CHAPTER

3

Separation Axioms

After the initiation of fuzzy topology by C. L. Chang in 1968, a number of works have been done in fuzzy topology. One of them is the study of separation axioms. This Chapter mainly deals with the separation axioms in our setting of fuzzy sequential topology.

3.1 Separation Axioms in a Fuzzy Sequential Topological Space

Definition 3.1.1 *Two fuzzy sequential points* $P_f(s) = (p^M_{fx},\ r)$ *and* $Q_f(s) = (p^N_{fy},\ t)$ *are said to be identical if* $x = y$, $M = N$ *and* $r = t$*; otherwise they are distinct.*

Definition 3.1.2 *A set* $M \subseteq \mathbb{N}$ *is said to be the base of a fuzzy sequential set* $U_f(s)$ *if* $U^n_f \neq \overline{0}\ \forall n \in M$ *and* $U^n_f = \overline{0}\ \forall n \in \mathbb{N} - M$.

Definition 3.1.3 *A fuzzy sequential set $B_f(s)$ (having base N) is said to be completely contained in a fuzzy sequential set $A_f(s)$ (having base M) if $M = N$ and $B_f^n \leq A_f^n$ for all $n \in N$.*

Definition 3.1.4 *A fuzzy sequential set $B_f(s)$ (having base N) is said to be totally reduced from the fuzzy sequential set $A_f(s)$ (having base M) if $N \subsetneqq M$ and $B_f^n \leq A_f^n$ $\forall n \in N$.*

Definition 3.1.5 *An FSTS $(X, \delta(s))$ is said to be an fs-T_0 space if for any two distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$, there exist a weak Q-nbd of one of $P_f(s)$ and $Q_f(s)$ which is not weakly quasi-coincident with the other.*

Theorem 3.1.1 *An FSTS $(X, \delta(s))$ is an fs-T_0 space if and only if for every pair of distinct fuzzy sequential points $P_f(s)$ and $Q_f(s)$, either $P_f(s)$ does not belong to the closure of $Q_f(s)$ or $Q_f(s)$ does not belong to the closure of $P_f(s)$.*

Proof. Suppose $(X, \delta(s))$ be fs-T_0. Then, $\exists$ a weak Q-nbd $U_f(s)$ of $P_f(s)$ (say) which is not weakly quasi-coincident with $Q_f(s)$. This implies that $P_f(s) \notin \overline{Q_f(s)}$. Conversely, suppose $P_f(s)$ and $Q_f(s)$ be any two distinct fuzzy sequential points such that $P_f(s) \notin \overline{Q_f(s)}$. This implies that $\exists$ a weak Q-nbd of $P_f(s)$ which is not weakly quasi-coincident with $Q_f(s)$. Hence $(X, \delta(s))$ is fs-T_0. ■

Corollary 3.1.1 *An FSTS $(X, \delta(s))$ is fs-T_0 space if and only if distinct fuzzy sequential points have distinct closures.*

Theorem 3.1.2 *A fuzzy topology $(X,\ \delta)$ is fuzzy T_0 if and only if the fuzzy sequential topology $(X,\ \delta^{\mathbb{N}})$ is fs-T_0.*

Proof. Suppose $(X,\ \delta)$ be fuzzy T_0. Let $P_f(s) = (p^M_{fx},\ r)$ and $Q_f(s) = (p^N_{fy},\ t)$ be any two distinct fuzzy sequential points where $r = \{r_n\}_n$ and $t = \{t_n\}_n$.
Case I. Suppose $x \neq y$. Then, for $p_x^{r_m} \neq p_y^{t_m}$ $(m \in M)$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.
Case II. Suppose $x = y$, $M \cap N = \phi$. Then for $p_x^{r_m} \neq p_x^{t_m}$ $(m \in M$ $)$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.
Case III. Suppose $x = y$, $N \subseteq M$. If $r_m \neq t_m$ for some $m \in N$, then for $p_x^{r_m} \neq p_x^{t_m}$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$. If $r_n = t_n$ $\forall$ $n \in N$, then for $p_x^{r_m} \neq p_x^{t_m}$ $(m \in M - N)$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.
Case IV. Suppose $x = y$ and neither $N \subseteq M$ nor $M \subseteq N$ nor $M \cap N = \phi$. Then, for $p_x^{r_m} \neq p_x^{t_m}$ $(m \in M,\ m \notin N\ )$, $\exists$ a Q-nbd U of $p_x^{r_m}$ which is not quasi-coincident with $p_x^{t_m}$.
In all the above cases, the fuzzy sequential set $U_f(s)$ where $U_m = U$ and $U_n = \overline{0}$ $\forall n \neq m$, is a weak Q-nbd of $P_f(s)$ which is not weakly quasi-coincident with $Q_f(s)$.

Conversely, suppose $(X,\ \delta^{\mathbb{N}})$ is fs-T_0. Let p_x^{λ} and p_y^{μ} be any two distinct fuzzy points in X. Then, for each $n \in \mathbb{N}$, fuzzy sequential points $(p^n_{fx},\ \lambda)$ and $(p^n_{fy},\ \mu)$ are distinct. So $\exists$ a weak Q-nbd $U_f(s)$ of one of $(p^n_{fx},\ \lambda)$ and $(p^n_{fy},\ \mu)$ which is not weakly quasi-coincident with the other. This implies, U_f^n is a Q-nbd of one of p_x^λ and p_y^μ which is not quasi-coincident with the other . ■

Theorem 3.1.3 *If an FSTS $(X,\ \delta(s))$ is fs-T_0, then the fuzzy topological space $(X,\ \delta_n)$ is fuzzy T_0 for each $n \in \mathbb{N}$, where $\delta_n = \{A_f^n;\ A_f(s) \in \delta(s)\}$.*

