CHAPTER

6

Some Nearly Open Sets

In this Chapter, some nearly open sets like fs-semiopen, fs-preopen, fs-regular open sets have been studied and interrelations among the continuities associated with them have been investigated.

6.1 FS-semiopen sets and FS-semicontinuity

This section is devoted to the study of fs-semiopen sets and fs-semicontinuity.

Definition 6.1.1 *An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiopen set if $A_f(s) \leq \overline{\overset{o}{A}_f(s)}$. An fs-set $A_f(s)$ in an FSTS, is said to be an fs-semiclosed set if its complement is fs-semiopen.*

Fundamental properties of fs-semiopen (fs-semiclosed) sets are:

- Any union (intersection) of fs-semiopen (fs-semiclosed) sets is fs-semiopen (fs-semiclosed).
- Every fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).
- Closure (interior) of an fs-open (fs-closed) set is fs-semiopen (fs-semiclosed).

Example 6.1.1 shows that an fs-semiopen (fs-semiclosed) set may not be fs-open (fs-closed), the intersection (union) of any two fs-semiopen (fs-semiclosed) sets need not be an fs-semiopen (fs-semiclosed) set. Unlike in a general topological space, the intersection of an fs-semiopen set with an fs-open set may fail to be an fs-semiopen set.

Example 6.1.1 *Consider the fs-sets $A_f(s)$, $B_f(s)$ and $C_f(s)$ in $X = [0,1]$, defined as follows:*

$$\begin{aligned} A_f^1(x) &= 0, \ \text{if } 0 \leq x \leq \frac{1}{2} \\ &= \frac{1}{4}, \ \text{if } \frac{1}{2} < x \leq 1 \\ \text{and } A_f^n &= \overline{1} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} B_f^1(x) &= \frac{1}{2}, \ \text{if } 0 \leq x \leq \frac{1}{2} \\ &= 0, \ \text{if } \frac{1}{2} < x \leq 1 \\ \text{and } B_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

$$C_f^1 = \overline{\frac{3}{8}} \text{ and } C_f^n = \overline{1} \text{ for all } n \neq 1.$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. *Then* $(X, \delta(s))$ *is an FSTS. Now,*

(i) $B_f(s)$ *and* $C_f(s)$ *are fs-semiopen sets but their intersection is not fs-semiopen.*

(ii) $C_f(s)$ *is fs-semiopen but is not fs-open.*

Theorem 6.1.1 *Let* $(X, \delta(s))$ *be an FSTS. An fs-set* $A_f(s)$ *is fs-semiopen if and only if there exists an fs-open set* $O_f(s)$ *in* X *such that* $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$.

Proof. Straightforward. ■

Theorem 6.1.2 *Let* $(X, \delta(s))$ *be an FSTS. An fs-set* $A_f(s)$ *is fs-semiclosed if and only if there exists an fs-closed set* $C_f(s)$ *in* X *such that* $\overset{o}{C}_f(s) \leq A_f(s) \leq C_f(s)$.

Proof. Straightforward. ■

We will denote the set of all fs-semiopen sets in X by $FSSO(X)$.

Theorem 6.1.3 *In an FSTS* $(X, \delta(s))$, *(i)* $\delta(s) \subseteq FSSO(X)$, *(ii) If* $A_f(s) \in FSSO(X)$ *and* $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, *then* $B_f(s) \in FSSO(X)$.

Proof. (i) Follows from definition.

(ii) Let $A_f(s) \in FSSO(X)$. Then, there exists an fs-open set $O_f(s)$ such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. So,

$$\begin{aligned} & O_f(s) \leq A_f(s) \leq B_f(s) \leq \overline{A_f(s)} \leq \overline{O_f(s)} \\ \Rightarrow \ & O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \end{aligned}$$

Since $O_f(s)$ is fs-open, $B_f(s)$ is fs-semiopen. ■

Theorem 6.1.4 *If* $\overset{o}{C}_f(s) \leq B_f(s) \leq C_f(s)$ *holds in an FSTS* $(X, \delta(s))$, *where* $C_f(s)$ *is fs-semiclosed and* $B_f(s)$ *is any fs-set, then* $B_f(s)$ *is also fs-semiclosed.*

Proof. Omitted. ■

Theorem 6.1.5 *Let* $\mathfrak{A} = \{A_{\alpha f}(s); \alpha \in \Lambda\}$ *be a collection of fs-sets in an FSTS* $(X, \delta(s))$ *such that* (i) $\delta(s) \subseteq \mathfrak{A}$ *and* (ii) *if* $A_f(s) \in \mathfrak{A}$ *and* $A_f(s) \leq B_f(s) \leq \overline{A_f(s)}$, *then* $B_f(s) \in \mathfrak{A}$. *Then,* $FSSO(X) \subseteq \mathfrak{A}$, *that is,* $FSSO(X)$ *is the smallest class of fs-sets in* X *satisfying* (i) *and* (ii).

Proof. Let $A_f(s) \in FSSO(X)$. Then, $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$ for some $O_f(s) \in \delta(s)$. By (i), $O_f(s) \in \mathfrak{A}$ and thus $A_f(s) \in \mathfrak{A}$ by (ii). ■

If $\mathfrak{A} = \{A_{\lambda f}(s); \lambda \in \Lambda\}$ be a collection of fs-sets in X, then $Int(\mathfrak{A})$ denotes the set $\{\overset{o}{A}_{\lambda f}(s); \lambda \in \Lambda\}$.

Theorem 6.1.6 *If* $(X, \delta(s))$ *be a fuzzy sequential topological space, then* $\delta(s) = Int(FSSO(X))$.

Proof. It is clear that $\delta(s) \subseteq Int(FSSO(X))$. Conversely, let $O_f(s) \in Int(FSSO(X))$. Then, $O_f(s) = \overset{o}{A}_f(s)$ for some $A_f(s) \in FSSO(X)$ and hence $O_f(s) \in \delta(s)$. ■

Definition 6.1.2 *Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. We define fs-semiclosure $_scl(A_f(s))$ and fs-semiinterior $_sint(A_f(s))$ of $A_f(s)$ by*

$$
\begin{aligned}
{}_scl(A_f(s)) &= \wedge\{B_f(s); A_f(s) \leq B_f(s) \text{ and } B_f^c(s) \in FSSO(X)\} \\
{}_sint(A_f(s)) &= \vee\{C_f(s); C_f(s) \leq A_f(s) \text{ and } C_f(s) \in FSSO(X)\}
\end{aligned}
$$

Obviously, $_scl(A_f(s))$ is the smallest fs-semiclosed set containing $A_f(s)$ and $_sint(A_f(s))$ is the largest fs-semiopen set contained in $A_f(s)$. Further,
(i) $A_f(s) \leq {}_scl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_sint(A_f(s)) \leq A_f(s)$.
(ii) $A_f(s)$ is fs-semiopen if and only if $A_f(s) = {}_sint(A_f(s))$
(iii) $A_f(s)$ is fs-semiclosed if and only if $A_f(s) = {}_scl(A_f(s))$
(iv) $A_f(s) \leq B_f(s) \Rightarrow {}_sint(A_f(s)) \leq {}_sint(B_f(s))$ and $_scl(A_f(s)) \leq_s cl(B_f(s))$.

