Academic Science

International Journal of Computer & Mathematical Sciences
IJCMS
ISSN 2347 – 8527
Volume 3, Issue 4
June 2014

FUZZY SEQUENTIAL TOPOLOGICAL SPACES

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ABSTRACT

Fuzzy sequential topology on a nonempty set X which is a sub-collection of $(\mathrm{I}^{\mathrm{X}})^{\mathbb{N}}$satisfying the conditions given in the definition is introduced. Many pleasant properties of a countable number of fuzzy topologies on X associated as components of a fuzzy sequential topology have been investigated. Finally, a variant of Yang's Theorem is established in this setting.

Keywords

Fuzzy sequential topology, fuzzy sequential point, quasi coincidence, Q-neighbourhood, fuzzy derived sequential set.

1. INTRODUCTION

In 1965 fuzzy sets were introduced by L. A. Zadeh [20] which is followed by the initiation of fuzzy topology in 1968 by C. L. Chang [5]. Till present, a variety of studies have been done in the theory of fuzzy topology. Some of them are fuzzy closure operators and fuzzy interior operators by Mashour and Ghanim [14], G. Gerla [7], Bandler and Kohout [1], R. Belohlavek [2], R. Belohlavek and T. Funiokova [3]; separation axioms by B. Hutton and I. Reilly [9]; fuzzy compactness by authors like C. L. Chang [5], J.A. Goguen [8], R. Lowen [11, 12, 13], T.E. Gantner, R.C. Steinlage and R.H. Warren [6], Wang Guojun [19], Gunther Jager [10] etc.

In 2002, M. K. Bose and I. Lahiri introduced the concept of sequential topological spaces [4] and studied some separation axioms in such spaces. Then N. Tamang, M. Singha and S. De Sarkar [18] extended this field by studying separation axioms in the light of reduced and augmented bases. Also the new operators namely, $K\Omega$ operators, Relative Closure Operators and Monotonic Sequential Operators in a class of sequential sets are studied by M. Singha and S. De sarkar [16, 17].

The purpose of this paper is to study the concept of sequential topological spaces in fuzzy setting. We begin with some basic definitions:

Let X be a nonempty set and $I = [0,1]$ be the closed unit interval in the set $\mathbb{R}$ of real numbers. Let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be sequences of fuzzy sets in X called fuzzy sequential sets in X and we define

i. $A_f(s) \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (union),
ii. $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (intersection),
iii. $A_f(s) \leq B_f(s)$ if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$; $\mathbb{N}$ being the set of positive integers,
iv. $A_f(s) \leq_W B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$,
v. $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$,
vi. $A_f(s)(\mathrm{x}) = \{A_f^n(x)\}_n, x \in X$,
vii. $A_f(s)(\mathrm{x}) \geq_M \mathrm{r}$ if and only if $A_f^n(\mathrm{x}) \geq \mathrm{r}_n$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular if $M = \mathbb{N}$, we write $A_f(s)(\mathrm{x}) \geq \mathrm{r}$,
viii. $X_f^l(s) = \{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$,
ix. $A_f^c(s) = \{1 - A_f^n\}_n = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$,
x. A fuzzy sequential set $P_f(s) = \{P_f^n\}_n$ is called a fuzzy sequential point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that

$$P_f^n(t) = r_n, \text{ if } t = x,$$
$$= 0, \text{ if } t \in X - \{x\},$$

for all $n \in \mathbb{N}$.

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$P_f^n(x) = r_n, \text{ whenever } n \in M,$$
$$= 0, \text{ whenever } n \in \mathbb{N} - M.$$

The point x is called the support, M is called base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (P_{fx}^M, r)$. If further $M = \{n\}$, $n \in \mathbb{N}$, then the fuzzy sequential point is called a simple fuzzy sequential point and it is denoted by (P_{fx}^n, r_n). A fuzzy sequential point is called complete if its base is the set of natural numbers. A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$ symbolically $P_f(s) \in_W A_f(s)$ if and only if there exists $n \in M$ such that $P_f^n(x) \leq A_f^n(x)$. If $R \subseteq M$ and s is the sequence in I same to r in R and vanishes outside R, then the fuzzy sequential point $P_{rf}(s) = (P_{fx}^R, s)$ is called a reduced fuzzy sequential point of $P_f(s) = (P_{fx}^M, r)$. A sequence $(x, L) = \{A_n\}_n$

International Journal of Computer & Mathematical Sciences
IJCMS
ISSN 2347 – 8527
Volume 3, Issue 4
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of subsets of X, where $A_n = \{x\}$, for all $n \in L$ and $A_n = \varphi =$ the null subset of X, for all $n \in \mathbb{N} - L$, is called a sequential point in X.
Throughout this paper we use Chang's definition of fuzzy topology [5].

2 Definitions and Results:

Definition 2.1 A family $\delta(s)$ of fuzzy sequential sets on a nonempty set X satisfying the properties:

i. $X_f^r(s) \in \delta(s)$ for all $r \in \{0,1\}$,

ii. $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$

iii. for any family $\{A_{fj}(s) \in \delta(s), j \in J\}$, $\vee_{j \in J} A_{fj}(s) \in \delta(s)$.

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 2.2 If $\delta_1(s)$ and $\delta_2(s)$ be two FSTs on X such that $\delta_1(s) \subset \delta_2(s)$, then we say that $\delta_2(s)$ is finer than $\delta_1(s)$ or $\delta_1(s)$ is weaker than $\delta_2(s)$

Proposition 2.1 If δ be a fuzzy topology (FT) on X, then $\delta^{\mathbb{N}}$ forms an FST on X.

Proof. Proof is straightforward. ■

We may construct different FSTs on X from a given FT δ on X, $\delta^{\mathbb{N}}$ is the finest of all these FSTs. Not only that, any FT δ on X can be considered as a component of some FST on X, one of them is $\delta^{\mathbb{N}}$, there are at least countably many FSTs on X weaker than $\delta^{\mathbb{N}}$ of which δ is a component. One of them is $\delta'(s) = \{A_f(s) = \{A_f^n\}_n;\ A_f^n = A \text{ for all } n \in \mathbb{N} \text{ and } A \in \delta\}$.

Proposition 2.2 If $(X, \delta(s))$ is an FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where $\delta_n = \{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$.
Proof. Proof is omitted. ■

Definition 2.3 In Proposition 2.2, (X, δ_n), where $n \in \mathbb{N}$, is called the n^{th} component fuzzy topological space of $(X, \delta(s))$.

Proposition 2.3 Let $A_f(s) = \{A_f^n\}_n$ be an open (closed) fuzzy sequential set in an FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in (X, δ_n) but the converse is not necessarily true.
Proof. Proof of the first part is omitted. For the converse part let us take the FSTS $(X, \delta(s))$ where X is any nonempty set and $\delta(s) = \{X_f^r(s), r \in I\}$. Let $\{r_n\}_n$ be a strictly increasing sequence in I and $A_f(s) = \{A_f^n\}_n$, where $A_f^n = r_n$ and $r_n(x) = r_n$ for all $x \in X$, $n \in \mathbb{N}$. Clearly for each $n \in \mathbb{N}$,

A_f^n is an open fuzzy set in (X, δ_n) but $A_f(s) = \{A_f^n\}_n$ is not an open fuzzy sequential set in $(X, \delta(s))$. ■

Definition 2.4 Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called quasi-coincident, denoted by $A_f(s)qB(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$, whenever A_f^n and B_f^n both are non zero. We write $A_f(s)\bar{q}B_f(s)$ to say that $A_f(s)$ and $A_f(s)$ are not quasi-coincident.

Definition 2.5 Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called weakly quasi-coincident, denoted by $A_f(s)q_wB_f(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$, for some $n \in \mathbb{N}$. We write $A_f(s)\overline{q_w}B_f(s)$ to mean that $A_f(s)$ and $B_f(s)$ are not weakly quasi-coincident.

Definition 2.6 A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is called quasi-coincident with $A_f(s) = \{A_f^n\}_n$, denoted by $P_f(s)qA_f(s)$ if and only if $P_f^n(x) > (A_f^n)^c(x)$ for all $n \in M$. If $P_f(s)$ is not quasi-coincident with $A_f(s)$, then we write $P_f(s)\bar{q}A_f(s)$.

Definition 2.7 A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is called weakly quasi-coincident with $A_f(s) = \{A_f^n\}_n$, denoted by $P_f(s)q_wA_f(s)$ if and only if $P_f^n(x) > (A_f^n)^c(x)$ for some $n \in M$. If $P_f(s)$ is not weakly quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q_w}A_f(s)$. If $P_f^n(x) > (A_f^n)^c(x)$ for some $n \in L \subseteq M$, then we say that $P_f(s)$ is weakly quasi-coincident with $A_f(s)$ at the sequential point (x, L).

Proposition 2.4 If the fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are quasi-coincident, then each pair of non zero fuzzy sets A_f^n and B_f^n is also so but the converse is not necessarily true.

Proof. Proof of the first part is omitted. For the second part, let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be fuzzy sequential sets on $\mathbb{R}$ where

$$A_f^1(x) = \tfrac{2}{3}, x \in (-\infty, 0), \quad = 1, x \in [0, \infty).$$

$$A_f^2(x) = \tfrac{1}{3}, x \in (-\infty, 0), \quad = \tfrac{2}{3}, x \in [0, \infty).$$

$$A_f^n(x) = \tfrac{3}{4}, x \in \mathbb{R} \text{ and } n \neq 1, 2.$$

$$B_f^1(x) = \tfrac{1}{2}, x \in (-\infty, 0), \quad = \tfrac{2}{3}, x \in [0, \infty).$$

$$B_f^2(x) = \tfrac{1}{4}, x \in (-\infty, 0), \quad = \tfrac{3}{7}, x \in [0, \infty).$$

$$B_f^n(x) = \tfrac{1}{2}, x \in \mathbb{R} \text{ and } n \neq 1, 2.$$

Clearly $A_f^n q B_f^n$ for all $n \in \mathbb{N}$ but $A_f(s)\bar{q}B_f(s)$. ■

Corollary 2.1 A fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is quasi-coincident with a fuzzy sequential set $A_f(s) = \{A_f^n\}_n$ if and only if P_f^n and A_f^n are so for each $n \in M$.
Proof. Proof is straightforward.

Definition 2.8 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a neighbourhood (in short nbd) of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) \in B_f(s) \leq A_f(s)$.

Definition 2.9 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a weak nbd of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) \in_w B_f(s) \leq A_f(s)$.

Definition 2.10 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a Q-nbd of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s)qB_f(s) \leq A_f(s)$.

Definition 2.11 A fuzzy sequential set $A_f(s)$ in an FSTS $(X, \delta(s))$ is called a weak Q-nbd of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s)q_w B_f(s) \leq A_f(s)$.

Proposition 2.5 $A_f(s) \leq_w (\leq) B_f(s)$ if and only if $A_f(s)$ and $(B_f(s))^c$ are not (weakly) quasi-coincident. In particular $P_f(s) \in_w (\in) A_f(s)$ if and only if $P_f(s)$ is not (weakly) quasi-coincident with $(A_f(s))^c$.
Proof. Proof is omitted. ■

Proposition 2.6 Let $\{A_{fj}(s), j \in J\}$ be a family of fuzzy sequential sets in X. Then a fuzzy sequential point $P_f(s)q_w(\bigvee_{j\in J} A_{fj}(s))$ if and only if $P_f(s)q_w A_{fj}(s)$ for some $j \in J$.
Proof. Let $P_f(s)q_w(\bigvee_{j\in J} A_{fj}(s))$ where $P_f(s) = (P_{fx}^M, r)$ and $A_{fj}(s) = \{A_{fj}^n\}_n$. This implies
$P_f^k(x) + S_f^k(x) > 1$ for some $n = k \in M$,
where $\bigvee_{j\in J} A_{fj}(s) = \{S_f^n\}_{\mathrm{n}}$. Therefore

$$S_f^k(x) = 1 - P_f^k(x) + \varepsilon_k \text{ where } \varepsilon_k > 0 \tag{1}$$

Also

$$S_f^k(x) - \varepsilon_k < A_{fj}^k(x) \text{ for some } j \in J \tag{2}$$

From (1) and (2) we have $P_f^k(x) + A_{fj}^k(x) > 1$, that is, $P_f(s)q_w A_{fj}(s)$ for some $j \in J$. Other implication is straightforward. ■

Corollary 2.2 If $P_f(s)qA_{fj}(s)$ for some $j \in J$, then $P_f(s)q(\bigvee_{j\in J} A_{fj}(s))$ where $\{A_{fj}(s), j \in J\}$ is a family of fuzzy sequential sets in X but not conversely.
Proof. Proof of the first part is omitted. For second part, let $A_{fj}(s) = \{A_{fj}^n\}_n$,

$j = 1,2$ be fuzzy sequential sets in $\mathbb{R}$, where

$$A^1_{f1}(x) = 0 \text{ for all } x \in \mathbb{R} - (0,1),$$
$$= \tfrac{1}{4} \text{ for all } x \in (0,1).$$

$$A^2_{f1}(x) = 0 \text{ for all } x \in \mathbb{R} - (\tfrac{1}{3}, \tfrac{2}{3}),$$
$$= \tfrac{2}{3} \text{ for all } x \in (\tfrac{1}{3}, \tfrac{2}{3})$$

$A^n_{f1}(x) = 0 = A^n_{f1}(x)$ for all $x \in \mathbb{R}$, $n \neq 1,2$.

$$A^1_{f2}(x) = 0 \text{ for all } x \in \mathbb{R} - (-\tfrac{1}{2}, 1),$$
$$= \tfrac{1}{3} \text{ for all } x \in (-\tfrac{1}{2}, 1),$$

$$A^2_{f2}(x) = 0 \text{ for all } x \in \mathbb{R} - (-\tfrac{1}{2}, 2),$$
$$= \tfrac{1}{5} \text{ for all } x \in \left(-\tfrac{1}{2}, 2\right).$$

The fuzzy sequential point $P_f(s) = (P^M_{f0.5}, r)$ where $M = \{1,2\}$ and $r = \{r_n\}_n$ with $r_1 = r_2 = \frac{7}{10}$, $r_n = 0$ for all $n \neq 1,2$ is quasi-coincident with $A_{f1}(s) \vee A_{f2}(s)$ but it is not so with any one of them. ■

Definition 2.12 A subfamily β of an FST $\delta(s)$ on X is called a base for $\delta(s)$ if and only if to every $A_f(s) \in \delta(s)$, there exists a subfamily $\{B_{fj}(s),\ j \in J\}$ of β such that $A_f(s) = \vee_{j \in J} B_{fj}(s)$.

Definition 2.13 A subfamily $S = \{S_{f\lambda}(s);\ \lambda \in \Lambda\}$ of an FST $\delta(s)$ on X is called a subbase for $\delta(s)$ if and only if $\{\wedge_{j \in J} S_{fj}(s);\ J = finite\ subset\ of\ \Lambda\}$ forms a base for $\delta(s)$.

Theorem 2.1 A subfamily β of an FST $\delta(s)$ on X is a base for $\delta(s)$ if and only if for each fuzzy sequential point $P_f(s)$ in $(X, \delta(s))$ and for every open weak Q nbd $A_f(s)$ of $P_f(s)$, there exists a member $B_f(s) \in \beta$ such that $P_f(s)\, q_w \mathrm{B}_f(s) \leq A_f(s)$.

Proof. The necessary part is straightforward. To prove its sufficiency, if possible let β be not a base for $\delta(s)$. Then there exists a member $A_f(s) \in \delta(s) - \beta$, such that $O_f(s) = \vee\{B_f(s) \in \beta;\ B_f(s) \leq A_f(s)\ and\ B_f(s) \neq A_f(s)\} \neq A_f(s)$, and hence there is an $x \in X$ and an $M \subset \mathbb{N}$ such that $O^n_f(x) < A^n_f(x)$ for all $n \in M$. Let $r = \{r_n\}_n$ where $r_n = 1 - O^n_f(x) > 0$ whenever $n \in M$ and $r_n = 0$ whenever $n \in \mathbb{N} - M$, then $A^n_f(x) + r_n > O^n_f(x) + r_n = 1$, for all $n \in M$ and $(P^M_{fx}, r) = P_f(s) q_w A_f(s)$. Therefore $A_f(s)$ is an open weak Q nbd of $P_f(s)$. Now

$B_f(s) = \{B^n_f\}_n \in \beta, B_f(s) \leq A_f(s)$

$\Rightarrow B_f(s) \leq O_f(s)$

$\Rightarrow B^n_f(x) + r_n \leq O^n_f(x) + r_n = 1$ for all $n \in M$

$\Rightarrow P_f(s) \overline{q_w} B_f(s)$ which is a contradiction. Hence the proof.■

Proposition 2.7 If β be a base for an FST $\delta(s)$ on X, then $\beta_n = \{B^n_f;\ B_f(s) = \{B^n_f\}_n \in \beta\}$ will form a base for the component FT δ_n on X for each $n \in \mathbb{N}$ but not conversely.

Proof. Proof of the first part is straightforward. For the converse part we consider the FSTS $(\mathbb{R}, \delta^{\mathbb{N}})$, where $\mathbb{R}$ is the set of real numbers and $\delta = \{r;\ r \in [0,1]\}$, $r(x) = r$ for all $x \in \mathbb{R}$, which is a FT on $\mathbb{R}$. Clearly $\beta_n = \{r;\ r \in (0,1) \cap \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers, is a base for the component FT $\delta_n^{\mathbb{N}}$ on X for each $n \in \mathbb{N}$ but $\beta(s) = \{X_f^r(s);\ r \in (0,1) \cap \mathbb{Q}\}$ is not a base for the FST $\delta^{\mathbb{N}}$ on X because $A_f(s) = \{A_f^n\}_n$ where $A_f^n = \overline{\frac{1}{n}}$ for all $n \in \mathbb{N}$ is an open fuzzy sequential set in $(\mathbb{R}, \delta^{\mathbb{N}})$, but cannot be written as a supremum of a subfamily of $\beta(s)$. ■

Definition 2.14 Let $A_f(s)$ be any fuzzy sequential set in an FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $A_f^\circ(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \le C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$A_f^\circ(s) = \vee\{O_f(s);\ O_f(s) \le A_f(s)),\ O_f(s) \in \delta(s)\}.$$

Proposition 2.8 If $A_f(s) = \{A_f^n\}_n$ in $(X, \delta(s))$, then $cl(A_f^n) \le A_f^n$ in (X, δ_n) for each $n \in \mathbb{N}$, where $cl(A_f^n)$ is the closure of A_f^n in (X, δ_n).

Proof. Proof is straightforward. ■

Here we cite an example where the equality in the proposition 2.8 does not hold. Let $X = [0,1]$ and $\delta(s) = \{X_f^r(s);\ r \in [0,1]\}$. If $A_f(s) = (P_{f\frac{1}{3}}^{\mathbb{N}}, r)$, $r = \{\overline{\frac{1}{2} - \frac{1}{3n}}\}_n$, then $\overline{A_f(s)} = X_f^{\frac{1}{2}}(s)$. Here $cl(A_f^n) = (\overline{\frac{1}{2} - \frac{1}{3n}})$, whereas $\overline{A_f^n} = \overline{\frac{1}{2}}$.

Definition 2.15 The dual of a fuzzy sequential point $P_f(s) = (P_{fx}^M, r)$ is a fuzzy sequential point $P_{df}(s) = (P_{fx}^M, t)$, where $r = \{r_n\}_n$, $t = \{t_n\}_n$ and

$$t_n = 1 - r_n \text{ for all } n \in M,$$
$$= 0 \text{ for all } n \in \mathbb{N} - M.$$

Theorem 2.2 Every Q nbd of a fuzzy sequential point $P_f(s)$ is weakly quasi-coincident with a fuzzy sequential set $A_f(s)$ implies $P_f(s) \in \overline{A_f(s)}$ implies every weak Q nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident.

Proof. Let $P_f(s) = (P_{fx}^M, r)$. $P_f(s) \in \overline{A_f(s)}$ if for every closed fuzzy sequential set $C_f(s) \ge A_f(s)$, $P_f(s) \in C_f(s)$, that is $p_f^n(x) \le C_f^n(x)$ for all $n \in M \Rightarrow P_f(s) \in \overline{A_f(s)}$ if for every open fuzzy sequential set $B_f(s) = \{B_f^n\}_n \le (A_f(s))^c$, $B_f^n(x) \le 1 - p_f^n(x)$ for all $n \in M$; that is $P_f(s) \in \overline{A_f(s)}$ if for every open fuzzy sequential set $B_f(s) = \{B_f^n\}_n$ satisfying $B_f^n(x) > 1 - P_f^n(x)$ for all $n \in M$, $B_f(s) \nleq (A_f(s))^c$, which implies the first part.

Now let $P_f(s) \in \overline{A_f(s)}$. If possible let there exists a weak Q nbd $N_f(s)$ of $P_f(s)$ such that $N_f(s)\overline{q_w}A_f(s)$. Then there exists an open fuzzy sequential set $B_f(s)$ such that $P_f(s)q_w B_f(s) \leq N_f(s)$. Now $N_f(s)\overline{q_w}A_f(s)$ and $B_f(s) \leq N_f(s) \Rightarrow B_f^n(x) + A_f^n(x) \leq 1$ for all $x \in X$, $n \in \mathbb{N} \Rightarrow A_f(s) \leq (B_f(s))^c \Rightarrow P_f(s) \in (B_f(s))^c \Rightarrow p_f^n(x) + B_f^n(x) \leq 1$ for all $n \in \mathbb{N}$. This contradicts the fact that $P_f(s)q_w B_f(s)$. Hence the result follows. ∎

Corollary 2.3 A fuzzy sequential point $P_f(s) \in \overline{A_f(s)}$ if and only if each nbd of its dual point $P_{df}(s)$ is weakly quasi-coincident with $A_f(s)$.

Proof. Proof is straightforward since Q nbd of a fuzzy sequential point is exactly the nbd of its dual point. ∎

Theorem 2.3 A fuzzy sequential point $P_f(s) \in A_f^\circ(s)$ if and only if its dual point $P_{df}(s) \notin \overline{(A_f(s))^c}$.

Proof. Let $P_f(s) \in A_f^\circ(s) \Rightarrow$ there exists an open fuzzy sequential set $B_f(s)$ such that $P_f(s) \in B_f(s) \leq A_f(s) \Rightarrow B_f(s)$ and $(A_f(s))^c$ are not weakly quasi-coincident $\Rightarrow P_{df}(s) \notin \overline{(A_f(s))^c}$. Conversely let $P_{df}(s) \notin \overline{(A_f(s))^c}$. Then there exists an open nbd $B_f(s)$ of $P_f(s)$ which is not weakly quasi-coincident with $(A_f(s))^c$

$$\Rightarrow P_f(s) \in B_f(s) \leq A_f(s)$$
$$\Rightarrow P_f(s) \in A_f^\circ(s). \blacksquare$$

Proposition 2.9 In an FSTS $(X, \delta(s))$, the following hold:
(i) $\overline{X_f^r(s)} = X_f^r(s)$, $r \in \{0,1\}$, (ii) $A_f(s)$ is closed if and only if $\overline{A_f(s)} = A_f(s)$, (iii) $\overline{\overline{A_f(s)}} = \overline{A_f(s)}$, (iv) $\overline{A_f(s) \vee B_f(s)} = \overline{A_f(s)} \vee \overline{B_f(s)}$, (v) $\overline{A_f(s) \wedge B_f(s)} \leq \overline{A_f(s)} \wedge \overline{B_f(s)}$, (vi) $(X_f^r(s))^\circ = X_f^r(s)$, $r \in \{0,1\}$, (vii) $A_f(s)$ is open if $A_f^\circ(s) = A_f(s)$, (viii) $(A_f^\circ(s))^\circ = A_f^\circ(s)$, (ix) $(A_f(s) \wedge B_f(s))^\circ = A_f^\circ(s) \wedge B_f^\circ(s)$, (x) $A_f^\circ(s) \vee B_f^\circ(s) \leq (A_f(s) \vee B_f(s))^\circ$, (xi) $A_f^\circ(s) = \overline{(A_f(s))^c}^c$, (xii) $\overline{A_f(s)} = ((A_f(s))^c)^o$, (xiii) $\overline{(A_f(s))^c} = ((A_f(s))^c)^\circ$, (xiv) $\overline{(A_f(s))^c} = (A_f^\circ(s))^c$.

Proof. Proof is straightforward. ∎

Definition 2.16 A fuzzy sequential point $P_f(s)$ is called an adherence point of a fuzzy sequential set $A_f(s)$ if and only if every weak Q nbd of $P_f(s)$ is weakly quasi-coincident with $A_f(s)$.

Definition 2.17 A fuzzy sequential point $P_f(s)$ is called an accumulation point of a fuzzy sequential set $A_f(s)$ if and only if $P_f(s)$ is an adherence point of $A_f(s)$ and every weak Q nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident at some fuzzy sequential point having different base or support from that of $P_f(s)$ whenever $P_f(s) \in A_f(s)$.

