopen in X for every $B_f(s) \in \delta'(s)$.
(ii) fs-semiopen if $g(A_f(s))$ is fs-semiopen in Y for every $A_f(s) \in \delta(s)$.

(iii) fs-semiclosed if $g(A_f(s))$ is fs-semiclosed in Y for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-semicontinuous (fs-semiopen, fs-semiclosed). That the converse may not be true, is shown by Example 6.1.2.

Example 6.1.2 *Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$. Define $g : (X, \delta(s)) \to (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-semicontinuous but not fs-continuous.

Again, the map $h : (X, \delta'(s)) \to (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-semiopen and fs-semiclosed but is neither fs-open nor fs-closed.

Theorem 6.1.7 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:*

(i) g is fs-semicontinuous.

(ii) the inverse image of an fs-closed set in Y under g is fs-semiclosed in X.

(iii) For any fs-set $A_f(s)$ in X, $g({}_scl(A_f(s))) \leq \overline{g(A_f(s))}$.

Proof. $(i) \Rightarrow (ii)$ Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous map and $B_f(s)$ be an fs-closed set in Y. Then,

$$\begin{aligned}
& B^c_f(s) \text{ is fs-open in } Y \\
\Rightarrow\ & (g^{-1}(B_f(s)))^c = g^{-1}(B^c_f(s)) \text{ is fs-semiopen in } X \\
\Rightarrow\ & g^{-1}(B_f(s)) \text{ is fs-semiclosed in } X.
\end{aligned}$$

$(ii) \Rightarrow (iii)$ Suppose $A_f(s)$ be an fs-set in X. Then by (ii), $g^{-1}(\overline{g(A_f(s))})$ is fs-semiclosed in X and hence $g^{-1}(\overline{g(A_f(s))}) = {}_scl(g^{-1}(\overline{g(A_f(s))}))$. Again,

$$\begin{aligned}
& A_f(s) \leq g^{-1}(g(A_f(s))) \\
\Rightarrow\ & {}_scl(A_f(s)) \leq {}_scl(g^{-1}(\overline{g(A_f(s))})) = g^{-1}(\overline{g(A_f(s))}) \\
\Rightarrow\ & g({}_scl(A_f(s))) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}.
\end{aligned}$$

$(iii) \Rightarrow (i)$ Let $B_f(s)$ be an fs-open set in Y. Then, for the fs-closed set $B^c_f(s)$, we have

$$g({}_scl(g^{-1}(B^c_f(s)))) \leq \overline{g(g^{-1}(B^c_f(s)))} \leq \overline{B^c_f(s)} = B^c_f(s)$$

Thus, ${}_scl(g^{-1}(B^c_f(s))) \leq g^{-1}(g({}_scl(g^{-1}(B^c_f(s))))) \leq g^{-1}(B^c_f(s))$. Therefore, ${}_scl(g^{-1}(B^c_f(s))) = g^{-1}(B^c_f(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B^c_f(s))$ is fs-semiclosed in X. ■

Theorem 6.1.8 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiopen set in Y, is fs-semiopen in X.*

Proof. Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} & O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \\ \Rightarrow \ & g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)}) \end{aligned}$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies, $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $g(U_f(s))$ is a weak open Q-nbd of $g(P_f(s))$. Therefore,

$$\begin{aligned} & g(U_f(s)) q_w O_f(s) \\ \Rightarrow \ & U_f(s) q_w g^{-1}(O_f(s)) \\ \Rightarrow \ & P_f(s) \in \overline{g^{-1}(O_f(s))}. \end{aligned}$$

Thus we have, $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Hence, $g^{-1}(O_f(s))$ being fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen. ■

Corollary 6.1.1 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiclosed set in Y is fs-semiclosed in X.*

Proof. The proof is omitted. ■

Corollary 6.1.2 *Composition of an fs-semicontinuous open map $g : (X, \delta(s)) \to (Y, \delta'(s))$ and an fs-semicontinuous map h :*

$(Y, \delta'(s)) \to (Z, \eta(s))$, *that is, the map* $hog : (X, \delta(s)) \to (Z, \eta(s))$ *is fs-semicontinuous.*

Proof. Let $C_f(s)$ be an fs-open set in Z, then $h^{-1}(C_f(s))$ is fs-semiopen in Y and hence $(hog)^{-1}(C_f(s)) = g^{-1}(h^{-1}(C_f(s)))$ is fs-semiopen in X by Theorem 6.1.8. ■

Theorem 6.1.9 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *be an fs-continuous open map. Then the* g*-image of an fs-semiopen set in* X *is fs-semiopen in* Y*.*

Proof. Let $A_f(s)$ be an fs-semiopen set in X. Then, there exists an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. This implies, $g(O_f(s)) \leq g(A_f(s)) \leq g(\overline{O_f(s)}) \leq \overline{g(O_f(s))}$. Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-semiopen in Y. ■

Corollary 6.1.3 *Semi-openness in an FSTS, is a topological property.*

Proof. Follows from Theorem 6.1.9. ■

Remark 6.1.1 *Theorem* 6.1.9 *does not hold if* g *is not fs-open. This is shown by Example* 6.1.3.

Example 6.1.3 *Let* $(X, \delta(s))$ *and* $(X, \delta'(s))$ *be two fuzzy sequential topological spaces, where* $\delta(s)$ *contains all the constant fs-sets in* X *and* $\delta'(s) = \{X_f^0(s), X_f^1(s)\}$*. Define a map* $g : (X, \delta(s)) \to (X, \delta'(s))$ *by* $g(x) = x$ *for all* $x \in X$*. Then,* g *is fs-continuous* *but not fs-open. Here, for the fs-semiopen set* $A_f(s) = \overline{\{\frac{1}{2}\}}_{n=1}^{\infty}$ *in* $(X, \delta(s))$*,* $g(A_f(s)) = \overline{\{\frac{1}{2}\}}_{n=1}^{\infty}$*, which is not fs-semiopen in* $(X, \delta'(s))$*.*

Theorem 6.1.10 *Let* $g : (X, \delta(s)) \to (Y, \delta'(s))$ *and* $h : (Y, \delta'(s)) \to (Z, \eta(s))$ *be two mappings and* $hog : (X, \delta(s)) \to (Z, \eta(s))$ *be an fs-semiclosed mapping. Then,* g *is fs-semiclosed if* h *is an injective fs-semicontinuous open mapping.*

Proof. Let $A_f(s)$ be an fs-closed set in X. Then $hog(A_f(s))$ is fs-semiclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-semiclosed in Y. ■

Theorem 6.1.11 *For an fs-semicontinuous map* $g : (X, \delta(s)) \to (Y, \delta'(s))$ *and an fs-continuous map* $h : (Y, \delta'(s)) \to (Z, \eta(s))$*, the map* $hog : (X, \delta(s)) \to (Z, \eta(s))$ *is fs-semicontinuous.*

Proof. Omitted. ■

6.2 FS-regular open sets

Definition 6.2.1 *An fs-set* $A_f(s)$ *in an FSTS* $(X, \delta(s))$*, is said to be fs-regular open if* $(\overline{A_f(s)})^o = A_f(s)$*. An fs-set is said to be fs-regular closed if its complement is fs-regular open.*

It is obvious that every fs-regular open (closed) set is fs-open (closed). That the converse need not be true, is shown by Example 6.2.1. Example 6.2.2 shows that the union (intersection) of any two fs-regular open (closed) sets need not be an fs-regular open (closed) set.

Example 6.2.1 *Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $A_f(s)$ is fs-open but not fs-regular open.

Example 6.2.2 *Consider the fs-sets $A_f(s)$, $B_f(s)$ and $C_f(s)$ in $X = [0, 1]$, defined as follows:*

$$\begin{aligned} A_f^1(x) &= 0, \text{ if } 0 \leq x \leq \frac{1}{2} \\ &= \frac{2}{3}, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } A_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} B_f^1(x) &= 1, \text{ if } 0 \leq x \leq \frac{1}{4} \\ &= \frac{1}{2} \text{ if } \frac{1}{4} < x \leq \frac{1}{2} \\ &= 0, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } B_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $A_f(s)$ and $B_f(s)$ are fs-regular open sets but their union is not fs-regular open.

Theorem 6.2.1 (a) *The intersection of any two fs-regular open sets is an fs-regular open set.*

(b) *The union of any two fs-regular closed sets is an fs-regular closed set.*

Proof. We prove only (a). Let $A_f(s)$ and $B_f(s)$ be two fs-regular open sets in X. Since $A_f(s) \wedge B_f(s)$ is fs-open, we have $A_f(s) \wedge B_f(s) \leq (\overline{A_f(s) \wedge B_f(s)})^o$. Now, $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{A_f(s)})^o = A_f(s)$ and $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{B_f(s)})^o = B_f(s)$, which implies $(\overline{A_f(s) \wedge B_f(s)})^o \leq A_f(s) \wedge B_f(s)$. Hence the result. ∎

Theorem 6.2.2 (a) *Closure of an fs-semiopen set is fs-regular closed.*

(b) *Interior of an fs-semiclosed set is fs-regular open.*

Proof. We prove only (a). Let $A_f(s)$ be an fs-semiopen set in X. Since $(\overline{A_f(s)})^o \leq \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \leq \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-semiopen, $A_f(s) \leq \overline{\overset{o}{A}_f(s)} \leq \overline{(\overline{A_f(s)})^o}$ and hence $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ is fs-regular closed. ∎

Definition 6.2.2 *A mapping* $g : (X, \delta(s)) \to (Y, \eta(s))$ *is called an fs-almost continuous mapping if* $g^{-1}(B_f(s)) \in \delta(s)$ *for each fs-regular open set* $B_f(s)$ *in* Y.

Theorem 6.2.3 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *be a mapping. Then the following are equivalent:*

(i) g *is fs-almost continuous.*

(ii) $g^{-1}(B_f(s))$ *is an fs-closed set for each fs-regular closed set* $B_f(s)$ *of* Y.

(iii) $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$ *for each fs-open set* $B_f(s)$ *of* Y.

(iv) $g^{-1}(\overline{\overset{o}{B}_f(s)}) \leq g^{-1}(B_f(s))$ *for each fs-closed set* $B_f(s)$ *of* Y.

Proof. Note that $g^{-1}(B^c_f(s)) = (g^{-1}(B_f(s)))^c$ for any fs-set $B_f(s)$ in Y.

$(i) \Leftrightarrow (ii)$ Follows from the fact that an fs-set is fs-regular open if and only if its complement is fs-regular closed.

$(i) \Rightarrow (iii)$ Let $B_f(s)$ be an fs-open set in Y. Then $B_f(s) \leq (\overline{B_f(s)})^o$ and hence $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$. By Theorem 6.3.4 (b), $(\overline{B_f(s)})^o$ is an fs-regular open set in Y. Therefore, $g^{-1}((\overline{B_f(s)})^o)$ is fs-open in X and thus $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$ $= (g^{-1}((\overline{B_f(s)})^o))^o$.

$(iii) \Rightarrow (i)$ Let $B_f(s)$ be an fs-regular open set in Y. Then by (iii), we have $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o = (g^{-1}(B_f(s)))^o$. Hence $g^{-1}(B_f(s))$ is an fs-open set in X.

$(ii) \Leftrightarrow (iv)$ are easy to prove. ■

Clearly, an fs-continuous map is an fs-almost continuous map but the converse need not be true, as is shown by Example 6.2.3.

Example 6.2.3 *Consider the fs-sets* $A_f(s)$, $B_f(s)$ *in a set* X, *defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$ *and* $\eta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s), X_f^1(s)\}$. *Then,* $(X, \delta(s))$ *and* $(X, \eta(s))$ *are fuzzy sequential topological spaces. Define a map* $g : (X, \delta(s)) \to (X, \eta(s))$ *by* $g(x) = x$ *for all* $x \in X$. *Then,* g *is fs-almost continuous but not fs-continuous. Again, since the inverse image of the fs-open set* $A_f(s)$ *of* $(X, \eta(s))$ *is not fs-semiopen in* $(X, \delta(s))$, g *is not fs-semicontinuous.*

Example 6.2.4 *Example to show that an fs-semicontinuous map may not be fs-almost continuous.*

Consider the fs-sets $A_f(s)$, $B_f(s)$ *in a set* X, *as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, \overline{0}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ *and* $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. *Then,* $(X, \delta(s))$ *and* $(X, \eta(s))$ *are fuzzy sequential topological spaces. Define a map* $g : (X, \delta(s)) \to (X, \eta(s))$ *by* $g(x) = x$ *for all* $x \in X$. *Then,* g *is fs-semicontinuous but not fs-almost continuous.*

Remark 6.2.1 *Example* 6.2.3 *and Example* 6.2.4 *shows that an* *fs-almost continuous mapping and an fs-semicontinuous mapping are independent notions.*

Definition 6.2.3 *An FSTS $(X, \delta(s))$ is called an fs-semiregular space if the collection of all fs-regular open sets in X forms a base for $\delta(s)$.*

Theorem 6.2.4 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping, where $(Y, \eta(s))$ is an fs-semiregular space. Then, g is fs-almost continuous if and only if g is fs-continuous.*

Proof. We need only to show that if g is fs-almost continuous, then it is fs-continuous.

Suppose g is fs-almost continuous. Let $B_f(s) \in \eta(s)$, then $B_f(s) = \bigvee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where $B_{\lambda f}(s)$'s are fs-regular open sets in Y. Then

$$\begin{aligned} g^{-1}(B_f(s)) &= \bigvee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \bigvee_{\lambda\in\Lambda} (g^{-1}((\overline{B_{\lambda f}(s)})^o))^o \\ &= \bigvee_{\lambda\in\Lambda} (g^{-1}(B_{\lambda f}(s)))^o \\ &\leq (\bigvee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)))^o \\ &= (g^{-1}(B_f(s)))^o, \end{aligned}$$

which shows that $g^{-1}(B_f(s)) \in \delta(s)$. ■

Theorem 6.2.5 *Let X, X_1 and X_2 be fuzzy sequential topological spaces and $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \to X_1 \times X_2$ is fs-almost continuous, then $\pi_i o g$ is also fs-almost continuous.*

Proof. Let g be an fs-almost continuous map and let $B_f(s)$ be an fs-regular open set in X_i. Since π_i is fs-continuous, we have, $\overline{\pi_i^{-1}(A_f(s))} \leq \pi_i^{-1}(\overline{A_f(s)})$ and $\pi_i^{-1}(\overset{o}{A}_f(s)) \leq (\pi_i^{-1}(A_f(s)))^o$ for any fs-set $A_f(s)$ in X_i. Now,

$$\begin{aligned}
& \pi_i((\pi_i^{-1}(A_f(s)))^o) \leq \pi_i(\pi_i^{-1}(A_f(s))) \leq A_f(s) \\
\Rightarrow\ & \pi_i((\pi_i^{-1}(A_f(s)))^o) \leq \overset{o}{A}_f(s) \\
\Rightarrow\ & (\pi_i^{-1}(A_f(s)))^o \leq \pi_i^{-1}(\pi_i((\pi_i^{-1}(A_f(s)))^o)) \leq \pi_i^{-1}(\overset{o}{A}_f(s)) \\
\Rightarrow\ & \pi_i^{-1}(\overset{o}{A}_f(s)) = (\pi_i^{-1}(A_f(s)))^o
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\pi_i og)^{-1}(B_f(s)) &= g^{-1}(\pi_i^{-1}(B_f(s))) \\
&\leq (g^{-1}((\overline{\pi_i^{-1}(B_f(s))})^o))^o \\
&\leq (g^{-1}((\pi_i^{-1}(\overline{B_f(s)}))^o))^o \\
&= (g^{-1}(\pi_i^{-1}((\overline{B_f(s)})^o)))^o \\
&= (g^{-1}(\pi_i^{-1}(B_f(s))))^o \\
&= ((\pi_i og)^{-1}(B_f(s)))^o
\end{aligned}$$

Hence the theorem. ■

Definition 6.2.4 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-weakly continuous mapping if for each fs-open set $B_f(s)$ in Y, $g^{-1}(B_f(s)) \leq (g^{-1}(\overline{B_f(s)}))^o$.*

Remark 6.2.2 *It is clear that every fs-continuous mapping is fs-weakly continuous. That the converse may not true, in general,* *is shown by Example* 6.2.5. *Example* 6.2.5 *also shows that an fs-weakly continuous mapping may neither be fs-semicontinuous nor fs-almost continuous. However, it is clear that an fs-almost continuous mapping is fs-weakly continuous.*

Example 6.2.5 *Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \ldots\ldots\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{3}}, \overline{\frac{1}{3}}, \overline{\frac{1}{3}}, \ldots\ldots\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)$ $\}$. Then, $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then, g is fs-weakly continuous but not fs-continuous. Since the inverse image of the fs-open set $B_f(s)$ of Y is not fs-semiopen in X, g is not fs-semicontinuous. Again, as the inverse image of the fs-regular open set $B_f(s)$ of Y is not fs-open in X, g is not fs-almost continuous.

Remark 6.2.3 *The map g defined in Example* 6.2.4, *is not fs-weakly continuous but is fs-semicontinuous.*

Remark 6.2.4 *Example* 6.2.5 *and Remark* 6.2.3 *shows that fs-semicontinuity and fs-weakly continuity are independent notions.*

Definition 6.2.5 *An FSTS $(X, \delta(s))$ is called an Ωfs-semiregular space if each fs-open set $A_f(s)$ of X is the union of fs-open sets $A_{\lambda f}(s)(\lambda \in \Lambda)$ of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$.*

Theorem 6.2.6 *An Ωfs-semiregular space is fs-semiregular.*

Proof. Let $(X, \delta(s))$ be an Ωfs-semiregular space and $A_f(s)$ be an fs-open set in X. Then $A_f(s) = \underset{\lambda\in\Lambda}{\vee} A_{\lambda f}(s)$, where $A_{\lambda f}(s)$ are fs-open sets of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$. Since $A_{\lambda f}(s) \leq (\overline{A_{\lambda f}(s)})^o \leq A_f(s)$ for each $\lambda \in \Lambda$, we have $A_f(s) = \underset{\lambda\in\Lambda}{\vee} (\overline{A_{\lambda f}(s)})^o$. Now, for each $\lambda \in \Lambda$, $(\overline{A_{\lambda f}(s)})^o$ is fs-regular open in X and thus $(X, \delta(s))$ is fs-semiregular. ■

Remark 6.2.5 *Example* 6.2.6 *shows that the converse of Theorem* 6.2.6 *may not be true.*

Example 6.2.6 *Consider the fuzzy sequential topology $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ on a set X, where $A_f(s) = \left\{\overline{\tfrac{1}{4}}, \overline{0}, \overline{0}, ...\right\}$. Then $(X, \delta(s))$ is an fs-semiregular space. If $\{A_{\lambda f}(s); \lambda \in \Lambda\} \subseteq \delta(s)$ such that $A_f(s) = \underset{\lambda\in\Lambda}{\vee} A_{\lambda f}(s)$. Then $A_{\lambda f}(s) = A_f(s)$ for some $\lambda \in \Lambda$. Since $\overline{A_f(s)}$ is not contained in $A_f(s)$, $(X, \delta(s))$ is not an Ωfs-semiregular space.*

Theorem 6.2.7 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping where $(X, \delta(s))$ is any FSTS and $(Y, \eta(s))$ is an Ωfs-semiregular space. Then, g is fs-weakly continuous if and only if g is fs-continuous.*

Proof. It suffices to show that if g is fs-weakly continuous, then it is fs-continuous. Let $B_f(s) \in \eta(s)$. Then $B_f(s) = \underset{\lambda\in\Lambda}{\vee} B_{\lambda f}(s)$, where for all $\lambda \in \Lambda$, $B_{\lambda f}(s) \in \eta(s)$ and $\overline{B_{\lambda f}(s)} \leq B_f(s)$. Since g

is fs- weakly continuous, we have

$$\begin{aligned} g^{-1}(B_f(s)) = g^{-1}(\bigvee_{\lambda\in\Lambda} B_{\lambda f}(s)) &= \bigvee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \bigvee_{\lambda\in\Lambda}(g^{-1}(\overline{B_{\lambda f}(s)}))^o \\ &\leq \bigvee_{\lambda\in\Lambda}(g^{-1}(B_f(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

and hence $g^{-1}(B_f(s))$ is fs-open in X. Thus, g is fs-continuous. ■

Theorem 6.2.8 *Let X, X_1 and X_2 be FSTS's and $\pi_i : X_1 \times X_2 \rightarrow X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \rightarrow X_1 \times X_2$ is fs-weakly continuous, then $\pi_i o g$ is also fs-weakly continuous.*

Proof. The proof is analogous to the proof of Theorem 6.2.5. ■

6.3 FS-preopen sets and FS-precontinuity

Definition 6.3.1 (*i*) *An fs-set $A_f(s)$ in an FSTS, is said to be an fs-preopen set if $A_f(s) \leq (\overline{A_f(s)})^o$.*
(*ii*) *An fs-set $A_f(s)$ in an FSTS, is said to be an fs-preclosed set if its complement is fs-preopen or equivalently if $\overline{\overset{o}{A}_f(s)} \leq A_f(s)$.*

If $A_f(s)$ is both fs-preopen and fs-preclosed, then it is called an fs-preclopen set.

Definition 6.3.2 *An fs-set $A_f(s)$ is called fs-dense in an FSTS $(X, \delta(s))$, if $\overline{A_f(s)} = X_f^1(s)$.*

Fundamental properties of fs-preopen (fs-preclosed) sets are:

- Every fs-open (fs-closed) set is fs-preopen (fs-preclosed).
- Arbitrary union (intersection) of fs-preopen (fs-preclosed) sets is fs-preopen (fs-preclosed).

Example 6.3.1 shows that an fs-preopen (fs-preclosed) set may not be fs-open (fs-closed), the intersection (union) of any two fs-preopen (fs-preclosed) sets need not be an fs-preopen (fs-preclosed) set. Unlike in a general topological space, the intersection of an fs-preopen set with an fs-open set may fail to be an fs-preopen set.

Example 6.3.1 *Consider the fs-sets $A_f(s)$, $B_f(s)$ and $C_f(s)$ in*

$X = [0, 1]$, *defined as follows:*

$$\begin{aligned} A_f^1(x) &= 0,\ \textit{if}\ 0 \le x \le \frac{1}{2} \\ &= \frac{1}{4},\ \textit{if}\ \frac{1}{2} < x \le 1 \\ \textit{and}\ A_f^n &= \overline{1}\ \textit{for all}\ n \neq 1. \end{aligned}$$

$$\begin{aligned} B_f^1(x) &= \frac{1}{2},\ \textit{if}\ 0 \le x \le \frac{1}{2} \\ &= 0,\ \textit{if}\ \frac{1}{2} < x \le 1 \\ \textit{and}\ B_f^n &= \overline{0}\ \textit{for all}\ n \neq 1. \end{aligned}$$

$$\begin{aligned} C_f^1(x) &= \frac{3}{4},\ \textit{if}\ 0 \le x \le \frac{1}{2} \\ &= 1,\ \textit{if}\ \frac{1}{2} < x \le 1 \\ \textit{and}\ C_f^n &= \overline{0}\ \textit{for all}\ n \neq 1. \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s),\ X_f^1(s)\}$. *Then* $(X, \delta(s))$ *is an FSTS. Now,*

(i) $A_f(s)$ *and* $C_f(s)$ *are fs-preopen sets but their intersection is not fs-preopen.*

(ii) $C_f(s)$ *is fs-preopen but is not fs-open.*

Theorem 6.3.1 *Let* $(X, \delta(s))$ *be an FSTS. An fs-set* $A_f(s)$ *is fs-preopen if and only if there exists an fs-open set* $O_f(s)$ *in* X *such that* $A_f(s) \le O_f(s) \le \overline{A_f(s)}$.

Proof. Straightforward. ∎

Corollary 6.3.1 *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-preclosed if and only if there exists an fs-closed set $C_f(s)$ in X such that $\overset{o}{\overline{A_f(s)}} \le C_f(s) \le A_f(s)$.*

Proof. Straightforward. ■

Theorem 6.3.2 *An fs-set is fs-clopen (both fs-closed and fs-open) if and only if it is fs-closed and fs-preopen.*

Proof. Proof is omitted. ■

Theorem 6.3.3 *In an FSTS, every fs-set is fs-preopen if and only if every fs-open set is fs-closed.*

Proof. Suppose every fs-set in an FSTS $(X, \delta(s))$, is fs-preopen and let $A_f(s)$ be an fs-open set. Then, $A^c_f(s) = \overline{A^c_f(s)}$ is fs-preopen and hence $\overline{A^c_f(s)} \le (\overline{\overline{A^c_f(s)}})^o = (\overline{A^c_f(s)})^o = (A^c_f(s))^o$. Thus, $A^c_f(s)$ is fs-open and hence $A_f(s)$ is fs-closed.

Conversely, suppose every fs-open set is fs-closed and let $A_f(s)$ be any fs-set. By the assumption, $\overline{A_f(s)} = (\overline{A_f(s)})^o$ and hence $A_f(s)$ is fs-preopen. ■

Theorem 6.3.4 (a) *Closure of an fs-preopen set is fs-regular closed.*

(b) *Interior of an fs-preclosed set is fs-regular open.*

Proof. We prove only (a). Let $A_f(s)$ be an fs-preopen set in X. Since $(\overline{A_f(s)})^o \le \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \le \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-preopen, $A_f(s) \le (\overline{A_f(s)})^o$ and hence $\overline{A_f(s)} \le \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ is fs-regular closed. ■

The set of all fs-preopen sets in X, is denoted by $FSPO(X)$.

Theorem 6.3.5 *In an FSTS* $(X, \delta(s))$, *(i)* $\delta(s) \subseteq FSPO(X)$, *(ii) If* $V_f(s) \in FSPO(X)$ *and* $U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$, *then* $U_f(s) \in FSPO(X)$.

Proof. (i) Follows from definition.

(ii) Let $V_f(s) \in FSPO(X)$, that is, $V_f(s) \leq (\overline{V_f(s)})^o$. We have,

$$U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$$

Therefore, $U_f(s) \leq V_f(s) \leq (\overline{V_f(s)})^o \leq (\overline{U_f(s)})^o$. Hence the result. ■

Definition 6.3.3 *An fs-set* $A_f(s)$ *in an FSTS, is called an fs-preneighbourhood of an fs-point* $P_f(s) = (p^M_{fx}, r)$, *if there exists an fs-preopen set* $B_f(s)$ *such that* $P_f(s) \leq B_f(s) \leq A_f(s)$.

Theorem 6.3.6 *For an fs-set* $A_f(s)$ *in an FSTS* $(X, \delta(s))$, *the following are equivalent:*

(i) $A_f(s)$ *is fs-preopen.*

(ii) There exists an fs-regular open set $B_f(s)$ *containing* $A_f(s)$ *such that* $\overline{A_f(s)} = \overline{B_f(s)}$.

(iii) ${}_scl(A_f(s)) = (\overline{A_f(s)})^o$.

(iv) The semi-closure of $A_f(s)$ *is fs-regular open.*

(v) $A_f(s)$ *is an fs-preneighbourhood of each of its fs-points.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be fs-preopen. This implies

$$\begin{aligned} & A_f(s) \leq (\overline{A_f(s)})^o \leq \overline{A_f(s)} \\ \Rightarrow\ & \overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o} \leq \overline{A_f(s)} \\ \Rightarrow\ & \overline{(\overline{A_f(s)})^o} = \overline{A_f(s)} \\ \Rightarrow\ & \overline{A_f(s)} = \overline{B_f(s)} \end{aligned}$$

where $B_f(s) = (\overline{A_f(s)})^o$ is an fs-regular open set containing $A_f(s)$.

$(ii) \Rightarrow (iii)$ Let $\overline{A_f(s)} = \overline{B_f(s)}$, where $B_f(s)$ is an fs-regular open set containing $A_f(s)$. Then,

$$A_f(s) \leq B_f(s) \ = \ (\overline{B_f(s)})^o = (\overline{A_f(s)})^o$$

Also, $(\overline{A_f(s)})^o$ is fs-semiclosed. Let $C_f(s)$ be an fs-semiclosed set containing $A_f(s)$. Thus,

$$(\overline{A_f(s)})^o \leq (\overline{C_f(s)})^o \leq C_f(s).$$

Hence ${}_s cl(A_f(s)) = (\overline{A_f(s)})^o$.

$(iii) \Rightarrow (iv)$ Obvious.

$(iv) \Rightarrow (i)$ Suppose ${}_s cl(A_f(s))$ is fs-regular open. Now,

$$\begin{aligned} & A_f(s) \leq \ {}_s cl(A_f(s)) \\ \Rightarrow\ & (\overline{A_f(s)})^o \leq (\overline{{}_s cl(A_f(s))})^o = \ {}_s cl(A_f(s)) \leq (\overline{A_f(s)})^o \\ \Rightarrow\ & A_f(s) \leq \ {}_s cl(A_f(s)) = (\overline{A_f(s)})^o \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

$(i) \Rightarrow (v)$ and $(v) \Rightarrow (i)$ are obvious. ■

Corollary 6.3.2 *An fs-set is fs-regular open if and only if it is fs-semiclosed and fs-preopen.*

Proof. Proof is omitted. ■

Theorem 6.3.7 *In an FSTS $(X, \delta(s))$, the following are equivalent:*

(i) $\overline{A_f(s)} \in \delta(s)$ *for all* $A_f(s) \in \delta(s)$.

(ii) Every fs-regular closed set in X *is fs-preopen.*

(iii) Every fs-semiopen set in X *is fs-preopen.*

(iv) The closure of every fs-preopen set in X *is fs-open.*

(v) The closure of every fs-preopen set in X *is fs-preopen.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be an fs-regular closed set, that is, $\overline{\overset{o}{A}_f(s)} = A_f(s)$. By (i), $A_f(s) \in \delta(s)$ and hence $A_f(s)$ is fs-preopen.

$(ii) \Rightarrow (iii)$ Let $A_f(s)$ be an fs-semiopen set, that is, $A_f(s) \leq \overline{\overset{o}{A}_f(s)}$. By (ii), $\overline{\overset{o}{A}_f(s)}$ is fs-preopen. Also, we have, $A_f(s) \leq \overline{\overset{o}{A}_f(s)} \leq \overline{A_f(s)}$. Thus, $A_f(s)$ is fs-preopen.

$(iii) \Rightarrow (iv)$ Let $A_f(s)$ be an fs-preopen set, that is, $A_f(s) \leq (\overline{A_f(s)})^o$. This implies, $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ being fs-semiopen, is fs-preopen and the result follows.

$(iv) \Rightarrow (v)$ Obvious.

$(v) \Rightarrow (i)$ Let $A_f(s) \in \delta(s)$. Then, $A_f(s)$ is fs-preopen and hence $\overline{A_f(s)}$ is fs-preopen. Therefore, $\overline{A_f(s)} \leq (\overline{A_f(s)})^o \leq \overline{A_f(s)}$ and hence $\overline{A_f(s)} \in \delta(s)$.

■

Theorem 6.3.8 *In an FSTS $(X, \delta(s))$, the following are equivalent:*
(i) Every non-zero fs-open set is fs-dense.
(ii) For every non-zero fs-preopen set $A_f(s)$, we have ${}_s cl(A_f(s)) = X_f^1(s)$.
(iii) Every non-zero fs-preopen set is fs-dense.

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be a non-zero fs-preopen set. By Theorem 6.3.6 (iii), ${}_s cl(A_f(s)) = (\overline{A_f(s)})^o$. Also, there exists an fs-open set $O_f(s)$ such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. By (i), $\overline{O_f(s)} = X_f^1(s)$. Therefore, $\overline{A_f(s)} = X_f^1(s)$ and hence ${}_s cl(A_f(s)) = X_f^1(s)$.