Definition 6.1.3 *A mapping $g : (X, \delta(s)) \to (Y, \delta'(s))$ is said to be*
(i) fs-semicontinuous if $g^{-1}(B_f(s))$ is fs-semiopen in X for every $B_f(s) \in \delta'(s)$.
(ii) fs-semiopen if $g(A_f(s))$ is fs-semiopen in Y for every $A_f(s) \in \delta(s)$.

(iii) fs-semiclosed if $g(A_f(s))$ is fs-semiclosed in Y for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-semicontinuous (fs-semiopen, fs-semiclosed). That the converse may not be true, is shown by Example 6.1.2.

Example 6.1.2 *Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$. Define $g : (X, \delta(s)) \to (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-semicontinuous but not fs-continuous.

Again, the map $h : (X, \delta'(s)) \to (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-semiopen and fs-semiclosed but is neither fs-open nor fs-closed.

Theorem 6.1.7 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:*

(i) g is fs-semicontinuous.

(ii) the inverse image of an fs-closed set in Y under g is fs-semiclosed in X.

(iii) For any fs-set $A_f(s)$ in X, $g({}_scl(A_f(s))) \leq \overline{g(A_f(s))}$.

Proof. $(i) \Rightarrow (ii)$ Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous map and $B_f(s)$ be an fs-closed set in Y. Then,

$$\begin{aligned}
& B_f^c(s) \text{ is fs-open in } Y \\
\Rightarrow\ & (g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s)) \text{ is fs-semiopen in } X \\
\Rightarrow\ & g^{-1}(B_f(s)) \text{ is fs-semiclosed in } X.
\end{aligned}$$

$(ii) \Rightarrow (iii)$ Suppose $A_f(s)$ be an fs-set in X. Then by (ii), $g^{-1}(\overline{g(A_f(s))})$ is fs-semiclosed in X and hence $g^{-1}(\overline{g(A_f(s))}) = {}_scl(g^{-1}(\overline{g(A_f(s))}))$. Again,

$$\begin{aligned}
& A_f(s) \leq g^{-1}(g(A_f(s))) \\
\Rightarrow\ & {}_scl(A_f(s)) \leq {}_scl(g^{-1}(\overline{g(A_f(s))})) = g^{-1}(\overline{g(A_f(s))}) \\
\Rightarrow\ & g({}_scl(A_f(s))) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}.
\end{aligned}$$

$(iii) \Rightarrow (i)$ Let $B_f(s)$ be an fs-open set in Y. Then, for the fs-closed set $B_f^c(s)$, we have

$$g({}_scl(g^{-1}(B_f^c(s)))) \leq \overline{g(g^{-1}(B_f^c(s)))} \leq \overline{B_f^c(s)} = B_f^c(s)$$

Thus, ${}_scl(g^{-1}(B_f^c(s))) \leq g^{-1}(g({}_scl(g^{-1}(B_f^c(s))))) \leq g^{-1}(B_f^c(s))$. Therefore, ${}_scl(g^{-1}(B_f^c(s))) = g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c$ $= g^{-1}(B_f^c(s))$ is fs-semiclosed in X. ■

Theorem 6.1.8 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiopen set in Y, is fs-semiopen in X.*

Proof. Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} & O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \\ \Rightarrow \;\; & g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)}) \end{aligned}$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies, $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $g(U_f(s))$ is a weak open Q-nbd of $g(P_f(s))$. Therefore,

$$\begin{aligned} & g(U_f(s)) q_w O_f(s) \\ \Rightarrow \;\; & U_f(s) q_w g^{-1}(O_f(s)) \\ \Rightarrow \;\; & P_f(s) \in \overline{g^{-1}(O_f(s))}. \end{aligned}$$

Thus we have, $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Hence, $g^{-1}(O_f(s))$ being fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen. ■

Corollary 6.1.1 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous open map. Then the inverse image of every fs-semiclosed set in Y is fs-semiclosed in X.*

Proof. The proof is omitted. ■

Corollary 6.1.2 *Composition of an fs-semicontinuous open map $g : (X, \delta(s)) \to (Y, \delta'(s))$ and an fs-semicontinuous map h :*

$(Y, \delta'(s)) \to (Z, \eta(s))$, *that is, the map* $hog : (X, \delta(s)) \to (Z, \eta(s))$ *is fs-semicontinuous.*

Proof. Let $C_f(s)$ be an fs-open set in Z, then $h^{-1}(C_f(s))$ is fs-semiopen in Y and hence $(hog)^{-1}(C_f(s)) = g^{-1}(h^{-1}(C_f(s)))$ is fs-semiopen in X by Theorem 6.1.8. ■

Theorem 6.1.9 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *be an fs-continuous open map. Then the g-image of an fs-semiopen set in* X *is fs-semiopen in* Y*.*

Proof. Let $A_f(s)$ be an fs-semiopen set in X. Then, there exists an fs-open set $O_f(s)$ in X such that $O_f(s) \leq A_f(s) \leq \overline{O_f(s)}$. This implies, $g(O_f(s)) \leq g(A_f(s)) \leq g(\overline{O_f(s)}) \leq \overline{g(O_f(s))}$. Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-semiopen in Y. ■

Corollary 6.1.3 *Semi-openness in an FSTS, is a topological property.*

Proof. Follows from Theorem 6.1.9. ■

Remark 6.1.1 *Theorem* 6.1.9 *does not hold if* g *is not fs-open. This is shown by Example* 6.1.3.

Example 6.1.3 *Let* $(X, \delta(s))$ *and* $(X, \delta'(s))$ *be two fuzzy sequential topological spaces, where* $\delta(s)$ *contains all the constant fs-sets in* X *and* $\delta'(s) = \{X_f^0(s), X_f^1(s)\}$*. Define a map* $g : (X, \delta(s)) \to (X, \delta'(s))$ *by* $g(x) = x$ *for all* $x \in X$*. Then,* g *is fs-continuous* *but not fs-open. Here, for the fs-semiopen set* $A_f(s) = \overline{\{\frac{1}{2}\}}_{n=1}^{\infty}$ *in* $(X, \delta(s))$, $g(A_f(s)) = \overline{\{\frac{1}{2}\}}_{n=1}^{\infty}$, *which is not fs-semiopen in* $(X, \delta'(s))$.