Proposition 2.10 Any reduced fuzzy sequential point of an accumulation point of a fuzzy sequential set is also an accumulation point of it.

Proof. Easy to prove. ■

From the proposition 2.10, we see that any simple reduced fuzzy sequential point of an accumulation point of a fuzzy sequential set is also an accumulation point of it but the converse is not true. For let $X = \{a, b\}$ and $\delta(s) = \{X_f^r(s),\ G_f(s);\ r \in \{0,1\}\}$, where $G_f(s) = \{G_f^n\}_n$, $G_f^n(a) = \frac{1}{2}$ and $G_f^n(b) = 0$ for all $n \in \mathbb{N}$. Let $A_f(s) = \{A_f^n\}_n$ where $A_f^n = \frac{\bar{2}}{3}$ for $n = 1,2$ and $A_f^n = 0$ otherwise. Then the fuzzy sequential point $P_f(s) = (P_{fa}^M, r)$ where $r = \{r_n\}_n$ with $r_1 = r_2 = \frac{2}{3}$ and $r_n = 0$ otherwise, is not an accumulation point of $A_f(s)$ though $(P_{fa}^1, \frac{2}{3})$ and $(P_{fa}^2, \frac{2}{3})$ both are accumulation point of $A_f(s)$.

Definition 2.18 The union of all accumulation points of a fuzzy sequential set $A_f(s)$ is called the fuzzy derived sequential set of $A_f(s)$ and it is denoted by $A_f^d(s)$.

Theorem 2.4 In an FSTS $(X, \delta(s))$, $\overline{A_f(s)} = A_f(s) \vee A_f^d(s)$.

Proof. Let $\Omega = \{P_f(s);\ P_f(s)\ is\ an\ adherence\ point\ of\ A_f(s)\}$. Then $\overline{A_f(s)} = \vee\Omega$. Now let $P_f(s) \in \Omega$, then two cases may arise, $P_f(s) \in A_f(s)$ or $P_f(s) \notin A_f(s)$. If $P_f(s) \notin A_f(s)$, then $P_f(s) \in A_f^d(s)$ and hence $P_f(s) \in A_f(s) \vee A_f^d(s)$. Therefore ,

$\overline{A_f(s)} = \vee\Omega \leq A_f(s) \vee A_f^d(s)$……..(1)

Again, $A_f(s) \leq \overline{A_f(s)}$ and since any accumulation point $P_f(s)$ of $A_f(s)$ belongs to $\overline{A_f(s)}$ which implies $A_f^d(s) \leq \overline{A_f(s)}$. Therefore,

$A_f(s) \vee A_f^d(s) \leq \overline{A_f(s)}$…………....(2).

From (1) and (2) the result follows.

Corollary 2.4 A fuzzy sequential set is closed in an FSTS $(X, \delta(s))$ if and only if it contains all its accumulation points.

Proof. Proof is straightforward.

Remark 2.1 The fuzzy derived sequential set of any fuzzy sequential set may not be closed as shown by example 2.1.

Example 2.1 Let $X = \{a, b\}$, $\delta(s)$ be the FST having base $\beta = \{X_f^1(s)\} \vee \{X_f^0(s)\} \vee \{P_f(s),\ G_f(s)\}$, where $G_f^n(b) = 1$, $G_f^n(a) = 0\ \ \forall\, n \in \mathbb{N}$ and $P_f(s) = (P_{fa}^M, r)$ where $M = \{1,2,3\}$, $r_1 = 0.5$, $r_2 = 1$, $r_3 = 0.3$, $r_n = 0\ \forall$ $n \neq 1,2,3$. Here the fuzzy derived sequential set of $(P_{fa}^3, 0.3)$ is not closed.

Proposition 2.11 The fuzzy derived sequential set of a fuzzy sequential point equals the union of the fuzzy derived sequential sets of all its simple reduced fuzzy sequential points.

Proof. The proof is omitted.

Proposition 2.12 If the fuzzy derived sequential set of each of the simple reduced fuzzy sequential points of a fuzzy sequential point is closed, then the derived sequential set of the fuzzy sequential point is closed.

Proof. Let $A_f(s) = (P_{fx}^M, r)$ be a fuzzy sequential point. Let $D_f(s)$ be the fuzzy derived sequential set of $A_f(s)$. Let $D_{nf}(s)$ be the fuzzy derived sequential set of $A_{nf}(s) = (p_{fx}^n,\ r_n)$, $n \in M$. Suppose $D_{nf}(s)$ is closed for all $n \in M$. Let $P_f(s)$ be an accumulation point of $D_f(s)$.

Now, $P_f(s) \notin D_f(s)$

$\Rightarrow P_f(s)$ is not an accumulation point of $A_f(s)$

$\Rightarrow\ \exists$ a weak Q-nbd $B_f(s)$ of $P_f(s)$ which is not weakly quasi coincident with $A_f(s)$

$\Rightarrow B_f(s)$ is not weakly quasi coincident with $(p_{fx}^n,\ r_n)\ \forall\, n \in M$

$\Rightarrow P_f(s) \notin D_{nf}(s)\ \forall\, n \in M$

$\Rightarrow P_f(s)$ is not an accumulation point of $D_{nf}(s)\ \forall\, n \in M$ (since $D_{nf}(s)$ is closed $\forall\, n \in M$)

$\Rightarrow P_f(s)$ is not an accumulation point of $\bigvee_{n\in M} D_{nf}(s) = D_f(s)$ a contradiction.

Hence proved. ■

Remark 2.2 Converse of proposition 2.12 is not true as shown by example 2.2.

Example 2.2 Let $X = \{a, b\}$, $\delta(s)$ be the FST having base $\beta = \{X_f^1(s)\} \vee \{X_f^0(s)\} \vee \{P_f(s),\ G_f(s)\}$, where $G_f^n(b) = 1$, $G_f^n(a) = 0\ \ \forall\, n \in \mathbb{N}$ and $P_f(s) = (P_{fa}^M, r)$ where $M = \{1,2,3\}$, $r_1 = 0.5$, $r_2 = 1$, $r_3 = 0.3$, $r_n = 0\ \forall n \neq 1,2,3$.

Here the fuzzy derived sequential set of $P_f(s)$ is closed but the fuzzy derived sequential set of $(P_{fa}^3, 0.3)$ is not closed.

We conclude the paper stating two necessary lemmas followed by a variant of Yang's Theorem in fuzzy sequential topological spaces.

Lemma 2.1 Let $A_f(s) = (P_{fx}^M, r)$ be a fuzzy sequential point in FSTS $(X, \delta(s))$. Then,

(i) For $y \neq x$, $\overline{A_f(s)}(y) = A_f^d(s)(y)$.

(ii) If $\overline{A_f(s)}(x) >_P r$, then $\overline{A_f(s)}(x) =_P A_f^d(s)(x)$, where $P \subset M$.

(iii) If $\overline{A_f(s)}(x) >_M r$, then $\overline{A_f(s)}(x) =_M A_f^d(s)(x)$.

(iv) If $A_f^d(s)(x) = 0 =$ sequence of real zeros, then $\overline{A_f(s)}(x) = r$.

(v) If $A_f(s)$ is simple then converse of (iv) is true.

Lemma 2.2 Let $A_f(s) = (P_{fx}^k, r_k)$ be a simple fuzzy sequential point in the FSTS $(X, \delta(s))$. Then,

(i) If $A_f^d(s)(x)$ is a non zero sequence, then $\overline{A_f(s)} = A_f^d(s)$.
(ii) If $A_f^d(s)(x) = 0 =$ sequence of real zeros, then If $A_f^d(s)$ is closed iff $\exists$ an open fuzzy sequential set $B_f^@(s)$ such that $B_f^@(s)(x) = 1$ and for $y \neq x$, $B_f^@(s)\,(y) = \{\overline{A_f(s)}\}^c(y) =$ $\{A_f^d(s)\}^c(y)$.
(iii) $A_f^d(s)(x) = 0 =$sequence of real zeros iff $\exists$ an open fuzzy sequential set $B_f(s)$ such that $B_f(s)(x) = 1 - r$ where $r = \{r_n\}_n$ and $r_n = 0$ if $n \neq k$, $r_n = r_k$ if $n = k$.

Theorem 2.5 The fuzzy derived sequential set of each fuzzy sequential set is closed iff the fuzzy derived sequential set of each simple fuzzy sequential point is closed.
Proof. The necessity is obvious. Conversely, suppose $H_f(s)$ is a fuzzy sequential set. We will show that $H_f^d(s) = D_f(s)$ is closed. Let $P_f(s) = (P_{fx}^k, r_k)$ be an accumulation point of $D_f(s)$. It is sufficient to show that $P_f(s) \in D_f(s)$. Let $r = \{r_n\}_n$ where $r_n = r_k$ for $n = k$ and $r_n = 0\ \forall\, n \neq k$. Now $P_f(s) \in \overline{D_f(s)} = \overline{H_f^d(s)} \leq \overline{\overline{H_f(s)}}$ $= \overline{H_f(s)}$. Therefore $P_f(s)$ is an adherence point of $H_f(s)$. If $P_f(s) \notin H_f(s)$, then $P_f(s)$ is an accumulation point of $H_f(s)$, that is $P_f(s) \in D_f(s)$ and we are done.
Let us assume $P_f(s) \in H_f(s)$

$$\Rightarrow r \leq H_f(s)(x) = \rho \text{ (say)}$$

$$\Rightarrow r_k \leq H_f^k(x) = \rho_k$$

Now consider the simple fuzzy sequential point $A_f(s) = (P_{fx}^k, \rho_k)$. Let $\rho' = \{\rho'_n\}_n$ where $\rho'_k = \rho_k$ and $\rho'_n = 0\ \forall\, n \neq k$. There are two possibilities concerning $A_f^d(s)$.
Case I. $A_f^d(s)(x) = \rho_1$ is a non zero sequence. Now
$\overline{A_f(s)}(x) \geq A_f(s)(x) = \rho'$
By lemma 2.1(v), $\overline{A_f(s)}(x) > \rho'$
$\Rightarrow A_f^d(s)(x) = \overline{A_f(s)}(x) > \rho'$

$$\Rightarrow \rho_1 > \rho'$$

$\Rightarrow \rho_{1k} > \rho_k = A_f^k(x) = H_f^k(x)$.

Hence the simple fuzzy sequential point $Q_f(s) = (p_{fx}^k, \rho_{1k}) \notin H_f(s)$ but since $Q_f(s) \in A_f^d(s) \leq \overline{A_f(s)} \leq \overline{H_f(s)}$, $Q_f(s)$ is an accumulation point of $H_f(s)$, that is $Q_f(s) \in D_f(s)$. Moreover $r_k \leq \rho_k < \rho_{1k}$
$\Rightarrow r_k < \rho_{1k}$
$\Rightarrow P_f(s) \in D_f(s)$.

Case II. $A_f^d(s)(x) = 0$. Let $B_f(s)$ be an arbitrary weak Q-nbd of $A_f(s)$ and hence of $P_f(s)$. In view of lemma 2.2(ii), $\exists$ an open fuzzy sequential set $B_f^@(s)$ such that $B_f^@(s)(x) = 1$ and for $y \neq x$, $B_f^@(s)\,(y) = \{\overline{A_f(s)}\}^c(y)$. Let Let $C_f(s) = B_f(s) \wedge B_f^@(s)$. Then $C_f(s)(x) = B_f(s)(x)$ which implies $C_f^k(x) = B_f^k(x) > 1 - r_k$. Thus $C_f(s)$ is a weak Q-nbd of $P_f(s)$. Hence $C_f(s)$ and $D_f(s)$ are weakly quasi coincident, that is $\exists$ a point z and $n \in \mathbb{N}$

such that $D_f^n(z) + C_f^n(z) > 1$. Owing to the fact that $D_f(s)$ is the union of all the accumulation points of $H_f(s)$, $\exists$ an accumulation point $P'_f(s) = (p_{fz}^n,\ \mu_n)$ such that $\mu_n + C_f^n(z) > 1$. Therefore $C_f(s)$ is a weak Q-nbd of $P'_f(s)$. Let $\mu = \{\mu_m\}_m$ where $\mu_n \neq 0$ and $\mu_m = 0$ for all $m \neq n$. The proof will be carried out, according to the following subcases:

Subcase I. When $n = k$.

(a) when $z = x$ and $\mu \leq \rho'$, then $P'_f(s) \in H_f(s)$. Since $P'_f(s)$ is an accumulation point of $H_f(s)$, every weak Q-nbd of $P'_f(s)$ (and hence $B_f(s)$) and $H_f(s)$ are weakly quasi coincident at some point having different base or different support than that of $P_f(s)$.

(b) When $z = x$ and $\mu > \rho'$, then $P'_f(s) \notin H_f(s)$. From lemma 2.2(iii), $\exists$ an open fuzzy sequential set $B'_f(s)$ such that $B'_f(s)(x) = 1 - \rho' > 1 - \mu$. Therefore $G_f(s) = C_f(s) \wedge B'_f(s)$ is also a weak Q-nbd of $P'_f(s)$. Hence $G_f(s)$ and $H_f(s)$ are weakly quasi coincident. Since $G_f(s)(x) \leq B'_f(s)(x) = 1 - \rho'$

$\Rightarrow G_f^k(x) \leq B'^k_f(x) = 1 - \rho_k$

Thus $G_f(s)$ (and hence $B_f(s)$) and $H_f(s)$ are weakly quasi coincident at some point having different base or different support than that of $P_f(s)$.

(c) When $z \neq x$.

We have $B_f^{@}(s)\ (z) = \{\overline{A_f(s)}\}^c(z)$. Also $\{\overline{A_f(s)}\}^c = ((A_f(s))^c)^\circ$. Since $((A_f(s))^c)^\circ(z) = B_f^{@}(s)\ (z) \geq C_f(s)(z)$ $\exists$ an open fuzzy sequential set $B''_f(s) \leq (A_f(s))^c$ such that $B''^k_f(z) \geq C_f^k(z) > 1 - \mu_k$. Therefore $G'_f(s) = B_f(s) \wedge B''_f(s)$ is also a weak Q-nbd $P'_f(s)$ and hence is weakly quasi coincident with $H_f(s)$. Since $B''_f(s) \leq (A_f(s))^c$

$\Rightarrow B''_f(s)(x) \leq 1 - A_f(s)(x)$

$\Rightarrow B''^k_f(x) \leq 1 - A_f^k(x) = 1 - H_f^k(x)$.

Thus $G'_f(s)$ (and hence $B_f(s)$) is weakly quasi coincident with $H_f(s)$ at some point having different base or different support than that of $P_f(s)$.

Subcase II. When $n \neq k$.

(a) Suppose $z = x$. We have $B'_f(s)(x) = 1 - \rho'$

$\Rightarrow B'^n_f(x) = 1 > 1 - \mu_n$.

So $B'_f(s)$ is a weak Q-nbd of $P'_f(s)$. Hence $G_f(s) = C_f(s) \wedge B'_f(s)$ is a weak Q-nbd of $P'_f(s)$ and so it is weakly quasi coincident with $H_f(s)$. Now $G_f(s)(x) \leq B'_f(s)(x) = 1 - \rho'$

$\Rightarrow G_f^k(x) \leq B'^k_f(x) = 1 - \rho_k = 1 - H_f^k(x)$.

So $H_f(s)$ and $G_f(s)$ are weakly quasi coincident at some point having different base or different support than that of $P_f(s)$.

(b) When $z \neq x$, the proof is same as Subcase I (c).

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Journal of **A**dvanced **S**tudies in **T**opology
*e*ISSN: 2090-388X
*p*ISSN: 2090-8288
Vol. 4, No. 1, 2013, 83–97

RESEARCH ARTICLE

Separation Axioms in Fuzzy Sequential Topological Spaces

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(*Received: 4 July 2012, Accepted: 11 October 2012*)

We develop the separation axioms in fuzzy sequential topological spaces and establish some results related to those axioms. Notions of various separation axioms in fuzzy sequential topological spaces are introduced and investigated the relations among them. Dependency of a component on another component of a fuzzy sequential topology plays the main role in this paper.

Keywords: Fuzzy sequential topological spaces; fs-T_0; fs-T_1; fs-Hausdorff; weakly fs-Hausdorff; fs-regular; weakly fs-regular; fs-normal and weakly fs-normal spaces.

AMS Subject Classification: 54A40.

1. Introduction and Preliminaries

In 1965, L. A. Zadeh introduced the concept of fuzzy sets [1] and fuzzy topology was introduced by C. L. Chang in 1968 [2]. A number of works on fuzzy topological spaces and fuzzy metric spaces have been appeared in the literature. In this paper, we study various separation axioms in fuzzy sequential topological spaces [3]. The key idea behind this work has been drawn from [4–6]. First we give some basic definitions, notations and results of [3] which will be used in the sequel.

Let X be a non empty set and $I = [0, 1]$ be the closed unit interval in the set $\mathbb{R}$ of real numbers. Let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be sequences of fuzzy sets in X called fuzzy sequential sets in X and we define:

(1) $A_f(s) \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (Union).
(2) $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (Intersection).
(3) $A_f(s) \leq B_f(s)$ if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers.
(4) $A_f(s) \leq_w B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$.
(5) $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$.
(6) $A_f(s)(x) = \{A_f^n(x)\}_n$, $x \in X$.
(7) $A_f(s)(x) \geq_M r$ if and only if $A_f^n(x) \geq r_n$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular, if $M = \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, we write $A_f(s)(x) \geq r$.
(8) $X_f^l(s) = \{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$.
(9) $A_f^c(s) = \{1 - A_f^n\}_n = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$.

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(10) a fuzzy sequential set $P_f(s) = \{p_f^n\}_n$ is called a fuzzy sequential point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that

$$p_f^n(t) = r_n, \text{ if } t = x,$$
$$= 0, \text{ if } t \in X - \{x\}, \text{ for all } n \in \mathbb{N}.$$

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$p_f^n(x) = r_n, \text{ whenever } n \in M,$$
$$= 0, \text{ whenever } n \in \mathbb{N} - M.$$

The point x is called the support, M is called base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (p_{fx}^M, r)$. If further $M = \{n\}$, $n \in \mathbb{N}$, then the fuzzy sequential point is called a simple fuzzy sequential point and it is denoted by (p_{fx}^n, r_n). A fuzzy sequential point is called complete if its base is the set of natural numbers. A fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$, symbolically $P_f(s) \in_w A_f(s)$ if and only if there exists $n \in M$ such that $p_f^n(x) \leq A_f^n(x)$. If $R \subseteq M$ and s is the sequence in I same to r in R and vanishes outside R then the fuzzy sequential point $P_{rf}(s) = (p_{fx}^R, s)$ is called a reduced fuzzy sequential point of $P_f(s) = (p_{fx}^M, r)$.

A sequence $(x, L) = \{A_n\}_n$ of subsets of X, where $A_n = \{x\}$, for all $n \in L$ and $A_n = \Phi =$ the null subset of X, for all $n \in \mathbb{N} - L$, is called a sequential point in X. A family $\delta(s)$ of fuzzy sequential sets on a non empty set X satisfying the following properties:

(1) $X_f^r(s) \in \delta(s)$ for all $r \in \{0, 1\}$,
(2) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and,
(3) for any family $\{A_{fj}(s) \in \delta(s),\ j \in J\}$, $\bigvee_{j \in J} A_{fj}(s) \in \delta(s)$,

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Compliment of an open fuzzy sequential set in X is called closed fuzzy sequential set in X. If $\delta_1(s)$ and $\delta_2(s)$ be two FSTs on X such that $\delta_1(s) \subset \delta_2(s)$, then we say that $\delta_2(s)$ is finer than $\delta_1(s)$ or $\delta_1(s)$ is weaker than $\delta_2(s)$. If δ be a fuzzy topology (FT) on X, then $\delta^{\mathbb{N}}$ forms a FST on X. We may construct different FSTs on X from a given FT δ on X, $\delta^{\mathbb{N}}$ is the finest of all these FSTs. Not only that, any FT δ on X can be considered as a component of some FST on X, one of them is $\delta^{\mathbb{N}}$, there are at least countably many FSTs on X weaker than δ^N of which δ is a component. One of them is $\delta'(s) = \{A_f^n(s) = \{A_f^n\}_n;\ A_f^n = A \text{ for all } n \in \mathbb{N} \text{ and } A \in \delta\}$.

If $(X, \delta(s))$ is a FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where $\delta_n = \{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$ and (X, δ_n) is called the n^{th} component FTS of the FSTS $(X, \delta(s))$. Let $A_f^n(s)$=$\{A_f^n\}_n$ be an open (closed) fuzzy sequential set in the FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in (X, δ_n) but the converse is not necessarily true.

Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called quasi-coincident, denoted by $A_f(s)qB_f(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$, whenever A_f^n and B_f^n both are not $\overline{0}$. We write $A_f(s)\overline{q}B_f(s)$ to say that $A_f(s)$ and $B_f(s)$ are not quasi-coincident. Fuzzy sequential sets $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ are called weakly quasi-coincident, denoted by $A_f(s)q_wB_f(s)$ if and only if there exists $x \in X$ such that $A_f^n(x) > (B_f^n)^c(x)$ for some $n \in \mathbb{N}$. We write $A_f(s)\overline{q}_wB_f(s)$ to mean that $A_f(s)$ and $B_f(s)$ are not weakly quasi-coincident.

A fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ is called quasi-coincident with $A_f(s) = \{A_f^n\}_n$, denoted by $P_f(s)qA_f(s)$ if and only if $P_f^n(x) > (A_f^n)^c(x)$ for all $n \in M$. If $P_f(s) = (p_{fx}^M, r)$ is not quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q}A_f(s)$. A fuzzy sequential point $P_f(s) = (p^M_{fx}, r)$ is called weakly quasi-coincident with $A_f(s) = \{A^n_f\}_n$, denoted by $P_f(s)q_wA_f(s)$ if and only if $P^n_f(x) > (A^n_f)^c(x)$ for some $n \in M$. If $P_f(s) = (p^M_{fx}, r)$ is not weakly quasi-coincident with $A_f(s)$, then we write $P_f(s)\overline{q}_wA_f(s)$. If $P^n_f(x) > (A^n_f)^c(x)$ for some $n \in L \subseteq M$, then we say that $P_f(s)$ is weakly quasi-coincident with $A_f(s)$ at the sequential point (x, L). If the fuzzy sequential sets $A_f(s) = \{A^n_f\}_n$ and $B_f(s) = \{B^n_f\}_n$ are quasi-coincident, then each pair of non $\overline{0}$ fuzzy sets A^n_f and B^n_f is also so but the converse is not necessarily true.

The fuzzy sequential point $P_f(s) = (p^M_{f0.5}, r)$ where $M = \{1, 2\}$, $r = \{r_n\}_n$ and $r_1 = r_2 = \frac{7}{10}$ is quasi-coincident with $A_{f1}(s) \vee A_{f2}(s)$ but it is not so with any one of them. A subfamily β of a FST $\delta(s)$ on X is called a base for $\delta(s)$ if and only if to every $A_f(s) \in \delta(s)$, there exists a subfamily $\{B_{fj}(s), j \in J\}$ of β such that $A_f(s) = \bigvee_{j\in J} B_{fj}(s)$. A subfamily $S = \{S_{f\lambda}(s);\ \lambda \in \Lambda\}$ of a FST $\delta(s)$ on X is called a subbase for $\delta(s)$ if and only if $\{\bigwedge_{j\in J} S_{fj}(s);\ J = \text{finite subset of } \Lambda\}$ forms a base for $\delta(s)$.

A subfamily β of a fuzzy sequential topology $\delta(s)$ on X is a base for $\delta(s)$ if and only if for each fuzzy sequential point $P_f(s)$ in $(X, \delta(s))$ and for every open weak Q nbd $A_f(s)$ of $P_f(s)$, there exists a member $B_f(s) \in \beta$ such that $P_f(s)q_wB_f(s) \leq A_f(s)$. If β be a base for the FST $\delta(s)$ on X, then $\beta_n = \{B^n_f;\ B_f(s) = \{B^n_f\}_n \in \beta\}$ forms a base for the component fuzzy topology δ_n on X for each $n \in \mathbb{N}$ but not conversely.

