CHAPTER

5

Compactness

Following the introduction of fuzzy sets by L. A. Zadeh [42], several authors studied various notions of fuzzy topological spaces. Among them, one of the most studied topics is compactness. Fuzzy compact spaces were first studied by C. L. Chang [7] in 1968 and then different kinds of fuzzy compactness were studied by various authors like J.A. Goguen [14], R. Lowen [21, 22], T.E. Gantner, R.C. Steinlage and R.H. Warren [12], Wang Guojun [15], Gunther Jager [18] etc. and they were compared in detail by R. Lowen [23].

Here we present a development of fuzzy sequential topology, which includes the introduction and study of the concepts of continuous functions and compact spaces. Section 5.1 deals with the study of continuous functions, where both nbds and Q-nbds have been used to characterize it and Section 5.2 deals with the notion of compactness, where two types of compactness have been discussed.

5.1 FS-continuity

Let $g : X \to Y$ be a map. For $A_f(s) \in (I^X)^{\mathbb{N}}$ and $B_f(s) \in (I^Y)^{\mathbb{N}}$, $g(A_f(s))$ is an fs-set in Y defined by

$$\begin{aligned} g(A_f(s))(y) &= \{\sup_{x \in g^{-1}(y)} A_f^n(x)\}_{n=1}^{\infty} \text{ if } g^{-1}(y) \neq \phi, \\ &= X_f^0(s)(y) \text{ if } g^{-1}(y) = \phi, \end{aligned}$$

where $y \in Y$ and $g^{-1}(B_f(s))$ is an fs-set in X defined by

$$g^{-1}(B_f(s))(x) = B_f(s)(g(x)) \ \forall x \in X.$$

If $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$ and $C_f(s), D_f(s) \in (I^Y)^{\mathbb{N}}$, then it is seen that,

(i) $g(A_f(s)) = \{g(A_f^n)\}_{n=1}^{\infty}$.

(ii) $g^{-1}(C_f(s)) = \{g^{-1}(C_f^n)\}_{n=1}^{\infty}$.

(iii) $A_f(s) q_w B_f(s)$ if and only if $g(A_f(s)) q_w g(B_f(s))$.

(iv) $C_f(s) q_w D_f(s)$ at some point $y \in Y$ such that $g^{-1}(y) \neq \phi$ if and only if $g^{-1}(C_f(s)) q_w g^{-1}(D_f(s))$.

Theorem 5.1.1 *Let $g : X \to Y$ be a map. For $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$ and $C_f(s)$, $D_f(s) \in (I^Y)^{\mathbb{N}}$,*

(i) $g^{-1}((C_f(s))^c) = (g^{-1}(C_f(s)))^c$.

(ii) $(g(A_f(s)))^c(y) \leq g((A_f(s))^c)(y)$ $\forall y \in Y$ *such that* $g^{-1}(y) \neq \phi$ *and* $(g(A_f(s)))^c = g((A_f(s))^c)$ *if* g *is bijective.*
(iii) $A_f(s) \leq B_f(s) \Rightarrow g(A_f(s)) \leq g(B_f(s))$.
(iv) $C_f(s) \leq D_f(s) \Rightarrow g^{-1}(C_f(s)) \leq g^{-1}(D_f(s))$.
(v) $g(g^{-1}(C_f(s))) \leq C_f(s)$ *and the equality holds if* g *is onto.*
(vi) $A_f(s) \leq g^{-1}(g(A_f(s)))$ *and the equality holds if* g *is one-one.*
(vii) If $h : Y \to Z$ *be another map. Then,* $(h \circ g)^{-1}(G_f(s)) = g^{-1}(h^{-1}(G_f(s)))$ *for any fs-set* $G_f(s)$ *in* Z*, where* $h \circ g$ *is the composition of* h *and* g.

Proof. Proof is omitted. ■

Definition 5.1.1 *A map* g *from an FSTS* $(X, \delta(s))$ *to an FSTS* $(Y, \eta(s))$ *is called fs-continuous if* $g^{-1}(B_f(s))$ *is an fs-open set in* $(X, \delta(s))$ *for every fs-open set* $B_f(s)$ *in* $(Y, \eta(s))$.

Definition 5.1.2 *Fuzzy sequential sets* $X_f^l(s)$ $(l \in I)$*, in a set* X*, are called constant fs-sets.*

Definition 5.1.3 *An fs-set is called a component constant fs-set if its each component is a constant fuzzy set.*

Clearly, each constant fs-set is component constant.

Remark 5.1.1 *A constant function from an FSTS to another FSTS, may not be fs-continuous, as shown by Example* 5.1.1.

Example 5.1.1 *Let $(X, \delta(s))$ and $(Y, \gamma(s))$ be two fuzzy sequential topological spaces, where X be any set, $\delta(s) = \{X_f^0(s), X_f^1(s)\}$, $Y = [0,1]$, $\gamma(s) = \{X_f^0(s), X_f^1(s), \{id_{[0,1]}\}_{n=1}^{\infty}\}$. Define $g : X \to Y$ by*

$$g(x) \;=\; \frac{1}{2} \text{ for all } x \in X.$$

Here, g is a constant function but not fs-continuous.

