CHAPTER

7

Decomposition of Continuity

In the last few decades, there has been interests in the study of generalized open sets and generalized continuity in a topological space and various authors studied different kinds of generalized open sets. In the fuzzy setting, fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad [1] (1981) and fuzzy pre-open sets [25] (1982) by A. S. Mashhour, I. A. Hasanein and S. N. EI-Deeb. Apart from these, other sets like fuzzy α-open, fuzzy locally closed, fuzzy δ-set etc. have also been studied in the past.

In **Chapter** 6, we studied generalized open sets like fs-semiopen, fs-preopen and fs-regular open sets in a fuzzy sequential topological space. Here, we study some more of such sets and the respective continuities. Finally, we establish a decomposition of fs-continuity. For our convenience, we denote the closure and interior by *cl* and *int* respectively.

7.1 Decomposition of fs-continuity

Definition 7.1.1 *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is called*

(i) fs α-open if $A_f(s) \leq int\ cl\ int A_f(s)$;

(ii) locally fs-closed if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-closed;

(iii) an fs $\mathcal{A}$-set if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed;

(iv) an fs δ-set if $int\ cl A_f(s) \leq cl\ int A_f(s)$;

(v) fs S-preopen if $A_f(s)$ is fs-preopen and $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $int V_f(s)$ is fs-regular open.

We denote the collection of all fs α-open sets, fs-semiopen sets, fs-preopen sets, fs $\mathcal{A}$-sets, fs S-preopen sets, locally fs-closed sets and fs δ-sets in an FSTS $(X, \delta(s))$, by $\alpha(X)$, $FSSO(X)$, $FSPO(X)$, $\mathcal{A}(X)$, $FSSPO(X)$, $FSLC(X)$ and $\delta(X)$ respectively.

The relationships among different fs-sets defined above, are given by the following diagram:

$$\begin{tikzcd}
 & \text{fs } S\text{-preopen} \arrow[dr] & \\
\text{locally fs-closed} & \text{fs-open} \arrow[l] \arrow[r] \arrow[u] \arrow[dl] \arrow[d] & \text{fs-preopen} \\
\text{fs } \mathcal{A}\text{-set} \arrow[u] & \text{fs } \alpha\text{-open} \arrow[ur] \arrow[d] & \\
 & \text{fs-semiopen} \arrow[d] & \\
 & \text{fs } \delta\text{-set} &
\end{tikzcd}$$

The implications in the above diagram are not reversible. To show this, here we give examples. In Sections 6.1 and 6.3 of **Chapter** 6 respectively, it is already shown that an fs-semiopen and an fs-preopen set may not be fs-open.

Example 7.1.1 *Example to show that an fs α-open set may not be fs-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $C_f(s)$ is fs α-open but is not fs-open.

Example 7.1.2 *Example to show that a locally fs-closed set may not be fs-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $B_f(s)$ is locally fs-closed but not fs-open.

Example 7.1.3 *Example to show that an fs $\mathcal{A}$-set may not be fs-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$, $D_f(s)$ in a set X, defined as follows:

$$\begin{aligned}
A_f(s) &= \left\{\overline{1}, \overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, ...\right\}\\
B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{\frac{1}{2}}\overline{0}, ...\right\}\\
C_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\}\\
D_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, \overline{1}, ...\right\}
\end{aligned}$$

Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), X_f^0(s), X_f^1(s)\}$. Here, $C_f(s) = A_f(s) \wedge D_f(s)$, where $A_f(s)$ is fs-open and $D_f(s)$ is fs-regular closed. Hence $C_f(s)$ is an fs $\mathcal{A}$-set but is not fs-open.

Example 7.1.4 *Example to show that a locally fs-closed set may not be an fs $\mathcal{A}$-set.*

Consider the FSTS $(X, \delta(s))$, given in Example 7.1.2. *Here, $B_f(s)$ is a locally fs-closed set but not an fs $\mathcal{A}$-set.*

Example 7.1.5 *Example to show that an fs-semiopen set may not be an fs $\mathcal{A}$-set.*

Consider the FSTS $(X, \delta(s))$, given in Example 7.1.1. *The fs-set $C_f(s)$ is an fs-semiopen set but not an fs $\mathcal{A}$-set.*

Example 7.1.6 *Example to show that an fs-semiopen set may not be fs α-open.*

In the FSTS, given in Example 7.1.3, *the fs-set $C_f(s)$ is fs-semiopen but not fs α-open.*

Example 7.1.7 *Example to show that an fs-preopen set may not be fs α-open.*

Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \ldots\ldots\ldots\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{5}}, \overline{\frac{1}{5}}, \overline{\frac{1}{5}}, \ldots\ldots.\right\} \end{aligned}$$

Then $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ is a fuzzy sequential topology on X. In this FSTS, $B_f(s)$ is fs-preopen but not fs α-open.