Let $A_f(s)$ be any fuzzy sequential set in a FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \bigwedge\{C_f(s);\ A_f(s) \leq C_f(s), C^c_f(s) \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \bigvee\{O_f(s);\ O_f(s) \leq A_f(s),\ O_f(s) \in \delta(s)\}.$$

If $\overline{A_f(s)} = \{\overline{A^n_f}\}_n$ in $(X, \delta(s))$, then $cl(A^n_f) \leq \overline{A^n_f}$ in (X, δ_n) for each $n \in \mathbb{N}$, where $cl(A^n_f)$ is the closure of A^n_f in (X, δ_n). The dual of a fuzzy sequential point $P_f(s) = (p^M_{fx}, r)$ is a fuzzy sequential point $P_{df}(s) = (p^M_{fx}, t)$, where $r = \{r_n\}_n$, $t = \{t_n\}_n$ and

$$\begin{aligned} t_n &= 1 - r_n \text{ for all } n \in M, \\ &= 0 \text{ for all } n \in \mathbb{N} - M. \end{aligned}$$

Every Q nbd of a fuzzy sequential point $P_f(s)$ is weakly quasi-coincident with a fuzzy sequential set $A_f(s)$ implies $P_f(s) \in \overline{A_f(s)}$ implies every weak Q nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident. A fuzzy sequential point $P_f(s) \in \overset{o}{A}_f(s)$ if and only if its dual point $P_{df}(s) \notin \overline{A^c_f(s)}$. In a FSTS $(X, \delta(s))$, the following hold:

(1) $\overline{X^r_f(s)} = X^r_f(s)$, $r \in \{0, 1\}$.
(2) $A_f(s)$ is closed if and only if $\overline{A_f(s)} = A_f(s)$.
(3) $\overline{\overline{A_f(s)}} = \overline{A_f(s)}$.
(4) $\overline{A_f(s) \vee B_f(s)} = \overline{A_f(s)} \vee \overline{B_f(s)}$.
(5) $\overline{A_f(s) \wedge B_f(s)} \subseteq \overline{A_f(s)} \wedge \overline{B_f(s)}$.
(6) $(X^r_f(s))^o = X^r_f(s)$, $r \in \{0,\ 1\}$.
(7) $A_f(s)$ is open if and only if $\overset{o}{A}_f(s) = A_f(s)$.
(8) $(\overset{o}{A}_f(s))^o = \overset{o}{A}_f(s)$.
(9) $(A_f(s) \wedge B_f(s))^o = \overset{o}{A}_f(s) \wedge \overset{o}{B}_f(s)$.

(10) $(A_f(s) \vee B_f(s))^o = \overset{o}{A}_f(s) \vee \overset{o}{B}_f(s)$.

(11) $\overset{o}{A}_f(s) = \overline{(\overline{A^c_f(s)})}^c$.

(12) $\overline{A_f(s)} = \overline{(A^c_f(s))^o}$.

(13) $(\overline{A_f(s)})^c = (A^c_f(s))^o$.

(14) $\overline{(A^c_f(s))} = (\overset{o}{A}_f(s))^c$.

A fuzzy sequential point $P_f(s)$ is called an adherence point of a fuzzy sequential set $A_f(s)$ if and only if every weak Q-nbd of $P_f(s)$ is weakly quasi-coincident with $A_f(s)$. A fuzzy sequential point $P_f(s)$ is called an accumulation point of a fuzzy sequential set $A_f(s)$ if and only if $P_f(s)$ is an adherence point of $A_f(s)$ and every weak Q-nbd of $P_f(s)$ and $A_f(s)$ are weakly quasi-coincident at some sequential point having different base or support from that of $P_f(s)$ whenever $P_f(s) \in A_f(s)$. Any reduced sequential point of an accumulation point of a fuzzy sequential set is also an accumulation point of it. The union of all accumulation points of a fuzzy sequential set $A_f(s)$ is called the fuzzy derived sequential set of $A_f(s)$ and it is denoted by $A^d_f(s)$.

In a FSTS $(X, \delta(s))$, $\overline{A_f(s)} = A_f(s) \vee A^d_f(s)$. A fuzzy sequential set is closed in a FSTS $(X, \delta(s))$ if and only if it contains all its accumulation points. The fuzzy derived sequential set of a fuzzy sequential point equals the union of the fuzzy derived sequential sets of all its simple reduced fuzzy sequential points. If the fuzzy derived sequential set of each of the reduced fuzzy sequential points of a fuzzy sequential point is closed, then the derived sequential set of the fuzzy sequential point is closed.

Let $A_f(s) = (p^k_{fx}, r)$ be a fuzzy sequential point in FSTS $(X, \delta(s))$, then:

(1) For $y \neq x$, $\overline{A_f(s)}(y) = A^d_f(s)(y)$.

(2) If $\overline{A_f(s)}(x) >_P r$, $\overline{A_f(s)}(x) =_P A^d_f(s)(x)$, where $P \subset M$.

(3) If $\overline{A_f(s)}(x) >_M r$, $\overline{A_f(s)}(x) = A^d_f(s)(x)$.

(4) If $A^d_f(s)(x) = 0 =$ sequence of real zeros, then $\overline{A_f(s)}(x) = r$.

(5) If $A_f(s)$ is simple then converse of (iv) is true.

Let $A_f(s) = (p^k_{fx}, r_k)$ be a simple fuzzy sequential point in FSTS $(X, \delta(s))$. Then:

(1) If $A^d_f(s)(x)$ is a non zero sequence, then $\overline{A_f(s)} = A^d_f(s)$.

(2) If $A^d_f(s)(x) = 0 =$sequence of real zeros, then $A^d_f(s)$ is closed iff there exists an open fuzzy sequential set $B^@_f(s)$ such that $B^@_f(s)(x) = 1$ and for $y \neq x$, $B^@_f(s)(y) = \{\overline{A_f(s)}\}^c(y) = \{A^d_f(s)\}^c(y)$.

(3) $A^d_f(s)(x) = 0 =$sequence of real zeros iff there exists an open fuzzy sequential set $B_f(s)$ such that $B_f(s)(x) = 1 - r$ where $r = \{r_n\}_n$ and $r_n = 0$ if $n \neq k$, $r_n = r_k$ if $n = k$.

It is observed that fuzzy derived sequential set of each fuzzy sequential set is closed if and only if the fuzzy derived sequential set of each simple fuzzy sequential point is closed. Books [5, 7–9] may provide a suitable background for the present work.

2. Main Definitions and Results

Definition 2.1 Two fuzzy sequential points $P_f(s) = (p^M_{fx}, r)$ and $Q_f(s) = (p^N_{fy}, t)$ are said to be identical if $x = y$, $M = N$ and $r = t$; otherwise they are distinct.

Definition 2.2 A set $M \subset \mathbb{N}$ is said to be base of a fuzzy sequential set $U_f(s) = \{U^n_f\}^\infty_{n=1}$ if $U^n_f \neq \overline{0}$ $\forall n \in M$ and $U^n_f = \overline{0}$ $\forall n \in \mathbb{N} - M$.

Definition 2.3 A fuzzy sequential set $B_f(s) = \{B^n_f\}^\infty_{n=1}$ (having base N) is said to be completely contained in a fuzzy sequential set $A_f(s) = \{A^n_f\}^\infty_{n=1}$ (having base M) if $M = N$ and $B^n_f \leq A^n_f$ for all $n \in \mathbb{N}$.

Definition 2.4 A fuzzy sequential set $B_f(s) = \{B^n_f\}^{\infty}_{n=1}$ (having base N) is said to be totally reduced from the fuzzy sequential set $A_f(s) = \{A^n_f\}^{\infty}_{n=1}$ (having base M) if $N \subsetneqq M$ and $B^n_f \leq A^n_f$ for all $n \in N$.

Definition 2.5 A FSTS $(X, \delta(s))$ is said to be fs-T_0 space if for any two distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$, there exists a weak Q-nbd of one of $P_f(s)$ and $Q_f(s)$ which is not weakly quasi coincident with the other.

Theorem 2.6 A FSTS $(X, \delta(s))$ is fs-T_0 space iff for every pair of distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$ either $P_f(s)$ does not belong to the closure of $Q_f(s)$ or $Q_f(s)$ does not belong to the closure of $P_f(s)$.

Proof Suppose $(X, \delta(s))$ is fs-T_0. Then there exists a weak Q-nbd $U_f(s)$ of $P_f(s)$ which is not weakly quasi coincident with $Q_f(s)$. This implies that $P_f(s) \notin \overline{Q_f(s)}$. Conversely, suppose $P_f(s)$ and $Q_f(s)$ be any two distinct fuzzy sequential points such that $P_f(s) \notin \overline{Q_f(s)}$. This implies that exists a weak Q-nbd of $P_f(s)$ which is not weakly quasi coincident with $Q_f(s)$. Hence $(X, \delta(s))$ is fs-T_0. ■

Corollary 2.1 A FSTS $(X, \delta(s))$ is fs-T_0 space iff distinct fuzzy sequential points have distinct closures.

Theorem 2.7 A FTS (X, δ) is fuzzy T_0 iff the FSTS $(X, \delta^{\mathbb{N}})$ is fs-T_0.

Proof Suppose (X, δ) is fuzzy T_0. Let $P_f(s) = (p^M_{fx}, r)$ and $Q_f(s) = (p^N_{fy}, t)$ be any two distinct fuzzy sequential points where $r = \{r_n\}^{\infty}_{n=1}$ and $t = \{t_n\}^{\infty}_{n=1}$.

Case I: Suppose $x \neq y$. Then for $p^{r_m}_x \neq p^{t_m}_y$ $(m \in M)$ there exists a Q-nbd U of $p^{r_m}_x$ which is not quasi coincident with $p^{t_m}_x$.

Case II: Suppose $x = y$, $M \cap N = \phi$. Then for $p^{r_m}_x \neq p^{t_m}_x$ $(m \in M)$ there exists a Q-nbd U of $p^{r_m}_x$ which is not quasi coincident with $p^{t_m}_x$.

Case III: Suppose $x = y$, $N \subset M$. If $r_m \neq t_m$ for some $m \in N$, then for $p^{r_m}_x \neq p^{t_m}_x$ there exists a Q-nbd U of $p^{r_m}_x$ which is not quasi coincident with $p^{t_m}_x$. If $r_n = t_n \ \forall \ n \in N$, then for $p^{r_m}_x \neq p^{t_m}_x$ $(m \in M - N)$ there exists a Q-nbd U of $p^{r_m}_x$ which is not quasi coincident with $p^{t_m}_x$.

Case IV: Suppose $x = y$ and neither $N \subset M$ nor $M \subset N$ nor $M \cap N = \phi$. Then for $p^{r_m}_x \neq p^{t_m}_x$ $(m \in M,\ m \notin N)$ there exists a Q-nbd U of $p^{r_m}_x$ which is not quasi coincident with $p^{t_n}_x$.

In all the above cases, the fuzzy sequential set $U_f(s) = \{U^n_f\}^{\infty}_{n=1}$ where $U_m = U$ and $U_n = \overline{0}\ \forall n \neq m$, is a weak Q-nbd of $P_f(s)$ which is not weakly quasi coincident with $Q_f(s)$.

Conversely, suppose $(X,\ \delta^{\mathbb{N}})$ is fs-T_0. Let p^{λ}_x and p^{μ}_y be any two distinct fuzzy points in X. Then for each $n \in \mathbb{N}$, fuzzy sequential points $(p^n_{fx},\ \lambda)$ and $(p^n_{fy},\ \mu)$ are distinct. So exists a weak Q-nbd $U_f(s) = \{U^n_f\}^{\infty}_{n=1}$ of one of $(p^n_{fx},\ \lambda)$ and $(p^n_{fy},\ \mu)$ which is not weakly quasi coincident with the other. This implies U^n_f is a Q-nbd of one of p^{λ}_x and p^{μ}_y which is not quasi coincident with the other. ■

Theorem 2.8 If a FSTS $(X, \delta(s))$ is fs-T_0, then the FTS (X, δ_n) where $\delta_n = \{A^n_f;\ A_f(s) = \{A^n_f\}^{\infty}_{n=1} \in \delta(s)\}$ is fuzzy T_0 for each $n \in \mathbb{N}$.

Proof Let p^{λ}_x and p^{μ}_y be any two distinct fuzzy points in X. Then for each $n \in \mathbb{N}$, fuzzy sequential points (p^n_{fx}, λ) and (p^n_{fy}, μ) are distinct. So there exists a weak Q-nbd $U_f(s) = \{U^n_f\}^{\infty}_{n=1}$ of one of (p^n_{fx}, λ) and (p^n_{fy}, μ) which is not weakly quasi coincident with the other. This implies U^n_f is a Q-nbd of one of p^{λ}_x and p^{μ}_y which is not quasi coincident with the other. ■

Remark 2.9 Converse of Theorem 2.8 is not true as shown by Example 2.10.

Example 2.10 Let (X, δ) be a FTS. For any $A \in \delta$ let $B_{fA}(s) = \{B^n_{fA}\}^{\infty}_{n=1}$, $C_{fA}(s) = \{C^n_{fA}\}^{\infty}_{n=1}$ and $D_{fA}(s) = \{D^n_{fA}\}^{\infty}_{n=1}$, where $B^n_{fA} = A$ for odd n, $B^n_{fA} = \overline{0}$ for even n, $C^n_{fA} = \overline{0}$ for odd n, $C^n_{fA} = A$ for even n and $D^n_{fA} = A$ for all $n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$ for all $A \in \delta$ forms a FST on X. If (X, δ) is fuzzy T_0, then the components of $(X, \delta(s))$ are fuzzy T_0 but $(X, \delta(s))$ is not fs-T_0.

Definition 2.11 Suppose $U_f(s) = \{U_f^n\}_{n=1}^{\infty}$ and $V_f(s) = \{V_f^n\}_{n=1}^{\infty}$ are two fuzzy sequential sets. If there exists an $M \subset \mathbb{N}$ such that $U_f^n q V_f^n$ for all $n \in M$, we say that $U_f(s)$ is M-quasi coincident with $V_f(s)$ and we write $U_f(s) q^M V_f(s)$. If $U_f^n q V_f^n$ for at least one $n \in M$, we say that $U_f(s)$ is weakly M-quasi coincident with $V_f(s)$ and we write $U_f(s) q_w^M V_f(s)$.

Definition 2.12 A FSTS $(X, \delta(s))$ is said to be a fs-T_1 space if every fuzzy sequential point in X is closed.

Remark 2.13 A fs-T_1 space is fs-T_0.

Theorem 2.14 A FTS (X, δ) is fuzzy T_1 iff the FSTS $(X, \delta^{\mathbb{N}})$ is fs-T_1.

Proof Proof is omitted. ■

Theorem 2.15 If a FSTS $(X, \delta(s))$ is fs-T_1, then the component FTS (X, δ_n) is fuzzy T_1 for each $n \in \mathbb{N}$.

Proof Proof is omitted. ■

Remark 2.16 Converse of Theorem 2.15 is not true as shown by Example 2.17.

Example 2.17 Let (X, δ) be a FTS. For any $A \in \delta$ let $B_{fA}(s) = \{B_{fA}^n\}_{n=1}^{\infty}$, $C_{fA}(s) = \{C_{fA}^n\}_{n=1}^{\infty}$ and $D_{fA}(s) = \{D_{fA}^n\}_{n=1}^{\infty}$, where $B_{fA}^n = A$ for odd n, $B_{fA}^n = \overline{0}$ for even n, $C_{fA}^n = \overline{0}$ for odd n, $C_{fA}^n = A$ for even n and $D_{fA}^n = A$ for all $n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$ for all $A \in \delta$ forms a FST on X. If (X, δ) is fuzzy T_1 then the components of $(X, \delta(s))$ are fuzzy T_1 but $(X, \delta(s))$ is not fs-T_1.

Theorem 2.18 A FSTS $(X, \delta(s))$ is fs-T_1 iff for each $x \in X$ and each sequence $r = \{r_n\}_{n=1}^{\infty}$ in $[0, 1]$, there exists $B_f(s) \in \delta(s)$ such that $B_f(s)(x) = \{1 - r_n\}_{n=1}^{\infty}$ and $B_f(s)(y) = \{1\}_{n=1}^{\infty}$ for $y \neq x$.

Proof Suppose $(X, \delta(s))$ is fs-T_1. If r is a zero sequence, then it is sufficient to take $B_f(s) = X_f^1(s)$. Suppose r is a non zero sequence. Let $M \subset \mathbb{N}$ such that $r_n \neq 0$ for all $n \in M$ and $r_n = 0$ for all $n \in \mathbb{N} - M$. Then $P_f(s) = (p_{fx}^M, r)$ is a fuzzy sequential point in X and $B_f(s) = X_f^1(s) - P_f(s)$ is the required open fuzzy sequential set.

Conversely, suppose $P_f(s) = (p_{fx}^M, r)$ is an arbitrary fuzzy sequential point in X. By hypothesis, there exists $B_f(s) \in \delta(s)$ such that $B_f(s)(x) = 1 - r$ and $B_f(s)(y) = 1$ for $y \neq x$. It follows that $P_f(s)$ is the complement of $B_f(s)$ and hence is closed. ■

Theorem 2.19 The fuzzy derived sequential set of every fuzzy sequential set on a fs-T_1 space is closed.

Proof The fuzzy derived sequential set of a fuzzy sequential point in a fs-T_1 space, itself being a fuzzy sequential point is closed. Hence the result follows from [6, Theorem 2.5]. ■

Definition 2.20 A FSTS $(X, \delta(s))$ is said to be fs-Hausdorff space or fs-T_2 space if for any two distinct fuzzy sequential points $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w^{M-N} U_f(s),\ Q_f(s) q_w V_f(s),\ P_f(s) \overline{q}_w^{M-N} \overline{V_f(s)},\ Q_f(s) \overline{q}_w \overline{U_f(s)}$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$, otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w U_f(s),\ Q_f(s) q_w V_f(s),\ P_f(s) \overline{q}_w \overline{V_f(s)},\ Q_f(s) \overline{q}_w \overline{U_f(s)}.$$

Definition 2.21 A FSTS $(X, \delta(s))$ is said to be weak fs-Hausdorff space or (w) fs-Hausdorff space if for any two distinct fuzzy sequential points $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$

such that

$$P_f(s) \in_w^{M-N} U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_w V_f(s)$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$, otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in_w U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_w V_f(s).$$

Theorem 2.22 A fs-Hausdorff space is a weak fs-Hausdorff space.

Proof Proof is omitted. ∎

Remark 2.23 Example 2.24 shows that a weak fs-Hausdorff space may not fs-Hausdorff space.

Example 2.24 Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$. Then $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff but not fs-Hausdorff.

Remark 2.25 A fs-T_2 space may not be fs-T_1, shown by Example 2.26.

Example 2.26 Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p_x^1, p_y^1\}$. Let $(X, \delta^{\mathbb{N}})$ is fs-T_2 but not fs-T_1.

Definition 2.27 A FSTS $(X, \delta(s))$ is said to be (w) fs-T_2 space if it is (w) fs- Hausdorff and fs-T_1.

Remark 2.28 A fs-T_2 space is weak fs-T_2.

Theorem 2.29 A FSTS $(X, \delta(s))$ is said to be fs-Hausdorff if for any two distinct fuzzy sequential points $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in^{M-N} G_f(s),\ Q_f(s)\ q_w H_f(s),\ G_f(s)\overline{q_w} H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X,\delta(s))$ such that

$$P_f(s) q_w^{M-N} D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s)\overline{q_w} D_f(s)$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$, otherwise there exist open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in G_f(s),\ Q_f(s) q_w H_f(s),\ G_f(s)\overline{q_w} H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s)\overline{q_w} D_f(s).$$

Proof Let $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other. Suppose $(X,\ \delta(s))$ is fs-Hausdorff.

Case I: Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. Then there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w U_f(s),\ Q_f(s) q_w V_f(s),\ P_f(s)\overline{q}_w \overline{V_f(s)},\ Q_f(s)\overline{q}_w \overline{U_f(s)}.$$

Now if we take $G_f(s) = X_f^1(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X_f^1(s) - \overline{U_f(s)}$, we are done.

Case II: Suppose one of $P_f(s)$ and $Q_f(s)$, say $Q_f(s)$ is totally reduced from $P_f(s)$. Then there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s)q_w^{M-N}U_f(s),\ Q_f(s)q_wV_f(s),\ P_f(s)\overline{q}_w^{M-N}\overline{V_f(s)},\ Q_f(s)\overline{q}_w\overline{U_f(s)}.$$

Now if we take $G_f(s) = X_f^1(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X_f^1(s) - \overline{U_f(s)}$, we are done.

Conversely, suppose the given conditions are true. In both the cases, if we take $U_f(s) = D_f(s)$ and $V_f(s) = H_f(s)$, we are done. ■

Theorem 2.30 A FSTS $(X,\ \delta(s))$ is fs-Hausdorff iff for any fuzzy sequential point $P_f(s)$ in X,

$$P_f(s) = \bigwedge\{\overline{N_f(s)} : N_f(s) \text{ is a nbd of } P_f(s)\}. \tag{1}$$

Proof Suppose $(X, \delta(s))$ is fs-Hausdorff. Let $P_f(s) = (p_{fx}^M, r)$ be any fuzzy sequential point in X and $Q_f(s) = (p_{fx}^N, t)$ be another fuzzy sequential point distinct from $P_f(s)$ and $Q_f(s) \notin P_f(s)$.

If $P_f(s)$ is totally reduced from $Q_f(s)$, there exists open fuzzy sequential set $V_f(s)$ in $(X, \delta(s))$ such that $Q_f(s)\ q_w^{N-M}V_f(s)$, $P_f(s)\overline{q}_w\overline{V_f(s)}$, otherwise there exists open fuzzy sequential set $V_f(s)$ in $(X, \delta(s))$ such that $Q_f(s)q_wV_f(s)$, $P_f(s)\overline{q}_w\overline{V_f(s)}$.

In both cases, if we take $U_f(s) = X_f^1(s) - \overline{V_f(s)}$, then $P_f(s) \in U_f(s)$ and $Q_f(s) \notin \overline{U_f(s)}$. Hence (1) is true.

Conversely, suppose (1) is true. Let $P_f(s) = (p_{fx}^M, r)$ and $Q_f(s) = (p_{fy}^N, t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other.

Case I: Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. By (1), there exist nbds $S_f(s)$ and $T_f(s)$ of $P_f(s)$ and $Q_f(s)$ respectively such that $P_f(s) \notin \overline{T_f(s)}$ and $Q_f(s) \notin \overline{S_f(s)}$. If we take $U_f(s) = X_f^1(s) - \overline{T_f(s)}$ and $V_f(s) = X_f^1(s) - \overline{S_f(s)}$, we are done.

Case II: Suppose one of $P_f(s)$ and $Q_f(s)$, $Q_f(s)$ (say) is totally reduced from $P_f(s)$. Then there exists nbds $S_f(s)$ and $T_f(s)$ of $P_f^{'}(s)$ and $Q_f(s)$ respectively such that $P_f(s) \notin^{M-N} \overline{T_f(s)}$ and $Q_f(s) \notin \overline{S_f(s)}$, where $P_f^{'}(s)$ is a reduced fuzzy sequential point of $P_f(s)$ with base $M - N$. If we take $U_f(s) = X_f^1(s) - \overline{T_f(s)}$ and $V_f(s) = X_f^1(s) - \overline{S_f(s)}$, we are done. ■

Theorem 2.31 If a FTS (X, δ) is fuzzy T_2, then the FSTS $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff.

Proof Proof is omitted. ■

Remark 2.32 Converse of Theorem 2.31 is not true as shown by Example 2.33.

Example 2.33 Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p_x^1, p_y^1\}$. Then $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff even though (X, δ) is not fuzzy T_2.

Remark 2.34 Example 2.35 shows that even if $(X,\ \delta)$ is fuzzy T_2, the FSTS $(X, \delta^{\mathbb{N}})$ may not be fs-Hausdorff.

Example 2.35 Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$. Then (X, δ) is fuzzy T_2 but $(X, \delta^{\mathbb{N}})$ is not fs-Hausdorff.

Remark 2.36 Example 2.37 shows that if a FSTS $(X, \delta(s))$ is fs-T_2, then the component FTS (X, δ_n) may not be fuzzy T_2 for each $n \in \mathbb{N}$.

Example 2.37 Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p_x^1, p_y^1\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then $(X, \delta(s))$ is fs-T_2 but $(X, \delta_1) = (X, \delta)$ is not fuzzy T_2.

Remark 2.38 Example 2.39 shows that even if all the component fuzzy topological spaces of a FSTS are fuzzy T_2, the FSTS may not be fs-T_2.

Example 2.39 Let (X, δ) be a FTS. For any $G \in \delta$, let $A_{fG}(s) = \{A^n_{fG}\}^\infty_{n=1}$, $B_{fG}(s) = \{B^n_{fG}\}^\infty_{n=1}$, $C_{fG}(s) = \{C^n_{fG}\}^\infty_{n=1}$ where $A^n_{fG} = G$ for odd n, $A^n_{fG} = \overline{0}$ for even n, $B^n_{fG} = \overline{0}$ for odd n, $B^n_{fG} = G$ for even n, $C^n_{fG} = G$ for all n. Then the collection $\delta(s)$ of all fs-sets(fuzzy sequential sets) $A_{fG}(s)$, $B_{fG}(s)$, $C_{fG}(s)$ for all $G \in \delta$ forms a fuzzy sequential topology on X. If (X, δ) is fuzzy T_2, then the components of $(X, \delta(s))$ are fuzzy T_2 but $(X, \delta(s))$ itself is not fs-T_2.