Theorem 6.1.10 *Let* $g : (X, \delta(s)) \to (Y, \delta'(s))$ *and* $h : (Y, \delta'(s)) \to (Z, \eta(s))$ *be two mappings and* $hog : (X, \delta(s)) \to (Z, \eta(s))$ *be an fs-semiclosed mapping. Then,* g *is fs-semiclosed if* h *is an injective fs-semicontinuous open mapping.*

Proof. Let $A_f(s)$ be an fs-closed set in X. Then $hog(A_f(s))$ is fs-semiclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-semiclosed in Y. ■

Theorem 6.1.11 *For an fs-semicontinuous map* $g : (X, \delta(s)) \to (Y, \delta'(s))$ *and an fs-continuous map* $h : (Y, \delta'(s)) \to (Z, \eta(s))$, *the map* $hog : (X, \delta(s)) \to (Z, \eta(s))$ *is fs-semicontinuous.*

Proof. Omitted. ■

6.2 FS-regular open sets

Definition 6.2.1 *An fs-set* $A_f(s)$ *in an FSTS* $(X, \delta(s))$, *is said to be fs-regular open if* $(\overline{A_f(s)})^o = A_f(s)$. *An fs-set is said to be fs-regular closed if its complement is fs-regular open.*

It is obvious that every fs-regular open (closed) set is fs-open (closed). That the converse need not be true, is shown by Example 6.2.1. Example 6.2.2 shows that the union (intersection) of any two fs-regular open (closed) sets need not be an fs-regular open (closed) set.

Example 6.2.1 *Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $A_f(s)$ is fs-open but not fs-regular open.

Example 6.2.2 *Consider the fs-sets $A_f(s)$, $B_f(s)$ and $C_f(s)$ in $X = [0,1]$, defined as follows:*

$$\begin{aligned} A_f^1(x) &= 0, \text{ if } 0 \leq x \leq \frac{1}{2} \\ &= \frac{2}{3}, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } A_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} B_f^1(x) &= 1, \text{ if } 0 \leq x \leq \frac{1}{4} \\ &= \frac{1}{2} \text{ if } \frac{1}{4} < x \leq \frac{1}{2} \\ &= 0, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } B_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $A_f(s)$ and $B_f(s)$ are fs-regular open sets but their union is not fs-regular open.

Theorem 6.2.1 (*a*) *The intersection of any two fs-regular open sets is an fs-regular open set.*
(*b*) *The union of any two fs-regular closed sets is an fs-regular closed set.*

Proof. We prove only (a). Let $A_f(s)$ and $B_f(s)$ be two fs-regular open sets in X. Since $A_f(s) \wedge B_f(s)$ is fs-open, we have $A_f(s) \wedge B_f(s) \leq (\overline{A_f(s) \wedge B_f(s)})^o$. Now, $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{A_f(s)})^o = A_f(s)$ and $(\overline{A_f(s) \wedge B_f(s)})^o \leq (\overline{B_f(s)})^o = B_f(s)$, which implies $(\overline{A_f(s) \wedge B_f(s)})^o \leq A_f(s) \wedge B_f(s)$. Hence the result. ■

Theorem 6.2.2 (*a*) *Closure of an fs-semiopen set is fs-regular closed.*
(*b*) *Interior of an fs-semiclosed set is fs-regular open.*

Proof. We prove only (a). Let $A_f(s)$ be an fs-semiopen set in X. Since $(\overline{A_f(s)})^o \leq \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \leq \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-semiopen, $A_f(s) \leq \overline{\overset{o}{A_f}(s)} \leq \overline{(\overline{A_f(s)})^o}$ and hence $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ is fs-regular closed. ■

Definition 6.2.2 *A mapping* $g : (X, \delta(s)) \to (Y, \eta(s))$ *is called an fs-almost continuous mapping if* $g^{-1}(B_f(s)) \in \delta(s)$ *for each fs-regular open set* $B_f(s)$ *in* Y.

Theorem 6.2.3 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *be a mapping. Then the following are equivalent:*
(*i*) g *is fs-almost continuous.*

(ii) $g^{-1}(B_f(s))$ *is an fs-closed set for each fs-regular closed set* $B_f(s)$ *of* Y.

(iii) $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o$ *for each fs-open set* $B_f(s)$ *of* Y.

(iv) $\overline{g^{-1}(\overline{\overset{o}{B}_f(s)})} \leq g^{-1}(B_f(s))$ *for each fs-closed set* $B_f(s)$ *of* Y.

Proof. Note that $g^{-1}(B^c_f(s)) = (g^{-1}(B_f(s)))^c$ for any fs-set $B_f(s)$ in Y.

$(i) \Leftrightarrow (ii)$ Follows from the fact that an fs-set is fs-regular open if and only if its complement is fs-regular closed.

$(i) \Rightarrow (iii)$ Let $B_f(s)$ be an fs-open set in Y. Then $B_f(s) \leq (\overline{B_f(s)})^o$ and hence $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$. By Theorem 6.3.4 (b), $(\overline{B_f(s)})^o$ is an fs-regular open set in Y. Therefore, $g^{-1}((\overline{B_f(s)})^o)$ is fs-open in X and thus $g^{-1}(B_f(s)) \leq g^{-1}((\overline{B_f(s)})^o)$ $= (g^{-1}((\overline{B_f(s)})^o))^o$.

$(iii) \Rightarrow (i)$ Let $B_f(s)$ be an fs-regular open set in Y. Then by (iii), we have $g^{-1}(B_f(s)) \leq (g^{-1}((\overline{B_f(s)})^o))^o = (g^{-1}(B_f(s)))^o$. Hence $g^{-1}(B_f(s))$ is an fs-open set in X.

$(ii) \Leftrightarrow (iv)$ are easy to prove. ■

Clearly, an fs-continuous map is an fs-almost continuous map but the converse need not be true, as is shown by Example 6.2.3.