Example 7.1.8 *Example to show that an fs δ-set may not be fs-semiopen.*

In the FSTS, given in Example 7.1.2, *the fs-set $B_f(s)$ is an fs δ-set but is not fs-semiopen.*

Example 7.1.9 *Example to show that an fs-preopen set may not be an fs S-preopen set.*

Consider the FSTS, given in Example 7.1.1, *the fs-set $C_f(s)$ is fs-preopen but not fs S-preopen.*

Example 7.1.10 *Example to show that an fs S-preopen set may not be an fs-open set.*

Consider the FSTS, given in Example 7.1.7, *the fs-set $B_f(s)$ is fs S-preopen but not fs-open.*

Definition 7.1.2 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called*
(i) fs α-continuous if $g^{-1}(B_f(s))$ is fs α-open in X, for every fs-open set $B_f(s)$ in Y.

(ii) fs lc-continuous if $g^{-1}(B_f(s))$ *is locally fs-closed in* X*, for every fs-open set* $B_f(s)$ *in* Y*.*
(iii) fs $\mathcal{A}$*-continuous if* $g^{-1}(B_f(s))$ *is an fs* $\mathcal{A}$*-set in* X*, for every fs-open set* $B_f(s)$ *in* Y*.*
(iv) fs δ*-continuous if* $g^{-1}(B_f(s))$ *is an fs* δ*-set in* X*, for every fs-open set* $B_f(s)$ *in* Y*.*
(v) fs S-precontinuous if $g^{-1}(B_f(s))$ *is fs S-preopen in* X*, for every fs-open set* $B_f(s)$ *in* Y*.*

Theorem 7.1.1 *An fs-set in an FSTS, is fs* α*-open if and only if it is fs-semiopen and fs-preopen.*

Proof. Let $A_f(s)$ be an fs α-open set, that is, $A_f(s) \leq int\ cl\ intA_f(s)$. Clearly, $A_f(s)$ is fs-semiopen. Also,

$$\begin{aligned} intA_f(s) \leq clA_f(s) &\Rightarrow cl\ intA_f(s) \leq clA_f(s) \\ &\Rightarrow int\ cl\ intA_f(s) \leq int\ clA_f(s) \\ &\Rightarrow A_f(s) \leq int\ clA_f(s) \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

Conversely, suppose $A_f(s)$ be fs-semiopen and fs-preopen, that is, $A_f(s) \leq cl\ intA_f(s)$, $A_f(s) \leq int\ clA_f(s)$. Then,

$$\begin{aligned} & int\ clA_f(s) \leq clA_f(s) \leq cl\ intA_f(s) \\ \Rightarrow\ & A_f(s) \leq int\ cl\ intA_f(s) \end{aligned}$$

Hence, $A_f(s)$ is fs α-open. ■

Corollary 7.1.1 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is fs α-continuous if and only it is fs-semicontinuous and fs-precontinuous.*

Definition 7.1.3 *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then αfs-closure ${}_{\alpha}clA_f(s)$ and αfs-interior ${}_{\alpha}intA_f(s)$ of $A_f(s)$ are defined by*

$$\begin{aligned} {}_{\alpha}clA_f(s) &= \wedge\{V_f(s);\ A_f(s) \leq V_f(s) \text{ and } V_f^c(s) \in \alpha(X)\} \\ {}_{\alpha}intA_f(s) &= \vee\{U_f(s);\ U_f(s) \leq A_f(s) \text{ and } U_f(s) \in \alpha(X)\} \end{aligned}$$

Complement of an fs α-open set is called an fs α-closed set. Hence, it is clear that ${}_{\alpha}cl(A_f(s))$ is the smallest fs α-closed set containing $A_f(s)$ and ${}_{\alpha}int(A_f(s))$ is the largest fs α-open set contained in $A_f(s)$. Further,

(*i*) $A_f(s) \leq {}_{\alpha}cl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_{\alpha}int(A_f(s)) \leq A_f(s)$.