Definition 2.40 A FSTS $(X, \delta(s))$ is said to be fs-regular if for any fuzzy sequential point $P_f(s) = (p^M_{fx}, r)$ and a non zero closed fuzzy sequential set $A_f(s)$ such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q^{M-N}_w U_f(s),\ A_f(s) q_w V_f(s),\ P_f(s) \overline{q}^{M-N}_w \overline{V_f(s)},\ A_f(s) \leq X^1_f(s) - \overline{U_f(s)}$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and having base N, otherwise there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) q_w U_f(s),\ A_f(s) q_w V_f(s),\ P_f(s) \overline{q}_w \overline{V_f(s)},\ A_f(s) \leq X^1_f(s) - \overline{U_f(s)}.$$

Definition 2.41 A FSTS $(X, \delta(s))$ is said to be weak fs-regular if for any fuzzy sequential point $P_f(s) = (p^M_{fx}, r)$ and its any open weak nbd $A_f(s)$, there exists a fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in^{M-N}_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A^c_f(s)$ is totally reduced from $P_f(s)$ and having base N, otherwise there exists open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

and $A_f(s)$ is a nbd of $B_f(s)$.

Remark 2.42 Example 2.43 shows that a fs-regular space may not be weak fs-regular.

Example 2.43 Let $X = \{a\}$, $\delta = \{\overline{0}, \overline{1}, p^{0.4}_a\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but not weak fs-regular.

Remark 2.44 A weak fs-regular space may not be fs-regular as shown by Example 2.45.

Example 2.45 Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$, then $(X, \delta^{\mathbb{N}})$ is weak fs-regular but is not fs-regular.

Remark 2.46 A fs-regular space may not be fs-T_1. This is shown by Example 2.47.

Example 2.47 Let $X = \{x\}$ and let $\delta = \{\overline{0}, \overline{1}, p^{0.5}_x\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but not fs-T_1.

Definition 2.48 A FSTS $(X, \delta(s))$ is said to be fs-T_3 if it is fs-regular and fs-T_1.

Remark 2.49 A fs-T_3 space is fs-T_2.

Theorem 2.50 A FSTS $(X, \delta(s))$ is fs-regular iff for any fuzzy sequential point $P_f(s) = (p^M_{fx}, r)$ and a non zero closed fuzzy sequential set $A_f(s)$ such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, there exist open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in^{M-N} G_f(s),\ A_f(s) q_w H_f(s),\ G_f(s) \overline{q_w} H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s)q_w^{M-N}D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\overline{q_w}D_f(s)$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in G_f(s),\ A_f(s)q_wH_f(s),\ G_f(s)\overline{q_w}H_f(s)$$

and there exist open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s)q_wD_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\overline{q_w}D_f(s).$$

Proof Proof is omitted. ■

Theorem 2.51 A FSTS $(X, \delta(s))$ is fs-regular iff for any fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ and an open fuzzy sequential set $G_f(s)$ such that $P_f(s)q_wG_f(s)$ (where $X_f^1(s) - G_f(s)$ is not completely contained in $P_f(s)$), there exists an open fuzzy sequential set $H_f(s)$ in $(X,\ \delta(s))$ such that $P_f(s) \in^{M-N} H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s)\ q_w^{M-N}B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$, whenever $X_f^1(s) - G_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists an open fuzzy sequential set $H_f(s)$ in $(X,\ \delta(s))$ such that $P_f(s) \in H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and $\exists$ an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s)\ q_wB_f(s)$, $\overline{B_f(s)} \leq G_f(s)$.

Proof Suppose $(X, \delta(s))$ is fs-regular. Let $P_f(s) = (p_{fx}^M, r)$ be any fuzzy sequential point and $G_f(s)$ be an open fuzzy sequential set such that $P_f(s)q_wG(s)$, i.e., $P_f(s) \notin X_f^1(s) - G_f(s) = A_f(s)$ (say). Then there exists open fuzzy sequential sets $U(s)$ and $V(s)$ in (X, τ) such that

$$P_f(s) \in^{M-N} U_f(s),\ A_f(s)q_wV_f(s),\ U_f(s)\overline{q_w}V_f(s)$$

and there exists open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s)q_w^{M-N}D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\overline{q_w}D_f(s)$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in U_f(s),\ A_f(s)q_wV_f(s),\ U_f(s)\overline{q_w}V_f(s)$$

and there exists open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s)q_wD_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\overline{q_w}D_f(s).$$

If we take $H_f(s) = U_f(s)$ and $B_f(s) = D_f(s)$, we are done.

Conversely, suppose given conditions are true. Let $P_f(s) = (p_{fx}^M, r)$ be any fuzzy sequential point and $A_f(s)$ be any closed fuzzy sequential set such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, i.e., $P_f(s)q_wX_f^1(s) - A_f(s) = G_f(s)$ (say). Then there exists an open fuzzy sequential set $H_f(s)$ in $(X, \delta(s))$ such that $P_f(s) \in^{M-N} H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s)q_w^{M-N}B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$, whenever $X_f^1(s) - G_f(s)$ is totally reduced from $P_f(s)$ and with base N, otherwise there exists an open fuzzy sequential set $H_f(s)$ in $(X, \delta(s))$ such that $P_f(s) \in H_f(s)$, $\overline{H_f(s)} <_w X_f^1(s) - G_f(s)$ and there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s) q_w B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$. If we take $U_f(s) = H_f(s)$, $V_f(s) = X_f^1(s) - \overline{H_f}(s)$, $D_f(s) = B_f(s)$ and $E_f(s) = X_f^1(s) - \overline{B_f}(s)$, then we are done. ■

Theorem 2.52 If $(X, \delta(s))$ is fs-regular, then for any closed fuzzy sequential set $A_f(s)$ which is not a fuzzy sequential point,

$$A_f(s) = \bigwedge \{N_f(s),\ N_f(s) \text{ is a closed nbd of } A_f(s)\}. \tag{2}$$

Proof Suppose $(X, \delta(s))$ is fs-regular and $A_f(s)$ be any closed fuzzy sequential set which is not a fuzzy sequential point. If $A_f(s) = X_f^0(s)$, then (1) is true. Suppose $A_f(s) \neq X_f^0(s)$. Let $P_f(s)$ be any fuzzy sequential point such that $P_f(s) \notin A_f(s)$. Let M and N be the bases of $P_f(s)$ and $A_f(s)$, respectively. We have $P_f(s) \notin A_f(s)$ i.e.,

$$P_f(s) q_w X_f^1(s) - A_f(s) = G_f(s) \quad \text{(say)}.$$

then there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s) q_w^{M-N} B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$ whenever $A_f(s)$ is totally reduced from $P_f(s)$, otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that $P_f(s) q_w B_f(s)$, $\overline{B_f(s)} \leq G_f(s)$. This implies $A_f(s) \leq X_f^1(s) - \overline{B_f(s)} = H_f(s)$ (say). Again

$$P_f(s) \notin X_f^1(s) - B_f(s) \Longrightarrow P_f(s) \notin \overline{H_f(s)}.$$

Thus (1) holds. ■

Remark 2.53 Example 2.54 shows that converse of Theorem 2.52 may not be true.

Example 2.54 Let X be any non empty set and $\delta = \{\overline{r},\ r \in [0, 1]\}$ Then the FSTS $(X, \delta^{\mathbb{N}})$ is not regular although for any closed fuzzy sequential set $A_f(s)$ in $(X, \delta^{\mathbb{N}})$, $A_f(s) = \wedge\{N_f(s);\ N_f(s)$ is a closed nbd of $A_f(s)\}$.

Remark 2.55 Example 2.56 shows that for a fuzzy sequential point, (1) in Theorem 2.52 may not hold.

Example 2.56 Let $X = \{x\}$ and $\delta = \{\overline{1}, \overline{0}, p_x^{0.2}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but for the closed fuzzy sequential point $A_f(s) = (p_{fx}^{\{1,\ 2\}}, 0.8) \neq \wedge\{N_f(s);\ N_f(s)$ is a closed nbd of $A_f(s)\}$.

Theorem 2.57 A FTS (X, δ) is fuzzy regular iff $(X, \delta^{\mathbb{N}})$ is weak fs-regular.

Proof Proof is omitted. ■

Remark 2.58 Even if $(X, \delta^{\mathbb{N}})$ is fs-regular, (X, δ) may not be fuzzy regular, shown by Example 2.59.

Example 2.59 Let $X = \{a\}$, $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-regular but (X, δ) is not fuzzy regular.

Remark 2.60 A FTS (X, δ) is fuzzy regular, it may not imply $(X, \delta^{\mathbb{N}})$ is fs-regular as shown by Example 2.61.

Example 2.61 Let $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0, 1]\}$, then (X, δ) is fuzzy regular but $(X, \delta^{\mathbb{N}})$ is not fs-regular.

Remark 2.62 A FSTS $(X, \delta(s))$ is fs-regular, it may not imply component fuzzy topological spaces (X, δ_n), $n \in \mathbb{N}$ is fuzzy regular as shown by Example 2.63.

Example 2.63 Let $X = \{a\}$, $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then $(X, \delta(s))$ is fs-regular but $(X, \delta_n) = (X, \delta)$ for all $n \in \mathbb{N}$ is not fuzzy regular.

Remark 2.64 A FSTS $(X, \delta(s))$ may not be fs-regular even if the component fuzzy topological spaces (X, δ_n) is fuzzy regular for all $n \in \mathbb{N}$, as shown by Example 2.65.

Example 2.65 Let $X = \{x, y\}$, $\delta = \{\overline{r},\, r \in [0,1]\}$. Let $\delta(s) = \delta^{\mathbb{N}}$. Then $(X, \delta(s))$ is not be fs-regular but all the component fuzzy topological spaces (X, δ_n), $n \in \mathbb{N}$, is fuzzy regular.

Definition 2.66 Fuzzy sequential sets $A_f(s)=\{A_f^n\}_{n=1}^{\infty}$ and $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ are said to be quasi discoincident if A_f^n and B_f^n are strong quasi discoincident for all n.

Definition 2.67 Fuzzy sequential sets $A_f(s) = \{A_f^n\}_{n=1}^{\infty}$ and $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ are said to be partially quasi discoincident if A_f^n and B_f^n are strong quasi discoincident for some $n \in \mathbb{N}$..

Definition 2.68 A FSTS $(X, \delta(s))$ is said to be fs-normal iff for any two partially quasi discoincident non zero closed fuzzy sequential sets $A_f(s)$ and $B_f(s)$ (having the respective bases M and N and none of which is completely contained in the other), there exists an open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) q_w^{M-N} U_f(s),\ B_f(s) q_w V_f(s),\ A_f(s) \leq^{M-N} X_f^1(s) - \overline{V_f(s)},\ B_f(s) \leq X_f^1(s) - \overline{U_f(s)}$$

whenever $B_f(s)$ is totally reduced from $A_f(s)$, otherwise there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$A_f(s) q_w U_f(s),\ B_f(s) q_w V_f(s),\ A_f(s) \leq X_f^1(s) - \overline{V_f(s)},\ B_f(s) \leq X_f^1(s) - \overline{U_f(s)}.$$

Definition 2.69 A FSTS $(X, \delta(s))$ is said to be weak fs-normal iff for any non zero closed fuzzy sequential set $C_f(s)$ and its any open weak nbd $A_f(s)$, there exists a fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$C_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is totally reduced from $C_f(s)$, otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X,\ \delta(s))$ such that

$$C_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

($A_f(s)$ is a nbd of $B_f(s)$, M and N being the respective bases of $C_f(s)$ and $A_f^c(s)$).

Remark 2.70 A fs-normal FSTS may not be weak fs-normal, which is shown by Example 2.71.

Example 2.71 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-normal but not weak fs-normal.

Remark 2.72 Example 2.73 shows that a weak fs-normal space may not be fs-normal.

Example 2.73 Let $X = \{x, y\}$ and $\beta = \{p_x^r;\ r \in [0,1]\} \vee \{p_y^r;\ r \in [0,1]\}$ be a base for some fuzzy topology δ on X. Then $(X, \delta^{\mathbb{N}})$ is weak fs-normal but not fs-normal.

Definition 2.74 A FSTS $(X, \delta(s))$ is said to be fs-T_4 space if it is fs-normal and fs-T_1.

Remark 2.75 A fs-normal FSTS may not be fs-T_1 as shown by Example 2.76.

Example 2.76 Let $X = \{a, b\}$, $\delta(s) = \{X_f^0(s), X_f^1(s), A_f(s), B_f(s)\}$ where $A_f^n(a) = 1$ for all n, $A_f^n(b) = 0$ for all n, $B_f^n(a) = 0$ for all n, $B_f^n(b) = 1$ for all n, then $(X, \delta(s))$ is fs-normal but not fs-T_1.

Remark 2.77 A fs-normal FSTS may not be fs-regular as shown by Example 2.78.

Example 2.78 Let $X = \{x, y\}$, $\delta = \{\overline{r},\, r \in [0,1]\}$, then $(X, \delta^{\mathbb{N}})$ is fs-normal but not fs-regular. Hence a fs-T_4 space may not be fs-T_3.

Theorem 2.79 A FSTS $(X, \delta(s))$ is fs-normal iff for any two partially quasi discoincident non zero closed fuzzy sequential sets $A_f(s)$ and $B_f(s)$ (none of which is completely contained in the other), there exists an open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) \in^{M-N} G_f(s),\ B_f(s) q_w H_f(s),\ G_f(s) \overline{q_w} H_f(s)$$

and there exists an open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) q_w^{M-N} D_f(s),\ B_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s)$$

whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$), otherwise there exists open fuzzy sequential sets $G_f(s)$ and $H_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) \in G_f(s),\ B_f(s) q_w H_f(s),\ G_f(s) \overline{q_w} H_f(s)$$

and there exists an open fuzzy sequential sets $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) q_w D_f(s),\ B_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s).$$

Proof Proof is omitted. ■

Theorem 2.80 If a FSTS $(X, \delta(s))$ is weak fs-normal, then for any two non-zero closed partially quasi discoincident fs-sets $A_f(s)$ and $B_f(s)$ (none of which is completely contained in the other), there exists open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that $A_f(s) \in_w^{M-N} U_f(s)$, $B_f(s) \in_w V_f(s)$, $U_f(s) \overline{q_w} V_f(s)$ whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$), otherwise there exists an open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that $A_f(s) \in_w G_f(s)$, $B_f(s) \in_w V_f(s)$, $U_f(s) \overline{q_w} V_f(s)$.

Proof The proof is omitted. ■

Remark 2.81 For a FSTS to be weak fs-normal, the condition given in Theorem 2.80 is only necessary but not sufficient as shown by Example 2.82.

Example 2.82 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is not weak fs-normal but the condition in Theorem 2.80 is satisfied.

Theorem 2.83 A weak fs-regular space $(X, \delta(s))$ is weak fs-normal when X is finite.

Proof Let $(X, \delta(s))$ be a weak fs-regular space. Let $C_f(s) = \{C_f^n\}_{n=1}^{\infty}$ be any non zero closed fuzzy sequential set in $(X, \delta(s))$ and $A_f(s)$ be its any open weak nbd. Let M and N be respectively the bases of $C_f(s)$ and $A_f^c(s)$. We choose $m \in M - N$ when $A_f^c(s)$ is totally reduced from $C_f(s)$ and we take $m \in M$ otherwise. Let $x \in X$ such that $C_f^m(x) \neq 0$ and let $C_f^m(x) = r_m$. Then for the fuzzy sequential point $p_{xf}(s) = (p_{f_x}^m,\ r_m)$, $A_f(s)$ is an open weak nbd. Hence there exists an open fuzzy sequential set $B_{xf}(s)$ in $(X, \delta(s))$ such that

$$p_{xf}(s) \in_w^{M-N} \overset{o}{B}_{xf}(s) \leq \overline{B_{xf}(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is a totally reduced fuzzy sequential set from $C_f(s)$, otherwise there exists open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$p_{xf}(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s) \quad (A_f(s) \text{ is a nbd of } B_{xf}(s)).$$

Corresponding to each $x \in X$ for which $C_f^m(x) \neq 0$, we get such open fs-set $B_{xf}(s)$. Since X is finite, there exists finitely many fs-sets say

$$B_{x_1 f}(s), B_{x_2 f}(s), \ldots, B_{x_k f}(s)$$

such that

$$p_{x_n f}(s) \in_w^{M-N} \overset{o}{B}_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s), \quad x_n \in X, \quad n = 1, 2, \ldots, k.$$

whenever $A_f^c(s)$ is a totally reduced fuzzy sequential set from $C_f(s)$, otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X, \delta(s))$ such that

$$p_{x_n f}(s) \in_w \overset{o}{B}_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s), \quad x_n \in X, \quad n = 1, 2, \ldots, k.$$

Now, let $B_f(s) = \bigcup_{n=1}^{k} B_{x_n f}(s)$. Then

$$C_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is totally reduced from $C_f(s)$, otherwise

$$C_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s).$$

Hence $(X, \delta(s))$ is weak fs-normal. ■

Theorem 2.84 A FTS (X, δ) is fuzzy normal iff $(X, \delta^{\mathbb{N}})$ is weak fs-normal.

Proof Proof is omitted. ■

Remark 2.85 Even if $(X, \delta^{\mathbb{N}})$ is fs-normal, (X, δ) may not be fuzzy normal, shown by Example 2.86.

Example 2.86 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Then $(X, \delta^{\mathbb{N}})$ is fs-normal but (X, δ) is not fuzzy normal.

Remark 2.87 Example 2.88 shows that if (X, δ) is fuzzy normal, it may not imply $(X, \delta^{\mathbb{N}})$ is fs-normal.

Example 2.88 Let $X = \{x, y\}$ and $\beta = \{p_x^r;\ r \in [0,1]\} \vee \{p_y^r;\ r \in [0,1]\}$ be a base for some fuzzy topology δ on X. Then (X, δ) is fuzzy normal but $(X, \delta^{\mathbb{N}})$ is not fs-normal.

Remark 2.89 If $(X, \delta(s))$ is fs-normal, then it may not imply (X, δ_n) is fuzzy normal for each n, shown by Example 2.90.

Example 2.90 Let $X = \{a\}$ and $\delta = \{\overline{0}, \overline{1}, p_a^{0.6}\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then $(X, \delta(s))$ is fs-normal but $(X, \delta_n) = (X, \delta)$ for all $n \in \mathbb{N}$, is not fuzzy normal.

Remark 2.91 A FSTS $(X, \delta(s))$ may not be fs-normal even if the component fuzzy topological spaces (X, δ_n) is fuzzy regular for all $n \in \mathbb{N}$, as shown by Example 2.92.

Example 2.92 Let $X = \{x, y\}$ and $\beta = \{p_x^r;\ r \in [0,1]\} \vee \{p_y^r;\ r \in [0,1]\}$ be a base for some fuzzy topology δ on X. Let $\delta(s) = \delta^{\mathbb{N}}$. Then $(X, \delta(s))$ is not be fs-normal but all the component fuzzy topological spaces (X, δ_n), $n \in \mathbb{N}$, is fuzzy normal.

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Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 6, No. 3, (November 2013), pp. 589–603
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

FS-closure operators and FS-interior operators

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Received 7 January 2013; Revised 26 February 2013; Accepted 25 March 2013

Abstract. Notions of FS-closure operators, FS-interior operators and their components are introduced. Various properties of FS-closure systems and FS-interior systems are studied and established a relation between them. A set of necessary and sufficient conditions under which an FS-closure operator and an FS-interior operator induce same fuzzy sequential topology on the underlying set have been obtained.

2010 AMS Classification: 54A40, 03E72

Keywords: Fuzzy sequential topological space, FS-closure operator, FS-interior operator, FS-closure system, FS-interior system.

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1. Introduction

After the introduction of fuzzy sets by L. A. Zadeh in 1965 ([17]), C. L. Chang introduced the concept of fuzzy topology on a non empty set in 1968 ([6]). The concept of fuzzy sequential topological spaces (FSTS) were introduced in ([13]). In fuzzy set theory, fuzzy closure operators and fuzzy closure systems have been studied by Mashour and Ghanim ([10]), G. Gerla ([8]), Bandler and Kohout ([1]), R. Belohlavek ([2]), whereas fuzzy interior operators and fuzzy interior systems have appeared in the studies of R. Belohlavek and T. Funiokova ([3]), Bandler and Kohout ([1]).

Closure and interior operators on an ordinary set belong to the very fundamental mathematical structures with direct applications on the many fields like topology, logic etc. Being motivated by the importance of closure and interior operators, we introduce the concept of FS-closure and FS-interior operators on a set. Books ([5], [7] [9], [11]) and the articles ([4], [12], [14], [15], [16]) may provide a suitable background for the present work as some basic ideas have been derived from these sources. We begin with some basic definitions and results of ([13]) and ([16]). Let X be a non empty set and $I = [0, 1]$ be the closed unit interval in the set of real numbers. Let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be sequences of fuzzy sets in X called fuzzy sequential sets in X and we define

(i) $A_f(s) \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (union),
(ii) $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (intersection),
(iii) $A_f(s) \leq B_f(s)$ if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$,
(iv) $A_f(s) \leq_w B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$,
(v) $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$,
(vi) $A_f(s)(x) = \{A_f^n(x)\}_n$, $x \in X$,
(vii) $A_f(s)(x) \geq_M r$ if and only if $A_f^n(x) \geq r_n$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular if $M = \mathbb{N}$, we write $A_f(s)(x) \geq r$,
(viii) $X_f^l(s) = \{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$,
(ix) $(A_f(s))^c = \{1 - A_f^n\}_n = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$,
(x) A fuzzy sequential set $P_f(s) = \{p_f^n\}_n$ is called a fuzzy sequential point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that

$$\begin{aligned} p_f^n(t) &= r_n, \text{ if } t = x, \\ &= 0, \text{ if } t \in X - \{x\}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$\begin{aligned} p_f^n(x) &= r_n, \text{ whenever } n \in M, \\ &= 0, \text{ whenever } n \in \mathbb{N} - M. \end{aligned}$$

The point x is called the support, M is called the base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (p_{fx}^M, r)$. If further $M = \{n\}$, $n \in \mathbb{N}$, then the fuzzy sequential point is called a simple fuzzy sequential point and it is denoted by (p_{fx}^n, r_n). A fuzzy sequential point is called complete if its base is the set of natural numbers. A fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$, symbolically $P_f(s) \in_w A_f(s)$, if and only if there exists $n \in M$ such that $p_f^n(x) \leq A_f^n(x)$.

Definition 1.1 ([13]). A family $\delta(s)$ of fuzzy sequential sets on a non empty set X satisfying the properties

(i) $X_f^r(s) \in \delta(s)$ for $r = 0$ and 1,
(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and
(iii) for any family $\{A_{fj}(s) \in \delta(s),\ j \in J\}$, $\underset{j \in J}{\vee} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 1.2 ([13]). If $(X, \delta(s))$ is an FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where $\delta_n = \{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$. (X, δ_n), where $n \in \mathbb{N}$, is called the n^{th} component FTS of the FSTS $(X, \delta(s))$.

Proposition 1.3 ([13]). *Let $A_f(s)=\{A_f^n\}_n$ be an open (closed) fuzzy sequential set in the FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in (X, δ_n) but the converse is not necessarily true.*

Proposition 1.4 ([13]). *If δ be a fuzzy topology (FT) on a non empty set X, then $\delta^{\mathbb{N}}$ forms an FST on X.*

Definition 1.5 ([13]). Let $A_f(s)$ be any fuzzy sequential set in an FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\, A_f(s) \leq C_f(s),\, (C_f(s))^c \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \vee\{O_f(s);\, O_f(s) \leq A_f(s),\, O_f(s) \in \delta(s)\}.$$

Definition 1.6 ([13]). A fuzzy sequential set $A_f(s)$ in an FSTS $(X,\, \delta(s))$ is called a neighbourhood (in short nbd) of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) \in B_f(s) \leq A_f(s)$. A nbd $A_f(s)$ is called open if and only if $A_f(s) \in \delta(s)$.

2. Definition and results

Definition 2.1. Let X be a non empty set. An operator $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-closure operator on X if it satisfies the following conditions:

(FSC1) $\mathbf{Cl}(X_f^0(s)) = X_f^0(s)$.

(FSC2) $A_f(s) \leq \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

(FSC3) $\mathbf{Cl}(\mathbf{Cl}(A_f(s))) = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

(FSC4) $\mathbf{Cl}(A_f(s) \vee B_f(s)) = \mathbf{Cl}(A_f(s)) \vee \mathbf{Cl}(B_f(s))$ for all $A_f(s),\, B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.2. For any FSTS $(X, \delta(s))$, closure of an fs-set (fuzzy sequential set) is an FS-closure operator on X.

Example 2.3. Let X be a non empty set. The operator $\mathbf{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $\mathbf{C}(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\mathbf{C}(X_f^0(s)) = X_f^0(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X, is an FS-closure operator on X.

Theorem 2.4. *If $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then*

(i) $\boldsymbol{Cl}$ is monotonic increasing, that is, $A_f(s) \leq B_f(s) \Rightarrow \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(B_f(s))$ for all $A_f(s),\, B_f(s) \in (I^X)^{\mathbb{N}}$.