Proof. Suppose $(X,\ \delta(s))$ be fs-T_0. Let p_x^λ and p_y^μ be any two distinct fuzzy points in X. Then for each $n \in \mathbb{N}$, fuzzy sequential points $(p_{fx}^n,\ \lambda)$ and $(p_{fy}^n,\ \mu)$ are distinct. So $\exists$ a weak Q-nbd $U_f(s)$ of one of $(p_{fx}^n,\ \lambda)$ and $(p_{fy}^n,\ \mu)$ which is not weakly quasi-coincident with the other. This implies, U_f^n is a Q-nbd of one of p_x^λ and p_y^μ which is not quasi-coincident with the other . ■

Converse of Theorem 3.1.3 may not be true, as shown by the following Example.

Example 3.1.1 *Let $(X,\ \delta)$ be a fuzzy topological space. For any $A \in \delta$, let us consider the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$, where $B_{fA}^n = A$ for odd n, $B_{fA}^n = \overline{0}$ for even n, $C_{fA}^n = \overline{0}$ for odd n, $C_{fA}^n = A$ for even n and $D_{fA}^n = A\ \forall n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$ for all $A \in \delta$, forms a fuzzy sequential topology on X. If $(X,\ \delta)$ is fuzzy T_0, then the components of $(X,\ \delta(s))$ are fuzzy T_0 but $(X,\ \delta(s))$ is not fs-T_0.*

Definition 3.1.6 *Suppose $U_f(s)$ and $V_f(s)$ be two fuzzy sequential sets. If there exists an $M \subseteq \mathbb{N}$ such that $U_f^n q V_f^n\ \forall n \in$*

M, we say that $U_f(s)$ is M-quasi coincident with $V_f(s)$ and we write $U_f(s)q^M V_f(s)$. If $U_f^n q V_f^n$ for at least one $n \in M$, we say that $U_f(s)$ is weakly M-quasi coincident with $V_f(s)$ and we write $U_f(s)q_w^M V_f(s)$.

Definition 3.1.7 *An FSTS $(X,\, \delta(s))$ is said to be an fs-T_1 space if every fuzzy sequential point in X is closed.*

Remark 3.1.1 *An fs-T_1 space is fs-T_0.*

Theorem 3.1.4 *A fuzzy topological space $(X,\, \delta)$ is fuzzy T_1 if and only if the fuzzy sequential topological space $(X,\, \delta^{\mathbb{N}})$ is fs-T_1.*

Proof. Proof is omitted. ■

Theorem 3.1.5 *If an FSTS $(X,\, \delta(s))$ is fs-T_1, then the component fuzzy topological space $(X,\, \delta_n)$ is fuzzy T_1 for each $n \in \mathbb{N}$.*

Proof. Proof is omitted. ■

The next example shows that the converse of Theorem 3.1.5 may not true.

Example 3.1.2 *Let $(X,\, \delta)$ be a fuzzy topological space. For any $A \in \delta$, let us consider the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$ and $D_{fA}(s)$, where $B_{fA}^n = A$ for odd n, $B_{fA}^n = \overline{0}$ for even n, $C_{fA}^n = \overline{0}$ for odd n, $C_{fA}^n = A$ for even n and $D_{fA}^n = A\ \forall n \in \mathbb{N}$. The collection $\delta(s)$ of all the fuzzy sequential sets $B_{fA}(s)$, $C_{fA}(s)$*

and $D_{fA}(s)$ for all $A \in \delta$, forms a fuzzy sequential topology on X. If $(X,\, \delta)$ is fuzzy T_1 then the components of $(X,\, \delta(s))$ are fuzzy T_1 but $(X,\, \delta(s))$ is not fs-T_1.

Theorem 3.1.6 *An FSTS $(X,\, \delta(s))$ is fs-T_1 if and only if for each $x \in X$ and each sequence $r = \{r_n\}_n$ in I, $\exists$ $B_f(s) \in \delta(s)$ such that $B_f(s)(x) = 1 - r$ and $B_f(s)(y) = 1$ for $y \neq x$.*

Proof. Suppose $(X,\, \delta(s))$ be fs-T_1. If r is a zero sequence, then it is sufficient to take $B_f(s) = X^1_f(s)$. Suppose r is a non zero sequence. Let $M \subseteq \mathbb{N}$ such that $r_n \neq 0$ $\forall n \in M$ and $r_n = 0$ $\forall n \in \mathbb{N} - M$. If $P_f(s) = (p^M_{fx},\, r)$, then $B_f(s) = X^1_f(s) - P_f(s)$ is the required open fuzzy sequential set.

Conversely, suppose $P_f(s) = (p^M_{fx},\, r)$ be an arbitrary fuzzy sequential point in X. By the given condition, there exists an open fuzzy sequential set $B_f(s)$ in X such that $B_f(s)(x) = 1 - r$ and $B_f(s)(y) = 1$ for $y \neq x$. It follows that $P_f(s)$ is the complement of $B_f(s)$ and hence is closed. ■

Theorem 3.1.7 *The fuzzy derived sequential set of every fuzzy sequential set on an fs-T_1 space is closed.*

Proof. The fuzzy derived sequential set of a fuzzy sequential point in an fs-T_1 space is a fuzzy sequential point and hence is closed. Thus the result follows from Theorem 2.1.5. ■

Definition 3.1.8 *An FSTS $(X,\, \delta(s))$ is said to be an fs-Hausdorff or an fs-T_2 space if for any two distinct fuzzy sequential points*