(*ii*) $A_f(s)$ is fs α-open if and only if $A_f(s) = {}_{\alpha}int(A_f(s))$

(*iii*) $A_f(s)$ is fs α-closed if and only if $A_f(s) = {}_{\alpha}cl(A_f(s))$

(*iv*) $A_f(s) \leq B_f(s) \Rightarrow {}_{\alpha}int(A_f(s)) \leq {}_{\alpha}int(B_f(s))$ and ${}_{\alpha}cl(A_f(s)) \leq {}_{\alpha}cl(B_f(s))$.

Theorem 7.1.2 *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then,*

(i) ${}_{\alpha}intA_f(s) = A_f(s) \wedge int\ cl\ intA_f(s)$.

(ii) if $A_f(s)$ is both fs-preopen and fs-preclosed, then $A_f(s) = int\ clA_f(s) \wedge A_f(s)$ and thus $A_f(s)$ is fs S-preopen;

(iii) if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $intV_f(s)$

is fs-regular open, then ${}_{\alpha}intA_f(s) = intA_f(s)$*;*

(iv) if $A_f(s)$ *is an fs* δ*-set, then* ${}_{\alpha}intA_f(s) = {}_{p}intA_f(s)$*.*

Proof. (i) Easy to prove.

(ii) Given $A_f(s) \leq int\ clA_f(s)$ and $cl\ intA_f(s) \leq A_f(s)$. Then,

$$A_f(s) = int\ clA_f(s) \wedge A_f(s).$$

Since $intA_f(s) = int\ cl\ intA_f(s)$, hence $A_f(s)$ is fs S-preopen.

(iii) We have $int\ cl\ intA_f(s) \leq int\ cl\ intV_f(s) = intV_f(s)$. Therefore,

$$\begin{aligned} {}_{\alpha}intA_f(s) &= A_f(s) \wedge int\ cl\ intA_f(s) \\ &\leq A_f(s) \wedge intV_f(s) \\ &= intA_f(s) \end{aligned}$$

Also, $intA_f(s) \leq {}_{\alpha}intA_f(s)$. Hence $intA_f(s) = {}_{\alpha}intA_f(s)$.

(iv) Given $int\ clA_f(s) \leq cl\ intA_f(s)$. Since ${}_{\alpha}intA_f(s)$ is an fs-preopen set contained in $A_f(s)$, we have

$${}_{\alpha}intA_f(s) \leq {}_{p}intA_f(s)$$

Now,

$${}_{p}intA_f(s) \leq int\ clA_f(s) \leq int\ cl\ intA_f(s)$$

Thus, ${}_{p}intA_f(s) \leq A_f(s) \wedge int\ cl\ intA_f(s) = {}_{\alpha}intA_f(s)$. Hence the result. ■

Lemma 7.1.1 *An fs-set* $A_f(s)$ *is locally fs-closed if and only if* $A_f(s) = U_f(s) \wedge cl(A_f(s))$*, where* $U_f(s)$ *is an fs-open set.*

Proof. Omitted. ■

Theorem 7.1.3 *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then $A_f(s)$ is an fs $\mathcal{A}$-set if it is fs-semiopen and locally fs-closed.*

Proof. Suppose $A_f(s)$ be fs-semiopen and locally fs-closed. Then, $A_f(s) \leq cl\ intA_f(s)$ and $A_f(s) = U_f(s) \wedge clA_f(s)$, where $U_f(s)$ is fs-open. Since $clA_f(s) = cl\ intA_f(s)$ is fs-regular closed, the result follows. ■

Corollary 7.1.2 *A mapping $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is fs $\mathcal{A}$-continuous if it is fs-semicontinuous and fs lc-continuous.*

Remark 7.1.1 *Unlike in a general topological space, the converse of Theorem* 7.1.3 *may not be true and it has been shown by the next Example.*

Example 7.1.11 *Let $X = \{x, y\}$. Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$, $D_f(s)$ and $E_f(s)$ in X, where*

$A_f^1 = \overline{0.3}$, $A_f^n(x) = 1$ *and* $A_f^n(y) = 0$ *for all* $n \neq 1$;

$B_f^1(x) = 0.4$, $B_f^1(y) = 0.7$, $B_f^n(x) = 0$ *and* $B_f^n(y) = 1$ *for all* $n \neq 1$;

$C_f^1 = \overline{0.7}$ *and* $C_f^n = \overline{1}$ *for all* $n \neq 1$;

$D_f^1(x) = 0.6$, $D_f^1(y) = 0.3$, $D_f^n(x) = 1$ *and* $D_f^n(y) = 0$ *for all* $n \neq 1$;

$E_f^1(x) = 0.4$, $E_f^1(y) = 0.3$ *and* $E_f^n = \overline{0}$ *for all* $n \neq 1$.