$(ii) \Rightarrow (iii)$ Easy to prove.

$(iii) \Rightarrow (i)$ Since every fs-open set is fs-preopen, the proof is straightforward. ■

Definition 6.3.4 *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. We define fs-preclosure ${}_p cl(A_f(s))$ and fs-preinterior ${}_p int(A_f(s))$ of $A_f(s)$ by*

$$\begin{aligned} {}_p cl(A_f(s)) &= \wedge\{B_f(s); A_f(s) \leq B_f(s) \text{ and } B_f^c(s) \in FSPO(X)\} \\ {}_p int(A_f(s)) &= \vee\{C_f(s); C_f(s) \leq A_f(s) \text{ and } C_f(s) \in FSPO(X)\} \end{aligned}$$

Clearly, ${}_p cl(A_f(s))$ is the smallest fs-preclosed set containing $A_f(s)$ and ${}_p int(A_f(s))$ is the largest fs-preopen set contained in $A_f(s)$. Further,
(i) $A_f(s) \leq {}_p cl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_p int(A_f(s)) \leq$

$A_f(s)$.
(ii) $A_f(s)$ is fs-preopen if and only if $A_f(s) = {}_pint(A_f(s))$
(iii) $A_f(s)$ is fs-preclosed if and only if $A_f(s) = {}_pcl(A_f(s))$
(iv) $A_f(s) \leq B_f(s) \Rightarrow {}_pint(A_f(s)) \leq {}_pint(B_f(s))$ and ${}_pcl(A_f(s)) \leq {}_pcl(B_f(s))$.

Definition 6.3.5 *A mapping $g : (X, \delta(s)) \to (Y, \delta'(s))$ is said to be*
(i) fs-precontinuous if $g^{-1}(B_f(s))$ is fs-preopen in X, for every $B_f(s) \in \delta'(s)$.
(ii) fs-preopen if $g(A_f(s))$ is fs-preopen in Y, for every $A_f(s) \in \delta(s)$.
(iii) fs-preclosed if $g(A_f(s))$ is fs-preclosed in Y, for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-precontinuous (fs-preopen, fs-preclosed). That the converse may not be true, is shown by Example 6.3.2.

Example 6.3.2 *Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$*. Then* $(X, \delta(s))$ *is an FSTS. Let* $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$ *and define* $g : (X, \delta(s)) \to (X, \delta'(s))$ *by* $g(x) = x$ *for all* $x \in X$*. The function* g *is fs-precontinuous but not fs-continuous.*

Again, the map $h : (X, \delta'(s)) \to (X, \delta(s))$ *defined by* $h(x) = x$ *for all* $x \in X$*, is both fs-preopen and fs-preclosed but neither fs-open nor fs-closed.*

Theorem 6.3.9 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *be a map. Then the following conditions are equivalent:*
(i) *g is fs-precontinuous.*
(ii) *the inverse image of an fs-closed set in* Y *under* g*, is fs-preclosed in* X*.*
(iii) *For any fs-set* $A_f(s)$ *in* X*,* $g({}_pcl(A_f(s))) \leq \overline{g(A_f(s))}$*.*

Proof. $(i) \Rightarrow (ii)$ Suppose g be an fs-precontinuous map and $B_f(s)$ be an fs-closed set in Y. Then,

$$\begin{aligned} & B_f^c(s) \text{ is fs-open in } Y \\ \Rightarrow\ & (g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s)) \text{ is fs-preopen in } X \\ \Rightarrow\ & g^{-1}(B_f(s)) \text{ is fs-preclosed in } X. \end{aligned}$$

$(ii) \Rightarrow (iii)$ Let $A_f(s)$ be an fs-set in X. Then, $g^{-1}(\overline{g(A_f(s))})$ is fs-preclosed in X and hence $g^{-1}(\overline{g(A_f(s))}) = {}_pcl(g^{-1}(\overline{g(A_f(s))}))$.

Again,

$$\begin{aligned} & A_f(s) \leq g^{-1}(g(A_f(s))) \\ \Rightarrow\ & {}_pcl(A_f(s)) \leq {}_pcl(g^{-1}(g(A_f(s)))) \leq g^{-1}(\overline{g(A_f(s))}) \\ \Rightarrow\ & g({}_pcl(A_f(s))) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}. \end{aligned}$$

$(iii) \Rightarrow (i)$ Let $B_f(s)$ be an fs-open set in Y. Then for the fs-closed set $B_f^c(s)$, we have

$$g({}_pcl(g^{-1}(B_f^c(s)))) \leq \overline{g(g^{-1}(B_f^c(s)))} \leq \overline{B_f^c(s)} = B_f^c(s)$$

Thus, ${}_pcl(g^{-1}(B_f^c(s))) \leq g^{-1}(B_f^c(s))$. Therefore, ${}_pcl(g^{-1}(B_f^c(s))) = g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s))$ is fs-preclosed in X. ■

In Theorem 6.1.8, it has been proved that the inverse image of an fs-semiopen set is fs-semiopen, under an fs-semicontinuous open map. The next Theorem shows that the result is true even if we take an fs-semicontinuous preopen map.

Theorem 6.3.10 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiopen set in Y under g, is fs-semiopen in X.*

Proof. Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} & O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \\ \Rightarrow\ & g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)}) \end{aligned}$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $\overline{g(U_f(s))}$ is a weak Q-nbd of $g(P_f(s))$. Therefore,

$$
\begin{aligned}
& \overline{g(U_f(s))} q_w O_f(s) \\
\Rightarrow\ & W_f(s) q_w O_f(s) \text{ where } W_f(s) = \overline{g(U_f(s))} \\
\Rightarrow\ & W_f^n(y) + O_f^n(y) > 1 \text{ for some } y \in Y \\
\Rightarrow\ & O_f(s) \text{ is a weak open Q-nbd of the fs-point } (p_{fy}^n, W_f^n(y)) \\
\Rightarrow\ & g(U_f(s)) q_w O_f(s) \\
\Rightarrow\ & U_f(s) q_w g^{-1}(O_f(s)) \\
\Rightarrow\ & P_f(s) \in \overline{g^{-1}(O_f(s))}.
\end{aligned}
$$

Thus $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Since $g^{-1}(O_f(s))$ is fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen. ■

Corollary 6.3.3 *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiclosed set in Y under g, is fs-semiclosed in X.*

Proof. The proof is omitted. ■

Corollary 6.3.4 *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-semicontinuous preopen map and $h : (Y, \eta(s)) \rightarrow (Z, \delta'(s))$ be fs-semicontinuous. Then hog is fs-semicontinuous.*

Proof. The proof is omitted. ■

Theorem 6.3.11 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preopen set in Y under g, is fs-preopen in X.*

Proof. Let $B_f(s)$ be an fs-preopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} & B_f(s) \leq O_f(s) \leq \overline{B_f(s)} \\ \Rightarrow\ & g^{-1}(B_f(s)) \leq g^{-1}(O_f(s)) \leq g^{-1}(\overline{B_f(s)}). \end{aligned}$$

As in Theorem 6.3.10, we have $g^{-1}(\overline{B_f(s)}) \leq \overline{g^{-1}(B_f(s))}$. Thus $g^{-1}(B_f(s)) \leq g^{-1}(O_f(s)) \leq \overline{g^{-1}(B_f(s))}$, where $g^{-1}(O_f(s))$ is fs-preopen. Hence $g^{-1}(B_f(s))$ is fs-preopen. ■

Corollary 6.3.5 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preclosed set in Y under g, is fs-preclosed in X.*

Proof. The proof is omitted. ■

Corollary 6.3.6 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen map and $h : (Y, \eta(s)) \to (Z, \delta'(s))$ be an fs-precontinuous map. Then hog is fs-precontinuous.*

Proof. The proof is omitted. ■

Theorem 6.3.12 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-continuous open map. Then the g-image of an fs-preopen set in X, is fs-preopen in Y.*

Proof. Let $A_f(s)$ be an fs-preopen set in X. Then there exists an fs-open set $O_f(s)$ in X such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. This implies $g(A_f(s)) \leq g(O_f(s)) \leq \overline{g(A_f(s))}$. Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-preopen. ■

Corollary 6.3.7 *Pre-openness in an FSTS, is a topological property.*

Proof. Proof follows from Theorem 6.3.12. ■

Theorem 6.3.13 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *and* $h : (Y, \eta(s)) \to (Z, \delta'(s))$ *be two mappings, such that hog is fs-preclosed. Then g is fs-preclosed if h is an injective fs-precontinuous preopen mapping.*

Proof. Let $A_f(s)$ be an fs-closed set in X. Then, $hog(A_f(s))$ is fs-preclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-preclosed in Y. ■

Theorem 6.3.14 *If* $g : (X, \delta(s)) \to (Y, \delta'(s))$ *be fs-precontinuous and* $h : (Y, \delta'(s)) \to (Z, \eta(s))$ *be fs-continuous, then* $hog : (X, \delta(s)) \to (Z, \eta(s))$ *is fs-precontinuous.*

Proof. Omitted. ■

Previously, we showed that the intersection of any two fs-preopen sets may not be fs-preopen and an fs-preopen set may not be fs-open. Now, we investigate and establish conditions, under which the intersection of any two fs-preopen sets is fs-preopen and conditions, under which an fs-preopen set is fs-open.

Theorem 6.3.15 *The intersection of any two fs-preopen sets is fs-preopen if the closure is preserved under finite intersection.*

Proof. Proof is simple and hence omitted. ∎

Theorem 6.3.16 *In an FSTS $(X, \delta(s))$, if every fs-set is either fs-open or fs-closed, then every fs-preopen set in X is fs-open.*

Proof. Let $A_f(s)$ be an fs-preopen set in X. If $A_f(s)$ is not fs-open, then it is fs-closed and hence $\overline{A_f(s)} = A_f(s)$. Therefore, $A_f(s) \leq (\overline{A_f(s)})^o = \overset{o}{A}_f(s)$ and hence the theorem. ∎

For a fuzzy sequential topological space $(X, \delta(s))$, $\delta^*(s)$ will denote the fuzzy sequential topology on X, obtained by taking $FSPO(X)$ as a subbase.

Definition 6.3.6 *A mapping $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is called strongly fs-precontinuous if the inverse image of each fs-preopen set in Y is fs-open in X.*

By the definition of a strong fs-precontinuous mapping, the following two results are obvious.

Proposition 6.3.1 *(i) A map $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is strongly fs-precontinuous if and only if $g : (X, \delta(s)) \rightarrow (Y, \eta^*(s))$ is fs-continuous.*

(ii) If $g : (X, \delta(s)) \to (Y, \eta(s))$ is strongly fs-precontinuous, then it is fs-continuous.

Remark 6.3.1 *Converse of (ii) of Proposition 6.3.1 may not be true, as is shown by the following Example.*

Example 6.3.3 *Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Consider the identity map $id : (X, \delta(s)) \to (X, \delta(s))$. Then, id is fs-continuous but not strongly fs-precontinuous, as the inverse image of fs-preopen set $C_f(s)$ is not fs-open.

We conclude the section with a necessary and sufficient condition for an fs-preopen set to be fs-open.

Theorem 6.3.17 *In an FSTS $(Y, \eta(s))$, the following are equivalent:*

(i) Every fs-preopen set in Y is fs-open.

(ii) Every fs-continuous function $g : (X, \delta(s)) \to (Y, \eta(s))$ is strongly fs-precontinuous, where $(X, \delta(s))$ is any FSTS.

Proof. $(i) \Rightarrow (ii)$ is straightforward.

$(ii) \Rightarrow (i)$ The identity map $g : (Y, \eta(s)) \to (Y, \eta(s))$ is fs-continuous and hence is strongly fs-precontinuous. Let $B_f(s)$ be an fs-preopen set in Y, then $B_f(s) = g^{-1}(B_f(s))$ is fs-open in Y. ■

CHAPTER

7

Decomposition of Continuity

In the last few decades, there has been interests in the study of generalized open sets and generalized continuity in a topological space and various authors studied different kinds of generalized open sets. In the fuzzy setting, fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad [1] (1981) and fuzzy pre-open sets [25] (1982) by A. S. Mashhour, I. A. Hasanein and S. N. EI-Deeb. Apart from these, other sets like fuzzy α-open, fuzzy locally closed, fuzzy δ-set etc. have also been studied in the past.

In **Chapter** 6, we studied generalized open sets like fs-semiopen, fs-preopen and fs-regular open sets in a fuzzy sequential topological space. Here, we study some more of such sets and the respective continuities. Finally, we establish a decomposition of fs-continuity. For our convenience, we denote the closure and interior by *cl* and *int* respectively.

7.1 Decomposition of fs-continuity

Definition 7.1.1 *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is called*

(i) fs α-open if $A_f(s) \leq int\ cl\ int A_f(s)$;

(ii) locally fs-closed if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-closed;

(iii) an fs $\mathcal{A}$-set if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed;

(iv) an fs δ-set if $int\ cl A_f(s) \leq cl\ int A_f(s)$;

(v) fs S-preopen if $A_f(s)$ is fs-preopen and $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $int V_f(s)$ is fs-regular open.

We denote the collection of all fs α-open sets, fs-semiopen sets, fs-preopen sets, fs $\mathcal{A}$-sets, fs S-preopen sets, locally fs-closed sets and fs δ-sets in an FSTS $(X, \delta(s))$, by $\alpha(X)$, $FSSO(X)$, $FSPO(X)$, $\mathcal{A}(X)$, $FSSPO(X)$, $FSLC(X)$ and $\delta(X)$ respectively.

The relationships among different fs-sets defined above, are given by the following diagram:

$$
\begin{tikzcd}
 & \text{fs } S\text{-preopen} \arrow[dr] & \\
\text{locally fs-closed} & \text{fs-open} \arrow[l] \arrow[r] \arrow[u] \arrow[dl] \arrow[d] & \text{fs-preopen} \\
\text{fs } \mathcal{A}\text{-set} \arrow[u] & \text{fs } \alpha\text{-open} \arrow[ur] \arrow[d] & \\
 & \text{fs-semiopen} \arrow[d] & \\
 & \text{fs } \delta\text{-set} &
\end{tikzcd}
$$

The implications in the above diagram are not reversible. To show this, here we give examples. In Sections 6.1 and 6.3 of **Chapter** 6 respectively, it is already shown that an fs-semiopen and an fs-preopen set may not be fs-open.

Example 7.1.1 *Example to show that an fs α-open set may not be fs-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $C_f(s)$ is fs α-open but is not fs-open.

Example 7.1.2 *Example to show that a locally fs-closed set may not be fs-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $B_f(s)$ is locally fs-closed but not fs-open.

Example 7.1.3 *Example to show that an fs $\mathcal{A}$-set may not be fs-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$, $D_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{1}, \overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{\frac{1}{2}}\overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\} \\ D_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, \overline{1}, ...\right\} \end{aligned}$$

Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), X_f^0(s), X_f^1(s)\}$. Here, $C_f(s) = A_f(s) \wedge D_f(s)$, where $A_f(s)$ is fs-open and $D_f(s)$ is fs-regular closed. Hence $C_f(s)$ is an fs $\mathcal{A}$-set but is not fs-open.

Example 7.1.4 *Example to show that a locally fs-closed set may not be an fs $\mathcal{A}$-set.*

Consider the FSTS $(X, \delta(s))$, given in Example 7.1.2. *Here, $B_f(s)$ is a locally fs-closed set but not an fs $\mathcal{A}$-set.*

Example 7.1.5 *Example to show that an fs-semiopen set may not be an fs $\mathcal{A}$-set.*

Consider the FSTS $(X, \delta(s))$, given in Example 7.1.1. *The fs-set $C_f(s)$ is an fs-semiopen set but not an fs $\mathcal{A}$-set.*

Example 7.1.6 *Example to show that an fs-semiopen set may not be fs α-open.*

In the FSTS, given in Example 7.1.3, *the fs-set $C_f(s)$ is fs-semiopen but not fs α-open.*

Example 7.1.7 *Example to show that an fs-preopen set may not be fs α-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \dots\dots\dots\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{5}}, \overline{\frac{1}{5}}, \overline{\frac{1}{5}}, \dots\dots\right\} \end{aligned}$$

Then $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ is a fuzzy sequential topology on X. In this FSTS, $B_f(s)$ is fs-preopen but not fs α-open.

Example 7.1.8 *Example to show that an fs δ-set may not be fs-semiopen.*

In the FSTS, given in Example 7.1.2*, the fs-set $B_f(s)$ is an fs δ-set but is not fs-semiopen.*

Example 7.1.9 *Example to show that an fs-preopen set may not be an fs S-preopen set.*

Consider the FSTS, given in Example 7.1.1*, the fs-set $C_f(s)$ is fs-preopen but not fs S-preopen.*

Example 7.1.10 *Example to show that an fs S-preopen set may not be an fs-open set.*

Consider the FSTS, given in Example 7.1.7*, the fs-set $B_f(s)$ is fs S-preopen but not fs-open.*

Definition 7.1.2 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called*
(i) fs α-continuous if $g^{-1}(B_f(s))$ is fs α-open in X, for every fs-open set $B_f(s)$ in Y.

(ii) fs lc-continuous if $g^{-1}(B_f(s))$ is locally fs-closed in X, for every fs-open set $B_f(s)$ in Y.
(iii) fs $\mathcal{A}$-continuous if $g^{-1}(B_f(s))$ is an fs $\mathcal{A}$-set in X, for every fs-open set $B_f(s)$ in Y.
(iv) fs δ-continuous if $g^{-1}(B_f(s))$ is an fs δ-set in X, for every fs-open set $B_f(s)$ in Y.
(v) fs S-precontinuous if $g^{-1}(B_f(s))$ is fs S-preopen in X, for every fs-open set $B_f(s)$ in Y.

Theorem 7.1.1 *An fs-set in an FSTS, is fs α-open if and only if it is fs-semiopen and fs-preopen.*

Proof. Let $A_f(s)$ be an fs α-open set, that is, $A_f(s) \leq int\ cl\ intA_f(s)$. Clearly, $A_f(s)$ is fs-semiopen. Also,

$$\begin{aligned} intA_f(s) \leq clA_f(s) &\Rightarrow cl\ intA_f(s) \leq clA_f(s) \\ &\Rightarrow int\ cl\ intA_f(s) \leq int\ clA_f(s) \\ &\Rightarrow A_f(s) \leq int\ clA_f(s) \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

Conversely, suppose $A_f(s)$ be fs-semiopen and fs-preopen, that is, $A_f(s) \leq cl\ intA_f(s)$, $A_f(s) \leq int\ clA_f(s)$. Then,

$$\begin{aligned} & int\ clA_f(s) \leq clA_f(s) \leq cl\ intA_f(s) \\ \Rightarrow\ & A_f(s) \leq int\ cl\ intA_f(s) \end{aligned}$$

Hence, $A_f(s)$ is fs α-open. ■

Corollary 7.1.1 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is fs α-continuous if and only it is fs-semicontinuous and fs-precontinuous.*

Definition 7.1.3 *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then αfs-closure ${}_{\alpha}clA_f(s)$ and αfs-interior ${}_{\alpha}intA_f(s)$ of $A_f(s)$ are defined by*

$$\begin{aligned} {}_{\alpha}clA_f(s) &= \wedge\{V_f(s);\ A_f(s) \leq V_f(s) \text{ and } V_f^c(s) \in \alpha(X)\} \\ {}_{\alpha}intA_f(s) &= \vee\{U_f(s);\ U_f(s) \leq A_f(s) \text{ and } U_f(s) \in \alpha(X)\} \end{aligned}$$

Complement of an fs α-open set is called an fs α-closed set. Hence, it is clear that ${}_{\alpha}cl(A_f(s))$ is the smallest fs α-closed set containing $A_f(s)$ and ${}_{\alpha}int(A_f(s))$ is the largest fs α-open set contained in $A_f(s)$. Further,

(i) $A_f(s) \leq {}_{\alpha}cl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_{\alpha}int(A_f(s)) \leq A_f(s)$.

(ii) $A_f(s)$ is fs α-open if and only if $A_f(s) = {}_{\alpha}int(A_f(s))$

(iii) $A_f(s)$ is fs α-closed if and only if $A_f(s) = {}_{\alpha}cl(A_f(s))$

(iv) $A_f(s) \leq B_f(s) \Rightarrow {}_{\alpha}int(A_f(s)) \leq {}_{\alpha}int(B_f(s))$ and ${}_{\alpha}cl(A_f(s)) \leq {}_{\alpha}cl(B_f(s))$.

Theorem 7.1.2 *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then,*

(i) ${}_{\alpha}intA_f(s) = A_f(s) \wedge int\ cl\ intA_f(s)$.

(ii) if $A_f(s)$ is both fs-preopen and fs-preclosed, then $A_f(s) = int\ clA_f(s) \wedge A_f(s)$ and thus $A_f(s)$ is fs S-preopen;

(iii) if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $intV_f(s)$

is fs-regular open, then ${}_{\alpha}intA_f(s) = intA_f(s)$*;*

(iv) if $A_f(s)$ *is an fs* δ*-set, then* ${}_{\alpha}intA_f(s) = {}_{p}intA_f(s)$*.*

Proof. (i) Easy to prove.

(ii) Given $A_f(s) \leq int\ clA_f(s)$ and $cl\ intA_f(s) \leq A_f(s)$. Then,

$$A_f(s) = int\ clA_f(s) \wedge A_f(s).$$

Since $intA_f(s) = int\ cl\ intA_f(s)$, hence $A_f(s)$ is fs S-preopen.

(iii) We have $int\ cl\ intA_f(s) \leq int\ cl\ intV_f(s) = intV_f(s)$. Therefore,

$$\begin{aligned} {}_{\alpha}intA_f(s) &= A_f(s) \wedge int\ cl\ intA_f(s) \\ &\leq A_f(s) \wedge intV_f(s) \\ &= intA_f(s) \end{aligned}$$

Also, $intA_f(s) \leq {}_{\alpha}intA_f(s)$. Hence $intA_f(s) = {}_{\alpha}intA_f(s)$.

(iv) Given $int\ clA_f(s) \leq cl\ intA_f(s)$. Since ${}_{\alpha}intA_f(s)$ is an fs-preopen set contained in $A_f(s)$, we have

$${}_{\alpha}intA_f(s) \leq {}_{p}intA_f(s)$$

Now,

$${}_{p}intA_f(s) \leq int\ clA_f(s) \leq int\ cl\ intA_f(s)$$

Thus, ${}_{p}intA_f(s) \leq A_f(s) \wedge int\ cl\ intA_f(s) = {}_{\alpha}intA_f(s)$. Hence the result. ■

Lemma 7.1.1 *An fs-set* $A_f(s)$ *is locally fs-closed if and only if* $A_f(s) = U_f(s) \wedge cl(A_f(s))$*, where* $U_f(s)$ *is an fs-open set.*

Proof. Omitted. ■

Theorem 7.1.3 *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then $A_f(s)$ is an fs $\mathcal{A}$-set if it is fs-semiopen and locally fs-closed.*

Proof. Suppose $A_f(s)$ be fs-semiopen and locally fs-closed. Then, $A_f(s) \leq cl\ intA_f(s)$ and $A_f(s) = U_f(s) \wedge clA_f(s)$, where $U_f(s)$ is fs-open. Since $clA_f(s) = cl\ intA_f(s)$ is fs-regular closed, the result follows. ■

Corollary 7.1.2 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is fs $\mathcal{A}$-continuous if it is fs-semicontinuous and fs lc-continuous.*

Remark 7.1.1 *Unlike in a general topological space, the converse of Theorem* 7.1.3 *may not be true and it has been shown by the next Example.*

Example 7.1.11 *Let $X = \{x, y\}$. Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$, $D_f(s)$ and $E_f(s)$ in X, where*

$A_f^1 = \overline{0.3}$, $A_f^n(x) = 1$ *and* $A_f^n(y) = 0$ *for all* $n \neq 1$;

$B_f^1(x) = 0.4$, $B_f^1(y) = 0.7$, $B_f^n(x) = 0$ *and* $B_f^n(y) = 1$ *for all* $n \neq 1$;

$C_f^1 = \overline{0.7}$ *and* $C_f^n = \overline{1}$ *for all* $n \neq 1$;

$D_f^1(x) = 0.6$, $D_f^1(y) = 0.3$, $D_f^n(x) = 1$ *and* $D_f^n(y) = 0$ *for all* $n \neq 1$;

$E_f^1(x) = 0.4$, $E_f^1(y) = 0.3$ *and* $E_f^n = \overline{0}$ *for all* $n \neq 1$.

Let $\delta(s) = \{A_f(s), B_f(s), C_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. In the FSTS $(X, \delta(s))$, $D_f(s)$ being an fs-regular closed set, the fs-set $E_f(s) = B_f(s) \wedge D_f(s)$ is an fs $\mathcal{A}$-set but not fs-semiopen.

Theorem 7.1.4 *Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. Then the following statements are equivalent:*
(i) $A_f(s)$ is an fs-open set.
(ii) $A_f(s)$ is fs α-open and locally fs-closed.
(iii) $A_f(s)$ is fs-preopen and locally fs-closed.
(iv) $A_f(s)$ is fs-preopen and an fs $\mathcal{A}$-set.
(v) $A_f(s)$ is fs S-preopen and an fs δ-set.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (iv)$ Let $A_f(s)$ be fs-preopen and locally fs-closed. Then,

$$A_f(s) \leq int\ clA_f(s) \ \text{ and } \ A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. $clA_f(s)$ being fs-regular closed, the result follows.

$(iv) \Rightarrow (i)$ Let $A_f(s)$ be an fs-preopen and an fs $\mathcal{A}$-set. Then,

$$A_f(s) \leq int\ clA_f(s) \ \text{ and } \ A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. Since $intA_f(s) = U_f(s) \wedge int\ clA_f(s)$, $A_f(s)$ is fs-open.

$(i) \Rightarrow (v)$ Obvious.

$(v) \Rightarrow (i)$ Let $A_f(s)$ be an fs S-preopen and an fs δ-set. Using Theorem 7.1.2, (iii) and (iv),

$$intA_f(s) = \ _{\alpha}intA_f(s) = \ _{p}intA_f(s) = A_f(s).$$

Hence, $A_f(s)$ is fs-open. ■

By Theorems 7.1.1, 7.1.3 and 7.1.4, we have the following relationships among the different classes of fs-sets of an FSTS $(X, \delta(s))$:
(i) $\alpha(X) = FSPO(X) \cap FSSO(X)$.
(ii) $\mathcal{A}(X) \supseteq FSSO(X) \cap FSLC(X)$.
(iii) $\delta(s) = \alpha(X) \cap FSLC(X)$.
(iv) $\delta(s) = FSPO(X) \cap FSLC(X)$.
(v) $\delta(s) = FSPO(X) \cap \mathcal{A}(X)$.
(vi) $\delta(s) = FSSPO(X) \cap \delta(X)$.

Theorem 7.1.5 *In an FSTS* $(X, \delta(s))$, *the following are equivalent:*
(i) $clA_f(s) \in \delta(s)$ *for every* $A_f(s) \in \delta(s)$.
(v) $\mathcal{A}(X) = \delta(s)$.

Proof. $(i) \Rightarrow (ii)$ It is obvious that $\delta(s) \subseteq \mathcal{A}(X)$. For the reverse inclusion, let $A_f(s) \in \mathcal{A}(X)$, then

$$A_f(s) = U_f(s) \wedge V_f(s),$$

where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed. By (i), $V_f(s) \in \delta(s)$ and hence $A_f(s) \in \delta(s)$.

$(ii) \Rightarrow (i)$ Suppose $\mathcal{A}(X) = \delta(s)$. Let $A_f(s) \in \delta(s)$, then $clA_f(s)$ is fs-regular closed and hence belongs to $\mathcal{A}(X) = \delta(s)$. ■

We conclude the chapter by stating the following decompositions of fs-continuity:

Theorem 7.1.6 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then g is fs-continuous if and only if*

(i) g is fs α-continuous and fs lc-continuous.

(ii) g is fs-precontinuous and fs lc-continuous.

(iii) g is fs-precontinuous and fs $\mathcal{A}$-continuous.

(iv) g is fs S-continuous and fs δ-continuous.

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FUZZY SEQUENTIAL TOPOLOGICAL SPACES

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ABSTRACT
Fuzzy sequential topology on a nonempty set X which is a sub-collection of $(I^X)^{\mathbb{N}}$satisfying the conditions given in the definition is introduced. Many pleasant properties of a countable number of fuzzy topologies on X associated as components of a fuzzy sequential topology have been investigated. Finally, a variant of Yang's Theorem is established in this setting.

Keywords
Fuzzy sequential topology, fuzzy sequential point, quasi coincidence, Q-neighbourhood, fuzzy derived sequential set.

1. INTRODUCTION
In 1965 fuzzy sets were introduced by L. A. Zadeh [20] which is followed by the initiation of fuzzy topology in 1968 by C. L. Chang [5]. Till present, a variety of studies have been done in the theory of fuzzy topology. Some of them are fuzzy closure operators and fuzzy interior operators by Mashour and Ghanim [14], G. Gerla [7], Bandler and Kohout [1], R. Belohlavek [2], R. Belohlavek and T. Funiokova [3]; separation axioms by B. Hutton and I. Reilly [9]; fuzzy compactness by authors like C. L. Chang [5], J.A. Goguen [8], R. Lowen [11, 12, 13], T.E. Gantner, R.C. Steinlage and R.H. Warren [6], Wang Guojun [19], Gunther Jager [10] etc.
In 2002, M. K. Bose and I. Lahiri introduced the concept of sequential topological spaces [4] and studied some separation axioms in such spaces. Then N. Tamang, M. Singha and S. De Sarkar [18] extended this field by studying separation axioms in the light of reduced and augmented bases. Also the new operators namely, $K\Omega$ operators, Relative Closure Operators and Monotonic Sequential Operators in a class of sequential sets are studied by M. Singha and S. De sarkar [16, 17].

The purpose of this paper is to study the concept of sequential topological spaces in fuzzy setting. We begin with some basic definitions:

Let X be a nonempty set and $I = [0,1]$ be the closed unit interval in the set $\mathbb{R}$ of real numbers. Let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be sequences of fuzzy sets in X called fuzzy sequential sets in X and we define

i. $A_f(s) \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (union),
ii. $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (intersection),
iii. $A_f(s) \leq B_f(s)$ if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$; $\mathbb{N}$ being the set of positive integers,
iv. $A_f(s) \leq_{\mathrm{W}} B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$,
v. $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$,
vi. $A_f(s)(\mathrm{x}) = \{A_f^n(x)\}_n, x \in X$,
vii. $A_f(s)(\mathrm{x}) \geq_{\mathrm{M}} \mathrm{r}$ if and only if $A_f^n(\mathrm{x}) \geq \mathrm{r_n}$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular if $M = \mathbb{N}$, we write $A_f(s)(\mathrm{x}) \geq \mathrm{r}$,
viii. $X_f^l(s) = \{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$,
ix. $A_f^c(s) = \{1 - A_f^n\}_n = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$,
x. A fuzzy sequential set $P_f(s) = \{P_f^n\}_n$ is called a fuzzy sequential point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that

$$\begin{aligned} P_f^n(t) &= r_n, \text{ if } t = x, \\ &= 0, \text{ if } t \in X - \{x\}, \end{aligned}$$

for all $n \in \mathbb{N}$.