$P_f(s) = (p^M_{fx},\ r)$ and $Q_f(s) = (p^N_{fy},\ t)$, none of which is completely contained in the other, $\exists$ open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s)q^{M-N}_w U_f(s),\ Q_f(s)q_w V_f(s),\ P_f(s)\ \overline{q}^{M-N}_w\ \overline{V_f(s)},$$
$$\text{and } Q_f(s)\ \overline{q}_w\ \overline{U_f(s)},$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$; otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s)q_w U_f(s),\ Q_f(s)q_w V_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)},\ Q_f(s)\ \overline{q}_w\ \overline{U_f(s)}.$$

Definition 3.1.9 *An FSTS $(X, \delta(s))$ is said to be a weak fs-Hausdorff space or (w) fs-Hausdorff space if for any two distinct fuzzy sequential points $P_f(s) = (p^M_{fx},\ r)$ and $Q_f(s) = (p^N_{fy},\ t)$, none of which is completely contained in the other, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that*

$$P_f(s) \in^{M-N}_w U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_w V_f(s),$$

whenever $Q_f(s)$ is a totally reduced fuzzy sequential point from $P_f(s)$; otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) \in_w U_f(s),\ Q_{df}(s) \in_w V_f(s),\ U_f(s)\overline{q}_w V_f(s).$$

Theorem 3.1.8 *An fs-Hausdorff space is a weak fs-Hausdorff space.*

Proof. Proof is omitted. ■

Example 3.1.3 shows that a weak fs-Hausdorff space may not be an fs-Hausdorff space.

Example 3.1.3 *Let* $X = \{x,\, y\}$, $\delta = \{\overline{r},\, r \in [0,\, 1]\}$. *Then* $(X,\, \delta^{\mathbb{N}})$ *is weak fs-Hausdorff but not fs-Hausdorff.*

An fs-T_2 space may not be an fs-T_1 space, as shown by the following Example.

Example 3.1.4 *Let* $X = \{x,\, y\}$, $\delta = \{\overline{1},\, \overline{0},\, p_x^1,\, p_y^1\}$. *Fuzzy sequential topological space* $(X,\, \delta^{\mathbb{N}})$ *is fs-*T_2 *but not fs-*T_1.

Definition 3.1.10 *An FSTS* $(X,\, \delta(s))$ *is said to be a weak fs-*T_2 *or* (w) *fs-*T_2 *space if it is* (w) *fs-Hausdorff and fs-*T_1.

Remark 3.1.2 *An fs-*T_2 *space is weak fs-*T_2.

Theorem 3.1.9 *An FSTS* $(X,\, \delta(s))$ *is fs-Hausdorff if and only if for any two distinct fuzzy sequential points* $P_f(s) = (p_{fx}^M,\, r)$ *and* $Q_f(s) = (p_{fy}^N,\, t)$, *none of which is completely contained in the other, there exist open fuzzy sequential sets* $G_f(s)$, $H_f(s)$, $D_f(s)$ *and* $E_f(s)$ *in* $(X,\, \delta(s))$ *such that*

$$P_f(s) \in^{M-N} G_f(s),\ Q_f(s) q_w H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s) q_w^{M-N} D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s) \overline{q_w} D_f(s),$$

whenever $Q_f(s)$ *is a totally reduced fuzzy sequential point from* $P_f(s)$; *otherwise there exist open fuzzy sequential sets* $G_f(s)$, $H_f(s)$,

$D_f(s)$ and $E_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in G_f(s),\ Q_f(s)q_w H_f(s),\ G_f(s)\overline{q_w}H_f(s),$$
$$P_f(s)q_w D_f(s),\ Q_f(s) \in E_f(s),\ E_f(s)\overline{q_w}D_f(s).$$

Proof. Let $P_f(s) = (p^M_{fx},\, r)$ and $Q_f(s) = (p^N_{fy},\, t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other. Suppose $(X,\, \delta(s))$ is fs-Hausdorff.

Case I. Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. Then, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s)q_w U_f(s),\ Q_f(s)q_w V_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)},\ Q_f(s)\ \overline{q}_w\ \overline{U_f(s)}.$$

If we take $G_f(s) = X^1_f(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X^1_f(s) - \overline{U_f(s)}$, we are done.

Case II. Suppose one of $P_f(s)$ and $Q_f(s)$, say $Q_f(s)$ is totally reduced from $P_f(s)$. Then, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s)q^{M-N}_w U_f(s),\ Q_f(s)\ q_w\ V_f(s),\ P_f(s)\ \overline{q}^{M-N}_w\ \overline{V_f(s)},$$
$$\text{and } Q_f(s)\ \overline{q}_w\ \overline{U_f(s)}.$$

Now if we take $G_f(s) = X^1_f(s) - \overline{V_f(s)}$, $H_f(s) = V_f(s)$, $D_f(s) = U_f(s)$ and $E_f(s) = X^1_f(s) - \overline{U_f(s)}$, we are done.

Conversely, suppose the given conditions are true. In both the cases, if we take $U_f(s) = D_f(s)$, $V_f(s) = H_f(s)$ and use the Definition 3.1.8, we are done. ∎

Theorem 3.1.10 *An FSTS* $(X,\, \delta(s))$ *is fs-Hausdorff if and only if for any fuzzy sequential point* $P_f(s)$ *in* X*,*

$$P_f(s) = \wedge\{\overline{N_f(s)};\ N_f(s) \text{ is a nbd of } P_f(s)\} \quad (3.1.1)$$

Proof. Suppose $(X,\, \delta(s))$ be fs-Hausdorff. Let $P_f(s) = (p^M_{fx},\, r)$ be any fuzzy sequential point in X and $Q_f(s) = (p^N_{fx},\, t)$ be another fuzzy sequential point distinct from $P_f(s)$ and $Q_f(s) \notin P_f(s)$.