Let $\delta(s) = \{A_f(s), B_f(s), C_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$*. In the FSTS* $(X, \delta(s))$*,* $D_f(s)$ *being an fs-regular closed set, the fs-set* $E_f(s) = B_f(s) \wedge D_f(s)$ *is an fs* $\mathcal{A}$*-set but not fs-semiopen.*

Theorem 7.1.4 *Let* $(X, \delta(s))$ *be an FSTS and* $A_f(s)$ *be an fs-set in* X*. Then the following statements are equivalent:*
(i) $A_f(s)$ *is an fs-open set.*
(ii) $A_f(s)$ *is fs* α*-open and locally fs-closed.*
(iii) $A_f(s)$ *is fs-preopen and locally fs-closed.*
(iv) $A_f(s)$ *is fs-preopen and an fs* $\mathcal{A}$*-set.*
(v) $A_f(s)$ *is fs S-preopen and an fs* δ*-set.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (iv)$ Let $A_f(s)$ be fs-preopen and locally fs-closed. Then,

$$A_f(s) \leq int\ clA_f(s) \quad \text{and} \quad A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. $clA_f(s)$ being fs-regular closed, the result follows.

$(iv) \Rightarrow (i)$ Let $A_f(s)$ be an fs-preopen and an fs $\mathcal{A}$-set. Then,

$$A_f(s) \leq int\ clA_f(s) \quad \text{and} \quad A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. Since $intA_f(s) = U_f(s) \wedge int\ clA_f(s)$, $A_f(s)$ is fs-open.

$(i) \Rightarrow (v)$ Obvious.

$(v) \Rightarrow (i)$ Let $A_f(s)$ be an fs S-preopen and an fs δ-set. Using Theorem 7.1.2, (iii) and (iv),

$$intA_f(s) = \ _{\alpha}intA_f(s) = \ _{p}intA_f(s) = A_f(s).$$

Hence, $A_f(s)$ is fs-open. ■

By Theorems 7.1.1, 7.1.3 and 7.1.4, we have the following relationships among the different classes of fs-sets of an FSTS $(X, \delta(s))$:
(i) $\alpha(X) = FSPO(X) \cap FSSO(X)$.
(ii) $\mathcal{A}(X) \supseteq FSSO(X) \cap FSLC(X)$.
(iii) $\delta(s) = \alpha(X) \cap FSLC(X)$.
(iv) $\delta(s) = FSPO(X) \cap FSLC(X)$.
(v) $\delta(s) = FSPO(X) \cap \mathcal{A}(X)$.
(vi) $\delta(s) = FSSPO(X) \cap \delta(X)$.

Theorem 7.1.5 *In an FSTS $(X, \delta(s))$, the following are equivalent:*
(i) *$clA_f(s) \in \delta(s)$ for every $A_f(s) \in \delta(s)$.*
(v) *$\mathcal{A}(X) = \delta(s)$.*

Proof. $(i) \Rightarrow (ii)$ It is obvious that $\delta(s) \subseteq \mathcal{A}(X)$. For the reverse inclusion, let $A_f(s) \in \mathcal{A}(X)$, then

$$A_f(s) = U_f(s) \wedge V_f(s),$$

where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed. By (i), $V_f(s) \in \delta(s)$ and hence $A_f(s) \in \delta(s)$.

$(ii) \Rightarrow (i)$ Suppose $\mathcal{A}(X) = \delta(s)$. Let $A_f(s) \in \delta(s)$, then $clA_f(s)$ is fs-regular closed and hence belongs to $\mathcal{A}(X) = \delta(s)$. ■

We conclude the chapter by stating the following decompositions of fs-continuity:

Theorem 7.1.6 *Let $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be a map. Then g is fs-continuous if and only if*

(i) g is fs α-continuous and fs lc-continuous.

(ii) g is fs-precontinuous and fs lc-continuous.

(iii) g is fs-precontinuous and fs $\mathcal{A}$-continuous.

(iv) g is fs S-continuous and fs δ-continuous.