Example 6.2.3 *Consider the fs-sets* $A_f(s)$, $B_f(s)$ *in a set* X, *defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$ *and* $\eta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s),\ A_f(s) \vee B_f(s),\ X_f^0(s), X_f^1(s)\}$*. Then,* $(X, \delta(s))$ *and* $(X, \eta(s))$ *are fuzzy sequential topological spaces. Define a map* $g : (X, \delta(s)) \to (X, \eta(s))$ *by* $g(x) = x$ *for all* $x \in X$*. Then,* g *is fs-almost continuous but not fs-continuous. Again, since the inverse image of the fs-open set* $A_f(s)$ *of* $(X, \eta(s))$ *is not fs-semiopen in* $(X, \delta(s))$*,* g *is not fs-semicontinuous.*

Example 6.2.4 *Example to show that an fs-semicontinuous map may not be fs-almost continuous.*

Consider the fs-sets $A_f(s)$*,* $B_f(s)$ *in a set* X*, as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, \overline{0}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ *and* $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)\}$*. Then,* $(X, \delta(s))$ *and* $(X, \eta(s))$ *are fuzzy sequential topological spaces. Define a map* $g : (X, \delta(s)) \to (X, \eta(s))$ *by* $g(x) = x$ *for all* $x \in X$*. Then,* g *is fs-semicontinuous but not fs-almost continuous.*

Remark 6.2.1 *Example* 6.2.3 *and Example* 6.2.4 *shows that an* *fs-almost continuous mapping and an fs-semicontinuous mapping are independent notions.*

Definition 6.2.3 *An FSTS $(X, \delta(s))$ is called an fs-semiregular space if the collection of all fs-regular open sets in X forms a base for $\delta(s)$.*

Theorem 6.2.4 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping, where $(Y, \eta(s))$ is an fs-semiregular space. Then, g is fs-almost continuous if and only if g is fs-continuous.*

Proof. We need only to show that if g is fs-almost continuous, then it is fs-continuous.

Suppose g is fs-almost continuous. Let $B_f(s) \in \eta(s)$, then $B_f(s) = \bigvee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where $B_{\lambda f}(s)$'s are fs-regular open sets in Y. Then

$$\begin{aligned} g^{-1}(B_f(s)) &= \bigvee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \bigvee_{\lambda\in\Lambda} (g^{-1}((\overline{B_{\lambda f}(s)})^o))^o \\ &= \bigvee_{\lambda\in\Lambda} (g^{-1}(B_{\lambda f}(s)))^o \\ &\leq (\bigvee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)))^o \\ &= (g^{-1}(B_f(s)))^o, \end{aligned}$$

which shows that $g^{-1}(B_f(s)) \in \delta(s)$. ■

Theorem 6.2.5 *Let X, X_1 and X_2 be fuzzy sequential topological spaces and $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \to X_1 \times X_2$ is fs-almost continuous, then $\pi_i o g$ is also fs-almost continuous.*

Proof. Let g be an fs-almost continuous map and let $B_f(s)$ be an fs-regular open set in X_i. Since π_i is fs-continuous, we have, $\overline{\pi_i^{-1}(A_f(s))} \leq \pi_i^{-1}(\overline{A_f(s)})$ and $\pi_i^{-1}(\overset{o}{A}_f(s)) \leq (\pi_i^{-1}(A_f(s)))^o$ for any fs-set $A_f(s)$ in X_i. Now,

$$\begin{aligned} & \pi_i((\pi_i^{-1}(A_f(s)))^o) \leq \pi_i(\pi_i^{-1}(A_f(s))) \leq A_f(s) \\ \Rightarrow\ & \pi_i((\pi_i^{-1}(A_f(s)))^o) \leq \overset{o}{A}_f(s) \\ \Rightarrow\ & (\pi_i^{-1}(A_f(s)))^o \leq \pi_i^{-1}(\pi_i((\pi_i^{-1}(A_f(s)))^o)) \leq \pi_i^{-1}(\overset{o}{A}_f(s)) \\ \Rightarrow\ & \pi_i^{-1}(\overset{o}{A}_f(s)) = (\pi_i^{-1}(A_f(s)))^o \end{aligned}$$

Therefore,

$$\begin{aligned} (\pi_i og)^{-1}(B_f(s)) &= g^{-1}(\pi_i^{-1}(B_f(s))) \\ &\leq (g^{-1}((\overline{\pi_i^{-1}(B_f(s))})^o))^o \\ &\leq (g^{-1}((\pi_i^{-1}(\overline{B_f(s)}))^o))^o \\ &= (g^{-1}(\pi_i^{-1}((\overline{B_f(s)})^o)))^o \\ &= (g^{-1}(\pi_i^{-1}(B_f(s))))^o \\ &= ((\pi_i og)^{-1}(B_f(s)))^o \end{aligned}$$

Hence the theorem. ■

Definition 6.2.4 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called an fs-weakly continuous mapping if for each fs-open set $B_f(s)$ in Y, $g^{-1}(B_f(s)) \leq (g^{-1}(\overline{B_f(s)}))^o$.*

Remark 6.2.2 *It is clear that every fs-continuous mapping is fs-weakly continuous. That the converse may not true, in general,* *is shown by Example* 6.2.5. *Example* 6.2.5 *also shows that an fs-weakly continuous mapping may neither be fs-semicontinuous nor fs-almost continuous. However, it is clear that an fs-almost continuous mapping is fs-weakly continuous.*

Example 6.2.5 *Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \ldots\ldots\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{3}}, \overline{\frac{1}{3}}, \overline{\frac{1}{3}}, \ldots\ldots\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ *and* $\eta(s) = \{B_f(s), X_f^0(s), X_f^1(s)$ $\}$. *Then,* $(X, \delta(s))$ *and* $(X, \eta(s))$ *are fuzzy sequential topological spaces. Define a map* $g : (X, \delta(s)) \to (X, \eta(s))$ *by* $g(x) = x$ *for all* $x \in X$. *Then, g is fs-weakly continuous but not fs-continuous. Since the inverse image of the fs-open set $B_f(s)$ of Y is not fs-semiopen in X, g is not fs-semicontinuous. Again, as the inverse image of the fs-regular open set $B_f(s)$ of Y is not fs-open in X, g is not fs-almost continuous.*

Remark 6.2.3 *The map g defined in Example* 6.2.4, *is not fs-weakly continuous but is fs-semicontinuous.*

Remark 6.2.4 *Example* 6.2.5 *and Remark* 6.2.3 *shows that fs-semicontinuity and fs-weakly continuity are independent notions.*

Definition 6.2.5 *An FSTS $(X, \delta(s))$ is called an Ωfs-semiregular space if each fs-open set $A_f(s)$ of X is the union of fs-open sets $A_{\lambda f}(s)(\lambda \in \Lambda)$ of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$.*