If $P_f(s)$ is totally reduced from $Q_f(s)$, there exist an open fuzzy sequential set $V_f(s)$ in $(X,\, \delta(s))$ such that

$$Q_f(s)\ q^{N-M}_w\ V_f(s),\ P_f(s)\ \overline{q}_w \overline{V_f(s)}\ ;$$

otherwise there exist an open fuzzy sequential set $V_f(s)$ in $(X,\, \delta(s))$ such that

$$Q_f(s)\ q_w\ V_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)}.$$

In both the cases, if we take $U_f(s) = X^1_f(s) - \overline{V_f(s)}$, then $P_f(s) \in U_f(s)$ and $Q_f(s) \notin \overline{U_f(s)}$. Hence 3.1.1 is true.

Conversely, suppose 3.1.1 holds. Let $P_f(s) = (p^M_{fx},\, r)$ and $Q_f(s) = (p^N_{fy},\, t)$ be any two distinct fuzzy sequential points in X, none of which is completely contained in the other.

Case I. Suppose none of $P_f(s)$ and $Q_f(s)$ is totally reduced from the other. By 3.1.1, there exist nbds $S_f(s)$ and $T_f(s)$ of $P_f(s)$ and $Q_f(s)$ respectively such that

$$P_f(s) \notin \overline{T_f(s)} \text{ and } Q_f(s) \notin \overline{S_f(s)}.$$

Case II. Suppose one of $P_f(s)$ and $Q_f(s)$, $Q_f(s)$ (say) is totally reduced from $P_f(s)$. Then, there exist nbds $S_f(s)$ and $T_f(s)$ of $P'_f(s)$ and $Q_f(s)$ respectively such that $P_f(s) \notin^{M-N} \overline{T_f(s)}$ and $Q_f(s) \notin \overline{S_f(s)}$, where $P'_f(s)$ is a reduced fuzzy sequential point of $P_f(s)$ with base $M - N$.
In both of the above two cases, if we take $U_f(s) = X^1_f(s) - \overline{T_f(s)}$, $V_f(s) = X^1_f(s) - \overline{S_f(s)}$ and use the Definition 3.1.8, we are done.
■

Theorem 3.1.11 *If a fuzzy topological space (X, δ) is fuzzy T_2, then the FSTS $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff.*

Proof. Proof is omitted. ■

That the converse of Theorem 3.1.11 is not true, is shown by the following Example.

Example 3.1.5 *Let $X = \{x, y\}$, $\delta = \{\overline{1}, \overline{0}, p^1_x, p^1_y\}$. Then, the fuzzy sequential topological space $(X, \delta^{\mathbb{N}})$ is weak fs-Hausdorff although the fuzzy topological space (X, δ) is not fuzzy T_2.*

Example 3.1.6 shows that even if (X, δ) is fuzzy T_2, the FSTS $(X, \delta^{\mathbb{N}})$ may not be fs-Hausdorff.

Example 3.1.6 *Let $X = \{x, y\}$, $\delta = \{\overline{r}, r \in [0, 1]\}$. Then, the fuzzy topological space (X, δ) is fuzzy T_2 but the FSTS $(X, \delta^{\mathbb{N}})$ is not fs-Hausdorff.*

Example 3.1.7 shows that if an FSTS $(X,\, \delta(s))$ is fs-T_2, then its component fuzzy topological space $(X,\, \delta_n)$ may not be fuzzy T_2 for each $n \in \mathbb{N}$.

Example 3.1.7 *Let $X = \{x,\, y\}$, $\delta = \{\overline{1},\, \overline{0},\, p_x^1,\, p_y^1\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then $(X,\, \delta(s))$ is fs-T_2 but $(X,\, \delta_n) = (X,\, \delta)$ $(\forall n \in \mathbb{N})$ is not fuzzy T_2.*

Example 3.1.8 shows that even if all the component fuzzy topological spaces of an FSTS are fuzzy T_2, the FSTS may not be fs-T_2.

Example 3.1.8 *Let $(X,\, \delta)$ be a fuzzy topological space. For any $G \in \delta$, consider the fs-sets $A_{fG}(s)$, $B_{fG}(s)$, $C_{fG}(s)$ where $A_{fG}^n = G$ for odd n, $A_{fG}^n = \overline{0}$ for even n; $B_{fG}^n = \overline{0}$ for odd n, $B_{fG}^n = G$ for even n; $C_{fG}^n = G$ $\forall n \in \mathbb{N}$. Then the collection $\delta(s)$ of all the fs-sets $A_{fG}(s)$, $B_{fG}(s)$, $C_{fG}(s)$ $\forall G \in \delta$, forms a fuzzy sequential topology on X. If $(X,\, \delta)$ is fuzzy T_2, then the components of $(X,\, \delta(s))$ are fuzzy T_2 but $(X,\, \delta(s))$ itself is not fs-T_2.*

Definition 3.1.11 *An FSTS $(X,\, \delta(s))$ is said to be fs-regular if for any fuzzy sequential point $P_f(s) = (p_{fx}^M,\, r)$ and a non zero closed fuzzy sequential set $A_f(s)$ such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\, \delta(s))$ such that*

$$P_f(s)\; q_w^{M-N}\; U_f(s),\;\; A_f(s)\; q_w\; V_f(s),\;\; P_f(s)\; \overline{q}_w^{M-N}\; \overline{V_f(s)}$$
$$\text{and } A_f(s) \leq X_f^1(s) - \overline{U_f(s)},$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s)\ q_w\ U_f(s),\ A_f(s)\ q_w\ V_f(s),\ P_f(s)\ \overline{q}_w\ \overline{V_f(s)},$$
$$\text{and } A_f(s) \le X_f^1(s) - \overline{U_f(s)}.$$