(ii) $A_f(s) \leq \boldsymbol{Cl}(B_f(s)) \Rightarrow \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(B_f(s))$ for all $A_f(s),\, B_f(s) \in (I^X)^{\mathbb{N}}$.

Proof. Proof is omitted. □

Theorem 2.5. *Let X be a non empty set and $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an operator on X satisfying $(FSC1)$, $(FSC2)$ and $(FSC4)$, then*

a) The collection $\delta'(\mathrm{s}) = \{(A_f(s))^c;\, A_f(s) \in (I^X)^{\mathbb{N}}$ and $\boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ forms an FST on X.

b) If $\boldsymbol{Cl}$ also satisfies $(FSC3)$, then for all $A_f(s) \in (I^X)^{\mathbf{N}}$ we have $\overline{A_f(s)} = \boldsymbol{Cl}(A_f(s))$, where $\overline{A_f(s)}$ is the closure of $A_f(s)$ in $\delta'(s)$.

Proof. Proof is omitted. □

Remark 2.6. From **Theorem** 2.5 it follows that if $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X then $\delta'(s) = \{(A_f(s))^c;\ A_f(s) \in (I^X)^{\mathbb{N}}$ and $\mathbf{Cl}(A_f(s)) = A_f(s)\}$ forms an FST on X. Also $\overline{A_f(s)} = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbf{N}}$, where $\overline{A_f(s)}$ is the closure of $A_f(s)$ in $\delta'(s)$. This FST $\delta'(s)$ is called the fuzzy sequential topology induced by the FS-closure operator $\mathbf{Cl}$ and we denote it by $\delta_{\mathbf{Cl}}(s)$.

Remark 2.7. **Example** 2.8 shows that if an operator $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a non empty set X, satisfies $(FSC1)$, $(FSC2)$ and $(FSC4)$ but does not satisfy $(FSC3)$, then $\delta_{\mathbf{Cl}}(s)$ forms an FST on X but $\overline{A_f(s)}$ may not be equal to $\mathbf{Cl}(A_f(s))$, $A_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.8. Let $X = \{a\}$. Let $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be defined by

$$\mathbf{Cl}(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}.$$

Then $\mathbf{Cl}$ is an operator on X satisfying $(FSC1)$, $(FSC2)$ and $(FSC4)$ and hence $(X, \delta_{\mathbf{Cl}}(s))$ forms an FSTS. Further $\mathbf{Cl}$ does not satisfy $(FSC3)$ and in $(X, \delta_{\mathbf{Cl}}(s))$, $\mathbf{Cl}(B_f(s)) \neq \overline{B_f(s)}$ if $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ where $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}\ \forall\ n \neq 1, 2, 3$

Definition 2.9. Let X be a non empty set and $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. A function $(\mathbf{Cl})_f^n \colon I^X \to I^X$ defined by $(\mathbf{Cl})_f^n(A) = n^{th}$ term of $\mathbf{Cl}({}_{nA}X_f^0(s))$, where ${}_{nA}X_f^0(s)$ denotes an fs-set whose n^{th} term is A and others are $\overline{0}$, is called the n^{th} component of $\mathbf{Cl}$, $n \in \mathbb{N}$.

Theorem 2.10. *Let X be a non empty set. If $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then each component $(\boldsymbol{Cl})_f^n \colon I^X \to I^X$, $n \in \mathbb{N}$ is a fuzzy closure operator. Also $(\delta_{\boldsymbol{Cl}})_n = \delta_{(\boldsymbol{Cl})_f^n}$ where $(\delta_{\boldsymbol{Cl}})_n$ is the n^{th} component fuzzy topology of FST $\delta_{\boldsymbol{Cl}}(s)$ and $\delta_{(\boldsymbol{Cl})_f^n}$ is the fuzzy topology induced by the component $(\boldsymbol{Cl})_f^n$ of $\boldsymbol{Cl}$.*

Proof. $(\mathbf{Cl})_f^n(\overline{0}) = \overline{0}$ by definition. Let $A \in I^X$, then ${}_{nA}X_f^0(s) \leq \mathbf{Cl}({}_{nA}X_f^0(s)) \Rightarrow A \leq (\mathbf{Cl})_f^n(A)$. Hence $(\mathbf{Cl})_f^n(A) \leq (\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A))$. Also

$$\begin{aligned} & \mathbf{Cl}(\mathbf{Cl}({}_{nA}X_f^0(s))) = \mathbf{Cl}({}_{nA}X_f^0(s)) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(\mathbf{Cl})_f^n(A)}X_f^0(s)) \leq \mathbf{Cl}({}_{nA}X_f^0(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A)) \leq (\mathbf{Cl})_f^n(A) \end{aligned}$$

Hence $(\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A)) = (\mathbf{Cl})_f^n(A)$.
Again let A, B $\in I^X$, then

$$\begin{aligned} & \mathbf{Cl}({}_{nA}X_f^0(s) \vee_{nB} X_f^0(s)) = \mathbf{Cl}({}_{nA}X_f^0(s)) \vee \mathbf{Cl}({}_{nB}X_f^0(s)) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(A \vee B)}X_f^0(s)) = \mathbf{Cl}({}_{nA}X_f^0(s)) \vee \mathbf{Cl}({}_{nB}X_f^0(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n(A \vee B) = (\mathbf{Cl})_f^n(A) \vee (\mathbf{Cl})_f^n(B) \end{aligned}$$

Thus $(\mathbf{Cl})_f^n$ is a fuzzy closure operator.
For the next part, Let A$\in (\delta_{\mathbf{Cl}})_n$, then $\overline{1}$-A is a closed fuzzy set in $(X, (\delta_{\mathbf{Cl}})_n)$. Let

$B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ be a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$ such that $B_f^n = \overline{1} - A$. Now,

$$\begin{aligned} & {}_{n(\overline{1}-A)}X_f^0(s) \leq B_f(s) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(\overline{1}-A)}X_f^0(s)) \leq \mathbf{Cl}(B_f(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n(\overline{1} - A) \leq B_f^n = \overline{1} - A \\ \Rightarrow\ & (\mathbf{Cl})_f^n(\overline{1} - A) = \overline{1} - A \\ \Rightarrow\ & A \in \delta_{(\mathbf{Cl})_f^n} \end{aligned}$$

Also A$\in \delta_{(\mathbf{Cl})_f^n}$ implies $(\mathbf{Cl})_f^n(\overline{1} - A) = \overline{1} - A$. Let $B_f(s) = \mathbf{Cl}({}_{n(\overline{1}-A)}X_f^o(s))$, then $B_f(s)$ is a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$ and its n^{th} component is $\overline{1} - A$. Therefore $A \in (\delta_{\mathbf{Cl}})_n$. Hence the theorem. □

Theorem 2.11. *Let* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-closure operator on a non empty set X and* $A \subset X$. *If* $Char(A)$ *denote the characteristic function of A, Then* $\boldsymbol{Cl}_A$: $(I^A)^{\mathbb{N}} \to (I^A)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{Cl}_A(B_f(s)) \quad = \quad \{Char(A)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(B_f(s)) \ \forall B_f(s) \in (I^A)^{\mathbb{N}}.$$

is an FS-closure operator on A and $(\boldsymbol{Cl}_A)_f^n(B) = Char(A) \wedge (\boldsymbol{Cl})_f^n(B)$ *for all* $B \in I^A$.

Proof. Let $B_f(s) \in (I^A)^{\mathbb{N}}$. Now

$$\begin{aligned} & \mathbf{Cl}_A(\mathbf{Cl}_A(B_f(s)) = \mathbf{Cl}_A(\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s))) \\ = \ & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s))) \\ \leq \ & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(\{Char(A)\}_{n=1}^{\infty}) \wedge \mathbf{Cl}(\mathbf{Cl}(B_f(s))) \\ = \ & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s)) \\ = \ & \mathbf{Cl}_A(B_f(s)) \end{aligned}$$

All the other conditions being straightforward, we an conclude that $\mathbf{Cl}_A$ is an FS-closure operator on A. Also for $B \in I^A$, $(\mathbf{Cl}_A)_f^n(B) = n^{th}$ component of $\mathbf{Cl}_A({}_{nB}X_f^0(s))$= n^{th} component of $\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}({}_{nB}X_f^0(s)) = Char(A) \wedge$ n^{th} component of $\mathbf{Cl}({}_{nB}X_f^0(s)) = Char(A) \wedge (\mathbf{Cl})_f^n(B)$. □

Theorem 2.12. *Let* $\{\boldsymbol{Cl}_\lambda : (I^{X_\lambda})^{\mathbb{N}} \to (I^{X_\lambda})^{\mathbb{N}}; \lambda \in \Lambda\}$ *be a family of FS-closure operators, where* $X_\lambda \wedge X_\mu = \phi$ *for all* $\lambda, \mu \in \Lambda$. *If* $X = \vee_{\lambda \in \Lambda} X_\lambda$ *and* $Char(X_\lambda)$ *denote the characteristic function of* X_λ, *then* $\boldsymbol{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by* $\boldsymbol{C}(A_f(s)) = \vee_{\lambda \in \Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))$ *is an FS-closure operator on X.*

Proof. For $A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} & \mathbf{C}(\mathbf{C}(A_f(s))) = \mathbf{C}(\vee_{\lambda \in \Lambda} \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \ & \vee_{\lambda \in \Lambda} \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge (\vee_{\lambda \in \Lambda} \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)))) \\ = \ & \vee_{\lambda \in \Lambda} \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge (\vee_{\lambda \in \Lambda}(\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty}) \wedge \mathbf{Cl}_\lambda(A_f(s))))) \\ = \ & \vee_{\lambda \in \Lambda} \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \ & \vee_{\lambda \in \Lambda} \mathbf{Cl}_\lambda(\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \ & \vee_{\lambda \in \Lambda} \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)) \\ = \ & \mathbf{C}(A_f(s)) \end{aligned}$$

Other conditions being straightforward, it follows that **C** is an FS-closure operator. □

Definition 2.13. A collection $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}}; \lambda \in \Lambda\}$ is called an FS-closure system if for each $A_f(s) \in (I^X)^{\mathbb{N}}$, $\wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s)$

Theorem 2.14. *$\zeta(s)$ is an FS-closure system iff $\zeta(s)$ is closed under arbitrary intersection.*

Proof. Suppose $\zeta(s)$ is closed under arbitrary intersection. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ such that $A_f(s) \leq A_{\lambda f}(s)$ $\forall \lambda \in \Lambda$. Then

$$\wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s)$$

Conversely, suppose $\zeta(s)$ is an FS-closure system. Let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ and let $A_f(s) = \wedge_{\lambda \in \Lambda} A_{\lambda f}(s)$. Then

$$\begin{aligned} & A_f(s) \leq A_{\lambda f}(s) \ \forall \lambda \in \Lambda \\ \Rightarrow \quad & \wedge_{\lambda \in \Lambda} A_{\lambda f}(s) = \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s) \end{aligned}$$

Hence $\zeta(s)$ is closed under arbitrary intersection. □

Lemma 2.15. *Let $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}}; \lambda \in \Lambda\}$ be an FS-closure system containing $X_f^0(s)$. Then $\boldsymbol{Cl}_{\zeta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s)) = \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \text{ and}$$

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s) \vee B_f(s)) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s)) \vee \boldsymbol{Cl}_{\zeta(s)}(B_f(s)) \ \forall A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}}$$

is an FS-closure operator. Moreover for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \zeta(s)$ iff $A_f(s) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s))$.

Proof. Since $\mathbf{Cl}_{\zeta(s)}(A_f(s)) \in \zeta(s)$ for $A_f(s) \in (I^X)^{\mathbb{N}}$, we have

$$\mathbf{Cl}_{\zeta(s)}(\mathbf{Cl}_{\zeta(s)}(A_f(s))) \quad = \quad \wedge_{\lambda \in \Lambda, \mathbf{Cl}_{\zeta(s)}(A_f(s)) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \leq \mathbf{Cl}_{\zeta(s)}(A_f(s))$$

Hence $\mathbf{Cl}_{\zeta(s)}$ is an FS-closure operator.
Now, if $A_f(s) \in \zeta(s)$, then $A_f(s) = A_{\lambda f}(s)$ for some $\lambda \in \Lambda$ and

$$\mathbf{Cl}_{\zeta(s)}(A_f(s)) \quad = \quad \wedge_{i \in \Lambda, A_f(s) \leq A_{if}(s)} A_{if}(s) \leq A_{\lambda f}(s) = A_f(s)$$

Also $A_f(s) \leq \mathbf{Cl}_{\zeta(s)}((A_f(s))$. Hence $\mathbf{Cl}_{\zeta(s)}(A_f(s)) = A_f(s)$. Converse part follows from the definition of $\mathbf{Cl}_{\zeta(s)}$. □

Lemma 2.16. *Let $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator. Then*

$$\zeta_{\boldsymbol{Cl}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \boldsymbol{Cl}(A_f(s))\}$$

is an FS-closure system.

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ such that $B_f(s) \leq A_{\lambda f}(s)$ $\forall \lambda \in \Lambda$. Let $D_f(s) = \wedge_{\lambda \in \Lambda, B_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s)$. We know, $D_f(s) \leq \mathbf{Cl}(D_f(s))$. Again

$$\begin{aligned} & D_f(s) \leq A_{\lambda f}(s) \ \forall \lambda \in \Lambda \\ \Rightarrow \quad & \mathbf{Cl}(D_f(s)) \leq \mathbf{Cl}(A_{\lambda f}(s)) \ \forall \lambda \in \Lambda \\ \Rightarrow \quad & \mathbf{Cl}(D_f(s)) \leq \wedge_{\lambda \in \Lambda, B_f(s) \leq A_{\lambda f}(s)} \mathbf{Cl}(A_{\lambda f}(s)) = \wedge_{\lambda \in \Lambda, B_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \\ & \qquad = D_f(s) \end{aligned}$$

Thus $D_f(s) = \mathbf{Cl}(D_f(s))$ and so $D_f(s) \in \zeta_{\mathbf{Cl}}(s)$. Hence $\zeta_{\mathbf{Cl}}(s)$ is an FS-closure system. □

Note 2.17. In **Lemma** 2.16, the FS-closure system $\zeta_{\mathbf{Cl}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \mathbf{Cl}(A_f(s))\}$ is called an FS-closure system generated by the FS-closure operator **Cl**.

Theorem 2.18. *Let* $\boldsymbol{Cl}$ *be an FS-closure operator and* $\zeta(s)$ *be an FS-closure system on* X *containing* $X_f^0(s)$*, then* $\zeta_{\boldsymbol{Cl}}(s)$ *and* $\boldsymbol{Cl}_{\zeta(s)}$ *are respectively FS-closure system and FS-closure operator on* X*. Also* $\boldsymbol{Cl} = \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) = \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$*, that is, the mappings* $\boldsymbol{Cl} \to \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) \to \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$ *are mutually inverse.*

Proof. The first part follows from **Lemma** 2.15 and **Lemma** 2.16. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta_{\mathbf{Cl}}(s)$ such that $A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda$. Then $\mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s)) = \wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s)$. Now,

$$\begin{aligned}
& A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\
\Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq \mathbf{Cl}(A_{\lambda f}(s))\ \forall \lambda \in \Lambda \\
\Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\
\Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq \wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) = \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s))
\end{aligned}$$

Again,

$$\begin{aligned}
& A_f(s) \leq \mathbf{Cl}(A_f(s)) \in \zeta_{\mathbf{Cl}}(s) \\
\Rightarrow\ & \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s)) = \wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \leq \mathbf{Cl}(A_f(s))
\end{aligned}$$

Hence $\mathbf{Cl} = \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}$.
Also,

$$\begin{aligned}
& A_f(s) \in \zeta_{\mathbf{Cl}_{\zeta(s)}}(s) \\
\Leftrightarrow\ & A_f(s) = \mathbf{Cl}_{\zeta(s)}(A_f(s)) \\
\Leftrightarrow\ & A_f(s) \in \zeta(s)
\end{aligned}$$

Thus $\zeta(s) = \zeta_{\mathbf{Cl}_{\zeta(s)}}(s)$. □

Definition 2.19. Let X be a non empty set. An operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-interior operator if it satisfies the following conditions:
(FSI1) $\mathbf{I}(X_f^1(s)) = X_f^1(s)$.
(FSI2) $\mathbf{I}(A_f(s)) \leq A_f(s)$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
(FSI3) $\mathbf{I}(\mathbf{I}(A_f(s))) = \mathbf{I}(A_f(s)))$ for all $A_f(s)) \in (I^X)^{\mathbb{N}}$.
(FSI4) $\mathbf{I}(A_f(s) \wedge B_f(s)) = \mathbf{I}(A_f(s)) \wedge \mathbf{I}(B_f(s))$ for all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.20. For any FSTS $(X, \delta(s))$, interior of an fs-set is an FS-interior operator on X.

Example 2.21. Let X be a non empty set. The operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $\mathbf{I}(A_f(s)) = A_f(s) \wedge D_f(s)$ whenever $A_f(s) \neq X_f^1(s)$ and $\mathbf{I}(X_f^1(s)) = X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X, is an FS-interior operator on X.

Theorem 2.22. *If* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-interior operator on* X*, then*

(i) $\boldsymbol{I}$ *is monotonic increasing, that is,* $A_f(s) \leq B_f(s) \Rightarrow \boldsymbol{I}(A_f(s)) \leq \boldsymbol{I}(B_f(s))$ *for all* $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.

(ii) $\boldsymbol{I}(A_f(s)) \leq B_f(s) \Rightarrow \boldsymbol{I}(A_f(s)) \leq \boldsymbol{I}(B_f(s))$ *for all* $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.

Proof. Proof is omitted. □

Theorem 2.23. *Let* X *be a non empty set and* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an operator satisfying* $(FSI1)$, $(FSI2)$ *and* $(FSI4)$*, then*

a) *the collection* $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FST on* X.

b) *if* $\boldsymbol{I}$ *also satisfies* $(FSI3)$*, then for all* $A_f(s) \in (I^X)^{\mathbf{N}}$ *we have* $\overset{o}{A}_f(s) = \boldsymbol{I}(A_f(s))$*, where* $\overset{o}{A}_f(s)$ *is the interior of* $A_f(s)$ *in* $\delta(s)$.

Proof. Proof is omitted. □

Remark 2.24. From **Theorem** 2.23 it follows that if $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X then $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}\,;\ \mathbf{I}(A_f(s)) = A_f(s)\}$ forms an FST on X. Also $\overset{o}{A}_f(s) = \mathbf{I}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbf{N}}$, where $\overset{o}{A}_f(s)$ is the interior of $A_f(s)$ in $\delta(s)$. This FST $\delta(s)$ is called the fuzzy sequential topology induced by the FS-interior operator $\mathbf{I}$ and we denote it by $\delta_{\mathbf{I}}(s)$.

Remark 2.25. **Example** 2.26 shows that if an operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a non empty set X, satisfies $(FSI1)$, $(FSI2)$ and $(FSI4)$ but does not satisfy $(FSI3)$, then $\delta_{\mathbf{I}}(s)$ forms an FST on X but $\overset{o}{A}_f(s)$ may not be equal to $\mathbf{I}(A_f(s))$, $A_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.26. Let $X = \{a\}$. Let $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be defined by $\mathbf{I}(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{I}$ is an operator on X satisfying $(FSI1)$, $(FSI2)$ and $(FSI4)$ and hence $(X, \delta_{\mathbf{I}}(s))$ forms an FSTS. Further $\mathbf{I}$ does not satisfy $(FSI3)$ and in $(X, \delta_{\mathbf{I}}(s))$, $\mathbf{I}(B_f(s)) \neq \overset{0}{B}_f(s)$ if $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ where $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}\ \forall\ n \neq 1, 2, 3$.

Definition 2.27. Let X be a non empty set and $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X. A function $(\mathbf{I})_f^n : I^X \to I^X$ defined by $(\mathbf{I})_f^n(A) = n^{th}$ term of $\mathbf{I}({}_{nA}X_f^1(s))$, where ${}_{nA}X_f^1(s)$ denotes an fs-set whose n^{th} term is A and others are $\overline{1}$, is called the n^{th} component of $\mathbf{I}$, $n \in \mathbb{N}$.

Theorem 2.28. *Let* X *be a non empty set. If* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-interior operator on* X*, then each component* $(\boldsymbol{I})_f^n : I^X \to I^X$, $n \in \mathbb{N}$ *is a fuzzy interior operator. Also* $(\delta_{\boldsymbol{I}})_n = \delta_{(\boldsymbol{I})_f^n}$ *where* $(\delta_{\boldsymbol{I}})_n$ *is the* n^{th} *component fuzzy topology of FST* $\delta_{\boldsymbol{I}}(s)$ *and* $\delta_{(\boldsymbol{I})_f^n}$ *is the fuzzy topology induced by the component* $(\boldsymbol{I})_f^n$ *of* $\boldsymbol{I}$.

Proof. $(\mathbf{I})_f^n(\overline{1}) = \overline{1}$ by definition. Let $A \in I^X$, then $\mathbf{I}({}_{nA}X_f^1(s)) \leq {}_{nA}X_f^1(s) \Rightarrow (\mathbf{I})_f^n(A) \leq A$. Hence $(\mathbf{I})_f^n((\mathbf{I})_f^n(A)) \leq (\mathbf{I})_f^n(A)$. Also

$$
\begin{aligned}
& \mathbf{I}(\mathbf{I}({}_{nA}X_f^1(s))) = \mathbf{I}({}_{nA}X_f^1(s)) \\
\Rightarrow\quad & \mathbf{I}({}_{nA}X_f^1(s)) \leq \mathbf{I}({}_{n(\mathbf{I})_f^n(A)}X_f^1(s)) \\
\Rightarrow\quad & (\mathbf{I})_f^n(A) \leq (\mathbf{I})_f^n((\mathbf{I})_f^n(A))
\end{aligned}
$$

Hence $(\mathbf{I})^n_f((\mathbf{I})^n_f(A)) = (\mathbf{I})^n_f(A)$.
Again let A, B $\in I^X$, then

$$\begin{aligned} & \mathbf{I}({}_{nA}X^1_f(s) \wedge {}_{nB}X^1_f(s)) = \mathbf{I}({}_{nA}X^1_f(s)) \wedge \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\ & \mathbf{I}({}_{n(A\wedge B)}X^1_f) = \mathbf{I}({}_{nA}X^1_f(s)) \wedge \mathbf{I}({}_{nB}X^1_f(s)) \\ \Rightarrow\ & (\mathbf{I})^n_f(A \wedge B) = (\mathbf{I})^n_f(A) \wedge (\mathbf{I})^n_f(B) \end{aligned}$$

Thus $(\mathbf{I})^n_f$ is a fuzzy interior operator.
For the next part, Let A$\in (\delta_{\mathbf{I}})_n$. Let $B_f(s) = \{B^n_f\}^\infty_{n=1}$ be an open fs-set in $(X, \delta_{\mathbf{I}}(s))$ such that $B^n_f = A$. Now,

$$\begin{aligned} & B_f(s) \leq {}_{nA}X^1_f(s) \\ \Rightarrow\ & \mathbf{I}(B_f(s)) \leq \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\ & B_f(s) \leq \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\ & A \leq (\mathbf{I})^n_f(A) \\ \Rightarrow\ & (\mathbf{I})^n_f(A) = A \\ \Rightarrow\ & A \in \delta_{(\mathbf{I})^n_f} \end{aligned}$$

Also $A \in \delta_{(\mathbf{I})^n_f}$ implies $(\mathbf{I})^n_f(A) = A$. Let $B_f(s) = \mathbf{I}({}_{nA}X^1_f(s))$, then $B_f(s)$ is an open fs-set in $(X, (\delta_{\mathbf{I}}(s))$ and its n^{th} component is A. Therefore $A \in (\delta_{\mathbf{I}})_n$. Hence the theorem. □

Theorem 2.29. *Let $\boldsymbol{I}\colon (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on a non empty set X and $A \subset X$. If $Char(A)$ denote the characteristic function of A, then $\boldsymbol{I}_A$: $(I^A)^{\mathbb{N}} \to (I^A)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I}_A(B_f(s)) \quad = \quad \{Char(A)\}^\infty_{n=1} \vee \boldsymbol{I}(B_f(s))\ \forall B_f(s) \in (I^A)^{\mathbb{N}}.$$

is an FS-interior operator on A and $(\boldsymbol{I}_A)^n_f(B) = Char(A) \vee (\boldsymbol{I})^n_f(B)$ for all $B \in I^A$.