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$\begin{aligned} P_f^n(x) &= r_n, \text{ whenever } n \in M, \\ &= 0, \text{ whenever } n \in \mathbb{N} - M. \end{aligned}$$

The point x is called the support, M is called base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (P_{fx}^M, r)$. If further $M = \{n\}$, $n \in \mathbb{N}$, then the fuzzy sequential point is called a simple fuzzy sequential point and it is denoted by (P_{fx}^n, r_n). A fuzzy sequential point is called complete if its base is the set of natural numbers. A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$ symbolically $P_f(s) \in_W A_f(s)$ if and only if there exists $n \in M$ such that $P_f^n(x) \leq A_f^n(x)$. If $R \subseteq M$ and s is the sequence in I same to r in R and vanishes outside R, then the fuzzy sequential point $P_{rf}(s) = (P_{fx}^R, s)$ is called a reduced fuzzy sequential point of $P_f(s) = (P_{fx}^M, r)$. A sequence $(x, L) = \{A_n\}_n$

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of subsets of X, where $A_n = \{x\}$, for all $n \in L$ and $A_n = \varphi =$ the null subset of X, for all $n \in \mathbb{N} - L$, is called a sequential point in X.

Throughout this paper we use Chang's definition of fuzzy topology [5].

2 Definitions and Results:

Definition 2.1 A family $\delta(s)$ of fuzzy sequential sets on a nonempty set X satisfying the properties:

i. $X_f^r(s) \in \delta(s)$ for all $r \in \{0,1\}$,

ii. $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$

iii. for any family $\{A_{fj}(s) \in \delta(s), j \in J\}$, $\vee_{j \in J} A_{fj}(s) \in \delta(s)$.

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 2.2 If $\delta_1(s)$ and $\delta_2(s)$ be two FSTs on X such that $\delta_1(s) \subset \delta_2(s)$, then we say that $\delta_2(s)$ is finer than $\delta_1(s)$ or $\delta_1(s)$ is weaker than $\delta_2(s)$

Proposition 2.1 If δ be a fuzzy topology (FT) on X, then $\delta^{\mathbb{N}}$ forms an FST on X.

Proof. Proof is straightforward. ∎

We may construct different FSTs on X from a given FT δ on X, $\delta^{\mathbb{N}}$ is the finest of all these FSTs. Not only that, any FT δ on X can be considered as a component of some FST on X, one of them is $\delta^{\mathbb{N}}$, there are at least countably many FSTs on X weaker than $\delta^{\mathbb{N}}$ of which δ is a component. One of them is $\delta'(s) = \{A_f(s) = \{A_f^n\}_n ; A_f^n = A \; for \; all \; n \in \mathbb{N} \; and \; A \in \delta\}$.

Proposition 2.2 If $(X, \delta(s))$ is an FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where $\delta_n = \{A_f^n; A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$.

Proof. Proof is omitted. ∎

Definition 2.3 In Proposition 2.2, (X, δ_n), where $n \in \mathbb{N}$, is called the n^{th} component fuzzy topological space of $(X, \delta(s))$.

Proposition 2.3 Let $A_f(s) = \{A_f^n\}_n$ be an open (closed) fuzzy sequential set in an FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in (X, δ_n) but the converse is not necessarily true.

Proof. Proof of the first part is omitted. For the converse part let us take the FSTS $(X, \delta(s))$ where X is any nonempty set and $\delta(s) = \{X_f^r(s), r \in I\}$. Let $\{r_n\}_n$ be a strictly increasing sequence in I and $A_f(s) = \{A_f^n\}_n$, where $A_f^n = r_n$ and $r_n(x) = r_n$ for all $x \in X$, $n \in \mathbb{N}$. Clearly for each $n \in \mathbb{N}$,

A_f^n is an open fuzzy set in (X, δ_n) but $A_f(s) = \{A_f^n\}_n$ is not an open fuzzy sequential set in $(X, \delta(s))$. ∎

Definition 2.4 Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called quasi-coincident, denoted by $A_f(s)qB(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$, whenever A_f^n and B_f^n both are non zero. We write $A_f(s)\bar{q}B_f(s)$ to say that $A_f(s)$ and $A_f(s)$ are not quasi-coincident.

Definition 2.5 Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called weakly quasi-coincident, denoted by $A_f(s)q_wB_f(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$, for some $n \in \mathbb{N}$. We write $A_f(s)\overline{q_w}B_f(s)$ to mean that $A_f(s)$ and $B_f(s)$ are not weakly quasi-coincident.

Definition 2.6 A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is called quasi-coincident with $A_f(s) = \{A_f^n\}_n$, denoted by $P_f(s)qA_f(s)$ if and only if $P_f^n(x) > (A_f^n)^c(x)$ for all $n \in M$. If $P_f(s)$ is not quasi-coincident with $A_f(s)$, then we write $P_f(s)\bar{q}A_f(s)$.

Definition 2.7 A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is called weakly quasi-coincident with $A_f(s) = \{A_f^n\}_n$, denoted by $P_f(s)q_wA_f(s)$ if and only if $P_f^n(x) > (A_f^n)^c(x)$ for some $n \in M$. If $P_f(s)$ is not weakly quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q_w}A_f(s)$. If $P_f^n(x) > (A_f^n)^c(x)$ for some $n \in L \subseteq M$, then we say that $P_f(s)$ is weakly quasi-coincident with $A_f(s)$ at the sequential point (x, L).

Proposition 2.4 If the fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are quasi-coincident, then each pair of non zero fuzzy sets A_f^n and B_f^n is also so but the converse is not necessarily true.

Proof. Proof of the first part is omitted. For the second part, let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be fuzzy sequential sets on $\mathbb{R}$ where

$$\begin{aligned} A_f^1(x) &= \tfrac{2}{3}, x \in (-\infty, 0), \\ &= 1, x \in [0, \infty). \end{aligned}$$

$$\begin{aligned} A_f^2(x) &= \tfrac{1}{3}, x \in (-\infty, 0), \\ &= \tfrac{2}{3}, x \in [0, \infty). \end{aligned}$$

$$A_f^n(x) = \tfrac{3}{4}, x \in \mathbb{R} \text{ and } n \neq 1, 2.$$

$$\begin{aligned} B_f^1(x) &= \tfrac{1}{2}, x \in (-\infty, 0), \\ &= \tfrac{2}{3}, x \in [0, \infty). \end{aligned}$$

$$\begin{aligned} B_f^2(x) &= \tfrac{1}{4}, x \in (-\infty, 0), \\ &= \tfrac{3}{7}, x \in [0, \infty). \end{aligned}$$

$$B_f^n(x) = \tfrac{1}{2}, x \in \mathbb{R} \text{ and } n \neq 1, 2.$$

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Clearly $A_f^n q B_f^n$ for all $n \in \mathbb{N}$ but $A_f(s)\bar{q}B_f(s)$. ∎

Corollary 2.1 A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is quasi-coincident with a fuzzy sequential set $A_f(s) = \{A_f^n\}_n$ if and only if P_f^n and A_f^n are so for each $n \in M$.
Proof. Proof is straightforward.

Definition 2.8 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a neighbourhood (in short nbd) of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) \in B_f(s) \le A_f(s)$.

Definition 2.9 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a weak nbd of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) \in_w B_f(s) \le A_f(s)$.

Definition 2.10 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a Q-nbd of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) q B_f(s) \le A_f(s)$.

Definition 2.11 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a weak Q-nbd of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) q_w B_f(s) \le A_f(s)$.

Proposition 2.5 $A_f(s) \le_w (\le) B_f(s)$ if and only if $A_f(s)$ and $(B_f(s))^c$ are not (weakly) quasi-coincident. In particular $P_f(s) \in_w (\in) A_f(s)$ if and only if $P_f(s)$ is not (weakly) quasi-coincident with $(A_f(s))^c$.
Proof. Proof is omitted. ∎

Proposition 2.6 Let $\{A_{fj}(s), j \in J\}$ be a family of fuzzy sequential sets in X. Then a fuzzy sequential point $P_f(s) q_w (\bigvee_{j \in J} A_{fj}(s))$ if and only if $P_f(s) q_w A_{fj}(s)$ for some $j \in J$.
Proof. Let $P_f(s) q_w (\bigvee_{j \in J} A_{fj}(s))$ where $P_f(s) = (P_{fx}^M, r)$ and $A_{fj}(s) = \{A_{fj}^n\}_n$. This implies
$P_f^k(x) + S_f^k(x) > 1$ for some $n = k \in M$,
where $\bigvee_{j \in J} A_{fj}(s) = \{S_f^n\}_n$. Therefore

$$S_f^k(x) = 1 - P_f^k(x) + \varepsilon_k \text{ where } \varepsilon_k > 0 \qquad (1)$$

Also

$$S_f^k(x) - \varepsilon_k < A_{fj}^k(x) \text{ for some } j \in J \qquad (2)$$

From (1) and (2) we have $P_f^k(x) + A_{fj}^k(x) > 1$, that is, $P_f(s) q_w A_{fj}(s)$ for some $j \in J$. Other implication is straightforward. ∎

Corollary 2.2 If $P_f(s) q A_{fj}(s)$ for some $j \in J$, then $P_f(s) q (\bigvee_{j \in J} A_{fj}(s))$ where $\{A_{fj}(s), j \in J\}$ is a family of fuzzy sequential sets in X but not conversely.
Proof. Proof of the first part is omitted. For second part, let $A_{fj}(s) = \{A_{fj}^n\}_n$,

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$j = 1,2$ be fuzzy sequential sets in $\mathbb{R}$, where

$$A^1_{f1}(x) = 0 \text{ for all } x \in \mathbb{R} - (0,1),$$
$$= \tfrac{1}{4} \text{ for all } x \in (0,1).$$

$$A^2_{f1}(x) = 0 \text{ for all } x \in \mathbb{R} - (\tfrac{1}{3}, \tfrac{2}{3}),$$
$$= \tfrac{2}{3} \text{ for all } x \in (\tfrac{1}{3}, \tfrac{2}{3})$$

$A^n_{f1}(x) = 0 = A^n_{f1}(x)$ for all $x \in \mathbb{R}$, $n \neq 1,2$.

$$A^1_{f2}(x) = 0 \text{ for all } x \in \mathbb{R} - (-\tfrac{1}{2}, 1),$$
$$= \tfrac{1}{3} \text{ for all } x \in (-\tfrac{1}{2}, 1),$$

$$A^2_{f2}(x) = 0 \text{ for all } x \in \mathbb{R} - (-\tfrac{1}{2}, 2),$$
$$= \tfrac{1}{5} \text{ for all } x \in \left(-\tfrac{1}{2}, 2\right).$$

The fuzzy sequential point $P_f(s) = (P^M_{f0.5}, r)$ where $M = \{1,2\}$ and $r = \{r_n\}_n$ with $r_1 = r_2 = \frac{7}{10}$, $r_n = 0$ for all $n \neq 1,2$ is quasi-coincident with $A_{f1}(s) \vee A_{f2}(s)$ but it is not so with any one of them. ■

Definition 2.12 A subfamily β of an FST $\delta(s)$ on X is called a base for $\delta(s)$ if and only if to every $A_f(s) \in \delta(s)$, there exists a subfamily $\{B_{fj}(s),\ j \in J\}$ of β such that $A_f(s) = \vee_{j\in J} B_{fj}(s)$.

Definition 2.13 A subfamily $S = \{S_{f\lambda}(s);\ \lambda \in \Lambda\}$ of an FST $\delta(s)$ on X is called a subbase for $\delta(s)$ if and only if $\{\wedge_{j\in J} S_{fj}(s);\ J = finite\ subset\ of\ \Lambda\}$ forms a base for $\delta(s)$.

Theorem 2.1 A subfamily β of an FST $\delta(s)$ on X is a base for $\delta(s)$ if and only if for each fuzzy sequential point $P_f(s)$ in $(X, \delta(s))$ and for every open weak Q nbd $A_f(s)$ of $P_f(s)$, there exists a member $B_f(s) \in \beta$ such that $P_f(s)\ q_w \mathrm{B}_f(s) \leq A_f(s)$.

Proof. The necessary part is straightforward. To prove its sufficiency, if possible let β be not a base for $\delta(s)$. Then there exists a member $A_f(s) \in \delta(s) - \beta$, such that $O_f(s) = \vee\{B_f(s) \in \beta;\ B_f(s) \leq A_f(s)\ and\ B_f(s) \neq A_f(s)\} \neq A_f(s)$, and hence there is an $x \in X$ and an $M \subset \mathbb{N}$ such that $O^n_f(x) < A^n_f(x)$ for all $n \in M$. Let $r = \{r_n\}_n$ where $r_n = 1 - O^n_f(x) > 0$ whenever $n \in M$ and $r_n = 0$ whenever $n \in \mathbb{N} - M$, then $A^n_f(x) + r_n > O^n_f(x) + r_n = 1$, for all $n \in M$ and $(P^M_{fx}, r) = P_f(s) q_w A_f(s)$. Therefore $A_f(s)$ is an open weak Q nbd of $P_f(s)$. Now

$B_f(s) = \{B^n_f\}_n \in \beta, B_f(s) \leq A_f(s)$
$\Rightarrow B_f(s) \leq O_f(s)$
$\Rightarrow B^n_f(x) + r_n \leq O^n_f(x) + r_n = 1$ for all $n \in M$
$\Rightarrow P_f(s) \overline{q_w} B_f(s)$ which is a contradiction. Hence the proof.■

Proposition 2.7 If β be a base for an FST $\delta(s)$ on X, then $\beta_n = \{B^n_f;\ B_f(s) = \{B^n_f\}_n \in \beta\}$ will form a base for the component FT δ_n on X for each $n \in \mathbb{N}$ but not conversely.

Proof. Proof of the first part is straightforward. For the converse part we consider the FSTS $(\mathbb{R}, \delta^{\mathbb{N}})$, where $\mathbb{R}$ is the set of real numbers and $\delta = \{r;\ r \in [0,1]\}$, $r(x) = r$ for all $x \in \mathbb{R}$, which is a FT on $\mathbb{R}$. Clearly $\beta_n = \{r;\ r \in (0,1) \cap \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers, is a base for the component FT $\delta_n^{\mathbb{N}}$ on X for each $n \in \mathbb{N}$ but $\beta(s) = \{X_f^r(s);\ r \in (0,1) \cap \mathbb{Q}\}$ is not a base for the FST $\delta^{\mathbb{N}}$ on X because $A_f(s) = \{A_f^n\}_n$ where $A_f^n = \overline{\frac{1}{n}}$ for all $n \in \mathbb{N}$ is an open fuzzy sequential set in $(\mathbb{R}, \delta^{\mathbb{N}})$, but cannot be written as a supremum of a subfamily of $\beta(s)$. ■

Definition 2.14 Let $A_f(s)$ be any fuzzy sequential set in an FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $A_f^{\circ}(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \leq C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$A_f^{\circ}(s) = \vee\{O_f(s);\ O_f(s) \leq A_f(s)),\ O_f(s) \in \delta(s)\}.$$

Proposition 2.8 If $A_f(s) = \{A_f^n\}_n$ in $(X, \delta(s))$, then $cl(A_f^n) \leq A_f^n$ in (X, δ_n) for each $n \in \mathbb{N}$, where $cl(A_f^n)$ is the closure of A_f^n in (X, δ_n).

Proof. Proof is straightforward. ■

Here we cite an example where the equality in the proposition 2.8 does not hold. Let $X = [0,1]$ and $\delta(s) = \{X_f^r(s);\ r \in [0,1]\}$. If $A_f(s) = (P_{f\frac{1}{3}}^{\mathbb{N}}, r)$, $r = \{\frac{1}{2} - \frac{1}{3n}\}_n$, then $\overline{A_f(s)} = X_f^{\overline{\frac{1}{2}}}(s)$. Here $cl(A_f^n) = (\overline{\frac{1}{2} - \frac{1}{3n}})$, whereas $\overline{A_f^n} = \overline{\frac{1}{2}}$.

Definition 2.15 The dual of a fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is a fuzzy sequential point $P_{df}(s) = (P_{fx}^M, t)$, where $r = \{r_n\}_n$, $t = \{t_n\}_n$ and

$t_n = 1 - r_n$ for all $n \in M$,
$\quad = 0$ for all $n \in \mathbb{N} - M$.

Theorem 2.2 Every Q nbd of a fuzzy sequential point $P_f(s)$ is weakly quasi-coincident with a fuzzy sequential set $A_f(s)$ implies $P_f(s) \in \overline{A_f(s)}$ implies every weak Q nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident.

Proof. Let $P_f(s) = (P_{fx}^M, r)$. $P_f(s) \in \overline{A_f(s)}$ if for every closed fuzzy sequential set $C_f(s) \geq A_f(s)$, $P_f(s) \in C_f(s)$, that is $p_f^n(x) \leq C_f^n(x)$ for all $n \in M \Rightarrow P_f(s) \in \overline{A_f(s)}$ if for every open fuzzy sequential set $B_f(s) = \{B_f^n\}_n \leq (A_f(s))^c$, $B_f^n(x) \leq 1 - p_f^n(x)$ for all $n \in M$; that is $P_f(s) \in \overline{A_f(s)}$ if for every open fuzzy sequential set $B_f(s) = \{B_f^n\}_n$ satisfying $B_f^n(x) > 1 - P_f^n(x)$ for all $n \in M$, $B_f(s) \nleq (A_f(s))^c$, which implies the first part.

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Now let $P_f(s) \in \overline{A_f(s)}$. If possible let there exists a weak Q nbd $N_f(s)$ of $P_f(s)$ such that $N_f(s)\overline{q_w}A_f(s)$. Then there exists an open fuzzy sequential set $B_f(s)$ such that $P_f(s)q_w B_f(s) \leq N_f(s)$. Now $N_f(s)\overline{q_w}A_f(s)$ and $B_f(s) \leq N_f(s) \Rightarrow B_f^n(x) + A_f^n(x) \leq 1$ for all $x \in X$, $n \in \mathbb{N} \Rightarrow A_f(s) \leq (B_f(s))^c \Rightarrow P_f(s) \in (B_f(s))^c \Rightarrow p_f^n(x) + B_f^n(x) \leq 1$ for all $n \in \mathbb{N}$. This contradicts the fact that $P_f(s)q_w B_f(s)$. Hence the result follows. ∎

Corollary 2.3 A fuzzy sequential point $P_f(s) \in \overline{A_f(s)}$ if and only if each nbd of its dual point $P_{df}(s)$ is weakly quasi-coincident with $A_f(s)$.

Proof. Proof is straightforward since Q nbd of a fuzzy sequential point is exactly the nbd of its dual point. ∎

Theorem 2.3 A fuzzy sequential point $P_f(s) \in A_f^\circ(s)$ if and only if its dual point $P_{df}(s) \notin \overline{(A_f(s))^c}$.

Proof. Let $P_f(s) \in A_f^\circ(s) \Rightarrow$ there exists an open fuzzy sequential set $B_f(s)$ such that $P_f(s) \in B_f(s) \leq A_f(s) \Rightarrow B_f(s)$ and $(A_f(s))^c$ are not weakly quasi-coincident $\Rightarrow P_{df}(s) \notin \overline{(A_f(s))^c}$. Conversely let $P_{df}(s) \notin \overline{(A_f(s))^c}$. Then there exists an open nbd $B_f(s)$ of $P_f(s)$ which is not weakly quasi-coincident with $(A_f(s))^c$

$$\Rightarrow P_f(s) \in B_f(s) \leq A_f(s)$$
$$\Rightarrow P_f(s) \in A_f^\circ(s).$$ ∎

Proposition 2.9 In an FSTS $(X, \delta(s))$, the following hold:
(i) $\overline{X_f^r(s)} = X_f^r(s)$, $r \in \{0,1\}$, (ii) $A_f(s)$ is closed if and only if $\overline{A_f(s)} = A_f(s)$, (iii) $\overline{\overline{A_f(s)}} = \overline{A_f(s)}$, (iv) $\overline{A_f(s) \vee B_f(s)} = \overline{A_f(s)} \vee \overline{B_f(s)}$, (v) $\overline{A_f(s) \wedge B_f(s)} \leq \overline{A_f(s)} \wedge \overline{B_f(s)}$, (vi) $(X_f^r(s))^\circ = X_f^r(s)$, $r \in \{0,1\}$, (vii) $A_f(s)$ is open if $A_f^\circ(s) = A_f(s)$, (viii) $(A_f^\circ(s))^\circ = A_f^\circ(s)$, (ix) $(A_f(s) \wedge B_f(s))^\circ = A_f^\circ(s) \wedge B_f^\circ(s)$, (x) $A_f^\circ(s) \vee B_f^\circ(s) \leq (A_f(s) \vee B_f(s))^\circ$, (xi) $A_f^\circ(s) = \overline{(A_f(s))^c}^c$, (xii) $\overline{A_f(s)} = ((A_f(s))^c)^o$, (xiii) $(\overline{A_f(s)})^c = ((A_f(s))^c)^\circ$, (xiv) $\overline{(A_f(s))^c} = (A_f^\circ(s))^c$.

Proof. Proof is straightforward. ∎

Definition 2.16 A fuzzy sequential point $P_f(s)$ is called an adherence point of a fuzzy sequential set $A_f(s)$ if and only if every weak Q nbd of $P_f(s)$ is weakly quasi-coincident with $A_f(s)$.

Definition 2.17 A fuzzy sequential point $P_f(s)$ is called an accumulation point of a fuzzy sequential set $A_f(s)$ if and only if $P_f(s)$ is an adherence point of $A_f(s)$ and every weak Q nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident at some fuzzy sequential point having different base or support from that of $P_f(s)$ whenever $P_f(s) \in A_f(s)$.

Proposition 2.10 Any reduced fuzzy sequential point of an accumulation point of a fuzzy sequential set is also an accumulation point of it.

Proof. Easy to prove. ■

From the proposition 2.10, we see that any simple reduced fuzzy sequential point of an accumulation point of a fuzzy sequential set is also an accumulation point of it but the converse is not true. For let $X = \{a, b\}$ and $\delta(s) = \{X_f^r(s),\ G_f(s);\ r \in \{0,1\}\}$, where $G_f(s) = \{G_f^n\}_n$, $G_f^n(a) = \frac{1}{2}$ and $G_f^n(b) = 0$ for all $n \in \mathbb{N}$. Let $A_f(s) = \{A_f^n\}_n$ where $A_f^n = \frac{2}{3}$ for $n = 1,2$ and $A_f^n = 0$ otherwise. Then the fuzzy sequential point $P_f(s) = (P_{fa}^M, r)$ where $r = \{r_n\}_n$ with $r_1 = r_2 = \frac{2}{3}$ and $r_n = 0$ otherwise, is not an accumulation point of $A_f(s)$ though $(P_{fa}^1, \frac{2}{3})$ and $(P_{fa}^2, \frac{2}{3})$ both are accumulation point of $A_f(s)$.

Definition 2.18 The union of all accumulation points of a fuzzy sequential set $A_f(s)$ is called the fuzzy derived sequential set of $A_f(s)$ and it is denoted by $A_f^d(s)$.

Theorem 2.4 In an FSTS $(X, \delta(s))$, $\overline{A_f(s)} = A_f(s) \vee A_f^d(s)$.

Proof. Let $\Omega = \{P_f(s);\ P_f(s)\ is\ an\ adherence\ point\ of\ A_f(s)\}$. Then $\overline{A_f(s)} = \vee\Omega$. Now let $P_f(s) \in \Omega$, then two cases may arise, $P_f(s) \in A_f(s)$ or $P_f(s) \notin A_f(s)$. If $P_f(s) \notin A_f(s)$, then $P_f(s) \in A_f^d(s)$ and hence $P_f(s) \in A_f(s) \vee A_f^d(s)$. Therefore ,

$\overline{A_f(s)} = \vee\Omega \leq A_f(s) \vee A_f^d(s)$……..(1)

Again, $A_f(s) \leq \overline{A_f(s)}$ and since any accumulation point $P_f(s)$ of $A_f(s)$ belongs to $\overline{A_f(s)}$ which implies $A_f^d(s) \leq \overline{A_f(s)}$. Therefore,

$A_f(s) \vee A_f^d(s) \leq \overline{A_f(s)}$…………....(2).

From (1) and (2) the result follows.

Corollary 2.4 A fuzzy sequential set is closed in an FSTS $(X, \delta(s))$ if and only if it contains all its accumulation points.

Proof. Proof is straightforward.

Remark 2.1 The fuzzy derived sequential set of any fuzzy sequential set may not be closed as shown by example 2.1.

Example 2.1 Let $X = \{a, b\}$, $\delta(s)$ be the FST having base $\beta = \{X_f^1(s)\} \vee \{X_f^0(s)\} \vee \{P_f(s),\ G_f(s)\}$, where $G_f^n(b) = 1$, $G_f^n(a) = 0\ \ \forall\, n \in \mathbb{N}$ and $P_f(s) = (P_{fa}^M, r)$ where $M = \{1,2,3\}$, $r_1 = 0.5$, $r_2 = 1$, $r_3 = 0.3$, $r_n = 0\ \forall\ n \neq 1,2,3$. Here the fuzzy derived sequential set of $(P_{fa}^3, 0.3)$ is not closed.

Proposition 2.11 The fuzzy derived sequential set of a fuzzy sequential point equals the union of the fuzzy

derived sequential sets of all its simple reduced fuzzy sequential points.
Proof. The proof is omitted.

Proposition 2.12 If the fuzzy derived sequential set of each of the simple reduced fuzzy sequential points of a fuzzy sequential point is closed, then the derived sequential set of the fuzzy sequential point is closed.
Proof. Let $A_f(s) = (P^M_{fx}, r)$ be a fuzzy sequential point. Let $D_f(s)$ be the fuzzy derived sequential set of $A_f(s)$. Let $D_{nf}(s)$ be the fuzzy derived sequential set of $A_{nf}(s) = (p^n_{fx},\ r_n)$, $n \in M$. Suppose $D_{nf}(s)$ is closed for all $n \in M$. Let $P_f(s)$ be an accumulation point of $D_f(s)$.
Now, $P_f(s) \notin D_f(s)$
$\Rightarrow P_f(s)$ is not an accumulation point of $A_f(s)$
$\Rightarrow \exists$ a weak Q-nbd $B_f(s)$ of $P_f(s)$ which is not weakly quasi coincident with $A_f(s)$
$\Rightarrow B_f(s)$ is not weakly quasi coincident with $(p^n_{fx},\ r_n)\ \forall\, n \in M$
$\Rightarrow P_f(s) \notin D_{nf}(s)\ \forall\, n \in M$
$\Rightarrow P_f(s)$ is not an accumulation point of $D_{nf}(s)\ \forall\, n \in M$ (since $D_{nf}(s)$ is closed $\forall\, n \in M$)
$\Rightarrow P_f(s)$ is not an accumulation point of $\bigvee_{n\in M} D_{nf}(s) = D_f(s)$ a contradiction.
Hence proved. ∎

Remark 2.2 Converse of proposition 2.12 is not true as shown by example 2.2.

Example 2.2 Let $X = \{a, b\}$, $\delta(s)$ be the FST having base $\beta = \{X^1_f(s)\} \vee \{X^0_f(s)\} \vee \{P_f(s),\ G_f(s)\}$, where $G^n_f(b) = 1,\ \ G^n_f(a) = 0\ \ \forall\, n \in \mathbb{N}$ and $P_f(s) = (P^M_{fa}, r)$ where $M = \{1,2,3\}$, $r_1 = 0.5,\ r_2 = 1,\ r_3 = 0.3,\ r_n = 0\ \forall n \neq 1,2,3$.
Here the fuzzy derived sequential set of $P_f(s)$ is closed but the fuzzy derived sequential set of $(P^3_{fa}, 0.3)$ is not closed.

We conclude the paper stating two necessary lemmas followed by a variant of Yang's Theorem in fuzzy sequential topological spaces.

Lemma 2.1 Let $A_f(s) = (P^M_{fx}, r)$ be a fuzzy sequential point in FSTS $(X, \delta(s))$. Then,

(i) For $y \neq x$, $\overline{A_f(s)}(y) = A^d_f(s)(y)$.
(ii) If $\overline{A_f(s)}(x) >_P r$, then $\overline{A_f(s)}(x) =_P A^d_f(s)(x)$, where $P \subset M$.
(iii) If $\overline{A_f(s)}(x) >_M r$, then $\overline{A_f(s)}(x) =_M A^d_f(s)(x)$.
(iv) If $A^d_f(s)(x) = 0 =$ sequence of real zeros, then $\overline{A_f(s)}(x) = r$.
(v) If $A_f(s)$ is simple then converse of (iv) is true.

Lemma 2.2 Let $A_f(s) = (P^k_{fx}, r_k)$ be a simple fuzzy sequential point in the FSTS $(X, \delta(s))$. Then,

(i) If $A_f^d(s)(x)$ is a non zero sequence, then $\overline{A_f(s)} = A_f^d(s)$.

(ii) If $A_f^d(s)(x) = 0 =$ sequence of real zeros, then If $A_f^d(s)$ is closed iff ∃ an open fuzzy sequential set $B_f^@(s)$ such that $B_f^@(s)(x) = 1$ and for $y \neq x$, $B_f^@(s)\,(y) = \{\overline{A_f(s)}\}^c(y) = \{A_f^d(s)\}^c(y)$.

(iii) $A_f^d(s)(x) = 0 =$sequence of real zeros iff ∃ an open fuzzy sequential set $B_f(s)$ such that $B_f(s)(x) = 1 - r$ where $r = \{r_n\}_n$ and $r_n = 0$ if $n \neq k$, $r_n = r_k$ if $n = k$.

Theorem 2.5 The fuzzy derived sequential set of each fuzzy sequential set is closed iff the fuzzy derived sequential set of each simple fuzzy sequential point is closed.

Proof. The necessity is obvious. Conversely, suppose $H_f(s)$ is a fuzzy sequential set. We will show that $H_f^d(s) = D_f(s)$ is closed. Let $P_f(s) = (P_{fx}^k, r_k)$ be an accumulation point of $D_f(s)$. It is sufficient to show that $P_f(s) \in D_f(s)$. Let $r = \{r_n\}_n$ where $r_n = r_k$ for $n = k$ and $r_n = 0\ \forall\, n \neq k$. Now $P_f(s) \in \overline{D_f(s)} = \overline{H_f^d(s)} \leq \overline{\overline{H_f(s)}} = \overline{H_f(s)}$. Therefore $P_f(s)$ is an adherence point of $H_f(s)$. If $P_f(s) \notin H_f(s)$, then $P_f(s)$ is an accumulation point of $H_f(s)$, that is $P_f(s) \in D_f(s)$ and we are done.

Let us assume $P_f(s) \in H_f(s)$

$$\Rightarrow r \leq H_f(s)(x) = \rho \text{ (say)}$$

$$\Rightarrow r_k \leq H_f^k(x) = \rho_k$$

Now consider the simple fuzzy sequential point $A_f(s) = (P_{fx}^k, \rho_k)$. Let $\rho' = \{\rho'_n\}_n$ where $\rho'_k = \rho_k$ and $\rho'_n = 0\ \forall\, n \neq k$. There are two possibilities concerning $A_f^d(s)$.

Case I. $A_f^d(s)(x) = \rho_1$ is a non zero sequence. Now

$\overline{A_f(s)}(x) \geq A_f(s)(x) = \rho'$

By lemma 2.1(v), $\overline{A_f(s)}(x) > \rho'$

$\Rightarrow A_f^d(s)(x) = \overline{A_f(s)}(x) > \rho'$

$$\Rightarrow \rho_1 > \rho'$$

$\Rightarrow \rho_{1k} > \rho_k = A_f^k(x) = H_f^k(x)$.

Hence the simple fuzzy sequential point $Q_f(s) = (p_{fx}^k, \rho_{1k}) \notin H_f(s)$ but since $Q_f(s) \in A_f^d(s) \leq \overline{A_f(s)} \leq \overline{H_f(s)}$, $Q_f(s)$ is an accumulation point of $H_f(s)$, that is $Q_f(s) \in D_f(s)$. Moreover

$r_k \leq \rho_k < \rho_{1k}$

$\Rightarrow r_k < \rho_{1k}$

$\Rightarrow P_f(s) \in D_f(s)$.