Definition 3.1.12 *An FSTS $(X,\, \delta(s))$ is said to be weak fs-regular if for any fuzzy sequential point $P_f(s) = (p_{fx}^M,\, r)$ and its any open weak nbd $A_f(s)$, there exists a fuzzy sequential set $B_f(s)$ in $(X,\, \delta(s))$ such that*

$$P_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \le \overline{B_f(s)} \le A_f(s),$$

whenever $A_f^c(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exists a fuzzy sequential set $B_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in_w \overset{o}{B}_f(s) \le \overline{B_f(s)} \le A_f(s).$$

Example 3.1.9 shows that an fs-regular space may not be weak fs-regular.

Example 3.1.9 *Let $X = \{a\}$, $\delta = \{\overline{0},\, \overline{1},\, p_a^{0.4}\}$. Then the FSTS $(X,\, \delta^{\mathbb{N}})$ is fs-regular but not weak fs-regular.*

A weak fs-regular space may not be fs-regular and it has been shown in the following example.

Example 3.1.10 *Let $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,\ 1]\}$, then the FSTS $(X,\ \delta^{\mathbb{N}})$ is weak fs-regular but not fs-regular.*

Example 3.1.11 shows that an fs-regular space may not be fs-T_1.

Example 3.1.11 *Let $X = \{x\}$ and let $\delta = \{\overline{0},\ \overline{1},\ p_x^{0.5}\}$. Then the FSTS $(X,\ \delta^{\mathbb{N}})$ is fs-regular but not fs-T_1.*

Definition 3.1.13 *An FSTS $(X,\ \delta(s))$ is said to be fs-T_3 if it is fs-regular and fs-T_1.*

Remark 3.1.3 *An fs-T_3 space is fs-T_2.*

Theorem 3.1.12 *An FSTS $(X,\ \delta(s))$ is fs-regular if and only if for any fuzzy sequential point $P_f(s) = (p_{fx}^M,\ r)$ and a non zero closed fuzzy sequential set $A_f(s)$ such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$, $D_f(s)$ and $E_f(s)$ in $(X,\ \delta(s))$ such that*

$$P_f(s) \in^{M-N} G_f(s),\ A_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s)\ q_w^{M-N} D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s),$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$, $D_f(s)$ and $E_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) \in G_f(s),\ A_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s)\ q_w\ D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s).$$

Proof. Proof is omitted. ■

Theorem 3.1.13 *An FSTS $(X,\ \delta(s))$ is fs-regular if and only if for any fuzzy sequential point $P_f(s) = (p^M_{fx},\ r)$ and an open fuzzy sequential set $C_f(s)$ such that $P_f(s)q_wC_f(s)$ (where $X^1_f(s) - C_f(s)$ is not completely contained in $P_f(s)$), there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\ \delta(s))$ such that*

$$P_f(s) \in^{M-N} O_f(s),\ \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s)q^{M-N}_w B_f(s),\ \overline{B_f(s)} \leq C_f(s),$$

whenever $X^1_f(s) - C_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\ \delta(s))$ such that

$$P_f(s) \in O_f(s),\ \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s)\ q_w\ B_f(s),\ \overline{B_f(s)} \leq C_f(s).$$

Proof. Suppose $(X,\ \delta(s))$ be fs-regular. Let $P_f(s) = (p^M_{fx},\ r)$ be any fuzzy sequential point and $C_f(s)$ be an open fuzzy sequential set such that $P_f(s)q_wC(s)$ i.e $P_f(s) \notin X^1_f(s) - C_f(s) = A_f(s)$ (say). Then, there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$, $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$P_f(s) \in^{M-N} G_f(s),\ A_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$P_f(s)q^{M-N}_w D_f(s),\ A_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s),$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$,

$D_f(s)$ and $E_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in G_f(s),\; A_f(s)\; q_w\; H_f(s),\; G_f(s)\; \overline{q_w}\; H_f(s),$$
$$P_f(s)\; q_w\; D_f(s),\; A_f(s) \in E_f(s),\; E_f(s)\; \overline{q_w}\; D_f(s).$$

If we take $O_f(s) = G_f(s)$ and $B_f(s) = D_f(s)$, we are done.

Conversely, suppose the given conditions hold. Let $P_f(s) = (p^M_{fx},\, r)$ be any fuzzy sequential point and $A_f(s)$ be any closed fuzzy sequential set such that $P_f(s) \notin A_f(s)$ and $A_f(s)$ is not completely contained in $P_f(s)$, that is, $P_f(s) q_w (X^1_f(s) - A_f(s)) = C_f(s)$ (say). Then, there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in^{M-N} O_f(s),\; \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s) q_w^{M-N} B_f(s),\; \overline{B_f(s)} \leq C_f(s),$$

whenever $X^1_f(s) - C_f(s)$ is totally reduced from $P_f(s)$ and has base N; otherwise there exist open fuzzy sequential sets $O_f(s)$ and $B_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) \in O_f(s),\; \overline{O_f(s)} \leq_w X^1_f(s) - C_f(s),$$
$$P_f(s)\; q_w\; B_f(s),\; \overline{B_f(s)} \leq C_f(s).$$