Proof. Let $B_f(s) \in (I^A)^{\mathbb{N}}$. Now

$$\begin{aligned} & \mathbf{I}_A(B_f(s))) = \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s)) \\ =\ & \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(\{Char(A)\}^\infty_{n=1}) \vee \mathbf{I}(\mathbf{I}(B_f(s))) \\ \leq\ & \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(\{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s))) \\ =\ & \mathbf{I}_A(\{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s))) \\ =\ & \mathbf{I}_A(\mathbf{I}_A(B_f(s))) \end{aligned}$$

All the other conditions being straightforward, we an conclude that $\mathbf{I}_A$ is an FS-interior operator. Also $(\mathbf{I}_A)^n_f(B) = n^{th}$ component of $\mathbf{I}_A({}_{nB}X^1_f(s)) = n^{th}$ component of $\{Char(A)\}^{\mathbb{N}} \vee \mathbf{I}({}_{nB}X^1_f(s)) = Char(A) \vee n^{th}$ component of $\mathbf{I}({}_{nB}X^1_f(s)) = Char(A) \vee (\mathbf{I})^n_f(B)$. □

Definition 2.30. A collection $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ is called an FS-interior system if for each $A_f(s) \in (I^X)^{\mathbb{N}}$, $\vee_{j \in J, A_{jf}(s) \leq A_f(s)} A_{jf}(s) \in \eta(s)$.

Theorem 2.31. *$\eta(s)$ is an FS-interior system iff $\eta(s)$ is closed under arbitrary union.*

Proof. Suppose $\eta(s)$ is closed under arbitrary union. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Let $A_{jf}(s) \leq A_f(s)\ \forall j \in J$ where $A_{jf}(s) \in \eta(s)\ \forall j \in J$. Then

$$\vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s)$$

Conversely, suppose $\eta(s)$ is an FS-interior system. Let $\{A_{jf}(s); j \in J\} \in \eta(s)$ and let $A_f(s) = \vee_{j\in J} A_{jf}(s)$. Then

$$\begin{aligned} & A_{jf}(s) \leq A_f(s)\ \forall j \in J \\ \Rightarrow\ & \vee_{j\in J} A_{jf}(s) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s) \end{aligned}$$

Hence $\eta(s)$ is closed under arbitrary union. □

Lemma 2.32. *Let $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ be an FS-interior system containing $X_f^1(s)$. Then $\boldsymbol{I}_{\eta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\begin{aligned} & \boldsymbol{I}_{\eta(s)}(A_f(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \text{ and} \\ & \boldsymbol{I}_{\eta(s)}(A_f(s) \wedge B_f(s)) = \boldsymbol{I}_{\eta(s)}(A_f(s)) \wedge \boldsymbol{I}_{\eta(s)}(B_f(s))\ \forall A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}} \end{aligned}$$

is an FS-interior operator. Moreover for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \eta(s)$ iff $A_f(s) = \boldsymbol{I}_{\eta(s)}(A_f(s))$.

Proof. Proof of the first part is straightforward.
Now, if $A_f(s) \in \eta(s)$, then $A_f(s) = A_{jf}(s)$ for some $j \in J$ and

$$\mathbf{I}_{\eta(s)}(A_f(s)) = \vee_{i\in J, A_{if}(s)\leq A_f(s)} A_{if}(s) = A_f(s)$$

Converse part follows from the definition of $\mathbf{I}_{\eta(s)}$. □

Lemma 2.33. *Let $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator. Then*

$$\eta_{\boldsymbol{I}}(s) \quad = \quad \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \boldsymbol{I}(A_f(s))\}$$

is an FS-interior system.

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$. Let $D_f(s) = \vee_{j\in J, A_{jf}(s)\leq B_f(s)} A_{jf}(s)$, where $A_{jf}(s) \in \eta_{\mathbf{I}}(s)\ \forall j \in J$. We know, $\mathbf{I}(D_f(s)) \leq D_f(s)$. Again,

$$\begin{aligned} & A_{jf}(s) \leq D_f(s)\ \forall j \in J \text{ and for all } A_{jf}(s) \leq B_f(s) \\ \Rightarrow\ & \mathbf{I}(A_{jf}(s)) \leq \mathbf{I}(D_f(s))\ \forall j \in J \text{ and for all } A_{jf}(s) \leq B_f(s) \\ \Rightarrow\ & \vee_{j\in J, A_{jf}(s)\leq B_f(s)} \mathbf{I}(A_{jf}(s)) = \vee_{j\in J, A_{jf}(s)\leq B_f(s)} A_{jf}(s) = D_f(s) \leq \mathbf{I}(D_f(s)) \end{aligned}$$

Thus $D_f(s) = \mathbf{I}(D_f(s))$ and so $D_f(s) \in \eta_{\mathbf{I}}(s)$. Hence $\eta_{\mathbf{I}}(s)$ is a FS-interior system. □

Note 2.34. In **Lemma** 2.33, the FS-interior system $\eta_{\mathbf{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \mathbf{I}(A_f(s))\}$ is called an FS-interior system generated by the FS-interior operator $\mathbf{I}$.

Theorem 2.35. *Let $\boldsymbol{I}$ be an FS-interior operator and $\eta(s)$ be an FS-interior system on X containing $X_f^1(s)$, then $\eta_{\boldsymbol{I}}(s)$ and $\boldsymbol{I}_{\eta(s)}$ are respectively FS-interior system and FS-interior operator on X. Also $\boldsymbol{I} = \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) = \eta_{\boldsymbol{I}_{\eta(s)}}(s)$, that is, the mappings $\boldsymbol{I} \to \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) \to \eta_{\boldsymbol{I}_{\eta(s)}}(s)$ are mutually inverse.*

Proof. The first part follows from **Lemma** 2.32 and **Lemma** 2.33. Let $A_f(s) \in (I^X)^{\mathbb{N}}$, and let $\{A_{jf}(s); j \in J\} \in \eta_{\mathbf{I}}(s)$ such that $A_{jf}(s) \leq A_f(s)\ \forall j \in J$. Then $\mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)(\in \eta_{\mathbf{I}}(s))}(A_{jf}(s))$. Now,

$$\begin{aligned} & A_{jf}(s) \leq A_f(s)\ \forall j \in J \\ \Rightarrow\ & \mathbf{I}(A_{jf}(s)) \leq \mathbf{I}(A_f(s))\ \forall j \in J \\ \Rightarrow\ & \vee_{j\in J, A_{jf}(s)\leq A_f(s)}\mathbf{I}(A_{jf}(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \\ & \qquad = \mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)) \leq \mathbf{I}(A_f(s)) \end{aligned}$$

Again,

$$\begin{aligned} & \mathbf{I}(A_f(s))(\leq A_f(s)) \in \eta_{\mathbf{I}}(s) \\ \Rightarrow\ & \mathbf{I}(A_f(s)) \leq \vee_{j\in J, A_{jf}(s)\leq A_f(s)(\in \eta_{\mathbf{I}}(s))} A_{jf}(s) = \mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)). \end{aligned}$$

Hence $\mathbf{I} = \mathbf{I}_{\eta_{\mathbf{I}}(s)}$.
Also,

$$\begin{aligned} & A_f(s) \in \eta_{\mathbf{I}_{\eta(s)}}(s) \\ \Leftrightarrow\ & A_f(s) = \mathbf{I}_{\eta(s)}(A_f(s)) \\ \Leftrightarrow\ & A_f(s) \in \eta(s). \end{aligned}$$

Thus $\eta(s) = \eta_{\mathbf{I}_{\eta(s)}}(s)$. □

Definition 2.36. If **I** be an FS-interior operator on a non empty set X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}}; \mathbf{I}(A_f(s)) = A_f(s)\}$ forms an FS-closure system on X and we call it to be an FS-closure system generated by the FS-interior operator **I**.

Definition 2.37. If **Cl** be an FS-closure operator on a non empty set X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}}; \mathbf{Cl}(A_f(s)) = A_f(s)\}$ forms an FS-interior system on X and we call it to be an FS-interior system generated by the FS-closure operator **Cl**.

Theorem 2.38. *Let* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-interior operator on* X*, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FST on* X*.*
(ii) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FS-interior system on* X*.*
(iii) $\{A_f(s);\ (A_f(s))^c \in \delta_{\boldsymbol{I}}(s)\}$ *forms an FS-closure system on* X*.*

Proof. Proof is omitted. □

Theorem 2.39. *Let* $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-closure operator on* X*, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ *forms an FST on* X*.*
(ii) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FS-interior system on* X*.*
(iii) $\{A_f(s);\ (A_f(s))^c \in \delta_{\boldsymbol{Cl}}(s)\}$ *forms an FS-closure system on* X*.*

Proof. Proof is omitted. □

Theorem 2.40. *Let X be a non empty set. If $\boldsymbol{Cl}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then the operator $\boldsymbol{I_{Cl}}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I_{Cl}}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-interior operator on X. Again, if $\boldsymbol{I}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then the operator $\boldsymbol{Cl_I}: (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$\boldsymbol{Cl_I}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{I}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-closure operator on X.

Proof. Proof is omitted. □

Note 2.41. It follows from **Theorem** 2.40 that given an FS-closure operator we can define an FS-interior operator and given an FS-interior operator we can define an FS-closure operator. In fact, there is a one to one correspondence between the collections of all FS-closure and FS-interior operators on a set (**Theorem** 2.42). We denote the collection of all FS-closure operators and the collection of all FS-interior operators on X by $\mathcal{C}_X$ and $\mathcal{I}_X$ respectively.

Theorem 2.42. *Let X be a non empty set, then there exists a one to one correspondence between $\mathcal{C}_X$ and $\mathcal{I}_X$.*

Proof. $t : \mathcal{C}_X \to \mathcal{I}_X$ by

$$t(\mathbf{Cl}) \quad = \quad \mathbf{I_{Cl}} \ \forall \mathbf{Cl} \in \mathcal{C}_X$$

Then t is a well defined map. Now, for $\mathbf{Cl}_1, \mathbf{Cl}_2 \in \mathcal{C}_X$ such that $t(\mathbf{Cl}_1) = t(\mathbf{Cl}_2)$, we have $\mathbf{I_{Cl_1}} = \mathbf{I_{Cl_2}}$. Hence $\forall A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} \mathbf{I_{Cl_1}}((A_f(s))^c) &= \mathbf{I_{Cl_2}}((A_f(s))^c) \\ X_f^1(s) - \mathbf{Cl}_1(A_f(s)) &= X_f^1(s) - \mathbf{Cl}_2(A_f(s)) \\ \mathbf{Cl}_1(A_f(s)) &= \mathbf{Cl}_2(A_f(s)) \end{aligned}$$

Thus t is injective. Again for $\mathbf{I} \in \mathcal{I}_X$, there is $\mathbf{Cl_I} \in \mathcal{C}_X$ such that $\forall A_f(s) \in (I^X)^{\mathbb{N}}$

$$\mathbf{Cl_I}((A_f(s))^c) \quad = \quad X_f^1(s) - \mathbf{I}(A_f(s))$$

Now, $\forall A_f(s) \in (I^X)^{\mathbb{N}}$

$$\begin{aligned} \mathbf{I_{Cl_I}}((A_f(s)) &= X_f^1(s) - \mathbf{Cl_I}((A_f(s))^c) \\ &= X_f^1(s) - (X_f^1(s) - \mathbf{I}(A_f(s))) \\ &= \mathbf{I}(A_f(s)) \end{aligned}$$

Therefore t is surjective and this completes the theorem. □

Note 2.43. If $\mathbf{I}$ is the t-image of $\mathbf{Cl}$ under the bijection t defined in **Theorem** 2.42, then $\mathbf{I}$ and $\mathbf{Cl}$ are called t-associated to each other.

Theorem 2.44. *The FST's induced by $\boldsymbol{Cl}$ and $\boldsymbol{I_{Cl}}$ are identical and the FST's induced by $\boldsymbol{I}$ and $\boldsymbol{Cl_I}$ are identical.*

Proof. Proof is omitted. □

Now, if we define an FS-interior and an FS-closure operator, separately, on a non empty set, they will induce two fuzzy sequential topologies which may not be identical in general. In view of **Theorem** 2.42 and **Theorem** 2.44, we give a necessary and sufficient condition that the two fuzzy sequential topologies induced by an FS-interior operator and an FS-closure operator are identical.

Theorem 2.45. *Let X be a non empty set. If $\boldsymbol{Cl} \in \mathcal{C}_X$ and $\boldsymbol{I} \in \mathcal{I}_X$, then $\delta_{\boldsymbol{Cl}}(s)$ and $\delta_{\boldsymbol{I}}(s)$ are identical iff $\boldsymbol{Cl}$ and $\boldsymbol{I}$ are t-associated to each other.*

Proof. Suppose $\mathbf{Cl}$ and $\mathbf{I}$ are t-associated to each other. Then $t(\mathbf{Cl}) = \mathbf{I}_{\mathbf{Cl}} = \mathbf{I}$. Now,

$$\begin{aligned} & A_f(s) \in \delta_{\mathbf{I}}(s) \\ \Leftrightarrow\ & \mathbf{I}(A_f(s)) = A_f(s) \\ \Leftrightarrow\ & \mathbf{I}_{\mathbf{Cl}}(A_f(s)) = A_f(s) \\ \Leftrightarrow\ & X^1_f(s) - \mathbf{Cl}((A_f(s))^c) = A_f(s) \\ \Leftrightarrow\ & \mathbf{Cl}((A_f(s))^c) = (A_f(s))^c \\ \Leftrightarrow\ & A_f(s) \in \delta_{\mathbf{Cl}}(s). \end{aligned}$$

Thus $\delta_{\mathbf{I}}(s)$ and $\delta_{\mathbf{Cl}}(s)$ are identical.
Conversely, suppose $\delta_{\mathbf{I}}(s)$ and $\delta_{\mathbf{Cl}}(s)$ are identical. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Then

$$\begin{aligned} & (\mathbf{Cl}((A_f(s))^c))^c \in \delta_{\mathbf{I}}(s) \\ \Rightarrow\ & \mathbf{I}((\mathbf{Cl}((A_f(s))^c))^c) = (\mathbf{Cl}((A_f(s))^c))^c = X^1_f(s) - \mathbf{Cl}((A_f(s))^c) \end{aligned}$$

Now,

$$\begin{aligned} & (A_f(s))^c \le \mathbf{Cl}((A_f(s))^c) \\ \Rightarrow\ & (\mathbf{Cl}((A_f(s))^c))^c \le A_f(s) \\ \Rightarrow\ & \mathbf{I}((\mathbf{Cl}((A_f(s))^c))^c) \le \mathbf{I}(A_f(s)) \\ \Rightarrow\ & X^1_f(s) - \mathbf{Cl}((A_f(s))^c) \le \mathbf{I}(A_f(s)). \end{aligned}$$

Again,

$$\begin{aligned} & \mathbf{I}(A_f(s)) \in \delta_{\mathbf{Cl}}(s) \\ \Rightarrow\ & \mathbf{Cl}((\mathbf{I}(A_f(s)))^c) = (\mathbf{I}(A_f(s)))^c = X^1_f(s) - \mathbf{I}(A_f(s)). \end{aligned}$$

Also,

$$\begin{aligned} & \mathbf{I}(A_f(s)) \le A_f(s) \\ \Rightarrow\ & (A_f(s))^c \le (\mathbf{I}(A_f(s)))^c \\ \Rightarrow\ & \mathbf{Cl}((A_f(s))^c) \le \mathbf{Cl}((\mathbf{I}(A_f(s)))^c) = X^1_f(s) - \mathbf{I}(A_f(s)) \\ \Rightarrow\ & \mathbf{I}(A_f(s)) \le X^1_f(s) - \mathbf{Cl}((A_f(s))^c). \end{aligned}$$

Thus $\mathbf{I}(A_f(s)) = X^1_f(s) - \mathbf{Cl}((A_f(s))^c) = \mathbf{I}_{\mathbf{Cl}}(A_f(s))\ \forall\ A_f(s) \in (I^X)^{\mathbb{N}}$. Hence $\mathbf{I} = \mathbf{I}_{\mathbf{Cl}} = t(\mathbf{Cl})$. □

Theorem 2.46. *Let X be a non empty set. If $\boldsymbol{Cl} \in \mathcal{C}_X$, $\boldsymbol{I} \in \mathcal{I}_X$, then the following conditions are equivalent:*

(i) *$\boldsymbol{I}$ and $\boldsymbol{Cl}$ are t-associated to each other.*

(ii) *The FST's* $\delta_{\boldsymbol{I}}(s)$ *and* $\delta_{\boldsymbol{Cl}}(s)$ *are identical.*
(iii) *FS-closure systems generated by* $\boldsymbol{Cl}$ *and* $\boldsymbol{I}$ *are identical.*
(iv) *FS-interior systems generated by* $\boldsymbol{Cl}$ *and* $\boldsymbol{I}$ *are identical.*

Proof. Proof is omitted. □

Note 2.47. Theorem 2.46 gives two more necessary and sufficient conditions ((iii) and (iv)), that the fuzzy sequential topologies induced by an FS-interior operator and an FS-closure operator are identical.

Acknowledgements. The authors are grateful to the referees for their valuable comments and suggestions towards the improvement of the paper.

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Palestine Journal of Mathematics

Vol. 4(1) (2015) , 37–43

Composition of fuzzy sequential operators with special emphasis on FS-connectors

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Communicated by Ayman Badawi

MSC 2010 Classifications: 54A40, 03E72.

Keywords and phrases: Fuzzy sequential topological spaces, FS-closure operator, FS-interior operator, relative FS-closure operator, FS-connector.

Abstract. FS-closure and FS-interior operators both induce fuzzy sequential topologies on the underlying set. Do the composition of FS-closure and that of FS-interior operators provide any topological structure? If so, is there any relation among the topologies induced by the composition and that induced by the participants to the composition? We consider these questions in this article and also study relative FS-closure operators and FS-connectors.

1 Introduction

In 1968 C. L. Chang [6] introduced the concept of fuzzy topology after the initiation of fuzzy sets by L. A. Zadeh [18]. Towards the development of fuzzy set theory, fuzzy closure operators and fuzzy interior operators have been studied by Mashour and Ghanim [10], G. Gerla [8], Bandler and Kohout [1], R. Belohlavek [2], R. Belohlavek and T. Funiokova [3]. Notions of fuzzy sequential topological spaces (FSTS) and notions of FS-closure and FS-interior operators were introduced in [13] and [17] respectively.

Our purpose is to introduce FS-connectors connecting two fuzzy topologies on a set and to study the composition of FS-closure and that of FS-interior operators.

Section 2 deals with the composition of FS-closure operators, composition of FS-interior operators and the relation between collections of FS-closure and FS-interior operators. Section 3 deals with the relative FS-closure operators and the functions connecting two fuzzy topologies on a set, so called FS-connectors. The basic ideas behind the present work have been taken from the books ([5], [7] [9], [11]) and the articles ([4], [12], [14], [15], [16]).

In this paper, X will denote a non-empty set, $I = [0, 1]$, the closed unit interval in the real line. Before entering into our work we recall the following definitions and results.

Definition 1.1. [13] A family $\delta(s)$ of fuzzy sequential sets on a set X satisfying the properties
(i) $X_f^r(s) \in \delta(s)$ for $r = 0$ and 1,
(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and
(iii) for any family $\{A_{fj}(s) \in \delta(s), j \in J\}$, $\underset{j\in J}{\vee} A_{fj}(s) \in \delta(s)$
is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 1.2. [13] If $(X, \delta(s))$ is an FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where δ_n=$\{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$. (X, δ_n), where $n \in \mathbb{N}$, is called the n^{th} component FTS of the FSTS $(X, \delta(s))$.

Proposition 1.3. [13] *Let $A_f(s)$=$\{A_f^n\}_n$ be an open $(closed)$ fuzzy sequential set in the FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open $(closed)$ fuzzy set in (X, δ_n).*

Proposition 1.4. [13] *If δ be a fuzzy topology (FT) on a set X, then $\delta^{\mathbb{N}}$ forms an FST on X.*

Definition 1.5. [13] Let $A_f(s)$ be a fuzzy sequential set (fs-set) in an FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \leq C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \vee\{O_f(s);\ O_f(s) \leq A_f(s),\ O_f(s) \in \delta(s)\}.$$

Definition 1.6. [17] An operator **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-closure operator on X if it satisfies the following conditions:
$(FSC1)$ $\mathbf{Cl}(X_f^0(s)) = X_f^0(s)$.
$(FSC2)$ $A_f(s) \leq \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC3)$ $\mathbf{Cl}(\mathbf{Cl}(A_f(s))) = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC4)$ $\mathbf{Cl}(A_f(s) \vee B_f(s)) = \mathbf{Cl}(A_f(s)) \vee \mathbf{Cl}(B_f(s))$ for all $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$.

Definition 1.7. [17] An operator **I** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-interior operator on X if it satisfies the following conditions:
$(FSI1)$ $\mathbf{I}(X_f^1(s)) = X_f^1(s)$.
$(FSI2)$ $\mathbf{I}(A_f(s)) \leq A_f(s)$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSI3)$ $\mathbf{I}(\mathbf{I}(A_f(s))) = \mathbf{I}(A_f(s)))$ for all $A_f(s)) \in (I^X)^{\mathbb{N}}$.
$(FSI4)$ $\mathbf{I}(A_f(s) \wedge B_f(s)) = \mathbf{I}(A_f(s)) \wedge \mathbf{I}(B_f(s))$ for all $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$.

Theorem 1.8. [17] *If* ***Cl*** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-closure operator on* X*, then the operator* $\boldsymbol{I_{Cl}} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{I_{Cl}}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \; \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-interior operator on X*. Again, if* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-interior operator on* X*, then the operator* $\boldsymbol{Cl_I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{Cl_I}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{I}((A_f(s))^c) \; \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-closure operator on X*.*

Theorem 1.9. [17] *The map* $t : \mathcal{C}_X \to \mathcal{I}_X$ *defined by*

$$t(\boldsymbol{Cl}) \quad = \quad \boldsymbol{I_{Cl}} \; \forall \boldsymbol{Cl} \in \mathcal{C}_X$$

is a bijection, where $\mathcal{C}_X$ *and* $\mathcal{I}_X$ *respectively, denote the collections of all FS-closure operators and all FS-interior operators on* X*.*

2 Composition of FS-closure and FS-interior operators

Definition 2.1. If $\mathbf{C}_1$, $\mathbf{C}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be two FS-closure operators on X, then the mapping $\mathbf{C}_2 \circ \mathbf{C}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$(\mathbf{C}_2 \circ \mathbf{C}_1)(A_f(s)) \quad = \quad \mathbf{C}_2(\mathbf{C}_1(A_f(s))) \; \forall A_f(s) \in (I^X)^{\mathbb{N}}$$

is called the composition of the FS-closure operators $\mathbf{C}_1$ and $\mathbf{C}_2$.

It is easy to see that composition of FS-closure operators is associative but it may not be commutative and it may not be idempotent, as shown by **Example** 2.2.

Example 2.2. Let us consider the FS-closure operator $\mathbf{C}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{C}_1(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\mathbf{C}_1(X_f^0(s)) = X_f^0(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also consider FS-closure operator $\mathbf{C}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{C}_2(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty} \; \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{C}_2 \circ \mathbf{C}_1 \neq \mathbf{C}_1 \circ \mathbf{C}_2$. and $(\mathbf{C}_2 \circ \mathbf{C}_1) \circ (\mathbf{C}_2 \circ \mathbf{C}_1) \neq (\mathbf{C}_2 \circ \mathbf{C}_1)$.

Theorem 2.3. *If* $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *be two FS-closure operators on* X*, then* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ *satisfies* $FSC1$*,* $FSC2$ *and* $FSC4$*. Further, it satisfies* $FSC3$ *if the composition is commutative, that is, under commutative composition,* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ *forms an FS-closure operator.*
Proof: *Proof is omitted.*

Theorem 2.4. *Let* $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *be two FS-closure operators on* X*. Under commutative composition,* $\delta_{\mathbf{C}_2 \circ \mathbf{C}_1}(s) = \delta_{\mathbf{C}_2}(s) \wedge \delta_{\mathbf{C}_1}(s)$*, where* $\delta_{\mathbf{C}_2 \circ \mathbf{C}_1}(s)$*,* $\delta_{\mathbf{C}_2}(s)$ *and* $\delta_{\mathbf{C}_1}(s)$ *respectively denote the FST's induced by* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$*,* $\boldsymbol{C}_2$ *and* $\boldsymbol{C}_1$*.*
Proof: *Let* $A_f(s) \in \delta_{\mathbf{C}_2 \circ \mathbf{C}_1}(s)$*, then*

$$(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c) \quad = \quad (A_f(s))^c$$

Now,

$$\begin{aligned}\boldsymbol{C}_1((A_f(s))^c) &= \boldsymbol{C}_1((\boldsymbol{C}_2\circ\boldsymbol{C}_1)((A_f(s))^c))\\ &= \boldsymbol{C}_1((\boldsymbol{C}_1\circ\boldsymbol{C}_2)((A_f(s))^c))\\ &= \boldsymbol{C}_1(\boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c)))\\ &= \boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c))\\ &= (\boldsymbol{C}_1\circ\boldsymbol{C}_2)((A_f(s))^c)\\ &= (A_f(s))^c.\end{aligned}$$

Similarly, $\boldsymbol{C}_2((A_f(s))^c)=(A_f(s))^c$. *Hence* $A_f(s)\in\delta_{\boldsymbol{C}_2}(s)\wedge\delta_{\boldsymbol{C}_1}(s)$.
Again, let $A_f(s)\in\delta_{\boldsymbol{C}_2}(s)\wedge\delta_{\boldsymbol{C}_1}(s)$, *then*

$$\boldsymbol{C}_1((A_f(s))^c) = (A_f(s))^c \text{ and } \boldsymbol{C}_2((A_f(s))^c)=(A_f(s))^c$$

Now,

$$\begin{aligned}(\boldsymbol{C}_2\circ\boldsymbol{C}_1)((A_f(s))^c) &= \boldsymbol{C}_2(\boldsymbol{C}_1((A_f(s))^c))\\ &= \boldsymbol{C}_2((A_f(s))^c)\\ &= (A_f(s))^c\end{aligned}$$

Thus $A_f(s)\in\delta_{\boldsymbol{C}_2\circ\boldsymbol{C}_1}(s)$ *and hence the theorem.*

Definition 2.5. If $\mathbf{I}_1$, $\mathbf{I}_2:(I^X)^{\mathbb{N}}\to(I^X)^{\mathbb{N}}$ be two FS-interior operators on X, then the mapping $\mathbf{I}_2\circ\mathbf{I}_1:(I^X)^{\mathbb{N}}\to(I^X)^{\mathbb{N}}$ defined by

$$(\mathbf{I}_2\circ\mathbf{I}_1)(A_f(s)) = \mathbf{I}_2(\mathbf{I}_1(A_f(s)))\ \forall A_f(s)\in(I^X)^{\mathbb{N}}$$

is called the composition of the FS-interior operators $\mathbf{I}_1$ and $\mathbf{I}_2$.