Theorem 6.2.6 *An Ωfs-semiregular space is fs-semiregular.*

Proof. Let $(X, \delta(s))$ be an Ωfs-semiregular space and $A_f(s)$ be an fs-open set in X. Then $A_f(s) = \bigvee_{\lambda\in\Lambda} A_{\lambda f}(s)$, where $A_{\lambda f}(s)$ are fs-open sets of X such that $\overline{A_{\lambda f}(s)} \leq A_f(s)$ for all $\lambda \in \Lambda$. Since $A_{\lambda f}(s) \leq (\overline{A_{\lambda f}(s)})^o \leq A_f(s)$ for each $\lambda \in \Lambda$, we have $A_f(s) = \bigvee_{\lambda\in\Lambda} (\overline{A_{\lambda f}(s)})^o$. Now, for each $\lambda \in \Lambda$, $(\overline{A_{\lambda f}(s)})^o$ is fs-regular open in X and thus $(X, \delta(s))$ is fs-semiregular. ■

Remark 6.2.5 *Example* 6.2.6 *shows that the converse of Theorem* 6.2.6 *may not be true.*

Example 6.2.6 *Consider the fuzzy sequential topology $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ on a set X, where $A_f(s) = \left\{\overline{\tfrac{1}{4}}, \overline{0}, \overline{0}, ...\right\}$. Then $(X, \delta(s))$ is an fs-semiregular space. If $\{A_{\lambda f}(s); \lambda \in \Lambda\} \subseteq \delta(s)$ such that $A_f(s) = \bigvee_{\lambda\in\Lambda} A_{\lambda f}(s)$. Then $A_{\lambda f}(s) = A_f(s)$ for some $\lambda \in \Lambda$. Since $\overline{A_f(s)}$ is not contained in $A_f(s)$, $(X, \delta(s))$ is not an Ωfs-semiregular space.*

Theorem 6.2.7 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a mapping where $(X, \delta(s))$ is any FSTS and $(Y, \eta(s))$ is an Ωfs-semiregular space. Then, g is fs-weakly continuous if and only if g is fs-continuous.*

Proof. It suffices to show that if g is fs-weakly continuous, then it is fs-continuous. Let $B_f(s) \in \eta(s)$. Then $B_f(s) = \bigvee_{\lambda\in\Lambda} B_{\lambda f}(s)$, where for all $\lambda \in \Lambda$, $B_{\lambda f}(s) \in \eta(s)$ and $\overline{B_{\lambda f}(s)} \leq B_f(s)$. Since g

is fs- weakly continuous, we have

$$\begin{aligned} g^{-1}(B_f(s)) = g^{-1}(\bigvee_{\lambda\in\Lambda} B_{\lambda f}(s)) &= \bigvee_{\lambda\in\Lambda} g^{-1}(B_{\lambda f}(s)) \\ &\leq \bigvee_{\lambda\in\Lambda} (g^{-1}(\overline{B_{\lambda f}(s)}))^o \\ &\leq \bigvee_{\lambda\in\Lambda} (g^{-1}(B_f(s)))^o \\ &= (g^{-1}(B_f(s)))^o \end{aligned}$$

and hence $g^{-1}(B_f(s))$ is fs-open in X. Thus, g is fs-continuous. ■

Theorem 6.2.8 *Let X, X_1 and X_2 be FSTS's and $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings from $X_1 \times X_2$ onto X_i. If $g : X \to X_1 \times X_2$ is fs-weakly continuous, then $\pi_i o g$ is also fs-weakly continuous.*

Proof. The proof is analogous to the proof of Theorem 6.2.5. ■

6.3 FS-preopen sets and FS-precontinuity

Definition 6.3.1 *(i) An fs-set $A_f(s)$ in an FSTS, is said to be an fs-preopen set if $A_f(s) \leq (\overline{A_f(s)})^o$.*
(ii) An fs-set $A_f(s)$ in an FSTS, is said to be an fs-preclosed set if its complement is fs-preopen or equivalently if $\overline{\overset{o}{A}_f(s)} \leq A_f(s)$.

If $A_f(s)$ is both fs-preopen and fs-preclosed, then it is called an fs-preclopen set.

Definition 6.3.2 *An fs-set $A_f(s)$ is called fs-dense in an FSTS $(X, \delta(s))$, if $\overline{A_f(s)} = X_f^1(s)$.*

Fundamental properties of fs-preopen (fs-preclosed) sets are:

- Every fs-open (fs-closed) set is fs-preopen (fs-preclosed).
- Arbitrary union (intersection) of fs-preopen (fs-preclosed) sets is fs-preopen (fs-preclosed).

Example 6.3.1 shows that an fs-preopen (fs-preclosed) set may not be fs-open (fs-closed), the intersection (union) of any two fs-preopen (fs-preclosed) sets need not be an fs-preopen (fs-preclosed) set. Unlike in a general topological space, the intersection of an fs-preopen set with an fs-open set may fail to be an fs-preopen set.

Example 6.3.1 *Consider the fs-sets $A_f(s)$, $B_f(s)$ and $C_f(s)$ in*

$X = [0,1]$, *defined as follows:*

$$\begin{aligned} A_f^1(x) &= 0, \text{ if } 0 \leq x \leq \frac{1}{2} \\ &= \frac{1}{4}, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } A_f^n &= \overline{1} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} B_f^1(x) &= \frac{1}{2}, \text{ if } 0 \leq x \leq \frac{1}{2} \\ &= 0, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } B_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} C_f^1(x) &= \frac{3}{4}, \text{ if } 0 \leq x \leq \frac{1}{2} \\ &= 1, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } C_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s),\ X_f^1(s)\}$. *Then* $(X, \delta(s))$ *is an FSTS. Now,*

(i) $A_f(s)$ *and* $C_f(s)$ *are fs-preopen sets but their intersection is not fs-preopen.*

(ii) $C_f(s)$ *is fs-preopen but is not fs-open.*

Theorem 6.3.1 *Let* $(X, \delta(s))$ *be an FSTS. An fs-set* $A_f(s)$ *is fs-preopen if and only if there exists an fs-open set* $O_f(s)$ *in* X *such that* $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$.

Proof. Straightforward. ■

Corollary 6.3.1 *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-preclosed if and only if there exists an fs-closed set $C_f(s)$ in X such that $\overset{o}{A}_f(s) \le C_f(s) \le A_f(s)$.*

Proof. Straightforward. ■

Theorem 6.3.2 *An fs-set is fs-clopen (both fs-closed and fs-open) if and only if it is fs-closed and fs-preopen.*

Proof. Proof is omitted. ■

Theorem 6.3.3 *In an FSTS, every fs-set is fs-preopen if and only if every fs-open set is fs-closed.*

Proof. Suppose every fs-set in an FSTS $(X, \delta(s))$, is fs-preopen and let $A_f(s)$ be an fs-open set. Then, $A^c_f(s) = \overline{A^c_f(s)}$ is fs-preopen and hence $\overline{A^c_f(s)} \le (\overline{\overline{A^c_f(s)}})^o = (\overline{A^c_f(s)})^o = (A^c_f(s))^o$. Thus, $A^c_f(s)$ is fs-open and hence $A_f(s)$ is fs-closed.