Case II. $A_f^d(s)(x) = 0$. Let $B_f(s)$ be an arbitrary weak Q-nbd of $A_f(s)$ and hence of $P_f(s)$. In view of lemma 2.2(ii), ∃ an open fuzzy sequential set $B_f^@(s)$ such that $B_f^@(s)(x) = 1$ and for $y \neq x$, $B_f^@(s)\,(y) = \{\overline{A_f(s)}\}^c(y)$. Let Let $C_f(s) = B_f(s) \wedge B_f^@(s)$. Then $C_f(s)(x) = B_f(s)(x)$ which implies $C_f^k(x) = B_f^k(x) > 1 - r_k$. Thus $C_f(s)$ is a weak Q-nbd of $P_f(s)$. Hence $C_f(s)$ and $D_f(s)$ are weakly quasi coincident, that is ∃ a point z and $n \in \mathbb{N}$

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such that $D_f^n(z) + C_f^n(z) > 1$. Owing to the fact that $D_f(s)$ is the union of all the accumulation points of $H_f(s)$, ∃ an accumulation point $P'_f(s) = (p_{fz}^n, \mu_n)$ such that $\mu_n + C_f^n(z) > 1$. Therefore $C_f(s)$ is a weak Q-nbd of $P'_f(s)$. Let $\mu = \{\mu_m\}_m$ where $\mu_n \neq 0$ and $\mu_m = 0$ for all $m \neq n$. The proof will be carried out, according to the following subcases:

Subcase I. When $n = k$.

(a) when $z = x$ and $\mu \leq \rho'$, then $P'_f(s) \in H_f(s)$. Since $P'_f(s)$ is an accumulation point of $H_f(s)$, every weak Q-nbd of $P'_f(s)$ (and hence $B_f(s)$) and $H_f(s)$ are weakly quasi coincident at some point having different base or different support than that of $P_f(s)$.

(b) When $z = x$ and $\mu > \rho'$, then $P'_f(s) \notin H_f(s)$. From lemma 2.2(iii), ∃ an open fuzzy sequential set $B'_f(s)$ such that $B'_f(s)(x) = 1 - \rho' > 1 - \mu$. Therefore $G_f(s) = C_f(s) \wedge B'_f(s)$ is also a weak Q-nbd of $P'_f(s)$. Hence $G_f(s)$ and $H_f(s)$ are weakly quasi coincident. Since $G_f(s)(x) \leq B'_f(s)(x) = 1 - \rho'$

$\Rightarrow G_f^k(x) \leq {B'}_f^k(x) = 1 - \rho_k$

Thus $G_f(s)$ (and hence $B_f(s)$) and $H_f(s)$ are weakly quasi coincident at some point having different base or different support than that of $P_f(s)$.

(c) When $z \neq x$.

We have $B_f^@(s)\,(z) = \{\overline{A_f(s)}\}^c(z)$. Also $\{\overline{A_f(s)}\}^c = ((A_f(s))^c)^\circ$. Since $((A_f(s))^c)^\circ(z) = B_f^@(s)\,(z) \geq C_f(s)(z)$ ∃ an open fuzzy sequential set $B''_f(s) \leq (A_f(s))^c$ such that ${B''}_f^k(z) \geq C_f^k(z) > 1 - \mu_k$. Therefore $G'_f(s) = B_f(s) \wedge B''_f(s)$ is also a weak Q-nbd $P'_f(s)$ and hence is weakly quasi coincident with $H_f(s)$. Since $B''_f(s) \leq (A_f(s))^c$

$\Rightarrow B''_f(s)(x) \leq 1 - A_f(s)(x)$

$\Rightarrow {B''}_f^k(x) \leq 1 - A_f^k(x) = 1 - H_f^k(x)$.

Thus $G'_f(s)$ (and hence $B_f(s)$) is weakly quasi coincident with $H_f(s)$ at some point having different base or different support than that of $P_f(s)$.

Subcase II. When $n \neq k$.

(a) Suppose $z = x$. We have $B'_f(s)(x) = 1 - \rho'$

$\Rightarrow {B'}_f^n(x) = 1 > 1 - \mu_n$.

So $B'_f(s)$ is a weak Q-nbd of $P'_f(s)$. Hence $G_f(s) = C_f(s) \wedge B'_f(s)$ is a weak Q-nbd of $P'_f(s)$ and so it is weakly quasi coincident with $H_f(s)$. Now $G_f(s)(x) \leq B'_f(s)(x) = 1 - \rho'$

$\Rightarrow G_f^k(x) \leq {B'}_f^k(x) = 1 - \rho_k = 1 - H_f^k(x)$.

So $H_f(s)$ and $G_f(s)$ are weakly quasi coincident at some point having different base or different support than that of $P_f(s)$.

(b) When $z \neq x$, the proof is same as Subcase I (c).

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RESEARCH ARTICLE

Separation Axioms in Fuzzy Sequential Topological Spaces

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We develop the separation axioms in fuzzy sequential topological spaces and establish some results related to those axioms. Notions of various separation axioms in fuzzy sequential topological spaces are introduced and investigated the relations among them. Dependency of a component on another component of a fuzzy sequential topology plays the main role in this paper.

Keywords: Fuzzy sequential topological spaces; fs-T_0; fs-T_1; fs-Hausdorff; weakly fs-Hausdorff; fs-regular; weakly fs-regular; fs-normal and weakly fs-normal spaces.

AMS Subject Classification: 54A40.

1. Introduction and Preliminaries

In 1965, L. A. Zadeh introduced the concept of fuzzy sets [1] and fuzzy topology was introduced by C. L. Chang in 1968 [2]. A number of works on fuzzy topological spaces and fuzzy metric spaces have been appeared in the literature. In this paper, we study various separation axioms in fuzzy sequential topological spaces [3]. The key idea behind this work has been drawn from [4–6]. First we give some basic definitions, notations and results of [3] which will be used in the sequel.

Let X be a non empty set and $I = [0, 1]$ be the closed unit interval in the set $\mathbb{R}$ of real numbers. Let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be sequences of fuzzy sets in X called fuzzy sequential sets in X and we define:

(1) $A_f(s) \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (Union).
(2) $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (Intersection).
(3) $A_f(s) \leq B_f(s)$ if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers.
(4) $A_f(s) \leq_w B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$.
(5) $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$.
(6) $A_f(s)(x) = \{A_f^n(x)\}_n$, $x \in X$.
(7) $A_f(s)(x) \geq_M r$ if and only if $A_f^n(x) \geq r_n$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular, if $M = \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, we write $A_f(s)(x) \geq r$.
(8) $X_f^l(s) = \{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$.
(9) $A_f^c(s) = \{1 - A_f^n\}_n = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$.

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(10) a fuzzy sequential set $P_f(s) = \{p_f^n\}_n$ is called a fuzzy sequential point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that

$$p_f^n(t) = r_n, \text{ if } t = x,$$
$$= 0, \text{ if } t \in X - \{x\}, \text{ for all } n \in \mathbb{N}.$$

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$p_f^n(x) = r_n, \text{ whenever } n \in M,$$
$$= 0, \text{ whenever } n \in \mathbb{N} - M.$$

The point x is called the support, M is called base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (p_{fx}^M, r)$. If further $M = \{n\}$, $n \in \mathbb{N}$, then the fuzzy sequential point is called a simple fuzzy sequential point and it is denoted by (p_{fx}^n, r_n). A fuzzy sequential point is called complete if its base is the set of natural numbers. A fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$, symbolically $P_f(s) \in_w A_f(s)$ if and only if there exists $n \in M$ such that $p_f^n(x) \leq A_f^n(x)$. If $R \subseteq M$ and s is the sequence in I same to r in R and vanishes outside R then the fuzzy sequential point $P_{rf}(s) = (p_{fx}^R, s)$ is called a reduced fuzzy sequential point of $P_f(s) = (p_{fx}^M, r)$.

A sequence $(x, L) = \{A_n\}_n$ of subsets of X, where $A_n = \{x\}$, for all $n \in L$ and $A_n = \Phi =$ the null subset of X, for all $n \in \mathbb{N} - L$, is called a sequential point in X. A family $\delta(s)$ of fuzzy sequential sets on a non empty set X satisfying the following properties:

(1) $X_f^r(s) \in \delta(s)$ for all $r \in \{0, 1\}$,
(2) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and,
(3) for any family $\{A_{fj}(s) \in \delta(s),\ j \in J\}$, $\bigvee_{j \in J} A_{fj}(s) \in \delta(s)$,

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Compliment of an open fuzzy sequential set in X is called closed fuzzy sequential set in X. If $\delta_1(s)$ and $\delta_2(s)$ be two FSTs on X such that $\delta_1(s) \subset \delta_2(s)$, then we say that $\delta_2(s)$ is finer than $\delta_1(s)$ or $\delta_1(s)$ is weaker than $\delta_2(s)$. If δ be a fuzzy topology (FT) on X, then $\delta^{\mathbb{N}}$ forms a FST on X. We may construct different FSTs on X from a given FT δ on X, $\delta^{\mathbb{N}}$ is the finest of all these FSTs. Not only that, any FT δ on X can be considered as a component of some FST on X, one of them is $\delta^{\mathbb{N}}$, there are at least countably many FSTs on X weaker than δ^N of which δ is a component. One of them is $\delta'(s) = \{A_f^n(s) = \{A_f^n\}_n;\ A_f^n = A$ for all $n \in \mathbb{N}$ and $A \in \delta\}$.

If $(X, \delta(s))$ is a FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where $\delta_n = \{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$ and (X, δ_n) is called the n^{th} component FTS of the FSTS $(X, \delta(s))$. Let $A_f^n(s)$=$\{A_f^n\}_n$ be an open (closed) fuzzy sequential set in the FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in (X, δ_n) but the converse is not necessarily true.

Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called quasi-coincident, denoted by $A_f(s) q B_f(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$, whenever A_f^n and B_f^n both are not $\overline{0}$. We write $A_f(s) \overline{q} B_f(s)$ to say that $A_f(s)$ and $B_f(s)$ are not quasi-coincident. Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called weakly quasi-coincident, denoted by $A_f(s) q_w B_f(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$ for some $n \in \mathbb{N}$. We write $A_f(s) \overline{q}_w B_f(s)$ to mean that $A_f(s)$ and $B_f(s)$ are not weakly quasi-coincident.

A fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ is called quasi-coincident with $A_f(s) = \{A_f^n\}_n$, denoted by $P_f(s) q A_f(s)$ if and only if $P_f^n(x) > (A_f^n)^c(x)$ for all $n \in M$. If $P_f(s) = (p_{fx}^M, r)$ is not quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q}A_f(s)$. A fuzzy sequential point $P_f(s) = (p^M_{fx}, r)$ is called weakly quasi-coincident with $A_f(s) = \{A^n_f\}_n$, denoted by $P_f(s)q_wA_f(s)$ if and only if $P^n_f(x) > (A^n_f)^c(x)$ for some $n \in M$. If $P_f(s) = (p^M_{fx}, r)$ is not weakly quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q}_wA_f(s)$. If $P^n_f(x) > (A^n_f)^c(x)$ for some $n \in L \subseteq M$, then we say that $P_f(s)$ is weakly quasi-coincident with $A_f(s)$ at the sequential point (x, L). If the fuzzy sequential sets $A_f(s) = \{A^n_f\}_n$ and $B_f(s) = \{B^n_f\}_n$ are quasi-coincident, then each pair of non $\overline{0}$ fuzzy sets A^n_f and B^n_f is also so but the converse is not necessarily true.

The fuzzy sequential point $P_f(s) = (p^M_{f0.5}, r)$ where $M = \{1, 2\}$, $r = \{r_n\}_n$ and $r_1 = r_2 = \frac{7}{10}$ is quasi-coincident with $A_{f1}(s) \vee A_{f2}(s)$ but it is not so with any one of them. A subfamily β of a FST $\delta(s)$ on X is called a base for $\delta(s)$ if and only if to every $A_f(s) \in \delta(s)$, there exists a subfamily $\{B_{fj}(s), j \in J\}$ of β such that $A_f(s) = \bigvee_{j\in J} B_{fj}(s)$. A subfamily $S = \{S_{f\lambda}(s);\ \lambda \in \Lambda\}$ of a FST $\delta(s)$ on X is called a subbase for $\delta(s)$ if and only if $\{\bigwedge_{j\in J} S_{fj}(s);\ J = \text{finite subset of } \Lambda\}$ forms a base for $\delta(s)$.

A subfamily β of a fuzzy sequential topology $\delta(s)$ on X is a base for $\delta(s)$ if and only if for each fuzzy sequential point $P_f(s)$ in $(X, \delta(s))$ and for every open weak Q nbd $A_f(s)$ of $P_f(s)$, there exists a member $B_f(s) \in \beta$ such that $P_f(s)q_wB_f(s) \le A_f(s)$. If β be a base for the FST $\delta(s)$ on X, then $\beta_n = \{B^n_f;\ B_f(s) = \{B^n_f\}_n \in \beta\}$ forms a base for the component fuzzy topology δ_n on X for each $n \in \mathbb{N}$ but not conversely.

Let $A_f(s)$ be any fuzzy sequential set in a FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \bigwedge\{C_f(s);\ A_f(s) \le C_f(s), C^c_f(s) \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \bigvee\{O_f(s);\ O_f(s) \le A_f(s),\ O_f(s) \in \delta(s)\}.$$

If $\overline{A_f(s)} = \{\overline{A^n_f}\}_n$ in $(X, \delta(s))$, then $cl(A^n_f) \le \overline{A^n_f}$ in (X, δ_n) for each $n \in \mathbb{N}$, where $cl(A^n_f)$ is the closure of A^n_f in (X, δ_n). The dual of a fuzzy sequential point $P_f(s) = (p^M_{fx}, r)$ is a fuzzy sequential point $P_{df}(s) = (p^M_{fx}, t)$, where $r = \{r_n\}_n$, $t = \{t_n\}_n$ and

$$\begin{aligned} t_n &= 1 - r_n \text{ for all } n \in M, \\ &= 0 \text{ for all } n \in \mathbb{N} - M. \end{aligned}$$

Every Q nbd of a fuzzy sequential point $P_f(s)$ is weakly quasi-coincident with a fuzzy sequential set $A_f(s)$ implies $P_f(s) \in \overline{A_f(s)}$ implies every weak Q nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident. A fuzzy sequential point $P_f(s) \in \overset{o}{A}_f(s)$ if and only if its dual point $P_{df}(s) \notin \overline{A^c_f(s)}$. In a FSTS $(X, \delta(s))$, the following hold:

(1) $\overline{X^r_f(s)} = X^r_f(s)$, $r \in \{0, 1\}$.
(2) $A_f(s)$ is closed if and only if $\overline{A_f(s)} = A_f(s)$.
(3) $\overline{\overline{A_f(s)}} = \overline{A_f(s)}$.
(4) $\overline{A_f(s) \vee B_f(s)} = \overline{A_f(s)} \vee \overline{B_f(s)}$.
(5) $\overline{A_f(s) \wedge B_f(s)} \subseteq \overline{A_f(s)} \wedge \overline{B_f(s)}$.
(6) $(X^r_f(s))^o = X^r_f(s)$, $r \in \{0,\ 1\}$.
(7) $A_f(s)$ is open if and only if $\overset{o}{A}_f(s) = A_f(s)$.
(8) $(\overset{o}{A}_f(s))^o = \overset{o}{A}_f(s)$.
(9) $(A_f(s) \wedge B_f(s))^o = \overset{o}{A}_f(s) \wedge \overset{o}{B}_f(s)$.

(10) $(A_f(s) \vee B_f(s))^o = \overset{o}{A}_f(s) \vee \overset{o}{B}_f(s)$.
(11) $\overset{o}{A}_f(s) = (\overline{A^c_f(s))}^c$.
(12) $\overline{A_f(s)} = \overline{(A^c_f(s))^o}$.
(13) $(\overline{A_f(s)})^c = (A^c_f(s))^o$.
(14) $\overline{(A^c_f(s))} = (\overset{o}{A}_f(s))^c$.

A fuzzy sequential point $P_f(s)$ is called an adherence point of a fuzzy sequential set $A_f(s)$ if and only if every weak Q-nbd of $P_f(s)$ is weakly quasi-coincident with $A_f(s)$. A fuzzy sequential point $P_f(s)$ is called an accumulation point of a fuzzy sequential set $A_f(s)$ if and only if $P_f(s)$ is an adherence point of $A_f(s)$ and every weak Q-nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident at some sequential point having different base or support from that of $P_f(s)$ whenever $P_f(s) \in A_f(s)$. Any reduced sequential point of an accumulation point of a fuzzy sequential set is also an accumulation point of it. The union of all accumulation points of a fuzzy sequential set $A_f(s)$ is called the fuzzy derived sequential set of $A_f(s)$ and it is denoted by $A^d_f(s)$.

In a FSTS $(X, \delta(s))$, $\overline{A_f(s)} = A_f(s) \vee A^d_f(s)$. A fuzzy sequential set is closed in a FSTS $(X, \delta(s))$ if and only if it contains all its accumulation points. The fuzzy derived sequential set of a fuzzy sequential point equals the union of the fuzzy derived sequential sets of all its simple reduced fuzzy sequential points. If the fuzzy derived sequential set of each of the reduced fuzzy sequential points of a fuzzy sequential point is closed, then the derived sequential set of the fuzzy sequential point is closed.

Let $A_f(s) = (p^k_{fx}, r)$ be a fuzzy sequential point in FSTS $(X, \delta(s))$, then:

(1) For $y \neq x$, $\overline{A_f(s)}(y) = A^d_f(s)(y)$.
(2) If $\overline{A_f(s)}(x) >_P r$, $\overline{A_f(s)}(x) =_P A^d_f(s)(x)$, where $P \subset M$.
(3) If $\overline{A_f(s)}(x) >_M r$, $\overline{A_f(s)}(x) = A^d_f(s)(x)$.
(4) If $A^d_f(s)(x) = 0 =$ sequence of real zeros, then $\overline{A_f(s)}(x) = r$.
(5) If $A_f(s)$ is simple then converse of (iv) is true.

Let $A_f(s) = (p^k_{fx}, r_k)$ be a simple fuzzy sequential point in FSTS $(X, \delta(s))$. Then:

(1) If $A^d_f(s)(x)$ is a non zero sequence, then $\overline{A_f(s)} = A^d_f(s)$.
(2) If $A^d_f(s)(x) = 0 =$sequence of real zeros, then $A^d_f(s)$ is closed iff there exists an open fuzzy sequential set $B^@_f(s)$ such that $B^@_f(s)(x) = 1$ and for $y \neq x$, $B^@_f(s)(y) = \{\overline{A_f(s)}\}^c(y) = \{A^d_f(s)\}^c(y)$.
(3) $A^d_f(s)(x) = 0 =$sequence of real zeros iff there exists an open fuzzy sequential set $B_f(s)$ such that $B_f(s)(x) = 1 - r$ where $r = \{r_n\}_n$ and $r_n = 0$ if $n \neq k$, $r_n = r_k$ if $n = k$.

It is observed that fuzzy derived sequential set of each fuzzy sequential set is closed if and only if the fuzzy derived sequential set of each simple fuzzy sequential point is closed. Books [5, 7–9] may provide a suitable background for the present work.

2. Main Definitions and Results

Definition 2.1 Two fuzzy sequential points $P_f(s) = (p^M_{fx}, r)$ and $Q_f(s) = (p^N_{fy}, t)$ are said to be identical if $x = y$, $M = N$ and $r = t$; otherwise they are distinct.

Definition 2.2 A set $M \subset \mathbb{N}$ is said to be base of a fuzzy sequential set $U_f(s) = \{U^n_f\}^\infty_{n=1}$ if $U^n_f \neq \overline{0}$ $\forall n \in M$ and $U^n_f = \overline{0}$ $\forall n \in \mathbb{N} - M$.

Definition 2.3 A fuzzy sequential set $B_f(s) = \{B^n_f\}^\infty_{n=1}$ (having base N) is said to be completely contained in a fuzzy sequential set $A_f(s) = \{A^n_f\}^\infty_{n=1}$ (having base M) if $M = N$ and $B^n_f \leq A^n_f$ for all $n \in \mathbb{N}$.

Definition 2.4 A fuzzy sequential set $B_f(s)=\{B_f^n\}_{n=1}^{\infty}$ (having base N) is said to be totally reduced from the fuzzy sequential set $A_f(s)=\{A_f^n\}_{n=1}^{\infty}$ (having base M) if $N \subsetneqq M$ and $B_f^n \le A_f^n$ for all $n \in N$.

Definition 2.5 A FSTS $(X, \delta(s))$ is said to be fs-T_0 space if for any two distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$, there exists a weak Q-nbd of one of $P_f(s)$ and $Q_f(s)$ which is not weakly quasi coincident with the other.

Theorem 2.6 A FSTS $(X, \delta(s))$ is fs-T_0 space iff for every pair of distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$ either $P_f(s)$ does not belong to the closure of $Q_f(s)$ or $Q_f(s)$ does not belong to the closure of $P_f(s)$.

Proof Suppose $(X, \delta(s))$ is fs-T_0. Then there exists a weak Q-nbd $U_f(s)$ of $P_f(s)$ which is not weakly quasi coincident with $Q_f(s)$. This implies that $P_f(s) \notin \overline{Q_f(s)}$. Conversely, suppose $P_f(s)$ and $Q_f(s)$ be any two distinct fuzzy sequential points such that $P_f(s) \notin \overline{Q_f(s)}$. This implies that exists a weak Q-nbd of $P_f(s)$ which is not weakly quasi coincident with $Q_f(s)$. Hence $(X, \delta(s))$ is fs-T_0. ■

Corollary 2.1 A FSTS $(X, \delta(s))$ is fs-T_0 space iff distinct fuzzy sequential points have distinct closures.

Theorem 2.7 A FTS (X, δ) is fuzzy T_0 iff the FSTS $(X, \delta^{\mathbb{N}})$ is fs-T_0.

Proof Suppose (X, δ) is fuzzy T_0. Let $P_f(s)=(p_{fx}^M, r)$ and $Q_f(s)=(p_{fy}^N, t)$ be any two distinct fuzzy sequential points where $r=\{r_n\}_{n=1}^{\infty}$ and $t=\{t_n\}_{n=1}^{\infty}$.

Case I: Suppose $x \ne y$. Then for $p_x^{r_m} \ne p_y^{t_m}$ $(m \in M)$ there exists a Q-nbd U of $p_x^{r_m}$ which is not quasi coincident with $p_x^{t_m}$.

Case II: Suppose $x = y$, $M \cap N = \phi$. Then for $p_x^{r_m} \ne p_x^{t_m}$ $(m \in M\ )$ there exists a Q-nbd U of $p_x^{r_m}$ which is not quasi coincident with $p_x^{t_m}$.

Case III: Suppose $x=y$, $N \subset M$. If $r_m \ne t_m$ for some $m \in N$, then for $p_x^{r_m} \ne p_x^{t_m}$ there exists a Q-nbd U of $p_x^{r_m}$ which is not quasi coincident with $p_x^{t_m}$. If $r_n = t_n\ \forall\ n \in N$, then for $p_x^{r_m} \ne p_x^{t_m}$ $(m \in M-N)$ there exists a Q-nbd U of $p_x^{r_m}$ which is not quasi coincident with $p_x^{t_m}$.

Case IV: Suppose $x=y$ and neither $N \subset M$ nor $M \subset N$ nor $M \cap N = \phi$. Then for $p_x^{r_m} \ne p_x^{t_m}$ $(m \in M,\ m \notin N)$ there exists a Q-nbd U of $p_x^{r_m}$ which is not quasi coincident with $p_x^{t_n}$.

In all the above cases, the fuzzy sequential set $U_f(s)=\{U_f^n\}_{n=1}^{\infty}$ where $U_m=U$ and $U_n=\overline{0}\ \forall n \ne m$, is a weak Q-nbd of $P_f(s)$ which is not weakly quasi coincident with $Q_f(s)$.

Conversely, suppose $(X,\ \delta^{\mathbb{N}})$ is fs-T_0. Let p_x^{λ} and p_y^{μ} be any two distinct fuzzy points in X. Then for each $n \in \mathbb{N}$, fuzzy sequential points $(p_{fx}^n,\ \lambda)$ and $(p_{fy}^n,\ \mu)$ are distinct. So exists a weak Q-nbd $U_f(s)=\{U_f^n\}_{n=1}^{\infty}$ of one of $(p_{fx}^n,\ \lambda)$ and $(p_{fy}^n,\ \mu)$ which is not weakly quasi coincident with the other. This implies U_f^n is a Q-nbd of one of p_x^{λ} and p_y^{μ} which is not quasi coincident with the other. ■

Theorem 2.8 If a FSTS $(X, \delta(s))$ is fs-T_0, then the FTS (X, δ_n) where $\delta_n=\{A_f^n;\ A_f(s)=\{A_f^n\}_{n=1}^{\infty} \in \delta(s)\}$ is fuzzy T_0 for each $n \in \mathbb{N}$.

Proof Let p_x^{λ} and p_y^{μ} be any two distinct fuzzy points in X. Then for each $n \in \mathbb{N}$, fuzzy sequential points (p_{fx}^n, λ) and (p_{fy}^n, μ) are distinct. So there exists a weak Q-nbd $U_f(s)=\{U_f^n\}_{n=1}^{\infty}$ of one of (p_{fx}^n, λ) and (p_{fy}^n, μ) which is not weakly quasi coincident with the other. This implies U_f^n is a Q-nbd of one of p_x^{λ} and p_y^{μ} which is not quasi coincident with the other. ■

Remark 2.9 Converse of Theorem 2.8 is not true as shown by Example 2.10.

Example 2.10 Let (X, δ) be a FTS. For any $A \in \delta$ let $B_{fA}(s)=\{B_{fA}^n\}_{n=1}^{\infty}$, $C_{fA}(s)=\{C_{fA}^n\}_{n=1}^{\infty}$ and $D_{fA}(s)=\{D_{fA}^n\}_{n=1}^{\infty}$, where $B_{fA}^n=A$ for odd n, $B_{fA}^n=\overline{0}$ for even n, $C_{fA}^n=\overline{0}$ for odd n, $C_{fA}^n=A$ for even n and $D_{fA}^n=A$ for all $n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$ for all $A \in \delta$ forms a FST on X. If (X, δ) is fuzzy T_0, then the components of $(X, \delta(s))$ are fuzzy T_0 but $(X, \delta(s))$ is not fs-T_0.

Definition 2.11 Suppose $U_f(s) = \{U_f^n\}_{n=1}^{\infty}$ and $V_f(s) = \{V_f^n\}_{n=1}^{\infty}$ are two fuzzy sequential sets. If there exists an $M \subset \mathbb{N}$ such that $U_f^n q V_f^n$ for all $n \in M$, we say that $U_f(s)$ is M-quasi coincident with $V_f(s)$ and we write $U_f(s) q^M V_f(s)$. If $U_f^n q V_f^n$ for at least one $n \in M$, we say that $U_f(s)$ is weakly M-quasi coincident with $V_f(s)$ and we write $U_f(s) q_w^M V_f(s)$.

Definition 2.12 A FSTS $(X, \delta(s))$ is said to be a fs-T_1 space if every fuzzy sequential point in X is closed.

Remark 2.13 A fs-T_1 space is fs-T_0.

Theorem 2.14 A FTS (X, δ) is fuzzy T_1 iff the FSTS $(X, \delta^{\mathbb{N}})$ is fs-T_1.

Proof Proof is omitted. ■

Theorem 2.15 If a FSTS $(X, \delta(s))$ is fs-T_1, then the component FTS (X, δ_n) is fuzzy T_1 for each $n \in \mathbb{N}$.

Proof Proof is omitted. ■

Remark 2.16 Converse of Theorem 2.15 is not true as shown by Example 2.17.

Example 2.17 Let (X, δ) be a FTS. For any $A \in \delta$ let $B_{fA}(s) = \{B_{fA}^n\}_{n=1}^{\infty}$, $C_{fA}(s) = \{C_{fA}^n\}_{n=1}^{\infty}$ and $D_{fA}(s) = \{D_{fA}^n\}_{n=1}^{\infty}$, where $B_{fA}^n = A$ for odd n, $B_{fA}^n = \overline{0}$ for even n, $C_{fA}^n = \overline{0}$ for odd n, $C_{fA}^n = A$ for even n and $D_{fA}^n = A$ for all $n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$ for all $A \in \delta$ forms a FST on X. If (X, δ) is fuzzy T_1 then the components of $(X, \delta(s))$ are fuzzy T_1 but $(X, \delta(s))$ is not fs-T_1.

Theorem 2.18 A FSTS $(X, \delta(s))$ is fs-T_1 iff for each $x \in X$ and each sequence $r = \{r_n\}_{n=1}^{\infty}$ in $[0, 1]$, there exists $B_f(s) \in \delta(s)$ such that $B_f(s)(x) = \{1 - r_n\}_{n=1}^{\infty}$ and $B_f(s)(y) = \{1\}_{n=1}^{\infty}$ for $y \neq x$.

Proof Suppose $(X, \delta(s))$ is fs-T_1. If r is a zero sequence, then it is sufficient to take $B_f(s) = X_f^1(s)$. Suppose r is a non zero sequence. Let $M \subset \mathbb{N}$ such that $r_n \neq 0$ for all $n \in M$ and $r_n = 0$ for all $n \in \mathbb{N} - M$. Then $P_f(s) = (p_{fx}^M, r)$ is a fuzzy sequential point in X and $B_f(s) = X_f^1(s) - P_f(s)$ is the required open fuzzy sequential set.

Conversely, suppose $P_f(s) = (p_{fx}^M, r)$ is an arbitrary fuzzy sequential point in X. By hypothesis, there exists $B_f(s) \in \delta(s)$ such that $B_f(s)(x) = 1 - r$ and $B_f(s)(y) = 1$ for $y \neq x$. It follows that $P_f(s)$ is the complement of $B_f(s)$ and hence is closed. ■

Theorem 2.19 The fuzzy derived sequential set of every fuzzy sequential set on a fs-T_1 space is closed.

Proof The fuzzy derived sequential set of a fuzzy sequential point in a fs-T_1 space, itself being a fuzzy sequential point is closed. Hence the result follows from [6, Theorem 2.5]. ■

Definition 2.20 A FSTS $(X, \delta(s))$ is said to be fs-Hausdorff space or fs-T_2 space if for any two distinct fuzzy sequential points $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w^{M-N} U_f(s),\ Q_f(s) q_w V_f(s),\ P_f(s) \overline{q}_w^{M-N} \overline{V_f(s)},\ Q_f(s) \overline{q}_w \overline{U_f(s)}$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$, otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w U_f(s),\ Q_f(s) q_w V_f(s),\ P_f(s) \overline{q}_w \overline{V_f(s)},\ Q_f(s) \overline{q}_w \overline{U_f(s)}.$$

Definition 2.21 A FSTS $(X, \delta(s))$ is said to be weak fs-Hausdorff space or (w) fs-Hausdorff space if for any two distinct fuzzy sequential points $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$

such that

$$P_f(s) \in_w^{M-N} U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_w V_f(s)$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$, otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in_w U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_w V_f(s).$$

Theorem 2.22 A fs-Hausdorff space is a weak fs-Hausdorff space.

Proof Proof is omitted. ■

Remark 2.23 Example 2.24 shows that a weak fs-Hausdorff space may not fs-Hausdorff space.

Example 2.24 Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$. Then $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff but not fs-Hausdorff.

Remark 2.25 A fs-T_2 space may not be fs-T_1, shown by Example 2.26.

Example 2.26 Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p_x^1, p_y^1\}$. Let $(X, \delta^{\mathbb{N}})$ is fs-T_2 but not fs-T_1.