If we take $G_f(s) = O_f(s)$, $H_f(s) = X^1_f(s) - \overline{O_f(s)}$, $D_f(s) = B_f(s)$, $E_f(s) = X^1_f(s) - \overline{B_f(s)}$ and use Theorem 3.1.12, then we are done. ■

Theorem 3.1.14 *If $(X,\, \delta(s))$ is fs-regular, then for any closed fuzzy sequential set $A_f(s)$ which is not a fuzzy sequential point,*

$$A_f(s) = \wedge\{N_f(s);\; N_f(s) \textit{ is a closed nbd of } A_f(s)\} \quad (3.1.2)$$

Proof. Suppose $(X,\, \delta(s))$ be fs-regular and $A_f(s)$ be any closed fuzzy sequential set which is not a fuzzy sequential point. If $A_f(s) = X^0_f(s)$, then 3.1.2 is true. Suppose $A_f(s) \neq X^0_f(s)$. Let $P_f(s)$ be any fuzzy sequential point such that $P_f(s) \notin A_f(s)$. Let M and N be the bases of $P_f(s)$ and $A_f(s)$ respectively. Then, $P_f(s) q_w (X^1_f(s) - A_f(s)) = G_f(s)$ (say) and hence there exists an open fuzzy sequential set $B_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s) q_w^{M-N} B_f(s),\ \overline{B_f(s)} \leq G_f(s)$$

whenever $A_f(s)$ is totally reduced from $P_f(s)$; otherwise there exists an open fuzzy sequential set $B_f(s)$ in $(X,\, \delta(s))$ such that

$$P_f(s)\ q_w\ B_f(s),\ \overline{B_f(s)} \leq G_f(s).$$

This implies, $A_f(s) \leq X^1_f(s) - \overline{B_f(s)} = H_f(s)$ (say). Again, $P_f(s) \notin X^1_f(s) - B_f(s) \Longrightarrow P_f(s) \notin \overline{H_f(s)}$. Thus 3.1.2 holds. ■

Example 3.1.12 shows that converse of Theorem 3.1.14 may not be true.

Example 3.1.12 *Consider a set X and $\delta = \{\overline{r},\ r \in [0,\ 1]\}$. Then the FSTS $(X,\, \delta^{\mathbb{N}})$ is not fs-regular although for any closed fuzzy sequential set $A_f(s)$ in $(X,\, \delta^{\mathbb{N}})$, $A_f(s) = \wedge\{N_f(s);\ N_f(s)$ is a closed nbd of $A_f(s)\}$.*

Example 3.1.13 shows that for a fuzzy sequential point, Theorem 3.1.14 may not hold.

Example 3.1.13 *Let* $X = \{x\}$ *and* $\delta = \{\overline{1},\ \overline{0},\ p_x^{0.2}\}$. *Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-regular but for the closed fuzzy sequential point* $A_f(s) = (p_{fx}^{\{1,\ 2\}},\ 0.8)$, $A_f(s) \neq \wedge\{N_f(s);\ N_f(s)$ *is a closed nbd of* $A_f(s)\}$.

Theorem 3.1.15 *A fuzzy topological space* $(X,\ \delta)$ *is fuzzy regular if and only if* $(X,\ \delta^{\mathbb{N}})$ *is weak fs-regular.*

Proof. Proof is omitted. ■

Even if the FSTS $(X,\ \delta^{\mathbb{N}})$ is fs-regular, $(X,\ \delta)$ may not be fuzzy regular, as shown by Example 3.1.14.

Example 3.1.14 *Let* $X = \{a\}$, $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$. *Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-regular but* $(X,\ \delta)$ *is not fuzzy regular.*

If $(X,\ \delta)$ be a fuzzy regular space, then it may not imply $(X,\ \delta^{\mathbb{N}})$ is fs-regular, as shown by Example 3.1.15.

Example 3.1.15 *Let* $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,\ 1]\}$, *then the fuzzy topological space* $(X,\ \delta)$ *is fuzzy regular but* $(X,\ \delta^{\mathbb{N}})$ *is not fs-regular.*

The component fuzzy topological spaces $(X,\ \delta_n)$, $n \in \mathbb{N}$, of an fs-regular space $(X,\ \delta(s))$ may not be fuzzy regular. This is shown by Example 3.1.16.

Example 3.1.16 *Let* $X = \{a\}$, $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$. *Let* $\delta(s) = \delta^{\mathbb{N}}$, *then* $(X,\ \delta(s))$ *is fs-regular but* $(X,\ \delta_n) = (X,\ \delta)\ \forall n \in \mathbb{N}$, *is not fuzzy regular.*

An FSTS $(X,\ \delta(s))$ may not be fs-regular even if its each component fuzzy topological space $(X,\ \delta_n)$, $n \in \mathbb{N}$, is fuzzy regular, as shown by Example 3.1.17.

Example 3.1.17 *Let* $X = \{x, y\}$, $\delta = \{\overline{r},\ r \in [0,\ 1]\}$. *Let* $\delta(s) = \delta^{\mathbb{N}}$. *Then* $(X,\ \delta(s))$ *is not be fs-regular but each of its component fuzzy topological spaces* $(X,\ \delta_n)$, $n \in \mathbb{N}$, *is fuzzy regular.*

Definition 3.1.14 *Fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ *are said to be strong quasi-discoincident if* A_f^n *and* B_f^n *are strong quasi-discoincident* $\forall n \in \mathbb{N}$.

Definition 3.1.15 *Fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ *are said to be partially quasi-discoincident if* A_f^n *and* B_f^n *are strong quasi-discoincident for some* $n \in \mathbb{N}$.