Theorem 5.1.2 *If every constant function from an FSTS $(X, \delta(s))$ to any other FSTS is fs-continuous, then $\delta(s)$ must contain all the constant fs-sets.*

Proof. Proof is simple and hence omitted. ■

Remark 5.1.2 *Unlike in case of fuzzy topological spaces, the converse of Theorem* 5.1.2 *may not true, as shown by Example* 5.1.2.

Example 5.1.2 *Let $(X, \delta(s))$ and $(Y, \gamma(s))$ be two fuzzy sequential topological spaces, where X be any set, $\delta(s) = \{X_f^r(s); r \in [0,1]\}$, $Y = [0,1]$, $\gamma(s) = \{X_f^0(s), X_f^1(s), G_f(s)\}$ with $G_f^n = \overline{\frac{1}{3}}$ for n odd and $G_f^n = \overline{\frac{1}{4}}$ for n even. Define $g : X \to Y$ by*

$$g(x) \;=\; \frac{1}{2} \text{ for all } x \in X.$$

Though $\delta(s)$ contains all the constant fs-sets, the constant function g is not fs-continuous.

Theorem 5.1.3 *Every constant function from an FSTS $(X, \delta(s))$ to any other FSTS is fs-continuous if and only if $\delta(s)$ contains all the component constant fs-sets.*

Proof. Proof is discarded. ■

Theorem 5.1.4 *If $g : X \to Y$ and $h : Y \to Z$ be fs-continuous functions, then $h \circ g$ is an fs-continuous function from X to Z.*

Proof. The proof is straightforward. ■

Theorem 5.1.5 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:*
(i) g is fs-continuous.
(ii) For each fs-set $A_f(s)$ in X, $g(\overline{A_f(s)}) \leq \overline{g(A_f(s))}$.
(iii) The inverse image of every fs-closed set under g is fs-closed.
(iv) For each fs-set $A_f(s)$ in X, the g inverse of every nbd of $g(A_f(s))$ is a nbd of $A_f(s)$.
(v) For each fs-set $A_f(s)$ in X and each nbd $V_f(s)$ of $g(A_f(s))$, there exists a nbd $W_f(s)$ of $A_f(s)$ such that $g(W_f(s)) \leq V_f(s)$.
(vi) For each fs-set $A_f(s)$ in X, the g inverse of every weak Q-nbd of $g(A_f(s))$ is a weak Q-nbd of $A_f(s)$.
(vii) $\overline{g^{-1}(A_f(s))} \leq g^{-1}(\overline{A_f(s)})$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be an fs-set in X and $P_f(s) \in \overline{A_f(s)}$ be a fuzzy sequential point. So $g(P_f(s)) \in g(\overline{A_f(s)})$. Let $V_f(s)$ be a weak Q-nbd of $g(P_f(s))$. So $g^{-1}(V_f(s))$ is a weak Q-nbd of $P_f(s)$ and hence $g^{-1}(V_f(s)) q_w A_f(s)$. This implies, $V_f(s) q_w g(A_f(s))$ and the result follows.
$(ii) \Rightarrow (iii)$ Let $B_f(s)$ be an fs-closed set in Y and let $A_f(s) = g^{-1}(B_f(s))$. Consider a fuzzy sequential point $P_f(s) \in \overline{A_f(s)}$.

Then, $g(P_f(s)) \in g(\overline{A_f(s)}) \leq \overline{g(A_f(s))} \leq \overline{B_f(s)} = B_f(s)$ so that $P_f(s) \in g^{-1}(B_f(s))$. Hence the result.

$(iii) \Rightarrow (i)$ is straightforward.

$(i) \Rightarrow (iv)$ Let $V_f(s)$ be a nbd of $g(A_f(s))$. So there exists an fs-open set $W_f(s)$ in $(Y, \eta(s))$ such that

$$g(A_f(s)) \leq W_f(s) \leq V_f(s).$$

Then, $g^{-1}(W_f(s))$ is an fs-open set in $(X, \delta(s))$ such that

$$A_f(s) \leq g^{-1}(W_f(s)) \leq g^{-1}(V_f(s)).$$

Hence, $g^{-1}(V_f(s))$ is a nbd of $A_f(s)$.

$(iv) \Rightarrow (i)$ Let $B_f(s)$ be an fs-open set in Y and let a fuzzy sequential point $P_f(s) \in g^{-1}(B_f(s))$. Then $g(P_f(s)) \in B_f(s)$. By (iv), $g^{-1}(B_f(s))$ is a nbd of $P_f(s)$ and thus there exists an fs-open set $O_f(s)$ in X such that $P_f(s) \in O_f(s) \leq g^{-1}(B_f(s))$. Hence the result.

$(iv) \Rightarrow (v)$ and $(v) \Rightarrow (iv)$ are easy to check.