Definition 2.27 A FSTS $(X, \delta(s))$ is said to be (w) fs-T_2 space if it is (w) fs- Hausdorff and fs-T_1.

Remark 2.28 A fs-T_2 space is weak fs-T_2.

Theorem 2.29 A FSTS $(X, \delta(s))$ is said to be fs-Hausdorff if for any two distinct fuzzy sequential points $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in^{M-N} G_f(s),\ Q_f(s)\ q_w H_f(s),\ G_f(s)\overline{q_w} H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X,\delta(s))$ such that

$$P_f(s) q_w^{M-N} D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s)\overline{q_w} D_f(s)$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$, otherwise there exist open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in G_f(s),\ Q_f(s) q_w H_f(s),\ G_f(s)\overline{q_w} H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s)\overline{q_w} D_f(s).$$

Proof Let $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other. Suppose $(X,\ \delta(s))$ is fs-Hausdorff.

Case I: Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. Then there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w U_f(s),\ Q_f(s) q_w V_f(s),\ P_f(s)\overline{q}_w \overline{V_f(s)},\ Q_f(s)\overline{q}_w \overline{U_f(s)}.$$

Now if we take $G_f(s) = X_f^1(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X_f^1(s) - \overline{U_f(s)}$, we are done.

Case II: Suppose one of $P_f(s)$ and $Q_f(s)$, say $Q_f(s)$ is totally reduced from $P_f(s)$. Then there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s)q_w^{M-N}U_f(s),\ Q_f(s)q_wV_f(s),\ P_f(s)\overline{q}_w^{M-N}\overline{V_f(s)},\ Q_f(s)\overline{q}_w\overline{U_f(s)}.$$

Now if we take $G_f(s) = X_f^1(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X_f^1(s) - \overline{U_f(s)}$, we are done.

Conversely, suppose the given conditions are true. In both the cases, if we take $U_f(s) = D_f(s)$ and $V_f(s) = H_f(s)$, we are done. ■

Theorem 2.30 A FSTS $(X,\ \delta(s))$ is fs-Hausdorff iff for any fuzzy sequential point $P_f(s)$ in X,

$$P_f(s) = \bigwedge\{\overline{N_f(s)} : N_f(s) \text{ is a nbd of } P_f(s)\}. \tag{1}$$

Proof Suppose $(X, \delta(s))$ is fs-Hausdorff. Let $P_f(s) = (p_{fx}^M, r)$ be any fuzzy sequential point in X and $Q_f(s) = (p_{fx}^N, t)$ be another fuzzy sequential point distinct from $P_f(s)$ and $Q_f(s) \notin P_f(s)$.

If $P_f(s)$ is totally reduced from $Q_f(s)$, there exists open fuzzy sequential set $V_f(s)$ in $(X, \delta(s))$ such that $Q_f(s)\ q_w^{N-M}V_f(s)$, $P_f(s)\overline{q}_w\overline{V_f(s)}$, otherwise there exists open fuzzy sequential set $V_f(s)$ in $(X, \delta(s))$ such that $Q_f(s)q_wV_f(s)$, $P_f(s)\overline{q}_w\overline{V_f(s)}$.

In both cases, if we take $U_f(s) = X_f^1(s) - \overline{V_f(s)}$, then $P_f(s) \in U_f(s)$ and $Q_f(s) \notin \overline{U_f(s)}$. Hence (1) is true.

Conversely, suppose (1) is true. Let $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other.

Case I: Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. By (1), there exist nbds $S_f(s)$ and $T_f(s)$ of $P_f(s)$ and $Q_f(s)$ respectively such that $P_f(s) \notin \overline{T_f(s)}$ and $Q_f(s) \notin \overline{S_f(s)}$. If we take $U_f(s) = X_f^1(s) - \overline{T_f(s)}$ and $V_f(s) = X_f^1(s) - \overline{S_f(s)}$, we are done.

Case II: Suppose one of $P_f(s)$ and $Q_f(s)$, $Q_f(s)$ (say) is totally reduced from $P_f(s)$. Then there exists nbds $S_f(s)$ and $T_f(s)$ of $P_f'(s)$ and $Q_f(s)$ respectively such that $P_f(s) \notin^{M-N} \overline{T_f(s)}$ and $Q_f(s) \notin \overline{S_f(s)}$, where $P_f'(s)$ is a reduced fuzzy sequential point of $P_f(s)$ with base $M-N$. If we take $U_f(s) = X_f^1(s) - \overline{T_f(s)}$ and $V_f(s) = X_f^1(s) - \overline{S_f(s)}$, we are done. ■

Theorem 2.31 If a FTS (X, δ) is fuzzy T_2, then the FSTS $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff.

Proof Proof is omitted. ■

Remark 2.32 Converse of Theorem 2.31 is not true as shown by Example 2.33.

Example 2.33 Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p_x^1, p_y^1\}$. Then $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff even though (X, δ) is not fuzzy T_2.

Remark 2.34 Example 2.35 shows that even if $(X,\ \delta)$ is fuzzy T_2, the FSTS $(X, \delta^{\mathbb{N}})$ may not be fs-Hausdorff.

Example 2.35 Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$. Then (X, δ) is fuzzy T_2 but $(X, \delta^{\mathbb{N}})$ is not fs-Hausdorff.

Remark 2.36 Example 2.37 shows that if a FSTS $(X, \delta(s))$ is fs-T_2, then the component FTS (X, δ_n) may not be fuzzy T_2 for each $n \in \mathbb{N}$.

Example 2.37 Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p_x^1, p_y^1\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then $(X, \delta(s))$ is fs-T_2 but $(X, \delta_1) = (X, \delta)$ is not fuzzy T_2.

Remark 2.38 Example 2.39 shows that even if all the component fuzzy topological spaces of a FSTS are fuzzy T_2, the FSTS may not be fs-T_2.

Example 2.39 Let (X, δ) be a FTS. For any $G \in \delta$, let $A_{fG}(s) = \{A_{fG}^n\}_{n=1}^{\infty}$, $B_{fG}(s) = \{B_{fG}^n\}_{n=1}^{\infty}$, $C_{fG}(s) = \{C_{fG}^n\}_{n=1}^{\infty}$ where $A_{fG}^n = G$ for odd n, $A_{fG}^n = \overline{0}$ for even n, $B_{fG}^n = \overline{0}$ for odd n, $B_{fG}^n = G$ for even n, $C_{fG}^n = G$ for all n. Then the collection $\delta(s)$ of all fs-sets(fuzzy sequential sets) $A_{fG}(s)$, $B_{fG}(s)$, $C_{fG}(s)$ for all $G \in \delta$ forms a fuzzy sequential topology on X. If (X, δ) is fuzzy T_2, then the components of $(X, \delta(s))$ are fuzzy T_2 but $(X, \delta(s))$ itself is not fs-T_2.

Definition 2.40 A FSTS $(X, \delta(s))$ is said to be fs-regular if for any fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ and a non zero closed fuzzy sequential set $A_f(s)$ such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w^{M-N} U_f(s),\ A_f(s) q_w V_f(s),\ P_f(s) \overline{q}_w^{M-N} \overline{V_f(s)},\ A_f(s) \leq X_f^1(s) - \overline{U_f(s)}$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and having base N, otherwise there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w U_f(s),\ A_f(s) q_w V_f(s),\ P_f(s) \overline{q}_w \overline{V_f(s)},\ A_f(s) \leq X_f^1(s) - \overline{U_f(s)}.$$

Definition 2.41 A FSTS $(X, \delta(s))$ is said to be weak fs-regular if for any fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ and its any open weak nbd $A_f(s)$, there exists a fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is totally reduced from $P_f(s)$ and having base N, otherwise there exists open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

and $A_f(s)$ is a nbd of $B_f(s)$.

Remark 2.42 Example 2.43 shows that a fs-regular space may not be weak fs-regular.

Example 2.43 Let $X = \{a\}$, $\delta = \{\overline{0}, \overline{1}, p_a^{0.4}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but not weak fs-regular.

Remark 2.44 A weak fs-regular space may not be fs-regular as shown by Example 2.45.

Example 2.45 Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$, then $(X, \delta^{\mathbb{N}})$ is weak fs-regular but is not fs-regular.

Remark 2.46 A fs-regular space may not be fs-T_1. This is shown by Example 2.47.

Example 2.47 Let $X = \{x\}$ and let $\delta = \{\overline{0}, \overline{1}, p_x^{0.5}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but not fs-T_1.

Definition 2.48 A FSTS $(X, \delta(s))$ is said to be fs-T_3 if it is fs-regular and fs-T_1.

Remark 2.49 A fs-T_3 space is fs-T_2.

Theorem 2.50 A FSTS $(X, \delta(s))$ is fs-regular iff for any fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ and a non zero closed fuzzy sequential set $A_f(s)$ such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, there exist open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in^{M-N} G_f(s),\ A_f(s) q_w H_f(s),\ G_f(s) \overline{q_w} H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w^{M-N} D_f(s),\ A_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s)$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in G_f(s),\ A_f(s) q_w H_f(s),\ G_f(s) \overline{q_w} H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w D_f(s),\ A_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s).$$

Proof Proof is omitted. ■

Theorem 2.51 A FSTS $(X, \delta(s))$ is fs-regular iff for any fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ and an open fuzzy sequential set $G_f(s)$ such that $P_f(s) q_w G_f(s)$ (where $X_f^1(s) - G_f(s)$ is not completely contained in $P_f(s)$), there exists an open fuzzy sequential set $H_f(s)$ in $(X, \delta(s))$ such that $P_f(s) \in^{M-N} H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s)\ q_w^{M-N} B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$, whenever $X_f^1(s) - G_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists an open fuzzy sequential set $H_f(s)$ in $(X,\ \delta(s))$ such that $P_f(s) \in H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and $\exists$ an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s)\ q_w B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$.

Proof Suppose $(X, \delta(s))$ is fs-regular. Let $P_f(s) = (p_{fx}^M, r)$ be any fuzzy sequential point and $G_f(s)$ be an open fuzzy sequential set such that $P_f(s) q_w G(s)$, i.e., $P_f(s) \notin X_f^1(s) - G_f(s) = A_f(s)$ (say). Then there exists open fuzzy sequential sets $U(s)$ and $V(s)$ in (X, τ) such that

$$P_f(s) \in^{M-N} U_f(s),\ A_f(s) q_w V_f(s),\ U_f(s) \overline{q_w} V_f(s)$$

and there exists open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w^{M-N} D_f(s),\ A_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s)$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in U_f(s),\ A_f(s) q_w V_f(s),\ U_f(s) \overline{q_w} V_f(s)$$

and there exists open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) q_w D_f(s),\ A_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s).$$

If we take $H_f(s) = U_f(s)$ and $B_f(s) = D_f(s)$, we are done.

Conversely, suppose given conditions are true. Let $P_f(s) = (p_{fx}^M, r)$ be any fuzzy sequential point and $A_f(s)$ be any closed fuzzy sequential set such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, i.e., $P_f(s) q_w X_f^1(s) - A_f(s) = G_f(s)$ (say). Then there exists an open fuzzy sequential set $H_f(s)$ in $(X, \delta(s))$ such that $P_f(s) \in^{M-N} H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s) q_w^{M-N} B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$, whenever $X_f^1(s) - G_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists an open fuzzy sequential set $H_f(s)$ in $(X, \delta(s))$ such that $P_f(s) \in H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s) q_w B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$. If we take $U_f(s) = H_f(s)$, $V_f(s) = X_f^1(s) - \overline{H_f}(s)$, $D_f(s) = B_f(s)$ and $E_f(s) = X_f^1(s) - \overline{B_f}(s)$, then we are done. ■

Theorem 2.52 If $(X, \delta(s))$ is fs-regular, then for any closed fuzzy sequential set $A_f(s)$ which is not a fuzzy sequential point,

$$A_f(s) = \bigwedge\{N_f(s),\ N_f(s) \text{ is a closed nbd of } A_f(s)\}. \tag{2}$$

Proof Suppose $(X, \delta(s))$ is fs-regular and $A_f(s)$ be any closed fuzzy sequential set which is not a fuzzy sequential point. If $A_f(s) = X_f^0(s)$, then (1) is true. Suppose $A_f(s) \neq X_f^0(s)$. Let $P_f(s)$ be any fuzzy sequential point such that $P_f(s) \notin A_f(s)$. Let M and N be the bases of $P_f(s)$ and $A_f(s)$, respectively. We have $P_f(s) \notin A_f(s)$ i.e.,

$$P_f(s) q_w X_f^1(s) - A_f(s) = G_f(s) \quad \text{(say)}.$$

then there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s) q_w^{M-N} B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$ whenever $A_f(s)$ is totally reduced from $P_f(s)$, otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s) q_w B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$. This implies $A_f(s) \leq X_f^1(s) - \overline{B_f(s)} = H_f(s)$ (say). Again

$$P_f(s) \notin X_f^1(s) - B_f(s) \Longrightarrow P_f(s) \notin \overline{H_f(s)}.$$

Thus (1) holds. ■

Remark 2.53 Example 2.54 shows that converse of Theorem 2.52 may not be true.

Example 2.54 Let X be any non empty set and $\delta = \{\overline{r},\ r \in [0,1]\}$ Then the FSTS $(X, \delta^{\mathbb{N}})$ is not regular although for any closed fuzzy sequential set $A_f(s)$ in $(X, \delta^{\mathbb{N}})$, $A_f(s) = \wedge\{N_f(s);\ N_f(s)$ is a closed nbd of $A_f(s)\}$.

Remark 2.55 Example 2.56 shows that for a fuzzy sequential point, (1) in Theorem 2.52 may not hold.

Example 2.56 Let $X = \{x\}$ and $\delta = \{\overline{1}, \overline{0}, p_x^{0.2}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but for the closed fuzzy sequential point $A_f(s) = (p_{fx}^{\{1,\ 2\}}, 0.8) \neq \wedge\{N_f(s);\ N_f(s)$ is a closed nbd of $A_f(s)\}$.

Theorem 2.57 A FTS (X, δ) is fuzzy regular iff $(X, \delta^{\mathbb{N}})$ is weak fs-regular.

Proof Proof is omitted. ■

Remark 2.58 Even if $(X, \delta^{\mathbb{N}})$ is fs-regular, (X, δ) may not be fuzzy regular, shown by Example 2.59.

Example 2.59 Let $X = \{a\}$, $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but (X, δ) is not fuzzy regular.

Remark 2.60 A FTS (X, δ) is fuzzy regular, it may not imply $(X, \delta^{\mathbb{N}})$ is fs-regular as shown by Example 2.61.

Example 2.61 Let $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,1]\}$, then (X, δ) is fuzzy regular but $(X, \delta^{\mathbb{N}})$ is not fs-regular.

Remark 2.62 A FSTS $(X, \delta(s))$ is fs-regular, it may not imply component fuzzy topological spaces (X, δ_n), $n \in \mathbb{N}$ is fuzzy regular as shown by Example 2.63.

Example 2.63 Let $X = \{a\}$, $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then $(X, \delta(s))$ is fs-regular but $(X, \delta_n) = (X, \delta)$ for all $n \in \mathbb{N}$ is not fuzzy regular.

Remark 2.64 A FSTS $(X, \delta(s))$ may not be fs-regular even if the component fuzzy topological spaces (X, δ_n) is fuzzy regular for all $n \in \mathbb{N}$, as shown by Example 2.65.

Example 2.65 Let $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,1]\}$. Let $\delta(s) = \delta^{\mathbb{N}}$. Then $(X, \delta(s))$ is not be fs-regular but all the component fuzzy topological spaces (X, δ_n), $n \in \mathbb{N}$, is fuzzy regular.

Definition 2.66 Fuzzy sequential sets $A_f(s)=\{A_f^n\}_{n=1}^{\infty}$ and $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ are said to be quasi discoincident if A_f^n and B_f^n are strong quasi discoincident for all n.

Definition 2.67 Fuzzy sequential sets $A_f(s) = \{A_f^n\}_{n=1}^{\infty}$ and $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ are said to be partially quasi discoincident if A_f^n and B_f^n are strong quasi discoincident for some $n \in \mathbb{N}$..

Definition 2.68 A FSTS $(X, \delta(s))$ is said to be fs-normal iff for any two partially quasi discoincident non zero closed fuzzy sequential sets $A_f(s)$ and $B_f(s)$ (having the respective bases M and N and none of which is completely contained in the other), there exists an open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) q_w^{M-N} U_f(s),\ B_f(s) q_w V_f(s),\ A_f(s) \leq^{M-N} X_f^1(s) - \overline{V_f(s)},\ B_f(s) \leq X_f^1(s) - \overline{U_f(s)}$$

whenever $B_f(s)$ is totally reduced from $A_f(s)$, otherwise there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$A_f(s) q_w U_f(s),\ B_f(s) q_w V_f(s),\ A_f(s) \leq X_f^1(s) - \overline{V_f(s)},\ B_f(s) \leq X_f^1(s) - \overline{U_f(s)}.$$

Definition 2.69 A FSTS $(X, \delta(s))$ is said to be weak fs-normal iff for any non zero closed fuzzy sequential set $C_f(s)$ and its any open weak nbd $A_f(s)$, there exists a fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$C_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is totally reduced from $C_f(s)$, otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X,\ \delta(s))$ such that

$$C_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

($A_f(s)$ is a nbd of $B_f(s)$, M and N being the respective bases of $C_f(s)$ and $A_f^c(s)$).

Remark 2.70 A fs-normal FSTS may not be weak fs-normal, which is shown by Example 2.71.

Example 2.71 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-normal but not weak fs-normal.

Remark 2.72 Example 2.73 shows that a weak fs-normal space may not be fs-normal.

Example 2.73 Let $X = \{x, y\}$ and $\beta = \{p_x^r;\ r \in [0,1]\} \vee \{p_y^r;\ r \in [0,1]\}$ be a base for some fuzzy topology δ on X. Then $(X, \delta^{\mathbb{N}})$ is weak fs-normal but not fs-normal.

Definition 2.74 A FSTS $(X, \delta(s))$ is said to be fs-T_4 space if it is fs-normal and fs-T_1.

Remark 2.75 A fs-normal FSTS may not be fs-T_1 as shown by Example 2.76.

Example 2.76 Let $X = \{a, b\}$, $\delta(s) = \{X_f^0(s), X_f^1(s), A_f(s), B_f(s)\}$ where $A_f^n(a) = 1$ for all n, $A_f^n(b) = 0$ for all n, $B_f^n(a) = 0$ for all n, $B_f^n(b) = 1$ for all n, then $(X, \delta(s))$ is fs-normal but not fs-T_1.

Remark 2.77 A fs-normal FSTS may not be fs-regular as shown by Example 2.78.

Example 2.78 Let $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,1]\}$, then $(X, \delta^{\mathbb{N}})$ is fs-normal but not fs-regular. Hence a fs-T_4 space may not be fs-T_3.

Theorem 2.79 A FSTS $(X, \delta(s))$ is fs-normal iff for any two partially quasi discoincident non zero closed fuzzy sequential sets $A_f(s)$ and $B_f(s)$ (none of which is completely contained in the other), there exists an open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) \in^{M-N} G_f(s),\ B_f(s) q_w H_f(s),\ G_f(s) \overline{q_w} H_f(s)$$

and there exists an open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) q_w^{M-N} D_f(s),\ B_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s)$$

whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$), otherwise there exists open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) \in G_f(s),\ B_f(s) q_w H_f(s),\ G_f(s) \overline{q_w} H_f(s)$$

and there exists an open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) q_w D_f(s),\ B_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s).$$

Proof Proof is omitted. ■

Theorem 2.80 If a FSTS $(X, \delta(s))$ is weak fs-normal, then for any two non-zero closed partially quasi discoincident fs-sets $A_f(s)$ and $B_f(s)$ (none of which is completely contained in the other), there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that $A_f(s) \in_w^{M-N} U_f(s)$, $B_f(s) \in_w V_f(s)$, $U_f(s) \overline{q_w} V_f(s)$ whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$), otherwise there exists an open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that $A_f(s) \in_w G_f(s)$, $B_f(s) \in_w V_f(s)$, $U_f(s) \overline{q_w} V_f(s)$.

Proof The proof is omitted. ■

Remark 2.81 For a FSTS to be weak fs-normal, the condition given in Theorem 2.80 is only necessary but not sufficient as shown by Example 2.82.

Example 2.82 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is not weak fs-normal but the condition in Theorem 2.80 is satisfied.

Theorem 2.83 A weak fs-regular space $(X, \delta(s))$ is weak fs-normal when X is finite.

Proof Let $(X, \delta(s))$ be a weak fs-regular space. Let $C_f(s) = \{C_f^n\}_{n=1}^{\infty}$ be any non zero closed fuzzy sequential set in $(X, \delta(s))$ and $A_f(s)$ be its any open weak nbd. Let M and N be respectively the bases of $C_f(s)$ and $A_f^c(s)$. We choose $m \in M - N$ when $A_f^c(s)$ is totally reduced from $C_f(s)$ and we take $m \in M$ otherwise. Let $x \in X$ such that $C_f^m(x) \neq 0$ and let $C_f^m(x) = r_m$. Then for the fuzzy sequential point $p_{xf}(s) = (p_{f_x}^m,\ r_m)$, $A_f(s)$ is an open weak nbd. Hence there exists an open fuzzy sequential set $B_{xf}(s)$ in $(X, \delta(s))$ such that

$$p_{xf}(s) \in_w^{M-N} \overset{o}{B}_{xf}(s) \leq \overline{B_{xf}(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is a totally reduced fuzzy sequential set from $C_f(s)$, otherwise there exists open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$p_{xf}(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s) \quad (A_f(s) \text{ is a nbd of } B_{xf}(s)).$$

Corresponding to each $x \in X$ for which $C_f^m(x) \neq 0$, we get such open fs-set $B_{xf}(s)$. Since X is finite, there exists finitely many fs-sets say

$$B_{x_1 f}(s), B_{x_2 f}(s), \ldots, B_{x_k f}(s)$$

such that

$$p_{x_n f}(s) \in_w^{M-N} \overset{o}{B}_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s), \quad x_n \in X, \quad n = 1, 2, \ldots, k.$$

whenever $A_f^c(s)$ is a totally reduced fuzzy sequential set from $C_f(s)$, otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$p_{x_n f}(s) \in_w \overset{o}{B}_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s), \quad x_n \in X, \quad n = 1, 2, \ldots, k.$$

Now, let $B_f(s) = \bigcup_{n=1}^{k} B_{x_n f}(s)$. Then

$$C_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is totally reduced from $C_f(s)$, otherwise

$$C_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s).$$

Hence $(X, \delta(s))$ is weak fs-normal. ■

Theorem 2.84 A FTS (X, δ) is fuzzy normal iff $(X, \delta^{\mathbb{N}})$ is weak fs-normal.

Proof Proof is omitted. ■

Remark 2.85 Even if $(X, \delta^{\mathbb{N}})$ is fs-normal, (X, δ) may not be fuzzy normal, shown by Example 2.86.

Example 2.86 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-normal but (X, δ) is not fuzzy normal.

Remark 2.87 Example 2.88 shows that if (X, δ) is fuzzy normal, it may not imply $(X, \delta^{\mathbb{N}})$ is fs-normal.

Example 2.88 Let $X = \{x, y\}$ and $\beta = \{p_x^r;\ r \in [0,1]\} \vee \{p_y^r;\ r \in [0,1]\}$ be a base for some fuzzy topology δ on X. Then (X, δ) is fuzzy normal but $(X, \delta^{\mathbb{N}})$ is not fs-normal.

Remark 2.89 If $(X, \delta(s))$ is fs-normal, then it may not imply (X, δ_n) is fuzzy normal for each n, shown by Example 2.90.

Example 2.90 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then $(X, \delta(s))$ is fs-normal but $(X, \delta_n) = (X, \delta)$ for all $n \in \mathbb{N}$, is not fuzzy normal.

Remark 2.91 A FSTS $(X, \delta(s))$ may not be fs-normal even if the component fuzzy topological spaces (X, δ_n) is fuzzy regular for all $n \in \mathbb{N}$, as shown by Example 2.92.

Example 2.92 Let $X = \{x, y\}$ and $\beta = \{p_x^r;\ r \in [0,1]\} \vee \{p_y^r;\ r \in [0,1]\}$ be a base for some fuzzy topology δ on X. Let $\delta(s) = \delta^{\mathbb{N}}$. Then $(X, \delta(s))$ is not be fs-normal but all the component fuzzy topological spaces (X, δ_n), $n \in \mathbb{N}$, is fuzzy normal.

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FS-closure operators and FS-interior operators

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Abstract. Notions of FS-closure operators, FS-interior operators and their components are introduced. Various properties of FS-closure systems and FS-interior systems are studied and established a relation between them. A set of necessary and sufficient conditions under which an FS-closure operator and an FS-interior operator induce same fuzzy sequential topology on the underlying set have been obtained.

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1. Introduction

After the introduction of fuzzy sets by L. A. Zadeh in 1965 ([17]), C. L. Chang introduced the concept of fuzzy topology on a non empty set in 1968 ([6]). The concept of fuzzy sequential topological spaces (FSTS) were introduced in ([13]). In fuzzy set theory, fuzzy closure operators and fuzzy closure systems have been studied by Mashour and Ghanim ([10]), G. Gerla ([8]), Bandler and Kohout ([1]), R. Belohlavek ([2]), whereas fuzzy interior operators and fuzzy interior systems have appeared in the studies of R. Belohlavek and T. Funiokova ([3]), Bandler and Kohout ([1]).

Closure and interior operators on an ordinary set belong to the very fundamental mathematical structures with direct applications on the many fields like topology, logic etc. Being motivated by the importance of closure and interior operators, we introduce the concept of FS-closure and FS-interior operators on a set. Books ([5], [7] [9], [11]) and the articles ([4], [12], [14], [15], [16]) may provide a suitable background for the present work as some basic ideas have been derived from these sources. We begin with some basic definitions and results of ([13]) and ([16]). Let X be a non empty set and $I = [0, 1]$ be the closed unit interval in the set of real numbers. Let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be sequences of fuzzy sets in X called fuzzy sequential sets in X and we define

(i) $A_f(s) \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (union),
(ii) $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (intersection),
(iii) $A_f(s) \leq B_f(s)$ if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$,
(iv) $A_f(s) \leq_w B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$,
(v) $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$,
(vi) $A_f(s)(x) = \{A_f^n(x)\}_n$, $x \in X$,
(vii) $A_f(s)(x) \geq_M r$ if and only if $A_f^n(x) \geq r_n$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular if $M = \mathbb{N}$, we write $A_f(s)(x) \geq r$,
(viii) $X_f^l(s) = \{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$,
(ix) $(A_f(s))^c = \{1 - A_f^n\}_n = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$,
(x) A fuzzy sequential set $P_f(s) = \{p_f^n\}_n$ is called a fuzzy sequential point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that

$$\begin{aligned} p_f^n(t) &= r_n, \text{ if } t = x, \\ &= 0, \text{ if } t \in X - \{x\}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$\begin{aligned} p_f^n(x) &= r_n, \text{ whenever } n \in M, \\ &= 0, \text{ whenever } n \in \mathbb{N} - M. \end{aligned}$$

The point x is called the support, M is called the base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (p_{fx}^M, r)$. If further $M = \{n\}$, $n \in \mathbb{N}$, then the fuzzy sequential point is called a simple fuzzy sequential point and it is denoted by (p_{fx}^n, r_n). A fuzzy sequential point is called complete if its base is the set of natural numbers. A fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$, symbolically $P_f(s) \in_w A_f(s)$, if and only if there exists $n \in M$ such that $p_f^n(x) \leq A_f^n(x)$.

Definition 1.1 ([13]). A family $\delta(s)$ of fuzzy sequential sets on a non empty set X satisfying the properties

(i) $X_f^r(s) \in \delta(s)$ for $r = 0$ and 1,
(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and
(iii) for any family $\{A_{fj}(s) \in \delta(s), j \in J\}$, $\underset{j \in J}{\vee} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 1.2 ([13]). If $(X, \delta(s))$ is an FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where $\delta_n = \{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$. (X, δ_n), where $n \in \mathbb{N}$, is called the n^{th} component FTS of the FSTS $(X, \delta(s))$.

Proposition 1.3 ([13]). *Let $A_f(s)=\{A_f^n\}_n$ be an open (closed) fuzzy sequential set in the FSTS $(X,\delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in (X,δ_n) but the converse is not necessarily true.*

Proposition 1.4 ([13]). *If δ be a fuzzy topology (FT) on a non empty set X, then $\delta^{\mathbb{N}}$ forms an FST on X.*

Definition 1.5 ([13]). Let $A_f(s)$ be any fuzzy sequential set in an FSTS $(X,\delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \leq C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \vee\{O_f(s);\ O_f(s) \leq A_f(s),\ O_f(s) \in \delta(s)\}.$$

Definition 1.6 ([13]). A fuzzy sequential set $A_f(s)$ in an FSTS $(X,\ \delta(s))$ is called a neighbourhood (in short nbd) of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) \in B_f(s) \leq A_f(s)$. A nbd $A_f(s)$ is called open if and only if $A_f(s) \in \delta(s)$.

2. Definition and results

Definition 2.1. Let X be a non empty set. An operator $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-closure operator on X if it satisfies the following conditions:

(FSC1) $\mathbf{Cl}(X_f^0(s)) = X_f^0(s)$.

(FSC2) $A_f(s) \leq \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

(FSC3) $\mathbf{Cl}(\mathbf{Cl}(A_f(s))) = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

(FSC4) $\mathbf{Cl}(A_f(s) \vee B_f(s)) = \mathbf{Cl}(A_f(s)) \vee \mathbf{Cl}(B_f(s))$ for all $A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.2. For any FSTS $(X,\delta(s))$, closure of an fs-set (fuzzy sequential set) is an FS-closure operator on X.

Example 2.3. Let X be a non empty set. The operator $\mathbf{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $\mathbf{C}(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\mathbf{C}(X_f^0(s)) = X_f^0(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X, is an FS-closure operator on X.

Theorem 2.4. *If $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then*

(i) $\boldsymbol{Cl}$ is monotonic increasing, that is, $A_f(s) \leq B_f(s) \Rightarrow \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(B_f(s))$ for all $A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}}$.

(ii) $A_f(s) \leq \boldsymbol{Cl}(B_f(s)) \Rightarrow \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(B_f(s))$ for all $A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}}$.

Proof. Proof is omitted. □

Theorem 2.5. *Let X be a non empty set and $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an operator on X satisfying $(FSC1)$, $(FSC2)$ and $(FSC4)$, then*

a) The collection $\delta'(s) = \{(A_f(s))^c;\ A_f(s) \in (I^X)^{\mathbb{N}}$ and $\boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ forms an FST on X.

b) If $\boldsymbol{Cl}$ also satisfies $(FSC3)$, then for all $A_f(s) \in (I^X)^{\mathbf{N}}$ we have $\overline{A_f(s)} = \boldsymbol{Cl}(A_f(s))$, where $\overline{A_f(s)}$ is the closure of $A_f(s)$ in $\delta'(s)$.

Proof. Proof is omitted. □

Remark 2.6. From **Theorem** 2.5 it follows that if $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X then $\delta'(s) = \{(A_f(s))^c;\ A_f(s) \in (I^X)^{\mathbb{N}}$ and $\mathbf{Cl}(A_f(s)) = A_f(s)\}$ forms an FST on X. Also $\overline{A_f(s)} = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbf{N}}$, where $\overline{A_f(s)}$ is the closure of $A_f(s)$ in $\delta'(s)$. This FST $\delta'(s)$ is called the fuzzy sequential topology induced by the FS-closure operator **Cl** and we denote it by $\delta_{\mathbf{Cl}}(s)$.