Conversely, suppose every fs-open set is fs-closed and let $A_f(s)$ be any fs-set. By the assumption, $\overline{A_f(s)} = (\overline{A_f(s)})^o$ and hence $A_f(s)$ is fs-preopen. ■

Theorem 6.3.4 (a) *Closure of an fs-preopen set is fs-regular closed.*

(b) *Interior of an fs-preclosed set is fs-regular open.*

Proof. We prove only (a). Let $A_f(s)$ be an fs-preopen set in X. Since $(\overline{A_f(s)})^o \le \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \le \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-preopen, $A_f(s) \le (\overline{A_f(s)})^o$ and hence $\overline{A_f(s)} \le \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ is fs-regular closed. ■

The set of all fs-preopen sets in X, is denoted by $FSPO(X)$.

Theorem 6.3.5 *In an FSTS* $(X, \delta(s))$, *(i)* $\delta(s) \subseteq FSPO(X)$, *(ii) If* $V_f(s) \in FSPO(X)$ *and* $U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$, *then* $U_f(s) \in FSPO(X)$.

Proof. (i) Follows from definition.

(ii) Let $V_f(s) \in FSPO(X)$, that is, $V_f(s) \leq (\overline{V_f(s)})^o$. We have,

$$U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$$

Therefore, $U_f(s) \leq V_f(s) \leq (\overline{V_f(s)})^o \leq (\overline{U_f(s)})^o$. Hence the result. ■

Definition 6.3.3 *An fs-set* $A_f(s)$ *in an FSTS, is called an fs-preneighbourhood of an fs-point* $P_f(s) = (p^M_{fx}, r)$, *if there exists an fs-preopen set* $B_f(s)$ *such that* $P_f(s) \leq B_f(s) \leq A_f(s)$.

Theorem 6.3.6 *For an fs-set* $A_f(s)$ *in an FSTS* $(X, \delta(s))$, *the following are equivalent:*

(i) $A_f(s)$ *is fs-preopen.*

(ii) There exists an fs-regular open set $B_f(s)$ *containing* $A_f(s)$ *such that* $\overline{A_f(s)} = \overline{B_f(s)}$.

(iii) ${}_scl(A_f(s)) = (\overline{A_f(s)})^o$.

(iv) The semi-closure of $A_f(s)$ *is fs-regular open.*

(v) $A_f(s)$ *is an fs-preneighbourhood of each of its fs-points.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be fs-preopen. This implies

$$\begin{aligned} & A_f(s) \leq (\overline{A_f(s)})^o \leq \overline{A_f(s)} \\ \Rightarrow\ & \overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o} \leq \overline{A_f(s)} \\ \Rightarrow\ & \overline{(\overline{A_f(s)})^o} = \overline{A_f(s)} \\ \Rightarrow\ & \overline{A_f(s)} = \overline{B_f(s)} \end{aligned}$$

where $B_f(s) = (\overline{A_f(s)})^o$ is an fs-regular open set containing $A_f(s)$.

$(ii) \Rightarrow (iii)$ Let $\overline{A_f(s)} = \overline{B_f(s)}$, where $B_f(s)$ is an fs-regular open set containing $A_f(s)$. Then,

$$A_f(s) \leq B_f(s) \ = \ (\overline{B_f(s)})^o = (\overline{A_f(s)})^o$$

Also, $(\overline{A_f(s)})^o$ is fs-semiclosed. Let $C_f(s)$ be an fs-semiclosed set containing $A_f(s)$. Thus,

$$(\overline{A_f(s)})^o \leq (\overline{C_f(s)})^o \leq C_f(s).$$

Hence ${}_scl(A_f(s)) = (\overline{A_f(s)})^o$.

$(iii) \Rightarrow (iv)$ Obvious.

$(iv) \Rightarrow (i)$ Suppose ${}_scl(A_f(s))$ is fs-regular open. Now,

$$\begin{aligned} & A_f(s) \leq \ {}_scl(A_f(s)) \\ \Rightarrow\ & (\overline{A_f(s)})^o \leq (\overline{{}_scl(A_f(s))})^o = \ {}_scl(A_f(s)) \leq (\overline{A_f(s)})^o \\ \Rightarrow\ & A_f(s) \leq \ {}_scl(A_f(s)) = (\overline{A_f(s)})^o \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

$(i) \Rightarrow (v)$ and $(v) \Rightarrow (i)$ are obvious. ■

Corollary 6.3.2 *An fs-set is fs-regular open if and only if it is fs-semiclosed and fs-preopen.*

Proof. Proof is omitted. ■

Theorem 6.3.7 *In an FSTS* $(X, \delta(s))$*, the following are equivalent:*

(i) $\overline{A_f(s)} \in \delta(s)$ *for all* $A_f(s) \in \delta(s)$*.*

(ii) *Every fs-regular closed set in* X *is fs-preopen.*

(iii) *Every fs-semiopen set in* X *is fs-preopen.*

(iv) *The closure of every fs-preopen set in* X *is fs-open.*

(v) *The closure of every fs-preopen set in* X *is fs-preopen.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be an fs-regular closed set, that is, $\overline{\overset{o}{A_f}(s)} = A_f(s)$. By (i), $A_f(s) \in \delta(s)$ and hence $A_f(s)$ is fs-preopen.

$(ii) \Rightarrow (iii)$ Let $A_f(s)$ be an fs-semiopen set, that is, $A_f(s) \leq \overline{\overset{o}{A_f}(s)}$. By (ii), $\overline{\overset{o}{A_f}(s)}$ is fs-preopen. Also, we have, $A_f(s) \leq \overline{\overset{o}{A_f}(s)} \leq \overline{A_f(s)}$. Thus, $A_f(s)$ is fs-preopen.

$(iii) \Rightarrow (iv)$ Let $A_f(s)$ be an fs-preopen set, that is, $A_f(s) \leq (\overline{A_f(s)})^o$. This implies, $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ being fs-semiopen, is fs-preopen and the result follows.

$(iv) \Rightarrow (v)$ Obvious.

$(v) \Rightarrow (i)$ Let $A_f(s) \in \delta(s)$. Then, $A_f(s)$ is fs-preopen and hence $\overline{A_f(s)}$ is fs-preopen. Therefore, $\overline{A_f(s)} \leq (\overline{A_f(s)})^o \leq \overline{A_f(s)}$ and hence $\overline{A_f(s)} \in \delta(s)$.