$(i) \Rightarrow (vi)$ Let $V_f(s)$ be a weak Q-nbd of $g(A_f(s))$. So there exists an fs-open set $O_f(s)$ in $(Y, \eta(s))$ such that

$$g(A_f(s)) q_w O_f(s) \leq V_f(s).$$

Then, $g^{-1}(O_f(s))$ is an fs-open set in $(X, \delta(s))$ such that

$$A_f(s) q_w g^{-1}(O_f(s)) \leq g^{-1}(V_f(s)).$$

Hence, $g^{-1}(V_f(s))$ is a weak Q-nbd of $A_f(s)$.

$(vi) \Rightarrow (ii)$, $(i) \Rightarrow (vii)$ and $(vii) \Rightarrow (iii)$ are straightforward. ■

Definition 5.1.4 *A mapping g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, is called an fs-open map if the image of an fs-open set in $(X, \delta(s))$ is an fs-open set in $(Y, \eta(s))$.*

Definition 5.1.5 *A mappping g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, is called an fs-closed map if the image of an fs-closed set in $(X, \delta(s))$ is an fs-closed set in $(Y, \eta(s))$.*

Definition 5.1.6 *A map g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, is called an fs-homeomorphism if g is bijective, g and g^{-1} are both fs-continuous. Further, two fuzzy sequential topological spaces are said to be fs-homeomorphic if there exists an fs-homeomorphism between them.*

Theorem 5.1.6 *A bijective map g from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$ is fs-open if and only if it is fs-closed.*

Proof. Proof is obvious. ■

Theorem 5.1.7 *(i) Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-closed map. Given any $B_f(s) \in (I^Y)^{\mathbb{N}}$ and any fs-open set $U_f(s)$ in $(X, \delta(s))$ containing $g^{-1}(B_f(s))$, there exists an fs-open set $V_f(s)$ in $(Y, \eta(s))$ containing $B_f(s)$ such that $g^{-1}(V_f(s)) \leq U_f(s)$.*
(ii) Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-open map. Given any $B_f(s) \in (I^Y)^{\mathbb{N}}$ and any fs-closed set $U_f(s)$ in $(X, \delta(s))$ containing $g^{-1}(B_f(s))$, there exists an fs-closed set $V_f(s)$ in $(Y, \eta(s))$ containing $B_f(s)$ such that $g^{-1}(V_f(s)) \leq U_f(s)$.

Proof. In both (i) and (ii), if we take $V_f(s) = Y_f^1(s) - g(X_f^1(s) - U_f(s))$, we are done. ■

Now, we characterize fs-open maps, fs-closed maps and fs-homeomorphisms, stating some theorems (Theorem 5.1.8 to Theorem 5.1.12), without proofs as the proofs are simple and straightforward.

Theorem 5.1.8 *Let g be a function from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then the following conditions are equivalent:*
(i) g is fs-open.
(ii) $g((A_f(s))^\circ) \leq (g(A_f(s)))^\circ$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
(iii) For each fs-set $A_f(s)$ in X, the g image of every nbd of $A_f(s)$ is a nbd of $g(A_f(s))$.

Theorem 5.1.9 *Let g be a function from an FSTS $(X, \delta(s))$ to an FSTS $(Y, \eta(s))$, then the following conditions are equivalent:*
(i) g is fs-closed.
(ii) $\overline{g(A_f(s))} \leq g(\overline{A_f(s)})$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

Theorem 5.1.10 *If $g : (X, \delta(s)) \to (Y, \eta(s))$ be a bijective map, then the following conditions are equivalent:*
(i) g is an fs-homeomorphism.
(ii) g is fs-continuous and fs-open.
(iii) g is fs-continuous and fs-closed
(iv) For each fs-set $A_f(s)$ in X, $g(\overline{A_f(s)}) = \overline{g(A_f(s))}$.

Theorem 5.1.11 *Two fuzzy topological spaces* (X, δ) *and* (Y, η) *are homeomorphic to each other if and only if* $(X, \delta^{\mathbb{N}})$ *and* $(Y, \eta^{\mathbb{N}})$ *are fs-homeomorphic.*

Theorem 5.1.12 *If an FSTS* $(X, \delta(s))$ *is fs-homeomorphic to an FSTS* $(Y, \eta(s))$*, then the component fuzzy topologies of* $(X, \delta(s))$ *are homeomorphic to the corresponding component fuzzy topologies of* $(Y, \eta(s))$*.*

Converse of Theorem 5.1.12 may not be true, which is shown by Example 5.2.1 in our next section.