Remark 2.7. **Example** 2.8 shows that if an operator $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a non empty set X, satisfies $(FSC1)$, $(FSC2)$ and $(FSC4)$ but does not satisfy $(FSC3)$, then $\delta_{\mathbf{Cl}}(s)$ forms an FST on X but $\overline{A_f(s)}$ may not be equal to $\mathbf{Cl}(A_f(s))$, $A_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.8. Let $X = \{a\}$. Let $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be defined by

$$\mathbf{Cl}(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}.$$

Then **Cl** is an operator on X satisfying $(FSC1)$, $(FSC2)$ and $(FSC4)$ and hence $(X, \delta_{\mathbf{Cl}}(s))$ forms an FSTS. Further **Cl** does not satisfy $(FSC3)$ and in $(X, \delta_{\mathbf{Cl}}(s))$, $\mathbf{Cl}(B_f(s)) \neq \overline{B_f(s)}$ if $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ where $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}\ \forall\ n \neq 1, 2, 3$

Definition 2.9. Let X be a non empty set and $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. A function $(\mathbf{Cl})_f^n \colon I^X \to I^X$ defined by $(\mathbf{Cl})_f^n(A) = n^{th}$ term of $\mathbf{Cl}({}_{nA}X_f^0(s))$, where ${}_{nA}X_f^0(s)$ denotes an fs-set whose n^{th} term is A and others are $\overline{0}$, is called the n^{th} component of **Cl**, $n \in \mathbb{N}$.

Theorem 2.10. *Let X be a non empty set. If $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then each component $(\mathbf{Cl})_f^n \colon I^X \to I^X$, $n \in \mathbb{N}$ is a fuzzy closure operator. Also $(\delta_{\mathbf{Cl}})_n = \delta_{(\mathbf{Cl})_f^n}$ where $(\delta_{\mathbf{Cl}})_n$ is the n^{th} component fuzzy topology of FST $\delta_{\mathbf{Cl}}(s)$ and $\delta_{(\mathbf{Cl})_f^n}$ is the fuzzy topology induced by the component $(\mathbf{Cl})_f^n$ of $\mathbf{Cl}$.*

Proof. $(\mathbf{Cl})_f^n(\overline{0}) = \overline{0}$ by definition. Let $A \in I^X$, then ${}_{nA}X_f^0(s) \leq \mathbf{Cl}({}_{nA}X_f^0(s)) \Rightarrow A \leq (\mathbf{Cl})_f^n(A)$. Hence $(\mathbf{Cl})_f^n(A) \leq (\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A))$. Also

$$\begin{aligned} & \mathbf{Cl}(\mathbf{Cl}({}_{nA}X_f^0(s))) = \mathbf{Cl}({}_{nA}X_f^0(s)) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(\mathbf{Cl})_f^n(A)}X_f^0(s)) \leq \mathbf{Cl}({}_{nA}X_f^0(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A)) \leq (\mathbf{Cl})_f^n(A) \end{aligned}$$

Hence $(\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A)) = (\mathbf{Cl})_f^n(A)$.
Again let A, B $\in I^X$, then

$$\begin{aligned} & \mathbf{Cl}({}_{nA}X_f^0(s) \vee_{nB} X_f^0(s)) = \mathbf{Cl}({}_{nA}X_f^0(s)) \vee \mathbf{Cl}({}_{nB}X_f^0(s)) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(A\vee B)}X_f^0(s)) = \mathbf{Cl}({}_{nA}X_f^0(s)) \vee \mathbf{Cl}({}_{nB}X_f^0(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n(A \vee B) = (\mathbf{Cl})_f^n(A) \vee (\mathbf{Cl})_f^n(B) \end{aligned}$$

Thus $(\mathbf{Cl})_f^n$ is a fuzzy closure operator.
For the next part, Let A$\in (\delta_{\mathbf{Cl}})_n$, then $\overline{1}$-A is a closed fuzzy set in $(X, (\delta_{\mathbf{Cl}})_n)$. Let

$B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ be a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$ such that $B_f^n = \overline{1} - A$. Now,

$$\begin{aligned} & {}_{n(\overline{1}-A)}X_f^0(s) \leq B_f(s) \\ \Rightarrow \quad & \mathbf{Cl}({}_{n(\overline{1}-A)}X_f^0(s)) \leq \mathbf{Cl}(B_f(s)) \\ \Rightarrow \quad & (\mathbf{Cl})_f^n(\overline{1} - A) \leq B_f^n = \overline{1} - A \\ \Rightarrow \quad & (\mathbf{Cl})_f^n(\overline{1} - A) = \overline{1} - A \\ \Rightarrow \quad & A \in \delta_{(\mathbf{Cl})_f^n} \end{aligned}$$

Also A$\in \delta_{(\mathbf{Cl})_f^n}$ implies $(\mathbf{Cl})_f^n(\overline{1} - A) = \overline{1} - A$. Let $B_f(s) = \mathbf{Cl}({}_{n(\overline{1}-A)}X_f^o(s))$, then $B_f(s)$ is a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$ and its n^{th} component is $\overline{1} - A$. Therefore $A \in (\delta_{\mathbf{Cl}})_n$. Hence the theorem. □

Theorem 2.11. *Let* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-closure operator on a non empty set* X *and* $A \subset X$. *If* $Char(A)$ *denote the characteristic function of A, Then* $\boldsymbol{Cl}_A$: $(I^A)^{\mathbb{N}} \to (I^A)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{Cl}_A(B_f(s)) \quad = \quad \{Char(A)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(B_f(s)) \ \forall B_f(s) \in (I^A)^{\mathbb{N}}.$$

is an FS-closure operator on A and $(\boldsymbol{Cl}_A)_f^n(B) = Char(A) \wedge (\boldsymbol{Cl})_f^n(B)$ *for all* $B \in I^A$.

Proof. Let $B_f(s) \in (I^A)^{\mathbb{N}}$. Now

$$\begin{aligned} & \mathbf{Cl}_A(\mathbf{Cl}_A(B_f(s)) = \mathbf{Cl}_A(\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s))) \\ = \quad & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s))) \\ \leq \quad & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(\{Char(A)\}_{n=1}^{\infty}) \wedge \mathbf{Cl}(\mathbf{Cl}(B_f(s))) \\ = \quad & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s)) \\ = \quad & \mathbf{Cl}_A(B_f(s)) \end{aligned}$$

All the other conditions being straightforward, we an conclude that $\mathbf{Cl}_A$ is an FS-closure operator on A. Also for $B \in I^A$, $(\mathbf{Cl}_A)_f^n(B) = n^{th}$ component of $\mathbf{Cl}_A({}_{nB}X_f^0(s))= n^{th}$ component of $\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}({}_{nB}X_f^0(s)) = Char(A) \wedge$ n^{th} component of $\mathbf{Cl}({}_{nB}X_f^0(s)) = Char(A) \wedge (\mathbf{Cl})_f^n(B)$. □

Theorem 2.12. *Let* $\{\boldsymbol{Cl}_\lambda : (I^{X_\lambda})^{\mathbb{N}} \to (I^{X_\lambda})^{\mathbb{N}}; \lambda \in \Lambda\}$ *be a family of FS-closure operators, where* $X_\lambda \wedge X_\mu = \phi$ *for all* $\lambda, \mu \in \Lambda$. *If* $X = \vee_{\lambda\in\Lambda} X_\lambda$ *and* $Char(X_\lambda)$ *denote the characteristic function of* X_λ, *then* $\boldsymbol{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by* $\boldsymbol{C}(A_f(s)) = \vee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))$ *is an FS-closure operator on* X.

Proof. For $A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} & \mathbf{C}(\mathbf{C}(A_f(s))) = \mathbf{C}(\vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \quad & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge (\vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)))) \\ = \quad & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge (\vee_{\lambda\in\Lambda}(\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty}) \wedge \mathbf{Cl}_\lambda(A_f(s))))) \\ = \quad & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \quad & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \quad & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)) \\ = \quad & \mathbf{C}(A_f(s)) \end{aligned}$$

Other conditions being straightforward, it follows that **C** is an FS-closure operator. □

Definition 2.13. A collection $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}}; \lambda \in \Lambda\}$ is called an FS-closure system if for each $A_f(s) \in (I^X)^{\mathbb{N}}$, $\wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s)$

Theorem 2.14. *$\zeta(s)$ is an FS-closure system iff $\zeta(s)$ is closed under arbitrary intersection.*

Proof. Suppose $\zeta(s)$ is closed under arbitrary intersection. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ such that $A_f(s) \leq A_{\lambda f}(s)\ \forall\lambda \in \Lambda$. Then

$$\wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s)$$

Conversely, suppose $\zeta(s)$ is an FS-closure system. Let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ and let $A_f(s) = \wedge_{\lambda\in\Lambda} A_{\lambda f}(s)$. Then

$$\begin{aligned} & A_f(s) \leq A_{\lambda f}(s)\ \forall\lambda \in \Lambda \\ \Rightarrow\ & \wedge_{\lambda\in\Lambda} A_{\lambda f}(s) = \wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s) \end{aligned}$$

Hence $\zeta(s)$ is closed under arbitrary intersection. □

Lemma 2.15. *Let $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}}; \lambda \in \Lambda\}$ be an FS-closure system containing $X_f^0(s)$. Then $\boldsymbol{Cl}_{\zeta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s)) = \wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \text{ and}$$

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s) \vee B_f(s)) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s)) \vee \boldsymbol{Cl}_{\zeta(s)}(B_f(s))\ \forall A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}}$$

is an FS-closure operator. Moreover for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \zeta(s)$ iff $A_f(s) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s))$.

Proof. Since $\mathbf{Cl}_{\zeta(s)}(A_f(s)) \in \zeta(s)$ for $A_f(s) \in (I^X)^{\mathbb{N}}$, we have

$$\mathbf{Cl}_{\zeta(s)}(\mathbf{Cl}_{\zeta(s)}(A_f(s))) \;=\; \wedge_{\lambda\in\Lambda, \mathbf{Cl}_{\zeta(s)}(A_f(s))\leq A_{\lambda f}(s)} A_{\lambda f}(s) \leq \mathbf{Cl}_{\zeta(s)}(A_f(s))$$

Hence $\mathbf{Cl}_{\zeta(s)}$ is an FS-closure operator.
Now, if $A_f(s) \in \zeta(s)$, then $A_f(s) = A_{\lambda f}(s)$ for some $\lambda \in \Lambda$ and

$$\mathbf{Cl}_{\zeta(s)}(A_f(s)) \;=\; \wedge_{i\in\Lambda, A_f(s)\leq A_{if}(s)} A_{if}(s) \leq A_{\lambda f}(s) = A_f(s)$$

Also $A_f(s) \leq \mathbf{Cl}_{\zeta(s)}((A_f(s))$. Hence $\mathbf{Cl}_{\zeta(s)}(A_f(s)) = A_f(s)$. Converse part follows from the definition of $\mathbf{Cl}_{\zeta(s)}$. □

Lemma 2.16. *Let $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator. Then*

$$\zeta_{\boldsymbol{Cl}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \boldsymbol{Cl}(A_f(s))\}$$

is an FS-closure system.

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ such that $B_f(s) \leq A_{\lambda f}(s)$ $\forall\lambda \in \Lambda$. Let $D_f(s) = \wedge_{\lambda\in\Lambda, B_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s)$. We know, $D_f(s) \leq \mathbf{Cl}(D_f(s))$. Again

$$\begin{aligned} & D_f(s) \leq A_{\lambda f}(s)\ \forall\lambda \in \Lambda \\ \Rightarrow\ & \mathbf{Cl}(D_f(s)) \leq \mathbf{Cl}(A_{\lambda f}(s))\ \forall\lambda \in \Lambda \\ \Rightarrow\ & \mathbf{Cl}(D_f(s)) \leq \wedge_{\lambda\in\Lambda, B_f(s)\leq A_{\lambda f}(s)} \mathbf{Cl}(A_{\lambda f}(s)) = \wedge_{\lambda\in\Lambda, B_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \\ & \qquad = D_f(s) \end{aligned}$$

Thus $D_f(s) = \mathbf{Cl}(D_f(s))$ and so $D_f(s) \in \zeta_{\mathbf{Cl}}(s)$. Hence $\zeta_{\mathbf{Cl}}(s)$ is an FS-closure system. $\square$

Note 2.17. In **Lemma** 2.16, the FS-closure system $\zeta_{\mathbf{Cl}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \mathbf{Cl}(A_f(s))\}$ is called an FS-closure system generated by the FS-closure operator **Cl**.

Theorem 2.18. *Let* $\boldsymbol{Cl}$ *be an FS-closure operator and* $\zeta(s)$ *be an FS-closure system on* X *containing* $X_f^0(s)$*, then* $\zeta_{\boldsymbol{Cl}}(s)$ *and* $\boldsymbol{Cl}_{\zeta(s)}$ *are respectively FS-closure system and FS-closure operator on* X*. Also* $\boldsymbol{Cl} = \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) = \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$*, that is, the mappings* $\boldsymbol{Cl} \to \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) \to \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$ *are mutually inverse.*

Proof. The first part follows from **Lemma** 2.15 and **Lemma** 2.16. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta_{\mathbf{Cl}}(s)$ such that $A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda$. Then $\mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s)) = \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s)$. Now,

$$\begin{aligned}
& A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\
\Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq \mathbf{Cl}(A_{\lambda f}(s))\ \forall \lambda \in \Lambda \\
\Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\
\Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) = \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s))
\end{aligned}$$

Again,

$$\begin{aligned}
& A_f(s) \leq \mathbf{Cl}(A_f(s)) \in \zeta_{\mathbf{Cl}}(s) \\
\Rightarrow\ & \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s)) = \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \leq \mathbf{Cl}(A_f(s))
\end{aligned}$$

Hence $\mathbf{Cl} = \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}$.
Also,

$$\begin{aligned}
& A_f(s) \in \zeta_{\mathbf{Cl}_{\zeta(s)}}(s) \\
\Leftrightarrow\ & A_f(s) = \mathbf{Cl}_{\zeta(s)}(A_f(s)) \\
\Leftrightarrow\ & A_f(s) \in \zeta(s)
\end{aligned}$$

Thus $\zeta(s) = \zeta_{\mathbf{Cl}_{\zeta(s)}}(s)$. $\square$

Definition 2.19. Let X be a non empty set. An operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-interior operator if it satisfies the following conditions:

(FSI1) $\mathbf{I}(X_f^1(s)) = X_f^1(s)$.
(FSI2) $\mathbf{I}(A_f(s)) \leq A_f(s)$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
(FSI3) $\mathbf{I}(\mathbf{I}(A_f(s))) = \mathbf{I}(A_f(s)))$ for all $A_f(s)) \in (I^X)^{\mathbb{N}}$.
(FSI4) $\mathbf{I}(A_f(s) \wedge B_f(s)) = \mathbf{I}(A_f(s)) \wedge \mathbf{I}(B_f(s))$ for all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.20. For any FSTS $(X, \delta(s))$, interior of an fs-set is an FS-interior operator on X.

Example 2.21. Let X be a non empty set. The operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $\mathbf{I}(A_f(s)) = A_f(s) \wedge D_f(s)$ whenever $A_f(s) \neq X_f^1(s)$ and $\mathbf{I}(X_f^1(s)) = X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X, is an FS-interior operator on X.

Theorem 2.22. *If $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then*

(i) *$\boldsymbol{I}$ is monotonic increasing, that is, $A_f(s) \leq B_f(s) \Rightarrow \boldsymbol{I}(A_f(s)) \leq \boldsymbol{I}(B_f(s))$ for all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.*

(ii) *$\boldsymbol{I}(A_f(s)) \leq B_f(s) \Rightarrow \boldsymbol{I}(A_f(s)) \leq \boldsymbol{I}(B_f(s))$ for all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.*

Proof. Proof is omitted. □

Theorem 2.23. *Let X be a non empty set and $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an operator satisfying $(FSI1)$, $(FSI2)$ and $(FSI4)$, then*

a) *the collection $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; \boldsymbol{I}(A_f(s)) = A_f(s)\}$ forms an FST on X.*

b) *if $\boldsymbol{I}$ also satisfies $(FSI3)$, then for all $A_f(s) \in (I^X)^{\mathbf{N}}$ we have $\overset{o}{A}_f(s) = \boldsymbol{I}(A_f(s))$, where $\overset{o}{A}_f(s)$ is the interior of $A_f(s)$ in $\delta(s)$.*

Proof. Proof is omitted. □

Remark 2.24. From **Theorem** 2.23 it follows that if $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X then $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}\,; \mathbf{I}(A_f(s)) = A_f(s)\}$ forms an FST on X. Also $\overset{o}{A}_f(s) = \mathbf{I}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbf{N}}$, where $\overset{o}{A}_f(s)$ is the interior of $A_f(s)$ in $\delta(s)$. This FST $\delta(s)$ is called the fuzzy sequential topology induced by the FS-interior operator $\mathbf{I}$ and we denote it by $\delta_{\mathbf{I}}(s)$.

Remark 2.25. Example 2.26 shows that if an operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a non empty set X, satisfies $(FSI1)$, $(FSI2)$ and $(FSI4)$ but does not satisfy $(FSI3)$, then $\delta_{\mathbf{I}}(s)$ forms an FST on X but $\overset{o}{A}_f(s)$ may not be equal to $\mathbf{I}(A_f(s))$, $A_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.26. Let $X = \{a\}$. Let $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be defined by $\mathbf{I}(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{I}$ is an operator on X satisfying $(FSI1)$, $(FSI2)$ and $(FSI4)$ and hence $(X, \delta_{\mathbf{I}}(s))$ forms an FSTS. Further $\mathbf{I}$ does not satisfy $(FSI3)$ and in $(X, \delta_{\mathbf{I}}(s))$, $\mathbf{I}(B_f(s)) \neq \overset{0}{B}_f(s)$ if $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ where $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}\ \forall\ n \neq 1, 2, 3$.

Definition 2.27. Let X be a non empty set and $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X. A function $(\mathbf{I})_f^n : I^X \to I^X$ defined by $(\mathbf{I})_f^n(A) = n^{th}$ term of $\mathbf{I}({}_{nA}X_f^1(s))$, where ${}_{nA}X_f^1(s)$ denotes an fs-set whose n^{th} term is A and others are $\overline{1}$, is called the n^{th} component of $\mathbf{I}$, $n \in \mathbb{N}$.

Theorem 2.28. *Let X be a non empty set. If $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then each component $(\boldsymbol{I})_f^n : I^X \to I^X$, $n \in \mathbb{N}$ is a fuzzy interior operator. Also $(\delta_{\boldsymbol{I}})_n = \delta_{(\boldsymbol{I})_f^n}$ where $(\delta_{\boldsymbol{I}})_n$ is the n^{th} component fuzzy topology of FST $\delta_{\boldsymbol{I}}(s)$ and $\delta_{(\boldsymbol{I})_f^n}$ is the fuzzy topology induced by the component $(\boldsymbol{I})_f^n$ of $\boldsymbol{I}$.*

Proof. $(\mathbf{I})_f^n(\overline{1}) = \overline{1}$ by definition. Let $A \in I^X$, then $\mathbf{I}({}_{nA}X_f^1(s)) \leq {}_{nA}X_f^1(s) \Rightarrow (\mathbf{I})_f^n(A) \leq A$. Hence $(\mathbf{I})_f^n((\mathbf{I})_f^n(A)) \leq (\mathbf{I})_f^n(A)$. Also

$$\begin{aligned}
& \mathbf{I}(\mathbf{I}({}_{nA}X_f^1(s))) = \mathbf{I}({}_{nA}X_f^1(s)) \\
\Rightarrow\ & \mathbf{I}({}_{nA}X_f^1(s)) \leq \mathbf{I}({}_{n(\mathbf{I})_f^n(A)}X_f^1(s)) \\
\Rightarrow\ & (\mathbf{I})_f^n(A) \leq (\mathbf{I})_f^n((\mathbf{I})_f^n(A))
\end{aligned}$$

Hence $(\mathbf{I})^n_f((\mathbf{I})^n_f(A)) = (\mathbf{I})^n_f(A)$.
Again let A, B $\in I^X$, then

$$\begin{aligned} & \mathbf{I}({}_{nA}X^1_f(s) \wedge \ {}_{nB}X^1_f(s)) = \mathbf{I}({}_{nA}X^1_f(s)) \wedge \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow \quad & \mathbf{I}({}_{n(A\wedge B)}X^1_f) = \mathbf{I}({}_{nA}X^1_f(s)) \wedge \mathbf{I}({}_{nB}X^1_f(s)) \\ \Rightarrow \quad & (\mathbf{I})^n_f(A \wedge B) = (\mathbf{I})^n_f(A) \wedge (\mathbf{I})^n_f(B) \end{aligned}$$

Thus $(\mathbf{I})^n_f$ is a fuzzy interior operator.
For the next part, Let A$\in (\delta_{\mathbf{I}})_n$. Let $B_f(s) = \{B^n_f\}^\infty_{n=1}$ be an open fs-set in $(X, \delta_{\mathbf{I}}(s))$ such that $B^n_f = A$. Now,

$$\begin{aligned} & B_f(s) \leq \ {}_{nA}X^1_f(s) \\ \Rightarrow \quad & \mathbf{I}(B_f(s)) \leq \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow \quad & B_f(s) \leq \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow \quad & A \leq (\mathbf{I})^n_f(A) \\ \Rightarrow \quad & (\mathbf{I})^n_f(A) = A \\ \Rightarrow \quad & A \in \delta_{(\mathbf{I})^n_f} \end{aligned}$$

Also $A \in \delta_{(\mathbf{I})^n_f}$ implies $(\mathbf{I})^n_f(A) = A$. Let $B_f(s) = \mathbf{I}({}_{nA}X^1_f(s))$, then $B_f(s)$ is an open fs-set in $(X, (\delta_{\mathbf{I}}(s))$ and its n^{th} component is A. Therefore $A \in (\delta_{\mathbf{I}})_n$. Hence the theorem. □

Theorem 2.29. *Let $\boldsymbol{I}\colon (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on a non empty set X and $A \subset X$. If $Char(A)$ denote the characteristic function of A, then $\boldsymbol{I}_A$: $(I^A)^{\mathbb{N}} \to (I^A)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I}_A(B_f(s)) \quad = \quad \{Char(A)\}^\infty_{n=1} \vee \boldsymbol{I}(B_f(s)) \ \forall B_f(s) \in (I^A)^{\mathbb{N}}.$$

is an FS-interior operator on A and $(\boldsymbol{I}_A)^n_f(B) = Char(A) \vee (\boldsymbol{I})^n_f(B)$ for all $B \in I^A$.

Proof. Let $B_f(s) \in (I^A)^{\mathbb{N}}$. Now

$$\begin{aligned} & \mathbf{I}_A(B_f(s))) = \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s)) \\ = \quad & \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(\{Char(A)\}^\infty_{n=1}) \vee \mathbf{I}(\mathbf{I}(B_f(s))) \\ \leq \quad & \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(\{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s))) \\ = \quad & \mathbf{I}_A(\{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s))) \\ = \quad & \mathbf{I}_A(\mathbf{I}_A(B_f(s))) \end{aligned}$$

All the other conditions being straightforward, we an conclude that $\mathbf{I}_A$ is an FS-interior operator. Also $(\mathbf{I}_A)^n_f(B) = n^{th}$ component of $\mathbf{I}_A({}_{nB}X^1_f(s)) = n^{th}$ component of $\{Char(A)\}^{\mathbb{N}} \vee \mathbf{I}({}_{nB}X^1_f(s)) = Char(A) \vee n^{th}$ component of $\mathbf{I}({}_{nB}X^1_f(s)) = Char(A) \vee (\mathbf{I})^n_f(B)$. □

Definition 2.30. A collection $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ is called an FS-interior system if for each $A_f(s) \in (I^X)^{\mathbb{N}}$, $\vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s)$.

Theorem 2.31. *$\eta(s)$ is an FS-interior system iff $\eta(s)$ is closed under arbitrary union.*

Proof. Suppose $\eta(s)$ is closed under arbitrary union. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Let $A_{jf}(s) \leq A_f(s)\ \forall j \in J$ where $A_{jf}(s) \in \eta(s)\ \forall j \in J$. Then

$$\vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s)$$

Conversely, suppose $\eta(s)$ is an FS-interior system. Let $\{A_{jf}(s); j \in J\} \in \eta(s)$ and let $A_f(s) = \vee_{j\in J} A_{jf}(s)$. Then

$$\begin{aligned} & A_{jf}(s) \leq A_f(s)\ \forall j \in J \\ \Rightarrow\ & \vee_{j\in J} A_{jf}(s) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s) \end{aligned}$$

Hence $\eta(s)$ is closed under arbitrary union. □

Lemma 2.32. *Let $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ be an FS-interior system containing $X_f^1(s)$. Then $\boldsymbol{I}_{\eta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\begin{aligned} & \boldsymbol{I}_{\eta(s)}(A_f(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \text{ and} \\ & \boldsymbol{I}_{\eta(s)}(A_f(s) \wedge B_f(s)) = \boldsymbol{I}_{\eta(s)}(A_f(s)) \wedge \boldsymbol{I}_{\eta(s)}(B_f(s))\ \forall A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}} \end{aligned}$$

is an FS-interior operator. Moreover for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \eta(s)$ iff $A_f(s) = \boldsymbol{I}_{\eta(s)}(A_f(s))$.

Proof. Proof of the first part is straightforward.
Now, if $A_f(s) \in \eta(s)$, then $A_f(s) = A_{jf}(s)$ for some $j \in J$ and

$$\mathbf{I}_{\eta(s)}(A_f(s)) = \vee_{i\in J, A_{if}(s)\leq A_f(s)} A_{if}(s) = A_f(s)$$

Converse part follows from the definition of $\mathbf{I}_{\eta(s)}$. □

Lemma 2.33. *Let $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator. Then*

$$\eta_{\boldsymbol{I}}(s) \quad = \quad \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \boldsymbol{I}(A_f(s))\}$$

is an FS-interior system.

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$. Let $D_f(s) = \vee_{j\in J, A_{jf}(s)\leq B_f(s)} A_{jf}(s)$, where $A_{jf}(s) \in \eta_{\mathbf{I}}(s)\ \forall j \in J$. We know, $\mathbf{I}(D_f(s)) \leq D_f(s)$. Again,

$$\begin{aligned} & A_{jf}(s) \leq D_f(s)\ \forall j \in J \text{ and for all } A_{jf}(s) \leq B_f(s) \\ \Rightarrow\ & \mathbf{I}(A_{jf}(s)) \leq \mathbf{I}(D_f(s))\ \forall j \in J \text{ and for all } A_{jf}(s) \leq B_f(s) \\ \Rightarrow\ & \vee_{j\in J, A_{jf}(s)\leq B_f(s)} \mathbf{I}(A_{jf}(s)) = \vee_{j\in J, A_{jf}(s)\leq B_f(s)} A_{jf}(s) = D_f(s) \leq \mathbf{I}(D_f(s)) \end{aligned}$$

Thus $D_f(s) = \mathbf{I}(D_f(s))$ and so $D_f(s) \in \eta_{\mathbf{I}}(s)$. Hence $\eta_{\mathbf{I}}(s)$ is a FS-interior system. □

Note 2.34. In **Lemma** 2.33, the FS-interior system $\eta_{\mathbf{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \mathbf{I}(A_f(s))\}$ is called an FS-interior system generated by the FS-interior operator $\mathbf{I}$.

Theorem 2.35. *Let $\boldsymbol{I}$ be an FS-interior operator and $\eta(s)$ be an FS-interior system on X containing $X_f^1(s)$, then $\eta_{\boldsymbol{I}}(s)$ and $\boldsymbol{I}_{\eta(s)}$ are respectively FS-interior system and FS-interior operator on X. Also $\boldsymbol{I} = \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) = \eta_{\boldsymbol{I}_{\eta(s)}}(s)$, that is, the mappings $\boldsymbol{I} \to \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) \to \eta_{\boldsymbol{I}_{\eta(s)}}(s)$ are mutually inverse.*

Proof. The first part follows from **Lemma** 2.32 and **Lemma** 2.33. Let $A_f(s) \in (I^X)^{\mathbb{N}}$, and let $\{A_{jf}(s); j \in J\} \in \eta_{\mathbf{I}}(s)$ such that $A_{jf}(s) \leq A_f(s)\ \forall j \in J$. Then $\mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)(\in \eta_{\mathbf{I}}(s))}(A_{jf}(s))$. Now,

$$\begin{aligned} & A_{jf}(s) \leq A_f(s)\ \forall j \in J \\ \Rightarrow\ & \mathbf{I}(A_{jf}(s)) \leq \mathbf{I}(A_f(s))\ \forall j \in J \\ \Rightarrow\ & \vee_{j\in J, A_{jf}(s)\leq A_f(s)}\mathbf{I}(A_{jf}(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)}A_{jf}(s) \\ & \qquad\qquad\qquad\qquad\qquad = \mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)) \leq \mathbf{I}(A_f(s)) \end{aligned}$$

Again,

$$\begin{aligned} & \mathbf{I}(A_f(s))(\leq A_f(s)) \in \eta_{\mathbf{I}}(s) \\ \Rightarrow\ & \mathbf{I}(A_f(s)) \leq \vee_{j\in J, A_{jf}(s)\leq A_f(s)(\in \eta_{\mathbf{I}}(s))}A_{jf}(s) = \mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)). \end{aligned}$$

Hence $\mathbf{I} = \mathbf{I}_{\eta_{\mathbf{I}}(s)}$.
Also,

$$\begin{aligned} & A_f(s) \in \eta_{\mathbf{I}_{\eta(s)}}(s) \\ \Leftrightarrow\ & A_f(s) = \mathbf{I}_{\eta(s)}(A_f(s)) \\ \Leftrightarrow\ & A_f(s) \in \eta(s). \end{aligned}$$

Thus $\eta(s) = \eta_{\mathbf{I}_{\eta(s)}}(s)$. □

Definition 2.36. If **I** be an FS-interior operator on a non empty set X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}}; \mathbf{I}(A_f(s)) = A_f(s)\}$ forms an FS-closure system on X and we call it to be an FS-closure system generated by the FS-interior operator **I**.

Definition 2.37. If **Cl** be an FS-closure operator on a non empty set X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}}; \mathbf{Cl}(A_f(s)) = A_f(s)\}$ forms an FS-interior system on X and we call it to be an FS-interior system generated by the FS-closure operator **Cl**.

Theorem 2.38. *Let* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-interior operator on* X*, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FST on* X*.*
(ii) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FS-interior system on* X*.*
(iii) $\{A_f(s);\ (A_f(s))^c \in \delta_{\boldsymbol{I}}(s)\}$ *forms an FS-closure system on* X*.*

Proof. Proof is omitted. □

Theorem 2.39. *Let* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-closure operator on* X*, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ *forms an FST on* X*.*
(ii) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FS-interior system on* X*.*
(iii) $\{A_f(s);\ (A_f(s))^c \in \delta_{\boldsymbol{Cl}}(s)\}$ *forms an FS-closure system on* X*.*

Proof. Proof is omitted. □

Theorem 2.40. *Let X be a non empty set. If $\boldsymbol{Cl}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then the operator $\boldsymbol{I_{Cl}}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I_{Cl}}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-interior operator on X. Again, if $\boldsymbol{I}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then the operator $\boldsymbol{Cl_I}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$\boldsymbol{Cl_I}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{I}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-closure operator on X.

Proof. Proof is omitted. □

Note 2.41. It follows from **Theorem** 2.40 that given an FS-closure operator we can define an FS-interior operator and given an FS-interior operator we can define an FS-closure operator. In fact, there is a one to one correspondence between the collections of all FS-closure and FS-interior operators on a set (**Theorem** 2.42). We denote the collection of all FS-closure operators and the collection of all FS-interior operators on X by $\mathcal{C}_X$ and $\mathcal{I}_X$ respectively.