■

Theorem 6.3.8 *In an FSTS* $(X, \delta(s))$*, the following are equivalent:*
(i) *Every non-zero fs-open set is fs-dense.*
(ii) *For every non-zero fs-preopen set* $A_f(s)$*, we have* ${}_scl(A_f(s)) = X_f^1(s)$.
(iii) *Every non-zero fs-preopen set is fs-dense.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be a non-zero fs-preopen set. By Theorem 6.3.6 (iii), ${}_scl(A_f(s)) = (\overline{A_f(s)})^o$. Also, there exists an fs-open set $O_f(s)$ such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. By (i), $\overline{O_f(s)} = X_f^1(s)$. Therefore, $\overline{A_f(s)} = X_f^1(s)$ and hence ${}_scl(A_f(s)) = X_f^1(s)$.

$(ii) \Rightarrow (iii)$ Easy to prove.

$(iii) \Rightarrow (i)$ Since every fs-open set is fs-preopen, the proof is straightforward. ■

Definition 6.3.4 *Let* $A_f(s)$ *be an fs-set in an FSTS* $(X, \delta(s))$*. We define fs-preclosure* ${}_pcl(A_f(s))$ *and fs-preinterior* ${}_pint(A_f(s))$ *of* $A_f(s)$ *by*

$$\begin{aligned} {}_pcl(A_f(s)) &= \wedge\{B_f(s); A_f(s) \leq B_f(s) \text{ and } B_f^c(s) \in FSPO(X)\} \\ {}_pint(A_f(s)) &= \vee\{C_f(s); C_f(s) \leq A_f(s) \text{ and } C_f(s) \in FSPO(X)\} \end{aligned}$$

Clearly, ${}_pcl(A_f(s))$ is the smallest fs-preclosed set containing $A_f(s)$ and ${}_pint(A_f(s))$ is the largest fs-preopen set contained in $A_f(s)$. Further,
(i) $A_f(s) \leq {}_pcl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_pint(A_f(s)) \leq$

$A_f(s)$.
(ii) $A_f(s)$ is fs-preopen if and only if $A_f(s) = {}_pint(A_f(s))$
(iii) $A_f(s)$ is fs-preclosed if and only if $A_f(s) = {}_pcl(A_f(s))$
(iv) $A_f(s) \leq B_f(s) \Rightarrow {}_pint(A_f(s)) \leq {}_pint(B_f(s))$ and ${}_pcl(A_f(s)) \leq {}_pcl(B_f(s))$.

Definition 6.3.5 *A mapping $g : (X, \delta(s)) \to (Y, \delta'(s))$ is said to be*
(i) fs-precontinuous if $g^{-1}(B_f(s))$ is fs-preopen in X, for every $B_f(s) \in \delta'(s)$.
(ii) fs-preopen if $g(A_f(s))$ is fs-preopen in Y, for every $A_f(s) \in \delta(s)$.
(iii) fs-preclosed if $g(A_f(s))$ is fs-preclosed in Y, for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-precontinuous (fs-preopen, fs-preclosed). That the converse may not be true, is shown by Example 6.3.2.

Example 6.3.2 *Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:*

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\}$$
$$B_f(s) = \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\}$$
$$C_f(s) = \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$*. Then* $(X, \delta(s))$ *is an FSTS. Let* $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$ *and define* $g : (X, \delta(s)) \to (X, \delta'(s))$ *by* $g(x) = x$ *for all* $x \in X$*. The function* g *is fs-precontinuous but not fs-continuous.*

Again, the map $h : (X, \delta'(s)) \to (X, \delta(s))$ *defined by* $h(x) = x$ *for all* $x \in X$*, is both fs-preopen and fs-preclosed but neither fs-open nor fs-closed.*

Theorem 6.3.9 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *be a map. Then the following conditions are equivalent:*

(i) g *is fs-precontinuous.*

(ii) *the inverse image of an fs-closed set in* Y *under* g*, is fs-preclosed in* X*.*

(iii) *For any fs-set* $A_f(s)$ *in* X*,* $g({}_pcl(A_f(s))) \leq \overline{g(A_f(s))}$*.*

Proof. $(i) \Rightarrow (ii)$ Suppose g be an fs-precontinuous map and $B_f(s)$ be an fs-closed set in Y. Then,

$$\begin{aligned} & B_f^c(s) \text{ is fs-open in } Y \\ \Rightarrow\ & (g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s)) \text{ is fs-preopen in } X \\ \Rightarrow\ & g^{-1}(B_f(s)) \text{ is fs-preclosed in } X. \end{aligned}$$

$(ii) \Rightarrow (iii)$ Let $A_f(s)$ be an fs-set in X. Then, $g^{-1}(\overline{g(A_f(s))})$ is fs-preclosed in X and hence $g^{-1}(\overline{g(A_f(s))}) = {}_pcl(g^{-1}(\overline{g(A_f(s))}))$.

Again,

$$
\begin{aligned}
& A_f(s) \leq g^{-1}(g(A_f(s))) \\
\Rightarrow\ & {}_pcl(A_f(s)) \leq {}_pcl(g^{-1}(g(A_f(s)))) \leq g^{-1}(\overline{g(A_f(s))}) \\
\Rightarrow\ & g({}_pcl(A_f(s))) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}.
\end{aligned}
$$

$(iii) \Rightarrow (i)$ Let $B_f(s)$ be an fs-open set in Y. Then for the fs-closed set $B_f^c(s)$, we have

$$g({}_pcl(g^{-1}(B_f^c(s)))) \leq \overline{g(g^{-1}(B_f^c(s)))} \leq \overline{B_f^c(s)} = B_f^c(s)$$

Thus, ${}_pcl(g^{-1}(B_f^c(s))) \leq g^{-1}(B_f^c(s))$. Therefore, ${}_pcl(g^{-1}(B_f^c(s))) = g^{-1}(B_f^c(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s))$ is fs-preclosed in X. ■

In Theorem 6.1.8, it has been proved that the inverse image of an fs-semiopen set is fs-semiopen, under an fs-semicontinuous open map. The next Theorem shows that the result is true even if we take an fs-semicontinuous preopen map.