Theorem 5.1.13 *Let* $g : (X, \delta(s)) \to (Y, \eta(s))$ *be an fs-closed bijection. If* $(X, \delta(s))$ *is fs-Hausdorff, so is* $(Y, \eta(s))$*.*

Proof. Let $P_f(s) = (p^M_{fx}, r)$ and $Q_f(s) = (p^N_{fy}, t)$ be any two distinct fs-points in Y, none of which is completely contained in the other. Then, $g^{-1}(P_f(s)) = P'_f(s)$ (say) and $g^{-1}(Q_f(s)) = Q'_f(s)$ (say) are distinct fs-points in X, with the respective bases M and N and none of which is completely contained in the other and thus there exist $U_f(s), V_f(s) \in \delta(s)$ such that

$$P'_f(s) q_w^{M-N} U_f(s),\ Q'_f(s) q_w V_f(s),\ P'_f(s) \overline{q}_w^{M-N} \overline{V_f(s)},\ Q'_f(s) \overline{q}_w \overline{U_f(s)},$$

whenever $Q'_f(s)$ is a totally reduced fuzzy sequential point from $P'_f(s)$; otherwise $\exists\ U_f(s), V_f(s) \in \delta(s)$ such that

$$P'_f(s) q_w U_f(s),\ Q'_f(s) q_w V_f(s),\ P'_f(s) \overline{q}_w \overline{V_f(s)},\ Q'_f(s) \overline{q}_w \overline{U_f(s)}.$$

If we take $U'_f(s) = g(U_f(s))$ and $V'_f(s) = g(V_f(s))$, then $U'_f(s), V'_f(s) \in \eta(s)$ such that

$P_f(s)q_w^{M-N}U'_f(s),\ Q_f(s)q_wV'_f(s),\ P_f(s)\overline{q}_w^{M-N}\overline{V'_f(s)},\ Q_f(s)\overline{q}_w\overline{U'_f(s)},$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$; otherwise

$$P_f(s)q_wU'_f(s),\ Q_f(s)q_wV'_f(s),\ P_f(s)\overline{q}_w\overline{V'_f(s)},\ Q_f(s)\overline{q}_w\overline{U'_f(s)}$$

Hence the theorem. ■

Theorem 5.1.14 *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-closed and an fs-continuous bijection. If $(X, \delta(s))$ is fs-normal, so is $(Y, \eta(s))$.*

Proof. Let $A_f(s)$ and $B_f(s)$ be any two partially quasi discoincident non zero fs-closed sets in Y, with the respective bases M and N and none of which is completely contained in the other. Then, $g^{-1}(A_f(s)) = A'_f(s)$ (say) and $g^{-1}(B_f(s)) = B'_f(s)$ (say) are partially quasi discoincident non zero fs-closed sets in X, with the respective bases M and N and none of which is completely contained in the other. Thus, there exist $U_f(s), V_f(s) \in \delta(s)$ such that

$$A'_f(s)q_w^{M-N}U_f(s),\ B'_f(s)q_wV_f(s),\ A'_f(s) \leq^{M-N} (\overline{V_f(s)})^c$$
$$\text{and } B'_f(s) \leq (\overline{U_f(s)})^c,$$

whenever $B'_f(s)$ is totally reduced from $A'_f(s)$; otherwise there exist $U_f(s), V_f(s) \in \delta(s)$ such that

$A'_f(s)q_wU_f(s),\ B'_f(s)q_wV_f(s),\ A'_f(s) \leq (\overline{V_f(s)})^c,\ B'_f(s) \leq (\overline{U_f(s)})^c.$

Now if we take $U'_f(s) = g(U_f(s))$ and $V'_f(s) = g(V_f(s))$, then $U'_f(s), V'_f(s) \in \eta(s)$ such that

$$A_f(s) q_w^{M-N} U'_f(s),\ B_f(s) q_w V'_f(s),\ A_f(s) \leq^{M-N} (\overline{V'_f(s)})^c$$
$$\text{and } B_f(s) \leq (\overline{U'_f(s)})^c,$$

whenever $B_f(s)$ is totally reduced from $A_f(s)$; otherwise

$$A_f(s) q_w U'_f(s),\ B_f(s) q_w V'_f(s),\ A_f(s) \leq (\overline{V'_f(s)})^c,\ B_f(s) \leq (\overline{U'_f(s)})^c.$$

Hence the theorem. ■

5.2 FS-compactness and ΩFS-compactness

In this section, fs-compact spaces are introduced and studied. It has also been proved that an arbitrary product of fs-compact spaces may not be fs-compact. For this, we introduce a modified version of fs-compactness so called Ωfs-compactness, where the said problem is solved.

Definition 5.2.1 *A family* $\mathfrak{B}$ *of fs-sets is said to be a cover of an fs-set* $A_f(s)$ *if* $A_f(s) \leq \bigvee\{B_f(s);\ B_f(s) \in \mathfrak{B}\}$. *If each member of* $\mathfrak{B}$ *is open, then it is called an open cover of* $A_f(s)$. *A subcover of* $A_f(s)$ *is a subfamily of* $\mathfrak{B}$ *which is also a cover of* $A_f(s)$.