Theorem 2.42. *Let X be a non empty set, then there exists a one to one correspondence between $\mathcal{C}_X$ and $\mathcal{I}_X$.*

Proof. $t : \mathcal{C}_X \to \mathcal{I}_X$ by

$$t(\mathbf{Cl}) \quad = \quad \mathbf{I_{Cl}} \ \forall \mathbf{Cl} \in \mathcal{C}_X$$

Then t is a well defined map. Now, for $\mathbf{Cl}_1, \mathbf{Cl}_2 \in \mathcal{C}_X$ such that $t(\mathbf{Cl}_1) = t(\mathbf{Cl}_2)$, we have $\mathbf{I}_{\mathbf{Cl}_1} = \mathbf{I}_{\mathbf{Cl}_2}$. Hence $\forall A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} \mathbf{I}_{\mathbf{Cl}_1}((A_f(s))^c) &= \mathbf{I}_{\mathbf{Cl}_2}((A_f(s))^c) \\ X_f^1(s) - \mathbf{Cl}_1(A_f(s)) &= X_f^1(s) - \mathbf{Cl}_2(A_f(s)) \\ \mathbf{Cl}_1(A_f(s)) &= \mathbf{Cl}_2(A_f(s)) \end{aligned}$$

Thus t is injective. Again for $\mathbf{I} \in \mathcal{I}_X$, there is $\mathbf{Cl_I} \in \mathcal{C}_X$ such that $\forall A_f(s) \in (I^X)^{\mathbb{N}}$

$$\mathbf{Cl_I}((A_f(s))^c) \quad = \quad X_f^1(s) - \mathbf{I}(A_f(s))$$

Now, $\forall A_f(s) \in (I^X)^{\mathbb{N}}$

$$\begin{aligned} \mathbf{I}_{\mathbf{Cl_I}}((A_f(s)) &= X_f^1(s) - \mathbf{Cl_I}((A_f(s))^c) \\ &= X_f^1(s) - (X_f^1(s) - \mathbf{I}(A_f(s))) \\ &= \mathbf{I}(A_f(s)) \end{aligned}$$

Therefore t is surjective and this completes the theorem. □

Note 2.43. If $\mathbf{I}$ is the t-image of $\mathbf{Cl}$ under the bijection t defined in **Theorem** 2.42, then $\mathbf{I}$ and $\mathbf{Cl}$ are called t-associated to each other.

Theorem 2.44. *The FST's induced by $\boldsymbol{Cl}$ and $\boldsymbol{I_{Cl}}$ are identical and the FST's induced by $\boldsymbol{I}$ and $\boldsymbol{Cl_I}$ are identical.*

Proof. Proof is omitted. □

Now, if we define an FS-interior and an FS-closure operator, separately, on a non empty set, they will induce two fuzzy sequential topologies which may not be identical in general. In view of **Theorem** 2.42 and **Theorem** 2.44, we give a necessary and sufficient condition that the two fuzzy sequential topologies induced by an FS-interior operator and an FS-closure operator are identical.

Theorem 2.45. *Let X be a non empty set. If $\mathbf{Cl} \in \mathcal{C}_X$ and $\mathbf{I} \in \mathcal{I}_X$, then $\delta_{\mathbf{Cl}}(s)$ and $\delta_{\mathbf{I}}(s)$ are identical iff $\mathbf{Cl}$ and $\mathbf{I}$ are t-associated to each other.*

Proof. Suppose $\mathbf{Cl}$ and $\mathbf{I}$ are t-associated to each other. Then $t(\mathbf{Cl}) = \mathbf{I}_{\mathbf{Cl}} = \mathbf{I}$. Now,

$$\begin{aligned} & A_f(s) \in \delta_{\mathbf{I}}(s) \\ \Leftrightarrow\ & \mathbf{I}(A_f(s)) = A_f(s) \\ \Leftrightarrow\ & \mathbf{I}_{\mathbf{Cl}}(A_f(s)) = A_f(s) \\ \Leftrightarrow\ & X_f^1(s) - \mathbf{Cl}((A_f(s))^c) = A_f(s) \\ \Leftrightarrow\ & \mathbf{Cl}((A_f(s))^c) = (A_f(s))^c \\ \Leftrightarrow\ & A_f(s) \in \delta_{\mathbf{Cl}}(s). \end{aligned}$$

Thus $\delta_{\mathbf{I}}(s)$ and $\delta_{\mathbf{Cl}}(s)$ are identical.
Conversely, suppose $\delta_{\mathbf{I}}(s)$ and $\delta_{\mathbf{Cl}}(s)$ are identical. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Then

$$\begin{aligned} & (\mathbf{Cl}((A_f(s))^c))^c \in \delta_{\mathbf{I}}(s) \\ \Rightarrow\ & \mathbf{I}((\mathbf{Cl}((A_f(s))^c))^c) = (\mathbf{Cl}((A_f(s))^c))^c = X_f^1(s) - \mathbf{Cl}((A_f(s))^c) \end{aligned}$$

Now,

$$\begin{aligned} & (A_f(s))^c \leq \mathbf{Cl}((A_f(s))^c) \\ \Rightarrow\ & (\mathbf{Cl}((A_f(s))^c))^c \leq A_f(s) \\ \Rightarrow\ & \mathbf{I}((\mathbf{Cl}((A_f(s))^c))^c) \leq \mathbf{I}(A_f(s)) \\ \Rightarrow\ & X_f^1(s) - \mathbf{Cl}((A_f(s))^c) \leq \mathbf{I}(A_f(s)). \end{aligned}$$

Again,

$$\begin{aligned} & \mathbf{I}(A_f(s)) \in \delta_{\mathbf{Cl}}(s) \\ \Rightarrow\ & \mathbf{Cl}((\mathbf{I}(A_f(s)))^c) = (\mathbf{I}(A_f(s)))^c = X_f^1(s) - \mathbf{I}(A_f(s)). \end{aligned}$$

Also,

$$\begin{aligned} & \mathbf{I}(A_f(s)) \leq A_f(s) \\ \Rightarrow\ & (A_f(s))^c \leq (\mathbf{I}(A_f(s)))^c \\ \Rightarrow\ & \mathbf{Cl}((A_f(s))^c) \leq \mathbf{Cl}((\mathbf{I}(A_f(s)))^c) = X_f^1(s) - \mathbf{I}(A_f(s)) \\ \Rightarrow\ & \mathbf{I}(A_f(s)) \leq X_f^1(s) - \mathbf{Cl}((A_f(s))^c). \end{aligned}$$

Thus $\mathbf{I}(A_f(s)) = X_f^1(s) - \mathbf{Cl}((A_f(s))^c) = \mathbf{I}_{\mathbf{Cl}}(A_f(s))\ \forall\ A_f(s) \in (I^X)^{\mathbb{N}}$. Hence $\mathbf{I} = \mathbf{I}_{\mathbf{Cl}} = t(\mathbf{Cl})$. □

Theorem 2.46. *Let X be a non empty set. If $\mathbf{Cl} \in \mathcal{C}_X$, $\mathbf{I} \in \mathcal{I}_X$, then the following conditions are equivalent:*

(i) *$\mathbf{I}$ and $\mathbf{Cl}$ are t-associated to each other.*

(ii) *The FST's* $\delta_{\boldsymbol{I}}(s)$ *and* $\delta_{\boldsymbol{Cl}}(s)$ *are identical.*
(iii) *FS-closure systems generated by* $\boldsymbol{Cl}$ *and* $\boldsymbol{I}$ *are identical.*
(iv) *FS-interior systems generated by* $\boldsymbol{Cl}$ *and* $\boldsymbol{I}$ *are identical.*

Proof. Proof is omitted. □

Note 2.47. Theorem 2.46 gives two more necessary and sufficient conditions ((iii) and (iv)), that the fuzzy sequential topologies induced by an FS-interior operator and an FS-closure operator are identical.

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Composition of fuzzy sequential operators with special emphasis on FS-connectors

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Abstract. FS-closure and FS-interior operators both induce fuzzy sequential topologies on the underlying set. Do the composition of FS-closure and that of FS-interior operators provide any topological structure? If so, is there any relation among the topologies induced by the composition and that induced by the participants to the composition? We consider these questions in this article and also study relative FS-closure operators and FS-connectors.

1 Introduction

In 1968 C. L. Chang [6] introduced the concept of fuzzy topology after the initiation of fuzzy sets by L. A. Zadeh [18]. Towards the development of fuzzy set theory, fuzzy closure operators and fuzzy interior operators have been studied by Mashour and Ghanim [10], G. Gerla [8], Bandler and Kohout [1], R. Belohlavek [2], R. Belohlavek and T. Funiokova [3]. Notions of fuzzy sequential topological spaces (FSTS) and notions of FS-closure and FS-interior operators were introduced in [13] and [17] respectively.

Our purpose is to introduce FS-connectors connecting two fuzzy topologies on a set and to study the composition of FS-closure and that of FS-interior operators.

Section 2 deals with the composition of FS-closure operators, composition of FS-interior operators and the relation between collections of FS-closure and FS-interior operators. Section 3 deals with the relative FS-closure operators and the functions connecting two fuzzy topologies on a set, so called FS-connectors. The basic ideas behind the present work have been taken from the books ([5], [7] [9], [11]) and the articles ([4], [12], [14], [15], [16]).

In this paper, X will denote a non-empty set, $I = [0, 1]$, the closed unit interval in the real line. Before entering into our work we recall the following definitions and results.

Definition 1.1. [13] A family $\delta(s)$ of fuzzy sequential sets on a set X satisfying the properties
(i) $X_f^r(s) \in \delta(s)$ for $r = 0$ and 1,
(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and
(iii) for any family $\{A_{fj}(s) \in \delta(s), j \in J\}$, $\underset{j\in J}{\vee} A_{fj}(s) \in \delta(s)$
is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 1.2. [13] If $(X, \delta(s))$ is an FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where δ_n=$\{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$. (X, δ_n), where $n \in \mathbb{N}$, is called the n^{th} component FTS of the FSTS $(X, \delta(s))$.

Proposition 1.3. [13] *Let $A_f(s)$=$\{A_f^n\}_n$ be an open $(closed)$ fuzzy sequential set in the FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open $(closed)$ fuzzy set in (X, δ_n).*

Proposition 1.4. [13] *If δ be a fuzzy topology (FT) on a set X, then $\delta^{\mathbb{N}}$ forms an FST on X.*

Definition 1.5. [13] Let $A_f(s)$ be a fuzzy sequential set (fs-set) in an FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s); A_f(s) \leq C_f(s), (C_f(s))^c \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \vee\{O_f(s); O_f(s) \leq A_f(s), O_f(s) \in \delta(s)\}.$$

Definition 1.6. [17] An operator $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-closure operator on X if it satisfies the following conditions:
$(FSC1)$ $\mathbf{Cl}(X_f^0(s)) = X_f^0(s)$.
$(FSC2)$ $A_f(s) \leq \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC3)$ $\mathbf{Cl}(\mathbf{Cl}(A_f(s))) = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC4)$ $\mathbf{Cl}(A_f(s) \vee B_f(s)) = \mathbf{Cl}(A_f(s)) \vee \mathbf{Cl}(B_f(s))$ for all $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$.

Definition 1.7. [17] An operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-interior operator on X if it satisfies the following conditions:
$(FSI1)$ $\mathbf{I}(X_f^1(s)) = X_f^1(s)$.
$(FSI2)$ $\mathbf{I}(A_f(s)) \leq A_f(s)$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSI3)$ $\mathbf{I}(\mathbf{I}(A_f(s))) = \mathbf{I}(A_f(s)))$ for all $A_f(s)) \in (I^X)^{\mathbb{N}}$.
$(FSI4)$ $\mathbf{I}(A_f(s) \wedge B_f(s)) = \mathbf{I}(A_f(s)) \wedge \mathbf{I}(B_f(s))$ for all $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$.

Theorem 1.8. [17] *If* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-closure operator on* X*, then the operator* $\boldsymbol{I_{Cl}} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{I_{Cl}}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \; \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-interior operator on X*. Again, if* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-interior operator on* X*, then the operator* $\boldsymbol{Cl_I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{Cl_I}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{I}((A_f(s))^c) \; \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-closure operator on X.

Theorem 1.9. [17] *The map* $t : \mathcal{C}_X \to \mathcal{I}_X$ *defined by*

$$t(\boldsymbol{Cl}) \quad = \quad \boldsymbol{I_{Cl}} \; \forall \boldsymbol{Cl} \in \mathcal{C}_X$$

is a bijection, where $\mathcal{C}_X$ *and* $\mathcal{I}_X$ *respectively, denote the collections of all FS-closure operators and all FS-interior operators on* X.

2 Composition of FS-closure and FS-interior operators

Definition 2.1. If $\mathbf{C}_1$, $\mathbf{C}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be two FS-closure operators on X, then the mapping $\mathbf{C}_2 \circ \mathbf{C}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$(\mathbf{C}_2 \circ \mathbf{C}_1)(A_f(s)) \quad = \quad \mathbf{C}_2(\mathbf{C}_1(A_f(s))) \; \forall A_f(s) \in (I^X)^{\mathbb{N}}$$

is called the composition of the FS-closure operators $\mathbf{C}_1$ and $\mathbf{C}_2$.

It is easy to see that composition of FS-closure operators is associative but it may not be commutative and it may not be idempotent, as shown by **Example** 2.2.

Example 2.2. Let us consider the FS-closure operator $\mathbf{C}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{C}_1(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\mathbf{C}_1(X_f^0(s)) = X_f^0(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also consider FS-closure operator $\mathbf{C}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{C}_2(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty} \; \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{C}_2 \circ \mathbf{C}_1 \neq \mathbf{C}_1 \circ \mathbf{C}_2$. and $(\mathbf{C}_2 \circ \mathbf{C}_1) \circ (\mathbf{C}_2 \circ \mathbf{C}_1) \neq (\mathbf{C}_2 \circ \mathbf{C}_1)$.

Theorem 2.3. *If* $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *be two FS-closure operators on* X*, then* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ *satisfies* $FSC1$*,* $FSC2$ *and* $FSC4$*. Further, it satisfies* $FSC3$ *if the composition is commutative, that is, under commutative composition,* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ *forms an FS-closure operator.*
Proof: *Proof is omitted.*

Theorem 2.4. *Let* $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *be two FS-closure operators on* X*. Under commutative composition,* $\delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s) = \delta_{\boldsymbol{C}_2}(s) \wedge \delta_{\boldsymbol{C}_1}(s)$*, where* $\delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$*,* $\delta_{\boldsymbol{C}_2}(s)$ *and* $\delta_{\boldsymbol{C}_1}(s)$ *respectively denote the FST's induced by* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$*,* $\boldsymbol{C}_2$ *and* $\boldsymbol{C}_1$.
Proof: *Let* $A_f(s) \in \delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$*, then*

$$(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c) \quad = \quad (A_f(s))^c$$

Now,

$$\begin{aligned}
\boldsymbol{C}_1((A_f(s))^c) &= \boldsymbol{C}_1((\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c)) \\
&= \boldsymbol{C}_1((\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c)) \\
&= \boldsymbol{C}_1(\boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c))) \\
&= \boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c)) \\
&= (\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c) \\
&= (A_f(s))^c.
\end{aligned}$$

Similarly, $\boldsymbol{C}_2((A_f(s))^c) = (A_f(s))^c$*. Hence* $A_f(s) \in \delta_{\boldsymbol{C}_2}(s) \wedge \delta_{\boldsymbol{C}_1}(s)$*.*
Again, let $A_f(s) \in \delta_{\boldsymbol{C}_2}(s) \wedge \delta_{\boldsymbol{C}_1}(s)$*, then*

$$\boldsymbol{C}_1((A_f(s))^c) = (A_f(s))^c \text{ and } \boldsymbol{C}_2((A_f(s))^c) = (A_f(s))^c$$

Now,

$$\begin{aligned}
(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c) &= \boldsymbol{C}_2(\boldsymbol{C}_1((A_f(s))^c)) \\
&= \boldsymbol{C}_2((A_f(s))^c) \\
&= (A_f(s))^c
\end{aligned}$$

Thus $A_f(s) \in \delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$ *and hence the theorem.*

Definition 2.5. If $\mathbf{I}_1$, $\mathbf{I}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be two FS-interior operators on X, then the mapping $\mathbf{I}_2 \circ \mathbf{I}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$(\mathbf{I}_2 \circ \mathbf{I}_1)(A_f(s)) = \mathbf{I}_2(\mathbf{I}_1(A_f(s))) \; \forall A_f(s) \in (I^X)^{\mathbb{N}}$$

is called the composition of the FS-interior operators $\mathbf{I}_1$ and $\mathbf{I}_2$.

It is easy to see that composition of FS-interior operators is associative but it may not be commutative and it may not be idempotent, as shown by **Example** 2.6.

Example 2.6. Let us consider the FS-interior operator $\mathbf{I}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{I}_1(A_f(s)) = A_f(s) \wedge D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\mathbf{I}_1(X_f^1(s)) = X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also consider FS-interior operator $\mathbf{I}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{I}_2(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty} \; \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{I}_2 \circ \mathbf{I}_1 \neq \mathbf{I}_1 \circ \mathbf{I}_2$ and $(\mathbf{I}_2 \circ \mathbf{I}_1) \circ (\mathbf{I}_2 \circ \mathbf{I}_1) \neq (\mathbf{I}_2 \circ \mathbf{I}_1)$.

Theorem 2.7. *If* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *be two FS-interior operators on* X*, then* $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$ *satisfies* $FSI1$*,* $FSI2$ *and* $FSI4$*. Further, it satisfies* $FSI3$ *if the composition is commutative, that is, under commutative composition,* $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$ *forms an FS-interior operator.*
Proof: *Proof is omitted.*

Theorem 2.8. *Let* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *be two FS-interior operators on* X*. Under commutative composition,* $\delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s) = \delta_{\boldsymbol{I}_2}(s) \wedge \delta_{\boldsymbol{I}_1}(s)$*, where* $\delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s)$*,* $\delta_{\boldsymbol{I}_2}(s)$ *and* $\delta_{\boldsymbol{I}_1}(s)$ *respectively denote the FST's induced by* $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$*,* $\boldsymbol{I}_2$ *and* $\boldsymbol{I}_1$*.*
Proof: *The proof is similar to that in case of FS-closure operators.*

Theorem 2.9. *Under commutative composition,* $(\mathcal{I}_X, \circ)$ *and* $(\mathcal{C}_X, \circ)$ *both form semigroups with identity. Further, there exists a semigroup isomorphism between them.*
Proof: *First part is easy to check. For the second part, define* $t : \mathcal{C}_X \to \mathcal{I}_X$ *by*

$$t(\boldsymbol{Cl}) = \boldsymbol{I}_{\boldsymbol{Cl}} \; \forall \boldsymbol{Cl} \in \mathcal{C}_X$$

From ***Theorem*** 1.9*,* t *is a bijection. Also for* $\boldsymbol{C}_1$*,* $\boldsymbol{C}_2 \in \mathcal{C}_X$ *and* $A_f(s) \in (I^X)^{\mathbb{N}}$

$$\begin{aligned}
(\boldsymbol{I}_{\boldsymbol{C}_1} \circ \boldsymbol{I}_{\boldsymbol{C}_2})(A_f(s)) &= \boldsymbol{I}_{\boldsymbol{C}_1}(X_f^1(s) - \boldsymbol{C}_2((A_f(s))^c)) \\
&= X_f^1(s) - \boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c)) \\
&= X_f^1(s) - (\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c) \\
&= \boldsymbol{I}_{\boldsymbol{C}_1 \circ \boldsymbol{C}_2}(A_f(s)).
\end{aligned}$$

Therefore

$$\begin{aligned}
t(\boldsymbol{C}_1 \circ \boldsymbol{C}_2) &= \boldsymbol{I}_{\boldsymbol{C}_1 \circ \boldsymbol{C}_2} \\
&= \boldsymbol{I}_{\boldsymbol{C}_1} \circ \boldsymbol{I}_{\boldsymbol{C}_2} \\
&= t(\boldsymbol{C}_1) \circ t(\boldsymbol{C}_2)
\end{aligned}$$

Hence t *is an isomorphism.*

3 Relative FS-closure Operators and FS-connectors

Definition 3.1. Let $A_f(s)$ be an fs-set in X and **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. A function $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ defined by $(\mathbf{Cl})^n_{A_f(s)}(B) = n^{th}$ term of $\mathbf{Cl}({}_{nB}A_f(s))$, where ${}_{nB}A_f(s)$ is the fs-set in X obtained from $A_f(s)$ replacing n^{th} term of it by B, is called n^{th} relative FS-closure operator of **Cl** with respect to $A_f(s)$.

If **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then it is obvious that $(\mathbf{Cl})^n_{X^0_f(s)} = (\mathbf{Cl})^n_f$ and consequently $\delta_{(\mathbf{Cl})^n_{X^0_f(s)}} = \delta_{(\mathbf{Cl})^n_f}$, $\delta_{(\mathbf{Cl})^n_{X^0_f(s)}}$ and $\delta_{(\mathbf{Cl})^n_f}$ being the fuzzy topologies induced by $(\mathbf{Cl})^n_{X^0_f(s)}$ and $(\mathbf{Cl})^n_f$ respectively. It is also true that the n^{th} relative FS-closure operator $(\mathbf{Cl})^n_{A_f(s)}$ of an FS-closure operator **Cl** with respect to an fs-set $A_f(s)$ satisfies $FSC2$, $FSC3$ and $FSC4$ but it may not satisfy $FSC1$ shown by **Example** 3.2. Hence $(\mathbf{Cl})^n_{A_f(s)}$ may not be a fuzzy operator.

Example 3.2. Define a function **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ by

$$\begin{aligned} \mathbf{Cl}(B_f(s)) &= X^1_f(s) \text{ if } B_f(s) \neq X^0_f(s), \\ &= X^0_f(s) \text{ if } B_f(s) = X^0_f(s) \end{aligned}$$

Then for any fs-set $A_f(s)$ in X, having at least two non zero components, $(\mathbf{Cl})^n_{A_f(s)}(\overline{0}) = \overline{1}$ for all $n \in \mathbb{N}$.

Theorem 3.3. *Let $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ be the n^{th} relative FS-closure operator of an FS-closure operator **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X with respect to an fs-set $A_f(s)$. Then $\delta_{(\mathbf{Cl})^n_{A_f(s)}} = \{\overline{1}, B;$ $B \in I^X$ and $(\mathbf{Cl})^n_{A_f(s)}(B^c) = B^c\}$ forms a fuzzy topology on X. Further, the closure in the FTS $(X, \delta_{(\mathbf{Cl})^n_{A_f(s)}})$ and $(\mathbf{Cl})^n_{A_f(s)}$ are identical on $I^X - \{\overline{0}\}$.*
Proof: *Proof is omitted.*

Definition 3.4. The fuzzy topology $\delta_{(\mathbf{Cl})^n_{A_f(s)}} = \{\overline{1}, B; B \in I^X$ and $(\mathbf{Cl})^n_{A_f(s)}(B^c) = B^c\}$ induced by the n^{th} relative FS-closure operator $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ is called the n^{th} relative fuzzy topology induced by the FS-closure operator **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ with respect to the fs-set $A_f(s)$.

Theorem 3.5. *Let $A_f(s) = \{A^n_f\}_{n=1}^{\infty}$ be an fs-set in a set X and **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. Let $(\mathbf{Cl})^n_f$, $n \in \mathbb{N}$ be the n^{th} component of **Cl**. Then*
(1) $\mathbf{Cl}(A_f(s)) \geq \{(\mathbf{Cl})^n_f(A^n_f)\}$ and the equality holds if $A_f(s)$ is a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$.
(2) If $\mathbf{Cl}(A_f(s)) = \{(\mathbf{Cl})^n_f(A^n_f)\}$ and A_n is closed in $(X, \delta_{(\mathbf{Cl})^n_f})$ for each $n \in \mathbb{N}$, then $A_f(s)$ is closed in $(X, \delta_{\mathbf{Cl}}(s))$.
(3) $\mathbf{Cl}(A_f(s)) = \{(\mathbf{Cl})^n_{A_f(s)}(A^n_f)\}$.
Proof: *Proof is omitted.*

In an FSTS $(X, \delta(s))$ if $A_f(s) = \{A^n_f\}_{n=1}^{\infty}$ is closed, then A^n_f is closed in (X, δ_n) for each $n \in \mathbb{N}$ but the converse is not true [13]. **Corollary** 3.6 provides a pair of if and only if conditions for an fs-set $A_f(s)$ to be closed in an FSTS.

Corollary 3.6. *In an FSTS $(X, \delta(s))$, an fs-set $A_f(s) = \{A^n_f\}_{n=1}^{\infty}$ is closed:*
(1) if and only if $\overline{A_f(s)} = \{B^n_f\}$ and A^n_f is closed in (X, δ_n) for each $n \in \mathbb{N}$, where $B^n_f = n^{th}$ component of $\overline{{}_{nA^n_f}X^0_f(s)}$.
(2) if and only if A^n_f is closed in $(X, \delta_{R^n_{A_f(s)}})$ for each $n \in \mathbb{N}$, where $R^n_{A_f(s)}$ is the n^{th} relative FS-closure operator of the closure operator in $(X, \delta(s))$ with respect to $A_f(s)$.

Theorem 3.7. *If $\{A_{\lambda f}(s); \lambda \in \Lambda\}$ be a chain of fs-sets in $((I^X)^{\mathbb{N}}, \leq)$, then $\{\delta_{(\mathbf{Cl})^n_{A_{\lambda f}(s)}}, \lambda \in \Lambda\}$ is a chain of fuzzy topologies on X for each $n \in \mathbb{N}$, where **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is an FS-closure*

operator on X.
Proof: *Let $A_{\lambda f}(s) \leq A_{\mu f}(s)$, λ, $\mu\in \Lambda$. It suffices to show that $\delta_{(\mathbf{Cl})^n_{A_{\mu f}(s)}} \leq \delta_{(\mathbf{Cl})^n_{A_{\lambda f}(s)}}$.*

$$\text{Let } B \in \delta_{(\mathbf{Cl})^n_{A_{\mu f}(s)}} \Rightarrow (\mathbf{Cl})^n_{A_{\mu f}(s)}(\overline{1}-B) = \overline{1}-B$$
$$\Rightarrow n^{th} \text{ term of } \mathbf{Cl}({}_{n(\overline{1}-B)}A_{\mu f}(s)) = \overline{1}-B$$

Therefore n^{th} term of $\mathbf{Cl}({}_{n(\overline{1}-B)}A_{\lambda f}(s)) \leq \overline{1}-B$
$\Rightarrow (\mathbf{Cl})^n_{A_{\lambda f}(s)}(\overline{1}-B) \leq \overline{1}-B$.

Hence $B \in \delta_{(\mathbf{Cl})^n_{A_{\lambda f}(s)}}$.

Definition 3.8. Each member except possibly $\overline{1}$ of $\delta_{(\mathbf{Cl})^n_{A_f(s)}}$ is contained in $\overline{1}-(\mathbf{Cl})^n_{A_f(s)}(\overline{0})$ and so $\delta_{(\mathbf{Cl})^n_{A_f(s)}}$ is called $(\overline{1}-(\mathbf{Cl})^n_{A_f(s)}(\overline{0}))$-cut of $\delta_{(\mathbf{Cl})^n_f}$.

Theorem 3.9. *Let $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators on X. Then the operator $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $C(A_f(s)) = \{C_n(A^n_f)\}$ for all $A_f(s) = \{A^n_f\}^\infty_{n=1} \in (I^X)^{\mathbb{N}}$ is an FS-closure operator on X.*
Proof: *The proof is omitted.*

Definition 3.10. Let $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators on X. The operator $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $C(A_f(s)) = \{C_n(A^n_f)\}$ for all $A_f(s) = \{A^n_f\}^\infty_{n=1} \in (I^X)^{\mathbb{N}}$ is called an FS-closure operator induced by a sequence $\{C_n : I^X \to I^X\}$ of fuzzy closure operators on X.

Definition 3.11. Let δ and δ' be two fuzzy topologies on a set X. A subset K_f of δ'^{δ} is called an FS-connector of δ to δ' if it satisfies the following conditions:
(1) $A_\lambda \in \delta$ and $f_\lambda \in K_f$, $\lambda \in \Lambda \Rightarrow$ there exist $f \in K_f$ so that $f(\vee_{\lambda\in\Lambda}A_\lambda) = \vee_{\lambda\in\Lambda}f_\lambda(A_\lambda)$,
(2) $A_i \in \delta$ and $f_i \in K_f$, $i = 1(1)n \Rightarrow$ there exist $f \in K_f$ so that $f(\wedge^n_{i=1}A_i) = \wedge^n_{i=1}f_i(A_i)$ and
(3) $\delta' = \vee_{f\in K_f}f(\delta)$.

Example 3.12. Let δ and δ' be two fuzzy topologies on a set X. A function $f : \delta \to \delta'$ defined by $f(A) = O$ for all $A \in \delta$, where O is a fixed element of δ', is called a constant function from δ into δ'. If K_f be the collection of all such constant functions from δ into δ', then K_f forms an FS-connector from δ to δ'.

Definition 3.13. Let δ and δ' be two fuzzy topologies on a set X. Then the collection of all constant functions from δ into δ' forms an FS-connector of δ to δ'. This is called the discrete FS-connector of δ to δ'.

If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X, then any sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, provides a unique FST on X **(Theorem** 3.14) which is denoted by $\delta(s) < \{\delta_n\}, \{K_n\} >$ such that the n^{th} components $(\delta < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$ and it is called the FST generated by $\{\delta_n\}$ and $\{K_n\}$. If further each K_n is the discrete FS-connector of δ_n to δ_{n+1}, then the FST is said to be generated by $\{\delta_n\}$ and is denoted by $\delta < \{\delta_n\} >$.

Theorem 3.14. *Let $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_n\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$, $n \in \mathbb{N}$. Also for any FSTS $(X,\delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} and $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$.*
Proof: *Let $K = \prod^\infty_{n=1} K_n$, $g = \{g_n\} \in K$ and $A \in \delta_1$. Define $H_1 = A$ and $H_n = g_{n-1}g_{n-2}.....g_2g_1A$, $n > 1$. Let $H^g_A(s) = \{H_n\} \in (I^X)^{\mathbb{N}}$ and consider $\delta(s) < \{\delta_n\}, \{K_n\} >= \{X^1_f(s), X^0_f(s)\} \vee \{H^g_A(s); g \in K \text{ and } A \in \delta_1\}$. Consider*

$$H_\lambda(s) = H^{g_\lambda}_{A_\lambda}(s) \in \delta(s),\ \lambda \in \Lambda$$

where Λ is an index set and

$$A = \vee_{\lambda\in\Lambda}A_\lambda \in \delta_1.$$

For $g_{\lambda 1} \in K_1$ and $A \in \delta_1$ there exist $g_1 \in K_1$ such that

$$g_1 A = \vee_{\lambda \in \Lambda} g_{\lambda 1} A_\lambda;\ g_{\lambda n} \in K_n$$

and for $g_{n-1} g_{n-2} ...... g_2 g_1 A \in \delta_n$ there exist $g_n \in K_n$ such that

$$g_n g_{n-1} ...... g_2 g_1 A = \vee_{\lambda \in \Lambda} g_{\lambda n} g_{\lambda (n-1)} .... g_{\lambda 2} g_{\lambda 1} A_\lambda.$$

Obviously,

$$\vee_{\lambda \in \Lambda} H_\lambda(s) = \vee_{\lambda \in \Lambda} H^{g_\lambda}_{A_\lambda}(s) = H^g_A(s) \in \delta(s) < \{\delta_n\}, \{K_n\} >$$

where $g = g_n$. Arguing in the same way it can be shown that $\delta(s) < \{\delta_n\}, \{K_n\} >$ is closed under finite intersection. Therefore, $(X,\ \delta(s) < \{\delta_n\}, \{K_n\} >)$ is a fuzzy sequential topological space. The third condition to be an FS-connector ensures that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$. For the next part, for each $n \in \mathbb{N}$ define a relation $R^{n,n+1}$ on $\delta(s)$ by $A_f(s) = \{A^n_f\} R^{n,n+1} B_f(s) = \{B^n_f\}$ if and only if $A^n_f = B^n_f$. Then $R^{n,n+1}$ defines a partition of $\delta(s)$ say

$$\{Cls(A_f(s));\ A_f(s) \in \delta^{n,n+1}(s) \subset \delta(s)\}$$

where $\delta^{n,n+1}(s)$ is a family of open fs-sets taking exactly one from each class of the partition of $\delta(s)$ by $R^{n,n+1}$ and $Cls(A_f(s))$ represents the class of $A_f(s)$. Let

$$K^{n,n+1} = \prod_{A_f(s) \in \delta^{n,n+1}(s)} Cls(A_f(s))$$

Then each $t \in K^{n,n+1}$ defines a function $g_t : \delta_n \to \delta_{n+1}$ and $K_n = \{g_t;\ t \in K^{n,n+1}\}$ is an FS-connector connecting δ_n to δ_{n+1} and properties of FS-connectors ensures that $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$.