It is easy to see that composition of FS-interior operators is associative but it may not be commutative and it may not be idempotent, as shown by **Example** 2.6.

Example 2.6. Let us consider the FS-interior operator $\mathbf{I}_1:(I^X)^{\mathbb{N}}\to(I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{I}_1(A_f(s))=A_f(s)\wedge D_f(s)$ whenever $A_f(s)\neq X_f^0(s)$ and $\mathbf{I}_1(X_f^1(s))=X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also consider FS-interior operator $\mathbf{I}_2:(I^X)^{\mathbb{N}}\to(I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{I}_2(A_f(s))=\{A_f^n\wedge A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s)=\{A_f^n\}_{n=1}^{\infty}\in(I^X)^{\mathbb{N}}$. Then $\mathbf{I}_2\circ\mathbf{I}_1\neq\mathbf{I}_1\circ\mathbf{I}_2$ and $(\mathbf{I}_2\circ\mathbf{I}_1)\circ(\mathbf{I}_2\circ\mathbf{I}_1)\neq(\mathbf{I}_2\circ\mathbf{I}_1)$.

Theorem 2.7. *If* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *be two FS-interior operators on* X, *then* $\boldsymbol{I}_2\circ\boldsymbol{I}_1$ *satisfies* $FSI1$, $FSI2$ *and* $FSI4$. *Further, it satisfies* $FSI3$ *if the composition is commutative, that is, under commutative composition,* $\boldsymbol{I}_2\circ\boldsymbol{I}_1$ *forms an FS-interior operator.*
Proof: *Proof is omitted.*

Theorem 2.8. *Let* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *be two FS-interior operators on* X. *Under commutative composition,* $\delta_{\boldsymbol{I}_2\circ\boldsymbol{I}_1}(s)=\delta_{\boldsymbol{I}_2}(s)\wedge\delta_{\boldsymbol{I}_1}(s)$, *where* $\delta_{\boldsymbol{I}_2\circ\boldsymbol{I}_1}(s)$, $\delta_{\boldsymbol{I}_2}(s)$ *and* $\delta_{\boldsymbol{I}_1}(s)$ *respectively denote the FST's induced by* $\boldsymbol{I}_2\circ\boldsymbol{I}_1$, $\boldsymbol{I}_2$ *and* $\boldsymbol{I}_1$.
Proof: *The proof is similar to that in case of FS-closure operators.*

Theorem 2.9. *Under commutative composition,* $(\mathcal{I}_X,\circ)$ *and* $(\mathcal{C}_X,\circ)$ *both form semigroups with identity. Further, there exists a semigroup isomorphism between them.*
Proof: *First part is easy to check. For the second part, define* $t:\mathcal{C}_X\to\mathcal{I}_X$ *by*

$$t(\boldsymbol{Cl}) = \boldsymbol{I}_{\boldsymbol{Cl}}\ \forall\boldsymbol{Cl}\in\mathcal{C}_X$$

From ***Theorem*** 1.9, t *is a bijection. Also for* $\boldsymbol{C}_1$, $\boldsymbol{C}_2\in\mathcal{C}_X$ *and* $A_f(s)\in(I^X)^{\mathbb{N}}$

$$\begin{aligned}(\boldsymbol{I}_{\boldsymbol{C}_1}\circ\boldsymbol{I}_{\boldsymbol{C}_2})(A_f(s)) &= \boldsymbol{I}_{\boldsymbol{C}_1}(X_f^1(s)-\boldsymbol{C}_2((A_f(s))^c))\\ &= X_f^1(s)-\boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c))\\ &= X_f^1(s)-(\boldsymbol{C}_1\circ\boldsymbol{C}_2)((A_f(s))^c)\\ &= \boldsymbol{I}_{\boldsymbol{C}_1\circ\boldsymbol{C}_2}(A_f(s)).\end{aligned}$$

Therefore

$$\begin{aligned}t(\boldsymbol{C}_1\circ\boldsymbol{C}_2) &= \boldsymbol{I}_{\boldsymbol{C}_1\circ\boldsymbol{C}_2}\\ &= \boldsymbol{I}_{\boldsymbol{C}_1}\circ\boldsymbol{I}_{\boldsymbol{C}_2}\\ &= t(\boldsymbol{C}_1)\circ t(\boldsymbol{C}_2)\end{aligned}$$

Hence t *is an isomorphism.*

3 Relative FS-closure Operators and FS-connectors

Definition 3.1. Let $A_f(s)$ be an fs-set in X and **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. A function $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ defined by $(\mathbf{Cl})^n_{A_f(s)}(B) = n^{th}$ term of $\mathbf{Cl}({}_{nB}A_f(s))$, where ${}_{nB}A_f(s)$ is the fs-set in X obtained from $A_f(s)$ replacing n^{th} term of it by B, is called n^{th} relative FS-closure operator of **Cl** with respect to $A_f(s)$.

If **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then it is obvious that $(\mathbf{Cl})^n_{X^0_f(s)} = (\mathbf{Cl})^n_f$ and consequently $\delta_{(\mathbf{Cl})^n_{X^0_f(s)}} = \delta_{(\mathbf{Cl})^n_f}$, $\delta_{(\mathbf{Cl})^n_{X^0_f(s)}}$ and $\delta_{(\mathbf{Cl})^n_f}$ being the fuzzy topologies induced by $(\mathbf{Cl})^n_{X^0_f(s)}$ and $(\mathbf{Cl})^n_f$ respectively. It is also true that the n^{th} relative FS-closure operator $(\mathbf{Cl})^n_{A_f(s)}$ of an FS-closure operator **Cl** with respect to an fs-set $A_f(s)$ satisfies $FSC2$, $FSC3$ and $FSC4$ but it may not satisfy $FSC1$ shown by **Example** 3.2. Hence $(\mathbf{Cl})^n_{A_f(s)}$ may not be a fuzzy operator.

Example 3.2. Define a function **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ by

$$\begin{aligned} \mathbf{Cl}(B_f(s)) &= X^1_f(s) \text{ if } B_f(s) \neq X^0_f(s), \\ &= X^0_f(s) \text{ if } B_f(s) = X^0_f(s) \end{aligned}$$

Then for any fs-set $A_f(s)$ in X, having at least two non zero components, $(\mathbf{Cl})^n_{A_f(s)}(\overline{0}) = \overline{1}$ for all $n \in \mathbb{N}$.

Theorem 3.3. *Let $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ be the n^{th} relative FS-closure operator of an FS-closure operator **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X with respect to an fs-set $A_f(s)$. Then $\delta_{(\mathbf{Cl})^n_{A_f(s)}} = \{\overline{1}, B;$ $B \in I^X$ and $(\mathbf{Cl})^n_{A_f(s)}(B^c) = B^c\}$ forms a fuzzy topology on X. Further, the closure in the FTS $(X, \delta_{(\mathbf{Cl})^n_{A_f(s)}})$ and $(\mathbf{Cl})^n_{A_f(s)}$ are identical on $I^X - \{\overline{0}\}$.*
Proof: *Proof is omitted.*

Definition 3.4. The fuzzy topology $\delta_{(\mathbf{Cl})^n_{A_f(s)}} = \{\overline{1}, B; B \in I^X$ and $(\mathbf{Cl})^n_{A_f(s)}(B^c) = B^c\}$ induced by the n^{th} relative FS-closure operator $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ is called the n^{th} relative fuzzy topology induced by the FS-closure operator **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ with respect to the fs-set $A_f(s)$.

Theorem 3.5. *Let $A_f(s) = \{A^n_f\}^{\infty}_{n=1}$ be an fs-set in a set X and **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. Let $(\mathbf{Cl})^n_f$, $n \in \mathbb{N}$ be the n^{th} component of **Cl**. Then*
(1) $\mathbf{Cl}(A_f(s)) \geq \{(\mathbf{Cl})^n_f(A^n_f)\}$ and the equality holds if $A_f(s)$ is a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$.
(2) If $\mathbf{Cl}(A_f(s)) = \{(\mathbf{Cl})^n_f(A^n_f)\}$ and A_n is closed in $(X, \delta_{(\mathbf{Cl})^n_f})$ for each $n \in \mathbb{N}$, then $A_f(s)$ is closed in $(X, \delta_{\mathbf{Cl}}(s))$.
(3) $\mathbf{Cl}(A_f(s)) = \{(\mathbf{Cl})^n_{A_f(s)}(A^n_f)\}$.
Proof: *Proof is omitted.*

In an FSTS $(X, \delta(s))$ if $A_f(s) = \{A^n_f\}^{\infty}_{n=1}$ is closed, then A^n_f is closed in (X, δ_n) for each $n \in \mathbb{N}$ but the converse is not true [13]. **Corollary** 3.6 provides a pair of if and only if conditions for an fs-set $A_f(s)$ to be closed in an FSTS.

Corollary 3.6. *In an FSTS $(X, \delta(s))$, an fs-set $A_f(s) = \{A^n_f\}^{\infty}_{n=1}$ is closed:*
(1) if and only if $\overline{A_f(s)} = \{B^n_f\}$ and A^n_f is closed in (X, δ_n) for each $n \in \mathbb{N}$, where $B^n_f = n^{th}$ component of $\overline{{}_{nA^n_f}X^0_f(s)}$.
(2) if and only if A^n_f is closed in $(X, \delta_{R^n_{A_f(s)}})$ for each $n \in \mathbb{N}$, where $R^n_{A_f(s)}$ is the n^{th} relative FS-closure operator of the closure operator in $(X, \delta(s))$ with respect to $A_f(s)$.

Theorem 3.7. *If $\{A_{\lambda f}(s); \lambda \in \Lambda\}$ be a chain of fs-sets in $((I^X)^{\mathbb{N}}, \leq)$, then $\{\delta_{(\mathbf{Cl})^n_{A_{\lambda f}(s)}}, \lambda \in \Lambda\}$ is a chain of fuzzy topologies on X for each $n \in \mathbb{N}$, where **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is an FS-closure*

operator on X.
Proof: *Let $A_{\lambda f}(s) \leq A_{\mu f}(s)$, λ, $\mu \in \Lambda$. It suffices to show that $\delta_{(\mathbf{Cl})^n_{A_{\mu f}(s)}} \leq \delta_{(\mathbf{Cl})^n_{A_{\lambda f}(s)}}$.*

$$\text{Let } B \in \delta_{(\mathbf{Cl})^n_{A_{\mu f}(s)}} \Rightarrow (\mathbf{Cl})^n_{A_{\mu f}(s)}(\overline{1} - B) = \overline{1} - B$$
$$\Rightarrow n^{th} \text{ term of } \mathbf{Cl}({}_{n(\overline{1}-B)}A_{\mu f}(s)) = \overline{1} - B$$

Therefore n^{th} term of $\mathbf{Cl}({}_{n(\overline{1}-B)}A_{\lambda f}(s)) \leq \overline{1} - B$
$\Rightarrow (\mathbf{Cl})^n_{A_{\lambda f}(s)}(\overline{1} - B) \leq \overline{1} - B$.

Hence $B \in \delta_{(\mathbf{Cl})^n_{A_{\lambda f}(s)}}$.

Definition 3.8. Each member except possibly $\overline{1}$ of $\delta_{(\mathbf{Cl})^n_{A_f(s)}}$ is contained in $\overline{1} - (\mathbf{Cl})^n_{A_f(s)}(\overline{0})$ and so $\delta_{(\mathbf{Cl})^n_{A_f(s)}}$ is called $(\overline{1} - (\mathbf{Cl})^n_{A_f(s)}(\overline{0}))$-cut of $\delta_{(\mathbf{Cl})^n_f}$.

Theorem 3.9. *Let $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators on X. Then the operator $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $C(A_f(s)) = \{C_n(A^n_f)\}$ for all $A_f(s) = \{A^n_f\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$ is an FS-closure operator on X.*
Proof: *The proof is omitted.*

Definition 3.10. Let $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators on X. The operator $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $C(A_f(s)) = \{C_n(A^n_f)\}$ for all $A_f(s) = \{A^n_f\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$ is called an FS-closure operator induced by a sequence $\{C_n : I^X \to I^X\}$ of fuzzy closure operators on X.

Definition 3.11. Let δ and δ' be two fuzzy topologies on a set X. A subset K_f of δ'^{δ} is called an FS-connector of δ to δ' if it satisfies the following conditions:
(1) $A_\lambda \in \delta$ and $f_\lambda \in K_f$, $\lambda \in \Lambda \Rightarrow$ there exist $f \in K_f$ so that $f(\vee_{\lambda \in \Lambda} A_\lambda) = \vee_{\lambda \in \Lambda} f_\lambda(A_\lambda)$,
(2) $A_i \in \delta$ and $f_i \in K_f$, $i = 1(1)n \Rightarrow$ there exist $f \in K_f$ so that $f(\wedge_{i=1}^n A_i) = \wedge_{i=1}^n f_i(A_i)$ and
(3) $\delta' = \vee_{f \in K_f} f(\delta)$.

Example 3.12. Let δ and δ' be two fuzzy topologies on a set X. A function $f : \delta \to \delta'$ defined by $f(A) = O$ for all $A \in \delta$, where O is a fixed element of δ', is called a constant function from δ into δ'. If K_f be the collection of all such constant functions from δ into δ', then K_f forms an FS-connector from δ to δ'.

Definition 3.13. Let δ and δ' be two fuzzy topologies on a set X. Then the collection of all constant functions from δ into δ' forms an FS-connector of δ to δ'. This is called the discrete FS-connector of δ to δ'.

If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X, then any sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, provides a unique FST on X (**Theorem** 3.14) which is denoted by $\delta(s) < \{\delta_n\}, \{K_n\} >$ such that the n^{th} components $(\delta < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$ and it is called the FST generated by $\{\delta_n\}$ and $\{K_n\}$. If further each K_n is the discrete FS-connector of δ_n to δ_{n+1}, then the FST is said to be generated by $\{\delta_n\}$ and is denoted by $\delta < \{\delta_n\} >$.

Theorem 3.14. *Let $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_n\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$, $n \in \mathbb{N}$. Also for any FSTS $(X,\delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} and $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$.*
Proof: *Let $K = \prod_{n=1}^{\infty} K_n$, $g = \{g_n\} \in K$ and $A \in \delta_1$. Define $H_1 = A$ and $H_n = g_{n-1}g_{n-2}.....g_2g_1A$, $n > 1$. Let $H^g_A(s) = \{H_n\} \in (I^X)^{\mathbb{N}}$ and consider $\delta(s) < \{\delta_n\}, \{K_n\} >= \{X^1_f(s), X^0_f(s)\} \vee \{H^g_A(s); g \in K$ and $A \in \delta_1\}$. Consider*

$$H_\lambda(s) = H^{g_\lambda}_{A_\lambda}(s) \in \delta(s),\ \lambda \in \Lambda$$

where Λ is an index set and

$$A = \vee_{\lambda \in \Lambda} A_\lambda \in \delta_1.$$

For $g_{\lambda 1} \in K_1$ and $A \in \delta_1$ there exist $g_1 \in K_1$ such that

$$g_1 A = \vee_{\lambda\in\Lambda} g_{\lambda 1} A_\lambda;\ g_{\lambda n} \in K_n$$

and for $g_{n-1} g_{n-2} \ldots\ldots g_2 g_1 A \in \delta_n$ there exist $g_n \in K_n$ such that

$$g_n g_{n-1} \ldots\ldots g_2 g_1 A = \vee_{\lambda\in\Lambda} g_{\lambda n} g_{\lambda(n-1)} \ldots . g_{\lambda 2} g_{\lambda 1} A_\lambda.$$

Obviously,

$$\vee_{\lambda\in\Lambda} H_\lambda(s) = \vee_{\lambda\in\Lambda} H^{g_\lambda}_{A_\lambda}(s) = H^g_A(s) \in \delta(s) < \{\delta_n\}, \{K_n\} >$$

where $g = g_n$. Arguing in the same way it can be shown that $\delta(s) < \{\delta_n\}, \{K_n\} >$ is closed under finite intersection. Therefore, $(X,\ \delta(s) < \{\delta_n\}, \{K_n\} >)$ is a fuzzy sequential topological space. The third condition to be an FS-connector ensures that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$. For the next part, for each $n \in \mathbb{N}$ define a relation $R^{n,n+1}$ on $\delta(s)$ by $A_f(s) = \{A^n_f\} R^{n,n+1} B_f(s) = \{B^n_f\}$ if and only if $A^n_f = B^n_f$. Then $R^{n,n+1}$ defines a partition of $\delta(s)$ say

$$\{Cls(A_f(s));\ A_f(s) \in \delta^{n,n+1}(s) \subset \delta(s)\}$$

where $\delta^{n,n+1}(s)$ is a family of open fs-sets taking exactly one from each class of the partition of $\delta(s)$ by $R^{n,n+1}$ and $Cls(A_f(s))$ represents the class of $A_f(s)$. Let

$$K^{n,n+1} = \prod_{A_f(s)\in\delta^{n,n+1}(s)} Cls(A_f(s))$$

Then each $t \in K^{n,n+1}$ defines a function $g_t : \delta_n \to \delta_{n+1}$ and $K_n = \{g_t;\ t \in K^{n,n+1}\}$ is an FS-connector connecting δ_n to δ_{n+1} and properties of FS-connectors ensures that $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$.

Corollary 3.15. *Let **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{Cl})^n_f}$ to $\delta_{(\boldsymbol{Cl})^{n+1}_f}$ for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >)_n = \delta_{(\boldsymbol{Cl})^n_f}$ and the components of the closure operator on $(X,\ \delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >)$ are $(\boldsymbol{Cl})^n_f$, $n \in \mathbb{N}$. Also for any FSTS $(X, \delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{Cl})^n_f}$ to $\delta_{(\boldsymbol{Cl})^{n+1}_f}$ and $\delta(s) = \delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >$.*

Corollary 3.16. *Let **I** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{I})^n_f}$ to $\delta_{(\boldsymbol{I})^{n+1}_f}$ for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >)_n = \delta_{(\boldsymbol{I})^n_f}$ and the components of the interior operator on $(X,\ \delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >)$ are $(\boldsymbol{I})^n_f$, $n \in \mathbb{N}$. Also for any FSTS $(X, \delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{I})^n_f}$ to $\delta_{(\boldsymbol{I})^{n+1}_f}$ and $\delta(s) = \delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >$.*

Corollary 3.17. *If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X such that $\delta_n = \delta$ for all $n \in \mathbb{N}$, then $\delta(s) < \{\delta_n\} >= \delta^{\mathbb{N}}$.*

Corollary 3.18. *If $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators and **C** be an FS-closure operator induced by $\{C_n\}$, then $\delta_{\boldsymbol{C}}(s) = \delta(s) < \{\delta_n\} >$ where δ_n is the fuzzy topology on X induced by C_n, $n \in \mathbb{N}$.*

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Received: July 13, 2013.

Accepted: October 20, 2013.

International Journal of Mathematical Archive-7(2), 2016, 120-128
Available online through www.ijma.info ISSN 2229 – 5046

SOME NEARLY OPEN SETS IN A FUZZY SEQUENTIAL TOPOLOGICAL SPACE

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(Received On: 19-01-16; Revised & Accepted On: 15-02-16)

ABSTRACT

The present article gives a study of fs-semiopen sets, fs-regular open sets and fs-semicontinuous functions in a fuzzy sequential topological space. Other studied notions are fs-almost continuous functions, fs-weakly continuous functions and it has been shown that both of these functions and fs-semicontinuous functions are independent notions. Further, many results relating these functions together with fs-continuous functions have been obtained.

Keywords and Phrases: *Fuzzy sequential topological spaces, fs-semiopen sets, fs-semicontinuous functions, fs-regular open sets, fs-almost continuous functions, fs-weakly continuous functions.*

***AMS Subject Classification**: 54A40.*

1. PRELIMINARIES

The introduction of fuzzy sets in 1965, by L.A. Zadeh [12] leads to the foundation of a new area of research called fuzzy mathematics. Since then, many researchers have been working in this area and related areas. As a generalization of a topological space, C. L. Chang [3] introduced the concept of fuzzy topological space in 1968. Fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad [1].

The purpose of this work is to study the concept of semi-open sets and semicontinuity in fuzzy sequential topological spaces.

Throughout the paper, X will denote a non empty set and I the unit interval [0, 1]. Sequences of fuzzy sets in X called fuzzy sequential sets (fs-sets) will be denoted by the symbols $A_f(s), B_f(s), C_f(s)$ etc. An fs-set $X_f^l(s)$ is a sequence of fuzzy sets $\{X_f^n\}_n$, where $l \in I$ and $X_f^n(x) = l$, for all $x \in X, n \in \mathbb{N}$.

A family $\delta(s)$ of fuzzy sequential sets on a non-empty set X satisfying the properties:

i. $X_f^r(s) \in \delta(s)$ for all $r \in \{0,1\}$,

ii. $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$

iii. for any family $\{A_{fj}(s);\ j \in J\} \subseteq \delta(s), \vee_{j\in J} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called a fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets. Complement of an open fuzzy sequential set is called closed fuzzy sequential set. In an FSTS $(X, \delta(s))$, the closure $\overline{A_f(s)}$ and interior $A_f^\circ(s)$ of any fs-set $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\ A_f(s) \le C_f(s),\ (C_f(s))^c \in \delta(s)\},$$

$$A_f^o(s) = \vee\{O_f(s);\ O_f(s) \le A_f(s)), O_f(s) \in \delta(s)\},$$

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[10] Let g be a mapping from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then g is called
(i) fs-continuous if $g^{-1}(B_f(s))$ is open in $(X, \delta(s))$ for every open fs-set $B_f(s)$ in $(Y, \eta(s))$.
(ii) fs-open if $g(A_f(s))$ is fs-open in Y for every fs-open set $A_f(s)$ in X.
(iii) fs-closed if $g(A_f(s))$ is fs-closed in Y for every fs-closed set $A_f(s)$ in X.

Section **2** deals with the introduction and study of fs-semiopen sets as well as fs-semicontinuity. Section **3** deals with the introduction of fs-regular open sets and functions like fs-almost continuous and fs-weakly continuous functions. In this section, the interrelations among these functions together with fs-continuous and fs-semicontinuous functions have been investigated.

2. FS-SEMIOPEN SETS AND FS-SEMICONTINUITY

Definition 2.1: An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiopen set if $A_f(s) \leq \overline{A_f^o(s)}$. An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiclosed set if its complement is fs-semiopen.

Fundamental properties of fs-semiopen (fs-semiclosed) sets are:
- Any union (intersection) of fs-semiopen (fs-semiclosed) sets is fs-semiopen (fs-semiclosed).
- Every fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).
- Closure (interior) of an fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).

Example 2.1 shows that an fs-semiopen (fs-semiclosed) set may not be fs open (fs-closed), the intersection (union) of any two fs-semiopen (fs semiclosed) sets need not be an fs-semiopen (fs-semoclosed) set. Unlike in a general topological space, the intersection of an fs-semiopen set with an fs open set may fail to be an fs-semiopen set.

Example 2.1: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots\dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots\dots\right\}$$
$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots\dots\right\}$$
$$D_f(s) = \left\{\overline{\frac{3}{8}}, \bar{0}, \bar{0}, \dots\dots\right\}$$

Consider $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Now,
(i) $B_f(s)$ is fs-open, hence fs-semiopen and $C_f(s)$ is fs-semiopen but their intersection $D_f(s)$ is not fs-semiopen.
(ii) $C_f(s)$ is fs-semiopen but is not fs-open.

Theorem 2.1: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiopen if and only if there exist an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$.

Proof: Straightforward.