Definition 5.2.2 *An fs-set* $A_f(s)$ *is said to be compact if its every open cover has a finite subcover.*

Definition 5.2.3 *An FSTS* $(X, \delta(s))$ *is called fs-compact if* $X_f^1(s)$ *is compact.*

Definition 5.2.4 *A family $\mathfrak{B}$ of fs-sets is said to have finite intersection property (FIP) if intersection of the members of each finite subfamily of $\mathfrak{B}$ is non zero.*

Theorem 5.2.1 *An FSTS $(X, \delta(s))$ is fs-compact if and only if each family of fs-closed sets which has the finite intersection property, has a non zero intersection.*

Proof. Suppose $(X, \delta(s))$ is fs-compact. Let $\mathfrak{B}$ be a family of fs-closed sets having the finite intersection property. Suppose further that,

$$\bigwedge\{B_f(s);\ B_f(s) \in \mathfrak{B}\} = X_f^0(s).$$

This implies, $\{(B_f(s))^c;\ B_f(s) \in \mathfrak{B}\}$ is an open cover of $X_f^1(s)$ and hence there exists $\{B_{1f}(s),\ B_{2f}(s), \ldots\ldots,\ B_{kf}(s)\} \subseteq \mathfrak{B}$ such that $\bigvee\limits_{i=1}^{k} (B_{if}(s))^c = X_f^1(s)$ - a contradiction.

Conversely, let $\mathfrak{B}$ be an open cover of $X_f^1(s)$ having no finite subcover. Then $\{(B_f(s))^c;\ B_f(s) \in \mathfrak{B}\}$ is a family of fs-closed sets having the FIP but zero intersection. Hence the result. ■

Theorem 5.2.2 *An fs-continuous image of an fs-compact space is fs-compact.*

Proof. Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-continuous onto map, where $(X, \delta(s))$ is fs-compact. Let $\mathfrak{B}$ be an open cover of $Y_f^1(s)$, that is, $Y_f^1(s) = \bigvee\limits_{B_f(s) \in \mathfrak{B}} B_f(s)$. Then, $\{g^{-1}(B_f(s)); B_f(s) \in$

$\mathfrak{B}\}$ is an open cover of $X_f^1(s)$ and hence there exist a finite number of fs-sets, say $B_{1f}(s)$, $B_{2f}(s)$,....., $B_{kf}(s) \in \mathfrak{B}$ such that

$$X_f^1(s) = \bigvee_{i=1}^{k} g^{-1}(B_{if}(s)) = g^{-1}(\bigvee_{i=1}^{k} B_{if}(s)).$$

Since g is onto, we have

$$Y_f^1(s) = g(X_f^1(s)) = g(g^{-1}(\bigvee_{i=1}^{k} B_{if}(s))) = \bigvee_{i=1}^{k} B_{if}(s)$$

Hence, $\{B_{if}(s); i = 1, 2, ..., k\}$ is a finite subfamily of $\mathfrak{B}$ covering $Y_f^1(s)$. ■

Corollary 5.2.1 *An fs-homeomorphic image of an fs-compact space is fs-compact.*

Example 5.2.1 *Let (X, δ) be a fuzzy topological space, where $\delta = \{\overline{0}, \overline{1}\}$. Consider the FSTS's $(X, \delta^{\mathbb{N}})$ and $(X, \delta(s))$, where $\delta(s) = \{X_f^0(s), X_f^1(s)\}$. Both the FSTS's have each component fuzzy topologies δ. Again, $(X, \delta(s))$ is fs-compact but $(X, \delta^{\mathbb{N}})$ is not. Thus, $(X, \delta^{\mathbb{N}})$ and $(X, \delta(s))$ are not fs-homeomorphic although their component fuzzy topologies are homeomorphic.*

Theorem 5.2.3 *If an FSTS $(X, \delta(s))$ is fs-compact, then the component fuzzy topological space (X, δ_n) is fuzzy compact for each $n \in \mathbb{N}$.*

Proof. Proof is omitted. ■

Theorem 5.2.4 *If $(X, \delta^{\mathbb{N}})$ if fs-compact, then (X, δ) is fuzzy compact.*

Proof. Let $\mathbb{A}$ be an open cover of $\overline{1}$. For each $A \in \mathbb{A}$, consider the fs-sets $B_{Af}(s) = \{B^n_{Af}\}_{n=1}^{\infty}$, where $B^n_{Af} = A$ for all $n \in \mathbb{N}$. Then, $\{B_{Af}(s); A \in \mathbb{A}\}$ forms an open cover of $X^1_f(s)$ in $(X, \delta^{\mathbb{N}})$ and hence there exist A_1, A_2,..., $A_k \in \mathbb{A}$ such that $X^1_f(s) = \bigvee_{i=1}^{k} B_{A_i f}(s)$. Hence, $\{A_1, A_2, ..., A_k\}$ is a finite subfamily of $\mathbb{A}$ covering $\overline{1}$. ■

Remark 5.2.1 *Converse of Theorem* 5.2.3 *and Theorem* 5.2.4 *may not be true, as shown by Example* 5.2.2.

Example 5.2.2 *Let* (X, δ) *be a fuzzy topological space, where* $\delta = \{\overline{1}, \overline{0}\}$. *Then,* (X, δ) *is fuzzy compact but* $(X, \delta^{\mathbb{N}})$ *is not fs-compact, since the family* $\{A_{kf}(s), k \in \mathbb{N}\}$, *where*

$$\begin{aligned} A^n_{kf} &= \overline{1} \text{ if } n = k \\ &= \overline{0} \text{ otherwise,} \end{aligned}$$

is an open cover of $X^1_f(s)$ *in* $(X, \delta^{\mathbb{N}})$, *having no finite subfamily covering* $X^1_f(s)$.