Corollary 3.15. *Let $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{Cl})^n_f}$ to $\delta_{(\boldsymbol{Cl})^{n+1}_f}$ for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >)_n = \delta_{(\boldsymbol{Cl})^n_f}$ and the components of the closure operator on $(X,\ \delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >)$ are $(\boldsymbol{Cl})^n_f$, $n \in \mathbb{N}$. Also for any FSTS $(X,\delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{Cl})^n_f}$ to $\delta_{(\boldsymbol{Cl})^{n+1}_f}$ and $\delta(s) = \delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >$.*

Corollary 3.16. *Let $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{I})^n_f}$ to $\delta_{(\boldsymbol{I})^{n+1}_f}$ for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >)_n = \delta_{(\boldsymbol{I})^n_f}$ and the components of the interior operator on $(X,\ \delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >)$ are $(\boldsymbol{I})^n_f$, $n \in \mathbb{N}$. Also for any FSTS $(X,\delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{I})^n_f}$ to $\delta_{(\boldsymbol{I})^{n+1}_f}$ and $\delta(s) = \delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >$.*

Corollary 3.17. *If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X such that $\delta_n = \delta$ for all $n \in \mathbb{N}$, then $\delta(s) < \{\delta_n\} >= \delta^{\mathbb{N}}$.*

Corollary 3.18. *If $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators and $\boldsymbol{C}$ be an FS-closure operator induced by $\{C_n\}$, then $\delta_{\boldsymbol{C}}(s) = \delta(s) < \{\delta_n\} >$ where δ_n is the fuzzy topology on X induced by C_n, $n \in \mathbb{N}$.*

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SOME NEARLY OPEN SETS IN A FUZZY SEQUENTIAL TOPOLOGICAL SPACE

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ABSTRACT

The present article gives a study of fs-semiopen sets, fs-regular open sets and fs-semicontinuous functions in a fuzzy sequential topological space. Other studied notions are fs-almost continuous functions, fs-weakly continuous functions and it has been shown that both of these functions and fs-semicontinuous functions are independent notions. Further, many results relating these functions together with fs-continuous functions have been obtained.

Keywords and Phrases: *Fuzzy sequential topological spaces, fs-semiopen sets, fs-semicontinuous functions, fs-regular open sets, fs-almost continuous functions, fs-weakly continuous functions.*

AMS Subject Classification*: 54A40.*

1. PRELIMINARIES

The introduction of fuzzy sets in 1965, by L.A. Zadeh [12] leads to the foundation of a new area of research called fuzzy mathematics. Since then, many researchers have been working in this area and related areas. As a generalization of a topological space, C. L. Chang [3] introduced the concept of fuzzy topological space in 1968. Fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad [1].

The purpose of this work is to study the concept of semi-open sets and semicontinuity in fuzzy sequential topological spaces.

Throughout the paper, X will denote a non empty set and I the unit interval [0, 1]. Sequences of fuzzy sets in X called fuzzy sequential sets (fs-sets) will be denoted by the symbols $A_f(s)$, $B_f(s)$, $C_f(s)$ etc. An fs-set $X_f^l(s)$ is a sequence of fuzzy sets $\{X_f^n\}_n$, where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$.

A family $\delta(s)$ of fuzzy sequential sets on a non-empty set X satisfying the properties:

i. $X_f^r(s) \in \delta(s)$ for all $r \in \{0, 1\}$,

ii. $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$

iii. for any family $\{A_{fj}(s);\ j \in J\} \subseteq \delta(s), \vee_{j \in J} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called a fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets. Complement of an open fuzzy sequential set is called closed fuzzy sequential set. In an FSTS $(X, \delta(s))$, the closure $\overline{A_f(s)}$ and interior $A_f^\circ(s)$ of any fs-set $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \le C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$A_f^o(s) = \vee\{O_f(s);\ O_f(s) \le A_f(s)), O_f(s) \in \delta(s)\},$$

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[10] Let g be a mapping from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then g is called
(i) fs-continuous if $g^{-1}(B_f(s))$ is open in $(X, \delta(s))$ for every open fs-set $B_f(s)$ in $(Y, \eta(s))$.
(ii) fs-open if $g(A_f(s))$ is fs-open in Y for every fs-open set $A_f(s)$ in X.
(iii) fs-closed if $g(A_f(s))$ is fs-closed in Y for every fs-closed set $A_f(s)$ in X.

Section **2** deals with the introduction and study of fs-semiopen sets as well as fs-semicontinuity. Section **3** deals with the introduction of fs-regular open sets and functions like fs-almost continuous and fs-weakly continuous functions. In this section, the interrelations among these functions together with fs-continuous and fs-semicontinuous functions have been investigated.

2. FS-SEMIOPEN SETS AND FS-SEMICONTINUITY

Definition 2.1: An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiopen set if $A_f(s) \leq \overline{A_f^o(s)}$. An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiclosed set if its complement is fs-semiopen.

Fundamental properties of fs-semiopen (fs-semiclosed) sets are:

- Any union (intersection) of fs-semiopen (fs-semiclosed) sets is fs-semiopen (fs-semiclosed).
- Every fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).
- Closure (interior) of an fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).

Example 2.1 shows that an fs-semiopen (fs-semiclosed) set may not be fs open (fs-closed), the intersection (union) of any two fs-semiopen (fs semiclosed) sets need not be an fs-semiopen (fs-semoclosed) set. Unlike in a general topological space, the intersection of an fs-semiopen set with an fs open set may fail to be an fs-semiopen set.

Example 2.1: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots\dots\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots\dots\right\}$$

$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots\dots\right\}$$

$$D_f(s) = \left\{\overline{\frac{3}{8}}, \bar{0}, \bar{0}, \dots\dots\right\}$$

Consider $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Now,
(i) $B_f(s)$ is fs-open, hence fs-semiopen and $C_f(s)$ is fs-semiopen but their intersection $D_f(s)$ is not fs-semiopen.
(ii) $C_f(s)$ is fs-semiopen but is not fs-open.

Theorem 2.1: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiopen if and only if there exist an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$.

Proof: Straightforward.

Theorem 2.2: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiclosed if and only if there exist an fs-closed set $C_f(s)$ in X such that $C_f^\circ(s) \leq A_f(s) \leq C_f(s)$.

Proof: Straightforward.

We will denote the set of all fs-semiopen sets in X by $FSSO(X)$.

Theorem 2.3: In an FSTS $(X, \delta(s))$, (i) $\delta(s) \subseteq FSSO(X)$. (ii) If $A_f(s) \in FSSO(X)$ and $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, then $B_f(s) \in FSSO(X)$.

Proof:
(i) Follows from definition.
(ii) Let $A_f(s) \in FSSO(X)$. Then there exists an fs-open set $O_f(s)$ such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. So,

$$O_f(s) \leq A_f(s) \leq B_f(s) \leq \overline{A_f(s)} \leq \overline{O_f(s)}$$

$$\Rightarrow O_f(s) \leq B_f(s) \leq \overline{O_f(s)}.$$

$O_f(s)$ being fs-open, $B_f(s)$ is fs-semiopen.

Theorem 2.4: If in a fuzzy sequential topological space, $C_f^o(s) \leq B_f(s) \leq C_f(s)$, where $C_f(s)$ is fs-semiclosed, then $B_f(s)$ is also fs-semiclosed.

Proof: Omitted.

Theorem 2.5: Let $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in an FSTS $(X, \delta(s))$ such that (i) $\delta(s) \subseteq \mho$ and (ii) if $A_f(s) \in \mho$ and $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, then $B_f(s) \in \mho$. Then $FSSO(X) \subseteq \mho$. that is, $FSSO(X)$ is the smallest class of fs-sets in X satisfying (i) and (ii).

Proof: Let $A_f(s) \in FSSO(X)$. Then $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$ for some $O_f(s) \in \delta(s)$. By (i), $O_f(s) \in \mho$ and thus $A_f(s) \in \mho$ by (ii).

If $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in X, then $Int\mho$ denotes the set $\{A_{\alpha f}^o(s);\ \alpha \in \Lambda\}$.

Theorem 2.6: If $(X, \delta(s))$ be a fuzzy sequential topological space, then $\delta(s) = Int(FSSO(X))$.

Proof: Every fs-open set being fs-semiopen, $\delta(s) \subseteq Int(FSSO(X))$. Conversely, let $O_f(s) \in Int(FSSO(X))$. Then $O_f(s) = A_f^o(s)$ for some $A_f(s) \in FSSO(X)$ and hence $O_f(s) \in \delta(S)$.

Definition 2.2: Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. We define semi-closure $sCl(A_f(s))$ and semi-interior $sInt(A_f(s))$ of $A_f(s)$ by

$$sCl\left(A_f(s)\right) = \wedge\{B_f(s);\ A_f(s) \leq B_f(s) \ and\ A_f^c(s) \in FSSO(X)\}$$
$$sInt\left(B_f^c(s)\right) = \vee\{C_f(s);\ C_f(s) \leq A_f(s) \ and\ C_f(s) \in FSSO(X)\}.$$

Obviously, $sCl\left(A_f(s)\right)$ is the smallest fs-semiclosed set containing $A_f(s)$ and $sInt\left(A_f(s)\right)$ is the largest fs-semiopen set contained in $A_f(s)$. Further,

(i) $A_f(s) \leq sCl\left(A_f(s)\right) \leq \overline{A_f(s)}$ and $A_f^o(s) \leq sInt\left(A_f(s)\right) \leq A_f(s)$.
(ii) $A_f(s)$ is fs-semiopen if and only if $A_f(s) = sInt\left(A_f(s)\right)$.
(iii) $A_f(s)$ is fs-semiclosed if and only if $A_f(s) = sCl\left(A_f(s)\right)$.
(iv) $A_f(s) \leq B_f(s)$ implies $sInt\left(A_f(s)\right) \leq sInt\left(B_f(s)\right)$ and $sCl\left(A_f(s)\right) \leq sCl\left(B_f(s)\right)$.

Definition 2.3: A mapping $g: (X, \delta(s)) \to (Y, \delta'(s))$ is said to be
(i) fs-semicontinuous if $g^{-1}(B_f(s))$ is fs-semiopen in X for every $B_f(s) \in \delta'(s)$.
(ii) fs-semiopen if $g(A_f(s))$ is fs-semiopen in Y for every $A_f(s) \in \delta(s)$.
(iii) fs-semiclosed if $g(A_f(s))$ is fs-semiclosed in Y for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-semicontinuous (fs-semiopen, fs-semiclosed). That the converse may not be true, is shown by Example 2.2.

Example 2.2: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots\dots\right\}$$
$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots\dots\right\}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$. Define $g: (X, \delta(s)) \to (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-semicontinuous but not fs-continuous.

Again the map $h: (X, \delta'(s)) \to (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-semiopen and fs-semiclosed but is neither fs-open nor fs-closed.

Now consider the map $t: (X, \eta(s)) \to (X, \delta(s))$ defined by $t(x) = x$ for all $x \in X$, where $\eta(s) = \{C_f^c(s), X_f^0(s), X_f^1(s)\}$. Then t is fs-semiclosed but not fs-closed.

Theorem 2.7: Let $g:(X,\delta(s))\rightarrow(Y,\eta(s))$ be a map. Then the following conditions are equivalent:

(i) g is fs-semicontinuous.

(ii) the inverse image of an fs-closed set in Yunder g is fs-semiclosed in X.

(iii) For any fs-set $A_f(s)$ in X, $g\left(sCl\left(A_f(s)\right)\right)\leq\overline{g\left(A_f(s)\right)}$.

Proof:

$(\boldsymbol{i})\Rightarrow(\boldsymbol{ii})$: Suppose $g:(X,\delta(s))\rightarrow(Y,\eta(s))$ be an fs-semicontinuous map and $B_f(s)$ be an fs-closed set in Y. Then

$B_f^c(s)$ is fs-open in Y

$$\Rightarrow\left(g^{-1}\left(B_f(s)\right)\right)^c=g^{-1}\left(B_f^c(s)\right)\text{ is fs-semiopen in }X$$
$$\Rightarrow g^{-1}\left(B_f(s)\right)\text{ is fs-semiclosed in }X.$$

$(\boldsymbol{ii})\Rightarrow(\boldsymbol{iii})$: Suppose $A_f(s)$ be an fs-set in X. Then by (ii), $g^{-1}(\overline{g(A_f(s))})$ is fs-semiclosed in X and hence $g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)=sCl(g^{-1}(\overline{g(A_f(s))})$). Again

$$A_f(s)\leq g^{-1}\left(g\left(A_f(s)\right)\right)$$
$$\Rightarrow sCl\left(A_f(s)\right)\leq sCl\left(g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)\right)=g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)$$
$$\Rightarrow g\left(sCl\left(A_f(s)\right)\right)\leq g(g^{-1}(\overline{g(A_f(s))}))\leq\overline{g(A_f(s))}$$

$(\boldsymbol{iii})\Rightarrow(\boldsymbol{i})$: Let $B_f(s)$ be an fs-open set in Y. Then for the fs-closed set $B_f^c(s)$, we have

$$g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right)\leq\overline{g\left(g^{-1}\left(B_f^c(s)\right)\right)}\leq\overline{B_f^c(s)}=B_f^c(s)$$

Thus $sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\leq g^{-1}\left(g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right)\right)\leq g^{-1}\left(B_f^c(s)\right)$.

Therefore $sCl\left(g^{-1}\left(B_f^c(s)\right)\right)=g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c=g^{-1}(B_f^c(s))$ is fs-semiclosed in X.

Theorem 2.8: Suppose $g:(X,\delta(s))\rightarrow(Y,\eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiopen set in Y is fs-semiopen in X.

Proof: Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Ysuch that

$$O_f(s)\leq B_f(s)\leq\overline{O_f(s)}$$
$$\Rightarrow g^{-1}(O_f(s))\leq g^{-1}(B_f(s))\leq g^{-1}(\overline{O_f(s)})$$

We claim that $g^{-1}(\overline{O_f(s)})\leq\overline{g^{-1}(O_f(s))}$. Let $P_f(s)\in g^{-1}(\overline{O_f(s)})$. This implies $g(P_f(s))\in\overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $g(U_f(s))$ is a weak open Q-nbd of $g(P_f(s))$. Therefore

$$g(U_f(s))\ q_w\ O_f(s)$$
$$\Rightarrow U_f(s)\ q_w\ g^{-1}(O_f(s))$$
$$\Rightarrow P_f(s)\in\overline{g^{-1}(O_f(s))}.$$

Thus we have, $g^{-1}(O_f(s))\leq g^{-1}(B_f(s))\leq\overline{g^{-1}(O_f(s))}$. Hence, $g^{-1}(O_f(s))$ being fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen.

Corollary 2.1: Suppose $g:(X,\delta(s))\rightarrow(Y,\eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiclosed set in Y is fs-semiclosed in X.

Proof: Proof is omitted.

Corollary 2.2: If $g:(X,\delta(s))\rightarrow(Y,\delta'(s))$ be an fs-semicontinuous open map and $h:(Y,\delta'(s))\rightarrow(Z,\eta(s))$ be an fs-semicontinuous map, then $hog:(X,\delta(s))\rightarrow(Z,\eta(s))$ is fs-semicontinuous.

Proof: Let $C_f(s)$ be an fs-open set in Z, then $h^{-1}(C_f(s))$ is fs-semiopen in $Y$and hence $(hog)^{-1}\left(C_f(s)\right)=g^{-1}(h^{-1}(C_f(s)))$ is fs-semiopen in X by Theorem 2.8.

Theorem 2.9: Let $g:(X,\delta(s)) \to (Y,\eta(s))$ be an fs-continuous open map. Then the g-image of an fs-semiopen set in X is fs-semiopen in Y.

Proof: Let $A_f(s)$ be an fs-semiopen set in X. Then there exists an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. This implies

$$g(O_f(s)) \leq g(A_f(s)) \leq g(\overline{O_f(s)}) \leq \overline{g(O_f(s))}.$$

Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-semiopen in Y.

Corollary 2.3: Semi-openness in an FSTS is a topological property.

Proof: Follows from Theorem 2.9.

Remark 2.1: Theorem 2.9 does not hold if g is not fs-open. This is shown by Example 2.3.

Example 2.3: Let $(X,\delta(s))$ and $(Y,\delta'(s))$ be two fuzzy sequential topological spaces, where $\delta(s)$ contains all the constant fs-sets in X, $Y = [0,1]$ and $\delta^{'}(s) = \{Y_f^0(s),\ Y_f^1(s)\}$. Define a map $g:(X,\delta(s)) \to (Y,\delta'(s))$ by $g(x) = \frac{1}{2}$ for all $x \in X$. Then g is fs-continuous but not fs-open. Here, for any fs-semiopen set $A_f(s)$ in X, $g(A_f(s)) = \left\{\overline{\frac{1}{2}}\right\}_{n=1}^{\infty}$ is not fs-semiopen in Y.

Remark 2.2: Converse of Theorem 2.9 holds if g is one-one.

Theorem 2.10: Let $g:(X,\delta(s)) \to (Y,\delta'(s))$ and $h:(Y,\delta'(s)) \to (Z,\eta(s))$ be two mappings and $hog:(X,\delta(s)) \to (Z,\eta(s))$ be an fs-semiclosed mapping. Then, g is fs-semiclosed if h is an injective fs-semicontinuous open mapping.

Proof: Let $A_f(s)$ be an fs-closed set in X. Then $hog(A_f(s))$ is fs-semiclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-semiclosed in Y.

Theorem 2.11: If $g:(X,\delta(s)) \to (Y,\delta'(s))$ is fs-semicontinuous and $h:(Y,\delta'(s)) \to (Z,\eta(s))$ is fs-continuous, then $hog:(X,\delta(s)) \to (Z,\eta(s))$ is fs-semicontinuous.

Proof: Omitted.

3. FS-REGULAR OPEN SETS

Definition 3.1 An fs-set $A_f(s)$ in an FSTS $(X,\delta(s))$, is said to be fs-regular open in X if $(\overline{A_f(s)})^o = A_f(s)$. An fs-set $A_f(s)$ is said to be fs-regular closed in X if its complement is fs-regular open.

It is obvious that every fs-regular open (closed) set is fs-open (closed). The converse need not be true, is shown by Example 3.1. Example 3.2 shows that the union (intersection) of any two fs-regular open (closed) sets need not be an fs-regular open (closed) set.

Example 3.1: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \ldots..\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \ldots\ldots\right\}$$

Let $\delta(s) = \{A_f(s),\ B_f(s),\ A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X,\delta(s))$ is an FSTS where $A_f(s)$ is fs-open but not fs-regular open.

Example 3.2: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \ldots..\right\}$$

$$B_f(s) = \left\{\overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}} \ldots\ldots\right\}$$

Let $\delta(s) = \{A_f(s),\ B_f(s),\ A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X,\delta(s))$ is an FSTS. Here $A_f(s)$ and $B_f(s)$ are fs-regular open sets but their union is not fs-regular open.

Theorem 3.1:
(a) The intersection of two fs-regular open sets is an fs-regular open set.
(b) The union of two fs-regular closed sets is an fs-regular closed set.

Proof: We prove only (a). Let $A_f(s)$ and $B_f(s)$ be two fs-regular open sets in X. Since $A_f(s) \wedge B_f(s)$ is fs-open, we have $A_f(s) \wedge B_f(s) \leq (\overline{A_f(s) \wedge B_f(s)})^o$.

Now, $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{A_f(s)})^o = A_f(s)$ and $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{B_f(s)})^o = B_f(s)$ implies $(\overline{A_f(s) \wedge B_f(s)})^o \leq$ $Af(s) \wedge Bf(s)$. Hence the result.

Theorem 3.2:
(a) The closure of an fs-open set is fs-regular closed.
(b) The interior of an fs-closed set is fs-regular open.

Proof: We prove only (a). Let $A_f(s)$ be an fs-open set in X. Since $(\overline{A_f(s)})^o \leq \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \leq \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-open, $A_f(s) \leq (\overline{A_f(s)})^o$ and hence $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus $\overline{A_f(s)}$ is fs-regular closed.

Definition 3.2: A mapping $g: (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-almost continuous mapping if $g^{-1}(B_f(s)) \in \delta(s)$ for each fs-regular open set $B_f(s)$ in Y.

Theorem 3.3: Let $g: (X, \delta(s)) \to (Y, \eta(s))$ be a mapping. Then the following are equivalent:

(i) g is fs-almost continuous.
(ii) $g^{-1}(B_f(s))$ is an fs-closed set for each fs-regular closed set $B_f(s)$ of Y .
(iii) $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$ for each fs-open set $B_f(s)$ of Y .
(iv) $\overline{g^{-1}(\overline{B_f^o(s)})} \leq g^{-1}(B_f(s))$ for each fs-closed set $B_f(s)$ of Y .

Proof: Here, we note that $g^{-1}\left(B_f^c(s)\right) = (g^{-1}(B_f(s)))^c$ for any fs-set $B_f(s)$ in Y.
$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Follows from the fact that an fs-set is fs-regular open if and only if its complement is fs-regular closed.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Let $B_f(s)$ be an fs-open set in Y. Then $B_f(s) \leq (\overline{B_f(s)})^o$ and hence $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$. By Theorem 3.2 (b), $(\overline{B_f(s)})^o$ is an fs-regular open set in Y. Therefore, $g^{-1}((\overline{B_f(s)})^o)$ is fs-open in X and thus

$$g^{-1}\left(B_f(s)\right) \leq g^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = (g^{-1}((\overline{B_f(s)})^o))^o.$$

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $B_f(s)$ be an fs-regular open set in Y. Then by (iii), we have $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$. Hence $g^{-1}(B_f(s))$ is an fs-open set in X.

$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iv})$: are easy to prove.

Clearly an fs-continuous map is an fs-almost continuous map but the converse may not be true, as is shown by Example 3.3.

Example 3.3: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \overline{1}, \quad \overline{1}, \quad \overline{1}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots\dots..\right\}$$

Let $\delta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), \quad A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-almost continuous but not fs-continuous. Again, since the inverse image of fs-open set $A_f(s)$ of $(X, \eta(s))$ is not fs-semiopen in $(X, \delta(s))$, g is not fs-semicontinuous.

Example 3.4: Example to show that an fs-semicontinuous map may not be fs-almost continuous. Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{0}, \quad \overline{0}, \quad \overline{0}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots\dots..\right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-semicontinuous but not fs-almost continuous.

Remark 3.1: Example 3.3 and Example 3.4 shows that an fs-almost continuous mapping and an fs-semicontinuous mapping are independent notions.

Definition 3.3: An FSTS $(X, \delta(s))$ is called an fs-semiregular space if the collection of all fs-regular open sets in X forms a base for $\delta(s)$.

Theorem 3.4: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping, where $(Y, \eta(s))$ is an fs-semiregular space. Then g is fs-almost continuous if and only if g is fs-continuous.

Proof: We need only to show that if g is fs-almost continuous, then it is fs-continuous. Suppose g is fs-almost continuous. Let $B_f(s) \in \eta(s)$, then $B_f(s) = \vee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where $B_{\lambda f}(s)$'s are fs-regular open sets in Y . Then

$$\begin{aligned} g^{-1}(B_f(s)) &= \vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\le \vee_{\lambda\in\Lambda}(g^{-1}((\overline{B_{\lambda f}(s)})^o))^o \\ &= \vee_{\lambda\in\Lambda}(g^{-1}(B_{\lambda f}(s)))^o \\ &\le (\vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

which shows $g^{-1}(B_f(s)) \in \delta(s)$.

Theorem 3.5: Let X, X_1 and X_2 be fuzzy sequential topological spaces and $\pi_i: X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g: X \to X_1 \times X_2$ is fs-almost continuous, then $\pi_i og$ is also fs-almost continuous.

Proof: Let g be an fs-almost continuous map and let $B_f(s)$ be an fs-regular open set in X_i. Since π_i is fs-continuous, we have $\overline{\pi_i^{-1}(B_f(s))} \le \pi_i^{-1}(\overline{B_f(s)})$ and since π_i is fs-open we have, $\pi_i^{-1}(B_f^o(s)) \le (\pi_i^{-1}(B_f(s)))^o$. Also $B_f(s) \le \pi_i^{-1}(\pi_i(B_f(s)))$ and $\pi_i(\pi_i^{-1}(B_f(s)) \le B_f(s)$. Thus

$$\pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right) \le \pi_i\left(\pi_i^{-1}\left(B_f(s)\right)\right) \le B_f(s)$$

$$\Rightarrow \pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right) \le B_f^o(s)$$

$$\Rightarrow \left(\pi_i^{-1}\left(B_f(s)\right)\right)^o \le \pi_i^{-1}(\pi_i\left(\left(\pi_i^{-1}\left(B_f(s)\right)\right)^o\right)) \le \pi_i^{-1}\left(B_f^o(s)\right) = \pi_i^{-1}\left(B_f(s)\right)$$

$$\Rightarrow \pi_i^{-1}\left(B_f(s)\right) = \left(\pi_i^{-1}\left(B_f(s)\right)\right)^o \le \left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o \le \left(\pi_i^{-1}(\overline{B_f(s)})\right)^o = \pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = \pi_i^{-1}\left(B_f(s)\right)$$

$$\Rightarrow \pi_i^{-1}\left(B_f(s)\right) = \left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o .$$

Therefore,

$$\begin{aligned} (\pi_i og)^{-1}\left(B_f(s)\right) &= g^{-1}\left(\pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right) \\ &= g^{-1}\left(\left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o\right) \\ &= \left(g^{-1}\left(\left(\overline{\pi_i^{-1}\left(B_f(s)\right)}\right)^o\right)\right)^o \\ &\le \left(g^{-1}\left(\left(\pi_i^{-1}(\overline{B_f(s)})\right)^o\right)\right)^o \\ &= \left(g^{-1}\left(\pi_i^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right)\right)^o \\ &= \left(g^{-1}\left(\pi_i^{-1}(B_f(s))\right)\right)^o \\ &= \left((\pi_i og)^{-1}\left(B_f(s)\right)\right)^o \end{aligned}$$

Hence the theorem.

Definition 3.4: A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-weakly continuous mapping if for each fs-open set $B_f(s)$ in Y, $g^{-1}(B_f(s)) \le (g^{-1}(\overline{B_f(s)}))^o$.

Remark 3.2: It is clear that every fs-continuous mapping is fs-weakly continuous. The converse is not true, in general, which is shown by Example 3.5. The Example also shows that an fs-weakly continuous mapping may neither be fs-semicontinuous nor fs-almost continuous. However, it is clear that an fs-almost continuous mapping is also fs-weakly continuous.

Example 3.5: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \ldots \ldots.. \right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \ldots \ldots.. \right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-weakly continuous but not fs-continuous. Since the inverse image of fs-open set $B_f(s)$ of Y is not fs-semiopen in X, hence g is not fs-semicontinuous. Again, as the inverse image of fs-regular open set $B_f(s)$ of Y is not fs-open in X, g is not fs-almost continuous.

Remark 3.3: The map g defined in Example 3.4, is fs-semicontinuous but not fs-weakly continuous.

Remark 3.4: Example 3.5 and Remark 3.3 shows that fs-semicontinuity and fs-weakly continuity are independent notions.

Definition 3.5: An FSTS $(X, \delta(s))$ is called an Ωfs-semiregular space if each fs-open set $A_f(s)$ of X is the union of fs-open sets $A_{\lambda f}(s)$ $(\lambda \in \Lambda)$ of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$.

Theorem 3.6: An Ωfs-semiregular space is fs-semiregular.

Proof: Let $(X, \delta(s))$ be an Ωfs-semiregular space and $A_f(s)$ be an fs-open set in X. Then $A_f(s) = \vee_{\lambda \in \Lambda} A_{\lambda f}(s)$, where $A_{\lambda f}(s)$ are fs-open sets of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$. Since $A_{\lambda f}(s) \leq (\overline{A_{\lambda f}(s)})^o \leq A_f(s)$, we have $A_f(s) = \vee_{\lambda \in \Lambda} (\overline{A_{\lambda f}(s)})^o$. Now, for each $\lambda \in \Lambda$, $(\overline{A_{\lambda f}(s)})^o$ is fs-regular open in X and thus $(X, \delta(s))$ is a fs-semiregular space.

Remark 3.5: Example 3.6 shows that the converse of Theorem 3.6 may not be true.

Example 3.6: Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee$ Bfs, Afs∧Bfs, Xf0s, Xf1s and where the fs-sets *Afs* and *Bfs* in *X*, are defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \bar{1}, \quad \bar{1}, \quad \bar{1}, \ldots \ldots \right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \bar{0}, \quad \bar{0}, \quad \bar{0}, \ldots \ldots.. \right\}$$

Then $(X, \delta(s))$ is an fs-semiregular space. Now, the only way of writing $A_f(s)$ as the union of fs-open sets is the union of itself and $\overline{A_f(s)}$ is not contained in $A_f(s)$. Hence $(X, \delta(s))$ is not an Ωfs-semiregular space.

Theorem 3.7: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping where $(X, \delta(s))$ is any FSTS and $(Y, \eta(s))$ is an Ωfs-semiregular space. Then g is fs-weakly continuous if and only if g is fs-continuous.

Proof: It suffices to show that if g is fs-weakly continuous, then it is fs-continuous. For this, let $B_f(s) \in \eta(s)$. Then $B_f(s) = \vee_{\lambda \in \Lambda} B_{\lambda f}(s)$, where for all $\lambda \in \Lambda$, $B_{\lambda f}(s) \in \eta(s)$ and $\overline{B_{\lambda f}(s)} \leq B_f(s)$. Since g is fs- weakly continuous, we have

$$\begin{aligned} g^{-1}\left(B_f(s)\right) &= g^{-1}\left(\vee_{\lambda \in \Lambda} B_{\lambda f}(s)\right) = \vee_{\lambda \in \Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \vee_{\lambda \in \Lambda} (g^{-1}(\overline{B_{\lambda f}(s)}))^o \\ &\leq \vee_{\lambda \in \Lambda} (g^{-1}(B_f(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

and hence $g^{-1}(B_f(s))$ is fs-open in X. Thus g is fs-continuous.

Theorem 3.8: Let X, X_1 and X_2 be FSTS's and $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \to X_1 \times X_2$ is fs-weakly continuous, then $\pi_i o g$ is also fs-weakly continuous.

Proof: The proof is analogous to the proof of Theorem 3.5.

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