Theorem 2.2: Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-semiclosed if and only if there exist an fs-closed set $C_f(s)$ in X such that $C_f^o(s) \leq A_f(s) \leq C_f(s)$.

Proof: Straightforward.

We will denote the set of all fs-semiopen sets in X by $FSSO(X)$.

Theorem 2.3: In an FSTS $(X, \delta(s))$, (i) $\delta(s) \subseteq FSSO(X)$. (ii) If $A_f(s) \in FSSO(X)$ and $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, then $B_f(s) \in FSSO(X)$.

Proof:
(i) Follows from definition.
(ii) Let $A_f(s) \in FSSO(X)$. Then there exists an fs-open set $O_f(s)$ such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. So,
$$O_f(s) \leq A_f(s) \leq B_f(s) \leq \overline{A_f(s)} \leq \overline{O_f(s)}$$
$$\Rightarrow O_f(s) \leq B_f(s) \leq \overline{O_f(s)}.$$
$O_f(s)$ being fs-open, $B_f(s)$ is fs-semiopen.

Theorem 2.4: If in a fuzzy sequential topological space, $C_f^o(s) \leq B_f(s) \leq C_f(s)$, where $C_f(s)$ is fs-semiclosed, then $B_f(s)$ is also fs-semiclosed.

Proof: Omitted.

Theorem 2.5: Let $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in an FSTS $(X, \delta(s))$ such that (i) $\delta(s) \subseteq \mho$ and (ii) if $A_f(s) \in \mho$ and $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, then $B_f(s) \in \mho$. Then $FSSO(X) \subseteq \mho$. that is, $FSSO(X)$ is the smallest class of fs-sets in X satisfying (i) and (ii).

Proof: Let $A_f(s) \in FSSO(X)$. Then $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$ for some $O_f(s) \in \delta(s)$. By (i), $O_f(s) \in \mho$ and thus $A_f(s) \in \mho$ by (ii).

If $\mho = \{A_{\alpha f}(s);\ \alpha \in \Lambda\}$ be a collection of fs-sets in X, then $Int\mho$ denotes the set $\{A_{\alpha f}^o(s);\ \alpha \in \Lambda\}$.

Theorem 2.6: If $(X, \delta(s))$ be a fuzzy sequential topological space, then $\delta(s) = Int(FSSO(X))$.

Proof: Every fs-open set being fs-semiopen, $\delta(s) \subseteq Int(FSSO(X))$. Conversely, let $O_f(s) \in Int(FSSO(X))$. Then $O_f(s) = A_f^o(s)$ for some $A_f(s) \in FSSO(X)$ and hence $O_f(s) \in \delta(S)$.

Definition 2.2: Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. We define semi-closure $sCl(A_f(s))$ and semi-interior $sInt(A_f(s))$ of $A_f(s)$ by

$$sCl\left(A_f(s)\right) = \wedge\{B_f(s);\ A_f(s) \leq B_f(s)\ and\ A_f^c(s) \in FSSO(X)\}$$
$$sInt\left(B_f^c(s)\right) = \vee\{C_f(s);\ C_f(s) \leq A_f(s)\ and\ C_f(s) \in FSSO(X)\}.$$

Obviously, $sCl\left(A_f(s)\right)$ is the smallest fs-semiclosed set containing $A_f(s)$ and $sInt\left(A_f(s)\right)$ is the largest fs-semiopen set contained in $A_f(s)$. Further,

(i) $A_f(s) \leq sCl\left(A_f(s)\right) \leq \overline{A_f(s)}$ and $A_f^o(s) \leq sInt\left(A_f(s)\right) \leq A_f(s)$.
(ii) $A_f(s)$ is fs-semiopen if and only if $A_f(s) = sInt\left(A_f(s)\right)$.
(iii) $A_f(s)$ is fs-semiclosed if and only if $A_f(s) = sCl\left(A_f(s)\right)$.
(iv) $A_f(s) \leq B_f(s)$ implies $sInt\left(A_f(s)\right) \leq sInt\left(B_f(s)\right)$ and $sCl\left(A_f(s)\right) \leq sCl\left(B_f(s)\right)$.

Definition 2.3: A mapping $g: (X, \delta(s)) \to (Y, \delta'(s))$ is said to be
(i) fs-semicontinuous if $g^{-1}(B_f(s))$ is fs-semiopen in X for every $B_f(s) \in \delta'(s)$.
(ii) fs-semiopen if $g(A_f(s))$ is fs-semiopen in Y for every $A_f(s) \in \delta(s)$.
(iii) fs-semiclosed if $g(A_f(s))$ is fs-semiclosed in Y for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-semicontinuous (fs-semiopen, fs-semiclosed). That the converse may not be true, is shown by Example 2.2.

Example 2.2: Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \bar{1}, \bar{1}, \dots \dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \bar{0}, \bar{0}, \dots \dots\right\}$$
$$C_f(s) = \left\{\overline{\frac{3}{8}}, \bar{1}, \bar{1}, \dots \dots\right\}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$. Define $g: (X, \delta(s)) \to (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-semicontinuous but not fs-continuous.

Again the map $h: (X, \delta'(s)) \to (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-semiopen and fs-semiclosed but is neither fs-open nor fs-closed.

Now consider the map $t: (X, \eta(s)) \to (X, \delta(s))$ defined by $t(x) = x$ for all $x \in X$, where $\eta(s) = \{C_f^c(s), X_f^0(s), X_f^1(s)\}$. Then t is fs-semiclosed but not fs-closed.

Theorem 2.7: Let $g: (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:
(i) g is fs-semicontinuous.
(ii) the inverse image of an fs-closed set in Yunder g is fs-semiclosed in X.
(iii) For any fs-set $A_f(s)$ in X, $g\left(sCl\left(A_f(s)\right)\right) \leq \overline{g\left(A_f(s)\right)}$.

Proof:
$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous map and $B_f(s)$ be an fs-closed set in Y. Then

$B_f^c(s)$ is fs-open in Y

$$\Rightarrow \left(g^{-1}\left(B_f(s)\right)\right)^c = g^{-1}\left(B_f^c(s)\right) \text{ is fs-semiopen in } X$$
$$\Rightarrow g^{-1}\left(B_f(s)\right) \text{ is fs-semiclosed in } X.$$

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Suppose $A_f(s)$ be an fs-set in X. Then by (ii), $g^{-1}(\overline{g(A_f(s))})$ is fs-semiclosed in X and hence $g^{-1}\left(\overline{g\left(A_f(s)\right)}\right) = sCl(g^{-1}(\overline{g(A_f(s))})$). Again

$$A_f(s) \leq g^{-1}\left(g\left(A_f(s)\right)\right)$$
$$\Rightarrow sCl\left(A_f(s)\right) \leq sCl\left(g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)\right) = g^{-1}\left(\overline{g\left(A_f(s)\right)}\right)$$
$$\Rightarrow g\left(sCl\left(A_f(s)\right)\right) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}$$

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $B_f(s)$ be an fs-open set in Y. Then for the fs-closed set $B_f^c(s)$, we have

$$g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right) \leq \overline{g\left(g^{-1}\left(B_f^c(s)\right)\right)} \leq \overline{B_f^c(s)} = B_f^c(s)$$

Thus $sCl\left(g^{-1}\left(B_f^c(s)\right)\right) \leq g^{-1}\left(g\left(sCl\left(g^{-1}\left(B_f^c(s)\right)\right)\right)\right) \leq g^{-1}\left(B_f^c(s)\right)$.

Therefore $sCl\left(g^{-1}\left(B_f^c(s)\right)\right) = g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s))$ is fs-semiclosed in X.

Theorem 2.8: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiopen set in Y is fs-semiopen in X.

Proof: Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Ysuch that

$$O_f(s) \leq B_f(s) \leq \overline{O_f(s)}$$
$$\Rightarrow g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)})$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $g(U_f(s))$ is a weak open Q-nbd of $g(P_f(s))$. Therefore

$$g(U_f(s))\, q_w\, O_f(s)$$
$$\Rightarrow U_f(s)\, q_w\, g^{-1}(O_f(s))$$
$$\Rightarrow P_f(s) \in \overline{g^{-1}(O_f(s))}.$$

Thus we have, $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Hence, $g^{-1}(O_f(s))$ being fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen.

Corollary 2.1: Suppose $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiclosed set in Y is fs-semiclosed in X.

Proof: Proof is omitted.

Corollary 2.2: If $g: (X, \delta(s)) \to (Y, \delta'(s))$ be an fs-semicontinuous open map and $h: (Y, \delta'(s)) \to (Z, \eta(s))$ be an fs-semicontinuous map, then $hog: (X, \delta(s)) \to (Z, \eta(s))$ is fs-semicontinuous.

Proof: Let $C_f(s)$ be an fs-open set in Z, then $h^{-1}(C_f(s))$ is fs-semiopen in $Y$and hence
$(hog)^{-1}\left(C_f(s)\right) = g^{-1}(h^{-1}(C_f(s)))$ is fs-semiopen in X by Theorem 2.8.

Theorem 2.9: Let $g: (X, \delta(s)) \to (Y, \eta(s))$ be an fs-continuous open map. Then the g-image of an fs-semiopen set in X is fs-semiopen in Y.

Proof: Let $A_f(s)$ be an fs-semiopen set in X. Then there exists an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. This implies

$$g(O_f(s)) \leq g(A_f(s)) \leq g(\overline{O_f(s)}) \leq \overline{g(O_f(s))}.$$

Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-semiopen in Y.

Corollary 2.3: Semi-openness in an FSTS is a topological property.

Proof: Follows from Theorem 2.9.

Remark 2.1: Theorem 2.9 does not hold if g is not fs-open. This is shown by Example 2.3.

Example 2.3: Let $(X, \delta(s))$ and $(Y, \delta'(s))$ be two fuzzy sequential topological spaces, where $\delta(s)$ contains all the constant fs-sets in X, $Y = [0, 1]$ and $\delta'(s) = \{Y_f^0(s),\ Y_f^1(s)\}$. Define a map $g: (X, \delta(s)) \to (Y, \delta'(s))$ by $g(x) = \frac{1}{2}$ for all $x \in X$. Then g is fs-continuous but not fs-open. Here, for any fs-semiopen set $A_f(s)$ in X, $g(A_f(s)) = \left\{\overline{\frac{1}{2}}\right\}_{n=1}^{\infty}$ is not fs-semiopen in Y.

Remark 2.2: Converse of Theorem 2.9 holds if g is one-one.

Theorem 2.10: Let $g: (X, \delta(s)) \to (Y, \delta'(s))$ and $h: (Y, \delta'(s)) \to (Z, \eta(s))$ be two mappings and $hog: (X, \delta(s)) \to (Z, \eta(s))$ be an fs-semiclosed mapping. Then, g is fs-semiclosed if h is an injective fs-semicontinuous open mapping.

Proof: Let $A_f(s)$ be an fs-closed set in X. Then $hog(A_f(s))$ is fs-semiclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-semiclosed in Y.

Theorem 2.11: If $g: (X, \delta(s)) \to (Y, \delta'(s))$ is fs-semicontinuous and $h: (Y, \delta'(s)) \to (Z, \eta(s))$ is fs-continuous, then $hog: (X, \delta(s)) \to (Z, \eta(s))$ is fs-semicontinuous.

Proof: Omitted.

3. FS-REGULAR OPEN SETS

Definition 3.1 An fs-set $A_f(s)$ in an FSTS $(X, \delta(s))$, is said to be fs-regular open in X if $(\overline{A_f(s)})^o = A_f(s)$. An fs-set $A_f(s)$ is said to be fs-regular closed in X if its complement is fs-regular open.

It is obvious that every fs-regular open (closed) set is fs-open (closed). The converse need not be true, is shown by Example 3.1. Example 3.2 shows that the union (intersection) of any two fs-regular open (closed) sets need not be an fs-regular open (closed) set.

Example 3.1: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, \ldots\ldots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \ldots\ldots\right\}$$

Let $\delta(s) = \{A_f(s),\ B_f(s),\ A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS where $A_f(s)$ is fs-open but not fs-regular open.

Example 3.2: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \ldots\ldots\right\}$$
$$B_f(s) = \left\{\overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}} \ldots\ldots\right\}$$

Let $\delta(s) = \{A_f(s),\ B_f(s),\ A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s),\ X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Here $A_f(s)$ and $B_f(s)$ are fs-regular open sets but their union is not fs-regular open.

Theorem 3.1:
(a) The intersection of two fs-regular open sets is an fs-regular open set.
(b) The union of two fs-regular closed sets is an fs-regular closed set.

Proof: We prove only (a). Let $A_f(s)$ and $B_f(s)$ be two fs-regular open sets in X. Since $A_f(s) \wedge B_f(s)$ is fs-open, we have $A_f(s) \wedge B_f(s) \leq (\overline{A_f(s) \wedge B_f(s)})^o$.

Now, $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{A_f(s)})^o = A_f(s)$ and $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{B_f(s)})^o = B_f(s)$ implies $(\overline{A_f(s) \wedge B_f(s)})^o \leq$ $Af(s)\wedge Bf(s)$. Hence the result.

Theorem 3.2:
(a) The closure of an fs-open set is fs-regular closed.
(b) The interior of an fs-closed set is fs-regular open.

Proof: We prove only (a). Let $A_f(s)$ be an fs-open set in X. Since $(\overline{A_f(s)})^o \leq \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \leq \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-open, $A_f(s) \leq (\overline{A_f(s)})^o$ and hence $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus $\overline{A_f(s)}$ is fs-regular closed.

Definition 3.2: A mapping $g:(X,\delta(s)) \to (Y,\eta(s))$ is called an fs-almost continuous mapping if $g^{-1}(B_f(s)) \in \delta(s)$ for each fs-regular open set $B_f(s)$ in Y.

Theorem 3.3: Let $g:(X,\delta(s)) \to (Y,\eta(s))$ be a mapping. Then the following are equivalent:
(i) g is fs-almost continuous.
(ii) $g^{-1}(B_f(s))$ is an fs-closed set for each fs-regular closed set $B_f(s)$ of Y .
(iii) $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$ for each fs-open set $B_f(s)$ of Y .
(iv) $\overline{g^{-1}(\overline{B_f^o(s)})} \leq g^{-1}(B_f(s))$ for each fs-closed set $B_f(s)$ of Y .

Proof: Here, we note that $g^{-1}\left(B_f^c(s)\right) = (g^{-1}(B_f(s))\,)^c$ for any fs-set $B_f(s)$ in Y.
$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Follows from the fact that an fs-set is fs-regular open if and only if its complement is fs-regular closed.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Let $B_f(s)$ be an fs-open set in Y. Then $B_f(s) \leq (\overline{B_f(s)})^o$ and hence $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$. By Theorem 3.2 (b), $(\overline{B_f(s)})^o$ is an fs-regular open set in Y. Therefore, $g^{-1}((\overline{B_f(s)})^o)$ is fs-open in X and thus

$$g^{-1}\left(B_f(s)\right) \leq g^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = (g^{-1}((\overline{B_f(s)})^o))^o.$$

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $B_f(s)$ be an fs-regular open set in Y. Then by (iii), we have $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$. Hence $g^{-1}(B_f(s))$ is an fs-open set in X.

$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iv})$: are easy to prove.

Clearly an fs-continuous map is an fs-almost continuous map but the converse may not be true, as is shown by Example 3.3.

Example 3.3: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \bar{1}, \quad \bar{1}, \quad \bar{1}, \dots \dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots \dots .\right\}$$

Let $\delta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), \quad A_f(s) \wedge B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X,\delta(s))$ and $(X,\eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X,\delta(s)) \to (X,\eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-almost continuous but not fs-continuous. Again, since the inverse image of fs-open set $A_f(s)$ of $(X,\eta(s))$ is not fs-semiopen in $(X,\delta(s))$, g is not fs-semicontinuous.

Example 3.4: Example to show that an fs-semicontinuous map may not be fs-almost continuous. Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \bar{0}, \quad \bar{0}, \quad \bar{0}, \dots \dots\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots \dots .\right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-semicontinuous but not fs-almost continuous.

Remark 3.1: Example 3.3 and Example 3.4 shows that an fs-almost continuous mapping and an fs-semicontinuous mapping are independent notions.

Definition 3.3: An FSTS $(X, \delta(s))$ is called an fs-semiregular space if the collection of all fs-regular open sets in X forms a base for $\delta(s)$.

Theorem 3.4: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping, where $(Y, \eta(s))$ is an fs-semiregular space. Then g is fs-almost continuous if and only if g is fs-continuous.

Proof: We need only to show that if g is fs-almost continuous, then it is fs-continuous. Suppose g is fs-almost continuous. Let $B_f(s) \in \eta(s)$, then $B_f(s) = \vee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where $B_{\lambda f}(s)$'s are fs-regular open sets in Y . Then

$$\begin{aligned} g^{-1}(B_f(s)) &= \vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \vee_{\lambda\in\Lambda}(g^{-1}((\overline{B_{\lambda f}(s)})^o))^o \\ &= \vee_{\lambda\in\Lambda}(g^{-1}(B_{\lambda f}(s)))^o \\ &\leq (\vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

which shows $g^{-1}(B_f(s)) \in \delta(s)$.

Theorem 3.5: Let X, X_1 and X_2 be fuzzy sequential topological spaces and $\pi_i: X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g: X \to X_1 \times X_2$ is fs-almost continuous, then $\pi_i o g$ is also fs-almost continuous.

Proof: Let g be an fs-almost continuous map and let $B_f(s)$ be an fs-regular open set in X_i. Since π_i is fs-continuous, we have $\overline{{\pi_i}^{-1}(B_f(s))} \leq {\pi_i}^{-1}(\overline{B_f(s)})$ and since π_i is fs-open we have, ${\pi_i}^{-1}(B_f^o(s)) \leq ({\pi_i}^{-1}(B_f(s)))^o$. Also $B_f(s) \leq {\pi_i}^{-1}(\pi_i(B_f(s)))$ and $\pi_i({\pi_i}^{-1}(B_f(s)) \leq B_f(s)$. Thus

$$\pi_i\left(\left({\pi_i}^{-1}\left(B_f(s)\right)\right)^o\right) \leq \pi_i\left({\pi_i}^{-1}\left(B_f(s)\right)\right) \leq B_f(s)$$

$$\Rightarrow \pi_i\left(\left({\pi_i}^{-1}\left(B_f(s)\right)\right)^o\right) \leq B_f^o(s)$$

$$\Rightarrow \left({\pi_i}^{-1}\left(B_f(s)\right)\right)^o \leq {\pi_i}^{-1}(\pi_i\left(\left({\pi_i}^{-1}\left(B_f(s)\right)\right)^o\right)) \leq {\pi_i}^{-1}\left(B_f^o(s)\right) = {\pi_i}^{-1}\left(B_f(s)\right)$$

$$\Rightarrow {\pi_i}^{-1}\left(B_f(s)\right) = \left({\pi_i}^{-1}\left(B_f(s)\right)\right)^o \leq \left(\overline{{\pi_i}^{-1}\left(B_f(s)\right)}\right)^o \leq \left({\pi_i}^{-1}(\overline{B_f(s)})\right)^o = \quad {\pi_i}^{-1}\left(\left(\overline{B_f(s)}\right)^o\right) = {\pi_i}^{-1}\left(B_f(s)\right)$$

$$\Rightarrow {\pi_i}^{-1}\left(B_f(s)\right) = \left(\overline{{\pi_i}^{-1}\left(B_f(s)\right)}\right)^o \quad .$$

Therefore,

$$\begin{aligned} (\pi_i o g)^{-1}\left(B_f(s)\right) &= g^{-1}\left({\pi_i}^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right) \\ &= g^{-1}\left(\left(\overline{{\pi_i}^{-1}\left(B_f(s)\right)}\right)^o\right) \\ &= \left(g^{-1}\left(\left(\overline{{\pi_i}^{-1}\left(B_f(s)\right)}\right)^o\right)\right)^o \\ &\leq \left(g^{-1}\left(\left({\pi_i}^{-1}(\overline{B_f(s)})\right)^o\right)\right)^o \\ &= \left(g^{-1}\left({\pi_i}^{-1}\left(\left(\overline{B_f(s)}\right)^o\right)\right)\right)^o \\ &= \left(g^{-1}\left({\pi_i}^{-1}(B_f(s))\right)\right)^o \\ &= \left((\pi_i o g)^{-1}\left(B_f(s)\right)\right)^o \end{aligned}$$

Hence the theorem.

Definition 3.4: A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-weakly continuous mapping if for each fs-open set $B_f(s)$ in Y, $g^{-1}(B_f(s)) \leq (g^{-1}(\overline{B_f(s)}))^o$.

Remark 3.2: It is clear that every fs-continuous mapping is fs-weakly continuous. The converse is not true, in general, which is shown by Example 3.5. The Example also shows that an fs-weakly continuous mapping may neither be fs-semicontinuous nor fs-almost continuous. However, it is clear that an fs-almost continuous mapping is also fs-weakly continuous.

Example 3.5: Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X as follows:

$$A_f(s) = \left\{\overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \quad \overline{\frac{1}{2}}, \dots\dots.. \right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \quad \overline{\frac{1}{3}}, \dots\dots.. \right\}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ and $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ and $(X, \eta(s))$ are fuzzy sequential topological spaces. Define a map $g : (X, \delta(s)) \to (X, \eta(s))$ by $g(x) = x$ for all $x \in X$. Then g is fs-weakly continuous but not fs-continuous. Since the inverse image of fs-open set $B_f(s)$ of Y is not fs-semiopen in X, hence g is not fs-semicontinuous. Again, as the inverse image of fs-regular open set $B_f(s)$ of Y is not fs-open in X, g is not fs-almost continuous.

Remark 3.3: The map g defined in Example 3.4, is fs-semicontinuous but not fs-weakly continuous.

Remark 3.4: Example 3.5 and Remark 3.3 shows that fs-semicontinuity and fs-weakly continuity are independent notions.

Definition 3.5: An FSTS $(X, \delta(s))$ is called an Ωfs-semiregular space if each fs-open set $A_f(s)$ of X is the union of fs-open sets $A_{\lambda f}(s)$ $(\lambda \in \Lambda)$ of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$.

Theorem 3.6: An Ωfs-semiregular space is fs-semiregular.

Proof: Let $(X, \delta(s))$ be an Ωfs-semiregular space and $A_f(s)$ be an fs-open set in X. Then $A_f(s) = \vee_{\lambda\in\Lambda} A_{\lambda f}(s)$, where $A_{\lambda f}(s)$ are fs-open sets of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$. Since $A_{\lambda f}(s) \leq (\overline{A_{\lambda f}(s)})^o \leq A_f(s)$, we have $A_f(s) = \vee_{\lambda\in\Lambda} (\overline{A_{\lambda f}(s)})^o$. Now, for each $\lambda \in \Lambda$, $(\overline{A_{\lambda f}(s)})^o$ is fs-regular open in X and thus $(X, \delta(s))$ is a fs-semiregular space.

Remark 3.5: Example 3.6 shows that the converse of Theorem 3.6 may not be true.

Example 3.6: Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee$ Bfs, Afs∧Bfs, Xf0s, Xf1s and where the fs-sets *Afs* and *Bfs* in *X*, are defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \quad \bar{1}, \quad \bar{1}, \quad \bar{1}, \dots\dots \right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \quad \bar{0}, \quad \bar{0}, \quad \bar{0}, \dots\dots.. \right\}$$

Then $(X, \delta(s))$ is an fs-semiregular space. Now, the only way of writing $A_f(s)$ as the union of fs-open sets is the union of itself and $\overline{A_f(s)}$ is not contained in $A_f(s)$. Hence $(X, \delta(s))$ is not an Ωfs-semiregular space.

Theorem 3.7: Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping where $(X, \delta(s))$ is any FSTS and $(Y, \eta(s))$ is an Ωfs-semiregular space. Then g is fs-weakly continuous if and only if g is fs-continuous.

Proof: It suffices to show that if g is fs-weakly continuous, then it is fs-continuous. For this, let $B_f(s) \in \eta(s)$. Then $B_f(s) = \vee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where for all $\lambda \in \Lambda$, $B_{\lambda f}(s) \in \eta(s)$ and $\overline{B_{\lambda f}(s)} \leq B_f(s)$. Since g is fs- weakly continuous, we have

$$\begin{aligned} g^{-1}\left(B_f(s)\right) &= g^{-1}\left(\vee_{\lambda\in\Lambda} B_{\lambda f}(s)\right) = \vee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \vee_{\lambda\in\Lambda} (g^{-1}(\overline{B_{\lambda f}(s)}))^o \\ &\leq \vee_{\lambda\in\Lambda} (g^{-1}(B_f(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

and hence $g^{-1}(B_f(s))$ is fs-open in X. Thus g is fs-continuous.

Theorem 3.8: Let X, X_1 and X_2 be FSTS's and $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \to X_1 \times X_2$ is fs-weakly continuous, then $\pi_i o g$ is also fs-weakly continuous.

Proof: The proof is analogous to the proof of Theorem 3.5.

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Source of support: Nil, Conflict of interest: None Declared

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