Definition 5.2.5 *For each* $i \in J$, *let* $(X_i, \delta_i(s))$ *be an FSTS. A product fuzzy sequential topology* $\delta(s)$ *on the product* $X = \prod_{i \in J} X_i$ *is the coarsest fuzzy sequential topology on X, making all the projection mappings* $\pi_i : X \to X_i$ *fs-continuous. If* $\delta(s)$ *is the product fuzzy sequential topology on* $X = \prod_{i \in J} X_i$, *then* $(X, \delta(s))$ *is called product fuzzy sequential topological space.*

Theorem 5.2.5 *For each* $i \in J$, *let* $(X_i, \delta_i(s))$ *be an FSTS. A subbase for the product fuzzy sequential topology* $\delta(s)$ *on* $X =$

$\prod_{i\in J} X_i$ *is given by* $\mathbb{S} = \{\pi_i^{-1}(O_{if}(s)); O_{if}(s) \in \delta_i(s), i \in J\}$, *so that a basis for* $\delta(s)$ *can be taken to be* $\mathfrak{B} = \{\bigwedge_{j=1}^{n} \pi_{i_j}^{-1}(O_{i_jf}(s));$ $O_{i_jf}(s) \in \delta_{i_j}(s), i_j \in J, n \in \mathbb{N}\}$.

Proof. Proof is omitted. ∎

Lemma 5.2.1 *If* $\mathbb{S}$ *be a subbase for a fuzzy sequential topology* $\delta(s)$ *on* X*, then* $(X, \delta(s))$ *is fs-compact if and only if every open cover of* $X_f^1(s)$ *by the members of* $\mathbb{S}$*, has a finite subcover.*

Proof. Proof is omitted. ∎

Definition 5.2.6 *A collection of fs-sets in an FSTS is said to have the finite union property (FUP) if none of its finite sub-collection covers* $X_f^1(s)$*.*

Theorem 5.2.6 *Let* n *be a positive integer. If* $(X_i, \delta_i(s))$ *be fs-compact spaces for each* i = 1, 2,.....n *and* $\delta(s)$ *be the product fuzzy sequential topology on* $X = \prod_{i\in J} X_i$*, then* $(X, \delta(s))$ *is fs-compact.*

Proof. We know that $\mathbb{S} = \{\pi_i^{-1}(O_{if}(s)); O_{if}(s) \in \delta_i(s), i = 1, 2, ..., n\}$ is a subbase for $\delta(s)$. By Lemma 5.2.1, it suffices to show that no sub-collection of $\mathbb{S}$ with FUP covers $X_f^1(s)$. Let $\mathbb{D}$ be a sub-collection of $\mathbb{S}$ with FUP. For each i = 1, 2,....., n, let $\mathbb{D}_i = \{O_f(s) \in \delta_i(s) : \pi_i^{-1}(O_f(s)) \in \mathbb{D}\}$. Then $\mathbb{D}_i$ is a collection of fs-open sets in $(X_i, \delta_i(s))$ with FUP. By fs-compactness of

$(X_i, \delta_i(s))$, $\mathbb{D}_i$ cannot cover $X^1_{if}(s)$. So, there exists $x_i \in X_i$ and $m \in \mathbb{N}$ such that

$$\text{the } m^{th} \text{ component of } (\bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s))(x_i) = a_i(\text{ say }) < 1$$

Now, if we consider the point $x = (x_1, x_2, ..., x_n) \in X$ and the collection $\mathbb{D}'_i = \{\pi_i^{-1}(O_f(s)); O_f(s) \in \delta_i(s)\} \cap \mathbb{D}$, then it follows that

$$\begin{aligned}
&(\bigvee_{O'_f(s) \in \mathbb{D}'_i} O'_f(s))(x) \\
&= \bigvee \{\pi_i^{-1}(O_f(s))(x); O_f(s) \in \delta_i(s) \text{ and } \pi_i^{-1}(O_f(s)) \in \mathbb{D}\} \\
&= \bigvee \{O_f(s)(x_i); O_f(s) \in \delta_i(s) \text{ and } \pi_i^{-1}(O_f(s)) \in \mathbb{D}\} \\
&= (\bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s))(x_i)
\end{aligned}$$

Further, noting that $\mathbb{D} = \bigcup\limits_{i=1}^{n} \mathbb{D}'_i$, we obtain

$$\begin{aligned}
(\bigvee_{O'_f(s) \in \mathbb{D}} O'_f(s))(x) &= \bigvee_{i=1}^{n} (\bigvee_{O'_f(s) \in \mathbb{D}'_i} O'_f(s))(x) \\
&= \bigvee_{i=1}^{n} (\bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s))(x_i)
\end{aligned}$$

Therefore, the m^{th} term of $(\bigvee\limits_{O'_f(s) \in \mathbb{D}} O'_f(s))(x)$ is $\bigvee\limits_{i=1}^{n} a_i$ which is less than 1 and hence the theorem. ■

Remark 5.2.2 *An arbitrary product of fs-compact spaces may not be fs-compact, as shown by Example* 5.2.3.