Definition 3.1.16 *An FSTS* $(X,\ \delta(s))$ *is said to be fs-normal if for any two partially quasi-discoincident non zero closed fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ *(having the respective bases* M *and* N *and none of which is completely contained in the other), there exist open fuzzy sequential sets* $U_f(s)$ *and* $V_f(s)$ *in* $(X,\ \delta(s))$

such that

$$A_f(s) q_w^{M-N} U_f(s),\ B_f(s) q_w V_f(s),\ A_f(s) \leq^{M-N} X_f^1(s) - \overline{V_f(s)},$$
$$\text{and } B_f(s) \leq X_f^1(s) - \overline{U_f(s)},$$

whenever $B_f(s)$ *is totally reduced from* $A_f(s)$*; otherwise there exist open fuzzy sequential sets* $U_f(s)$ *and* $V_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$A_f(s)\ q_w\ U_f(s),\ B_f(s)\ q_w\ V_f(s),\ A_f(s) \leq X_f^1(s) - \overline{V_f(s)},$$
$$\text{and } B_f(s) \leq X_f^1(s) - \overline{U_f(s)}.$$

Definition 3.1.17 *An FSTS* $(X,\ \delta(s))$ *is said to be weak fs-normal if for any non zero closed fuzzy sequential set* $C_f(s)$ *and its any open weak nbd* $A_f(s)$*, there exists a fuzzy sequential set* $B_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$C_f(s) \in_w^{M-N} \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s),$$

whenever $A_f^c(s)$ *is totally reduced from* $C_f(s)$*; otherwise there exists a fuzzy sequential set* $B_f(s)$ *in* $(X,\ \delta(s))$ *such that*

$$C_f(s) \in_w \overset{o}{B}_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

(M *and* N *being the respective bases of* $C_f(s)$ *and* $A_f^c(s)$).

An fs-normal space may not be weak fs-normal, which is shown by Example 3.1.18.

Example 3.1.18 *Let* $X = \{a\}$ *and* $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$*. Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is fs-normal but not weak fs-normal.*

Example 3.1.19 shows that a weak fs-normal space may not be fs-normal.

Example 3.1.19 *Let* $X = \{x,\, y\}$ *and* $\beta = \{p_x^r;\, r \in [0,\, 1]\} \cup \{p_y^r;\, r \in [0,\, 1]\}$ *be a basis for some fuzzy topology* δ *on* X. *Then the FSTS* $(X,\, \delta^{\mathbb{N}})$ *is weak fs-normal but not fs-normal.*

Definition 3.1.18 *An FSTS* $(X,\, \delta(s))$ *is said to be an fs-*T_4 *space if it is fs-normal and fs-*T_1.

An fs-normal space may not be fs-T_1, as shown by Example 3.1.20.

Example 3.1.20 *Let* $X = \{a,\, b\}$ *and* $\delta(s) = \{X_f^0(s),\, X_f^1(s),\, A_f(s),\, B_f(s)\}$, *where* $\forall n \in \mathbb{N}$, $A_f^n(a) = 1$, $A_f^n(b) = 0$, $B_f^n(a) = 0$, $B_f^n(b) = 1$. *Then the FSTS* $(X,\, \delta(s))$ *is fs-normal but not fs-*T_1.

An fs-normal space may not be fs-regular as shown by Example 3.1.21.

Example 3.1.21 *Let* $X = \{x, y\}$ *and* $\delta = \{\overline{r},\, r \in [0,\, 1]\}$. *Then the FSTS* $(X,\, \delta^{\mathbb{N}})$ *is fs-normal but not fs-regular.*

Theorem 3.1.16 *An FSTS* $(X,\, \delta(s))$ *is fs-normal if and only if for any two partially quasi-discoincident non zero closed fuzzy sequential sets* $A_f(s)$ *and* $B_f(s)$ *(none of which is completely contained in the other) there exist open fuzzy sequential sets* $G_f(s)$,

$H_f(s)$, $D_f(s)$ and $E_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) \in^{M-N} G_f(s),\ B_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$A_f(s) q_w^{M-N} D_f(s),\ B_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s),$$

whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$); otherwise there exist open fuzzy sequential sets $G_f(s)$, $H_f(s)$, $D_f(s)$ and $E_f(s)$in $(X, \delta(s))$ such that

$$A_f(s) \in G_f(s),\ B_f(s)\ q_w\ H_f(s),\ G_f(s)\ \overline{q_w}\ H_f(s),$$
$$A_f(s)\ q_w\ D_f(s),\ B_f(s) \in E_f(s),\ E_f(s)\ \overline{q_w}\ D_f(s).$$

Proof. Proof is omitted. ■

Theorem 3.1.17 *If an FSTS $(X, \delta(s))$ is weak fs-normal, then for any two non-zero closed partially quasi-discoincident fs-sets $A_f(s)$ and $B_f(s)$ (none of which is completely contained in the other), there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that*

$$A_f(s) \in_w^{M-N} U_f(s),\ B_f(s) \in_w V_f(s),\ U_f(s)\ \overline{q_w}\ V_f(s),$$

whenever $B_f(s)$ is a totally reduced fuzzy sequential set from $A_f(s)$ (M and N are respectively the bases of $A_f(s)$ and $B_f(s)$); otherwise there exist open fuzzy sequential sets $U_f(s)$ and $V_f(s)$ in $(X, \delta(s))$ such that

$$A_f(s) \in_w G_f(s),\ B_f(s) \in_w V_f(s),\ U_f(s)\ \overline{q_w}\ V_f(s).$$

Proof. The proof is omitted. ■

Remark 3.1.4 *For an FSTS to be weak fs-normal, the condition given in Theorem* 3.1.17*, is only necessary but not sufficient, as shown by Example* 3.1.22*.*