Theorem 6.3.10 *Suppose $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiopen set in Y under g, is fs-semiopen in X.*

Proof. Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$
\begin{aligned}
& O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \\
\Rightarrow\ & g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)})
\end{aligned}
$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $\overline{g(U_f(s))}$ is a weak Q-nbd of $g(P_f(s))$. Therefore,

$$
\begin{aligned}
& \overline{g(U_f(s))} q_w O_f(s) \\
\Rightarrow\ & W_f(s) q_w O_f(s) \text{ where } W_f(s) = \overline{g(U_f(s))} \\
\Rightarrow\ & W_f^n(y) + O_f^n(y) > 1 \text{ for some } y \in Y \\
\Rightarrow\ & O_f(s) \text{ is a weak open Q-nbd of the fs-point } (p_{fy}^n, W_f^n(y)) \\
\Rightarrow\ & g(U_f(s)) q_w O_f(s) \\
\Rightarrow\ & U_f(s) q_w g^{-1}(O_f(s)) \\
\Rightarrow\ & P_f(s) \in \overline{g^{-1}(O_f(s))}.
\end{aligned}
$$

Thus $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Since $g^{-1}(O_f(s))$ is fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen. ■

Corollary 6.3.3 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiclosed set in Y under g, is fs-semiclosed in X.*

Proof. The proof is omitted. ■

Corollary 6.3.4 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous preopen map and $h : (Y, \eta(s)) \to (Z, \delta'(s))$ be fs-semicontinuous. Then hog is fs-semicontinuous.*

Proof. The proof is omitted. ■

Theorem 6.3.11 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preopen set in Y under g, is fs-preopen in X.*

Proof. Let $B_f(s)$ be an fs-preopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} & B_f(s) \leq O_f(s) \leq \overline{B_f(s)} \\ \Rightarrow \; & g^{-1}(B_f(s)) \leq g^{-1}(O_f(s)) \leq g^{-1}(\overline{B_f(s)}). \end{aligned}$$

As in Theorem 6.3.10, we have $g^{-1}(\overline{B_f(s)}) \leq \overline{g^{-1}(B_f(s))}$. Thus $g^{-1}(B_f(s)) \leq g^{-1}(O_f(s)) \leq \overline{g^{-1}(B_f(s))}$, where $g^{-1}(O_f(s))$ is fs-preopen. Hence $g^{-1}(B_f(s))$ is fs-preopen. ■

Corollary 6.3.5 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preclosed set in Y under g, is fs-preclosed in X.*

Proof. The proof is omitted. ■

Corollary 6.3.6 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen map and $h : (Y, \eta(s)) \to (Z, \delta'(s))$ be an fs-precontinuous map. Then hog is fs-precontinuous.*

Proof. The proof is omitted. ■

Theorem 6.3.12 *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-continuous open map. Then the g-image of an fs-preopen set in X, is fs-preopen in Y.*

Proof. Let $A_f(s)$ be an fs-preopen set in X. Then there exists an fs-open set $O_f(s)$ in X such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. This implies $g(A_f(s)) \leq g(O_f(s)) \leq \overline{g(A_f(s))}$. Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-preopen. ■

Corollary 6.3.7 *Pre-openness in an FSTS, is a topological property.*

Proof. Proof follows from Theorem 6.3.12. ■

Theorem 6.3.13 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ and $h : (Y, \eta(s)) \to (Z, \delta'(s))$ be two mappings, such that hog is fs-preclosed. Then g is fs-preclosed if h is an injective fs-precontinuous preopen mapping.*

Proof. Let $A_f(s)$ be an fs-closed set in X. Then, $hog(A_f(s))$ is fs-preclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-preclosed in Y. ■

Theorem 6.3.14 *If $g : (X, \delta(s)) \to (Y, \delta'(s))$ be fs-precontinuous and $h : (Y, \delta'(s)) \to (Z, \eta(s))$ be fs-continuous, then $hog : (X, \delta(s)) \to (Z, \eta(s))$ is fs-precontinuous.*

Proof. Omitted. ■

Previously, we showed that the intersection of any two fs-preopen sets may not be fs-preopen and an fs-preopen set may not be fs-open. Now, we investigate and establish conditions, under which the intersection of any two fs-preopen sets is fs-preopen and conditions, under which an fs-preopen set is fs-open.

Theorem 6.3.15 *The intersection of any two fs-preopen sets is fs-preopen if the closure is preserved under finite intersection.*

Proof. Proof is simple and hence omitted. ■

Theorem 6.3.16 *In an FSTS $(X, \delta(s))$, if every fs-set is either fs-open or fs-closed, then every fs-preopen set in X is fs-open.*

Proof. Let $A_f(s)$ be an fs-preopen set in X. If $A_f(s)$ is not fs-open, then it is fs-closed and hence $\overline{A_f(s)} = A_f(s)$. Therefore, $A_f(s) \leq (\overline{A_f(s)})^o = \overset{o}{A}_f(s)$ and hence the theorem. ■

For a fuzzy sequential topological space $(X, \delta(s))$, $\delta^*(s)$ will denote the fuzzy sequential topology on X, obtained by taking $FSPO(X)$ as a subbase.

Definition 6.3.6 *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called strongly fs-precontinuous if the inverse image of each fs-preopen set in Y is fs-open in X.*

By the definition of a strong fs-precontinuous mapping, the following two results are obvious.

Proposition 6.3.1 *(i) A map $g : (X, \delta(s)) \to (Y, \eta(s))$ is strongly fs-precontinuous if and only if $g : (X, \delta(s)) \to (Y, \eta^*(s))$ is fs-continuous.*

(ii) If $g : (X, \delta(s)) \to (Y, \eta(s))$ is strongly fs-precontinuous, then it is fs-continuous.

Remark 6.3.1 *Converse of (ii) of Proposition 6.3.1 may not be true, as is shown by the following Example.*

Example 6.3.3 *Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:*

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Consider the identity map $id : (X, \delta(s)) \to (X, \delta(s))$. Then, id is fs-continuous but not strongly fs-precontinuous, as the inverse image of fs-preopen set $C_f(s)$ is not fs-open.

We conclude the section with a necessary and sufficient condition for an fs-preopen set to be fs-open.

Theorem 6.3.17 *In an FSTS $(Y, \eta(s))$, the following are equivalent:*
(i) Every fs-preopen set in Y is fs-open.
(ii) Every fs-continuous function $g : (X, \delta(s)) \to (Y, \eta(s))$ is strongly fs-precontinuous, where $(X, \delta(s))$ is any FSTS.

Proof. $(i) \Rightarrow (ii)$ is straightforward.

$(ii) \Rightarrow (i)$ The identity map $g : (Y, \eta(s)) \to (Y, \eta(s))$ is fs-continuous and hence is strongly fs-precontinuous. Let $B_f(s)$ be an fs-preopen set in Y, then $B_f(s) = g^{-1}(B_f(s))$ is fs-open in Y.

■