Example 5.2.3 *For each $i \in \mathbb{N}$, let $X_i = \mathbb{N}$. Let $A_{if}(s)$ be an fs-set in X_i such that $A^n_{if}(x_i) = \frac{i-1}{i}$ $\forall x_i \in X_i$ and $\forall n \in \mathbb{N}$. Let $\delta_i(s) = \{X^0_f(s), X^1_f(s), A_{if}(s)\} \cup \{A_{if}(s) \times \chi_{\{1,2,....,n\}}(s); n \in \mathbb{N}\}$, where $\chi_{\{1,2,....,n\}}(s)$ is an fs-set whose each component is the characteristic function of the set $\{1, 2, ..., n\}$. Then $(X_i, \delta_i(s))$ is an FSTS. Further, if $\{O_{\lambda f}(s); \lambda \in \Lambda\}$ be an open cover of $X^1_{if}(s)$ in $(X_i, \delta_i(s))$, then $O_{\lambda f}(s) = X^1_{if}(s)$ for some $\lambda \in \Lambda$. This implies that $(X_i, \delta_i(s))$ is fs-compact for all $i \in \mathbb{N}$.*

Now, let $\delta(s)$ be the product fuzzy sequential topology on $X = \prod_{i\in\mathbb{N}} X_i$. For $(i, n) \in \mathbb{N} \times \mathbb{N}$,

$$\begin{aligned} \pi_i^{-1}(A_{if}(s) \times \chi_{\{1,2,....,n\}}(s)) &= A_{if}(s) \times \chi_{\{1,2,....,n\}}(s) \circ \pi_i \\ &= B_{if}(s) \ (say) \end{aligned}$$

is a member of $\delta(s)$. Let $x = (x_i)_{i\in\mathbb{N}} \in X$. Then

$$B_{if}(s)(x) = A_{if}(s) \times \chi_{\{1,2,....,n\}}(s)(x_i)$$

which implies, $\forall\ n \in \mathbb{N}$,

$$B^n_{if}(x) = (A^n_{if} \times \chi_{\{1,2,....,n\}})(x_i) = \begin{cases} \frac{i-1}{i} \ if\ x_i \le n \\ 0 \ if\ x_i > n \end{cases}$$

Given $\epsilon > 0$, we can find i with $1 - \epsilon < \frac{i-1}{i}$, which gives $B^n_{if}(x) > 1 - \epsilon\ \forall\ n \ge x_i$. So $\bigvee_{(i,n)\in\mathbb{N}\times\mathbb{N}} B^n_{if}(x) = 1$ for all $n \in \mathbb{N}$, that is, $\bigvee_{(i,n)\in\mathbb{N}\times\mathbb{N}} B_{if}(s) = X^1_f(s)$. If L be a finite subset of $\mathbb{N} \times \mathbb{N}$, then we can find $N \in \mathbb{N}$ such that whenever $(i, n) \in L$, $n < N$. It follows that, for $x = (N, N, N, ...)$, we have $B^n_{if}(x) = 0$ for all

$(i,n) \in L$ and certainly $\bigvee_{(i,n)\in L} B^n_{if}(x) = 0$. Thus, $\bigvee_{(i,n)\in L} B_{if}(s) \neq X^1_f(s)$ and hence $(X, \delta(s))$ is not fs-compact.

Definition 5.2.7 *A fuzzy sequential topology is called Ω fuzzy sequential topology if it contains all the component constant fs-sets.*

Definition 5.2.8 *An fs-set $A_f(s)$ in an FSTS $(X, \delta(s))$ is said to be Ω-compact if for any open cover $\{B_{if}(s); i \in J\}$ of $A_f(s)$ and for any positive sequence $\epsilon = \{\epsilon_n\}$ of real numbers, there exist finitely many $B_{if}(s)$'s say $B_{i_1f}(s)$, $B_{i_2f}(s)$,......,$B_{i_kf}(s)$, such that*

$$\bigvee_{j=1}^{k} B_{i_jf}(s)(x) \geq A_f(s)(x) - \epsilon \text{ for all } x \in X.$$

Definition 5.2.9 *An Ω fuzzy sequential topological space or $\Omega-FSTS$ is called Ωfs-compact if every component constant fs-set is Ω-compact.*

Theorem 5.2.7 *An fs-continuous image of an Ωfs-compact space is Ωfs-compact.*

Proof. Proof is omitted. ■

Lemma 5.2.2 *Let $\mathbb{S}$ be a subbase for an Ω fuzzy sequential topology $\delta(s)$ on X. Then $(X, \delta(s))$ is Ωfs-compact if and only if for any component constant fs-set $\alpha_f(s)$ with $\bigvee_{i\in J} O_{if}(s) \geq \alpha_f(s)$, where $O_{if}(s) \in \delta(s)$ for all $i \in J$, and for any positive sequence* *of real numbers $\epsilon = \{\epsilon_n\}$, there are finitely many indices say i_1, i_2,....,$i_k \in J$ such that*

$$\bigvee_{j=1}^{k} O_{i_j f}(s)(x) \ \geq \ \alpha_f(s)(x) - \epsilon \text{ for all } x \in X.$$