Example 3.1.22 *Let* $X = \{a\}$ *and* $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$*. Then the FSTS* $(X,\ \delta^{\mathbb{N}})$ *is not weak fs-normal but the condition in Theorem* 3.1.17 *is satisfied.*

Theorem 3.1.18 *A weak fs-regular space is weak fs-normal, when* X *is finite.*

Proof. Let $(X,\ \delta(s))$ be a weak fs-regular space, where X is finite. Let $C_f(s)$ be any non zero closed fuzzy sequential set in $(X,\ \delta(s))$ and $A_f(s)$ be its any open weak nbd. Let M and N be respectively the bases of $C_f(s)$ and $A_f^c(s)$. We choose $m \in M - N$ when $A_f^c(s)$ is totally reduced from $C_f(s)$ and we take $m \in M$ otherwise. Let $x \in X$ such that $C_f^m(x) \neq 0$ and let $C_f^m(x) = r_m$. Then, for the fuzzy sequential point $P_{xf}(s) = (p_{fx}^m,\ r_m)$, $A_f(s)$ is an open weak nbd. Hence, there exists an open fuzzy sequential set $B_{xf}(s)$ in $(X,\ \delta(s))$ such that

$$P_{xf}(s) \in_w^{M-N} B_{xf}(s) \leq \overline{B_{xf}(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is a totally reduced fuzzy sequential set from $C_f(s)$; otherwise there exists an open fuzzy sequential set $B_{xf}(s)$ in $(X,\ \delta(s))$ such that

$$P_{xf}(s) \in_w B_{xf}(s) \leq \overline{B_{xf}(s)} \leq A_f(s)$$

Corresponding to each $x \in X$ for which $C_f^m(x) \neq 0$, we get such fs-open set $B_{xf}(s)$. Taking $X = \{x_1, x_2, ..., x_k\}$, we get fs-open sets $B_{x_1 f}(s)$, $B_{x_2 f}(s)$,..........., $B_{x_k f}(s)$ such that
$P_{x_n f}(s) \in_w^{M-N} B_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s),\ \forall n = 1, 2, .., k,$
whenever $A_f^c(s)$ is totally reduced from $C_f(s)$; otherwise

$$P_{x_n f}(s) \in_w B_{x_n f}(s) \leq \overline{B_{x_n f}(s)} \leq A_f(s).$$

Now, let $B_f(s) = \bigvee_{n=1}^{k} B_{x_n f}(s)$. Then

$$C_f(s) \in_w^{M-N} B_f(s) \leq \overline{B_f(s)} \leq A_f(s)$$

whenever $A_f^c(s)$ is totally reduced from $C_f(s)$; otherwise

$$C_f(s) \in_w B_f(s) \leq \overline{B_f(s)} \leq A_f(s).$$

Hence $(X,\, \delta(s))$ is weak fs-normal.

■

Theorem 3.1.19 *A fuzzy topological space $(X,\, \delta)$ is fuzzy normal if and only if $(X,\, \delta^{\mathbb{N}})$ is weak fs-normal.*

Proof. Proof is omitted. ■

Fuzzy topological space $(X,\, \delta)$ may not be fuzzy normal even if $(X,\, \delta^{\mathbb{N}})$ is fs-normal, as shown by Example 3.1.23.

Example 3.1.23 *Let $X = \{a\}$ and $\delta = \{\overline{0},\, \overline{1},\, p_a^{0.6}\}$. Then the FSTS $(X,\, \delta^{\mathbb{N}})$ is fs-normal but $(X,\, \delta)$ is not fuzzy normal.*

Example 3.1.24 shows that if $(X,\ \delta)$ is fuzzy normal, it may not imply $(X,\ \delta^{\mathbb{N}})$ is fs-normal.

Example 3.1.24 *Let $X = \{x,\ y\}$ and $\beta = \{p_x^r;\ r \in [0,\ 1]\} \cup \{p_y^r;\ r \in [0,\ 1]\}$ be a basis for some fuzzy topology δ on X. Then the fuzzy topological space $(X,\ \delta)$ is fuzzy normal but $(X,\ \delta^{\mathbb{N}})$ is not fs-normal.*

If an FSTS $(X,\ \delta(s))$ is fs-normal, then it may not imply $(X,\ \delta_n)$ is fuzzy normal for each $n \in \mathbb{N}$ and it is shown by Example 3.1.25.

Example 3.1.25 *Let $X = \{a\}$ and $\delta = \{\overline{0},\ \overline{1},\ p_a^{0.6}\}$. Let $\delta(s) = \delta^{\mathbb{N}}$, then the FSTS $(X,\ \delta(s))$ is fs-normal but $(X,\ \delta_n) = (X,\ \delta)$ $(\forall n \in \mathbb{N})$, is not fuzzy normal.*

An FSTS $(X,\ \delta(s))$ may not be fs-normal even if its each component fuzzy topological space $(X,\ \delta_n)$ is fuzzy normal, as shown by Example 3.1.26 .

Example 3.1.26 *Let $X = \{x,\ y\}$, $\beta = \{p_x^r;\ r \in [0,\ 1]\} \cup \{p_y^r;\ r \in [0,\ 1]\}$ be a basis for some fuzzy topology δ on X and let $\delta(s) = \delta^{\mathbb{N}}$. Then the FSTS $(X,\ \delta(s))$ is not fs-normal but each of the component fuzzy topological spaces $(X,\ \delta_n)$, $n \in \mathbb{N}$, is fuzzy normal.*