Proof. Proof is omitted. ■

Definition 5.2.10 *For an fs-set $\alpha_f(s) = \{\overline{\alpha}_n\}_{n=1}^{\infty}$ and for a positive real sequence $\epsilon = \{\epsilon_n\}$ with $\epsilon_n < \alpha_n$ for each $n \in \mathbb{N}$, we say that a collection of fs-sets has ϵ-FUP for $\alpha_f(s)$ if none of its finite sub-collection covers $\alpha_f(s) - \epsilon$.*

Theorem 5.2.8 *Let $(X_i, \delta_i(s))$ be Ωfs-compact spaces for all $i \in J$ (J being an index set). Then the product fuzzy sequential topological space $(X, \delta(s))$, where $X = \prod_{i \in J} X_i$, is Ωfs-compact.*

Proof. Let $\alpha_f(s) = \{\overline{\alpha}_n\}_{n=1}^{\infty}$ be a component constant fs-set in X. We wish to show that $\alpha_f(s)$ is Ω-compact. A subbase for $\delta(s)$ is $\mathbb{S} = \{\pi_i^{-1}(O_{if}(s)); O_{if}(s) \in \delta_i(s), i \in J\}$. Let $\epsilon = \{\epsilon_n\}$ be any positive sequence of real numbers with $\epsilon_n < \alpha_n$ for all $n \in \mathbb{N}$ and let $\mathbb{D}$ be a sub-collection of $\mathbb{S}$ with ϵ-FUP for $\alpha_f(s)$. By Lemma 5.2.2, it is sufficient to show that $\mathbb{D}$ does not cover $\alpha_f(s)$.

For each $i \in J$, set $\mathbb{D}_i = \{O_f(s) \in \delta_i(s) : \pi_i^{-1}(O_f(s)) \in \mathbb{D}\}$. Let $O_{i_1 f}(s)$, $O_{i_2 f}(s)$,...,$O_{i_k f}(s) \in \mathbb{D}_i$. Then $\{\pi_i^{-1}(O_{i_j f}(s))\ ; j = 1, 2, ..., k\}$ is a finite sub-collection of $\mathbb{D}$, whence there exists a point $x = (x_i)_{i \in J} \in X$ and $r \in \mathbb{N}$ such that

$$\text{the } r^{th} \text{ component of } \bigvee_{j=1}^{k} \pi_i^{-1}(O_{i_j f}(s))(x) \ < \ \alpha_r - \epsilon_r.$$

It then follows that

$$\begin{aligned} & \text{the } r^{th} \text{ component of } \bigvee_{j=1}^{k} O_{i_j f}(s)(x_i) \\ = & \text{ the } r^{th} \text{ component of } \bigvee_{j=1}^{k} O_{i_j f}(s)(\pi_i(x)) \\ = & \text{ the } r^{th} \text{ component of } \bigvee_{j=1}^{k} \pi_i^{-1}(O_{i_j f}(s))(x) \\ < & \ \alpha_r - \epsilon_r \\ = & \ (\alpha_r - \epsilon_r/2) - \epsilon_r/2 \end{aligned}$$

This implies, $\mathbb{D}_i$ is a collection of fs-open sets in $(X_i, \delta_i(s))$ with $\{\epsilon_n/2\}$-FUP for $\alpha_f(s) - \{\epsilon_n/2\}$.

By Ωfs-compactness of $(X_i, \delta_i(s))$, $\mathbb{D}_i$ cannot cover $\alpha_f(s) - \{\epsilon_n/2\}$. So, there exists $y_i \in X_i$ and $m \in \mathbb{N}$ such that

$$\text{the } m^{th} \text{ component of } (\bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s))(y_i) \ < \ \alpha_m - \epsilon_m/2$$

Having done this for each $i \in J$, set $y = (y_i)_{i \in J}$. If we set $\mathbb{D}'_i = \{\pi_i^{-1}(O_f(s)); O_f(s) \in \delta_i(s)\} \cap \mathbb{D}$, then as in Theorem 5.2.6, $\mathbb{D} = \bigcup_{i \in J} \mathbb{D}'_i$ and

$$\bigvee_{O'_f(s) \in \mathbb{D}'_i} O'_f(s)(y) \ = \ \bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s)(y_i)$$

so that

$$\begin{aligned} (\bigvee_{O'_f(s) \in \mathbb{D}} O'_f(s))(y) \ &= \ \bigvee_{i \in J} (\bigvee_{O'_f(s) \in \mathbb{D}'_i} O'_f(s))(y) \\ &= \ \bigvee_{i \in J} (\bigvee_{O_f(s) \in \mathbb{D}_i} O_f(s))(y_i) \end{aligned}$$

Therefore, the m^{th} component of $(\bigvee_{O'_f(s)\in\mathbb{D}} O'_f(s))(y) \leq \alpha_m - \epsilon_m/2$, which is less than α_m. Thus $\mathbb{D}$ cannot cover $\alpha_f(s)$. ■