Chapter 3

Further Characterizations for Interval Tournaments[1]

A *tournament* is a complete oriented graph, that is, a digraph with no loops such that $u \to v$ iff $v \not\to u$. A tournament with n vertices is an *n-tournament.* A tournament that is an interval digraph is an *interval tournament* and an interval tournament with n-vertices is an *interval n-tournament.* Interval tournaments were introduced and characterized by Brown et. al [7]. They have characterized them by a complete list of forbidden subtournaments given in Figure 3.1. They have also proved that an n-tournament is an interval tournament if and only if it has a transitive $(n-1)$-subtournament.

A.H.Busch [10] has shown that the vertices of every loopless interval digraph can be partitioned into two acyclic digraphs from which he has shown that every interval tournament can be partitioned into two transitive tournaments. He then uses this result to provide a short proof of the result that every interval n-tournament has a transitive $(n-1)$-tournament.

As an extension of the study of interval tournaments, Drust, Das Gupta and

[1]A part of this chapter has appeared in *J. Indian Math. Soc.* (2011)**78**(1–4)15–26.

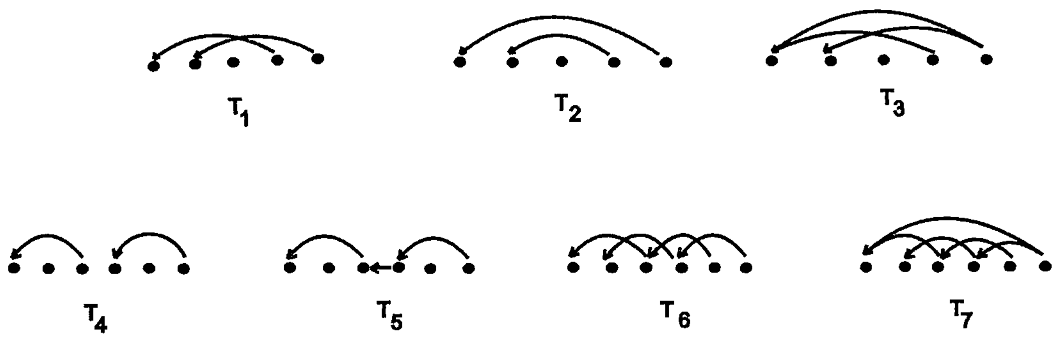

Figure 3.1: Complete list of forbidden tournaments for interval tournament. (The arcs which are not shown are from left to right)

Lundgren [24] started with an oriented digraph D which is not a tournament and explored the restrictions to be placed on D to gurantee that it is an interval digraph. In particular, they investigated a broader class of oriented graphs on n-vertices that contain a transitive $(n-2)$-tournament as a subdigraph.

It is well known [63] that a cycle of length 6 or more is not an interval bigraph. This example will be used very often in the present chapter.

The present chapter is actually an extended study of the paper by Brown et al. [7]. We first apply the zero partition of an interval digraph to obtain an alternative proof of the main theorem 3 of [7]. The importance of this alternative approach is that during the process we have obtained several additional characterizations of an interval tournament. We prove that a tournament is an interval tournament if and only if all its cycles have a common vertex. Then we show that the three digraphs D_1, D_2, D_3 in Figure 3.2 are the only three forbidden subdigraphs of an interval tournament. Next we show that a tournament T is an interval tournament if and only if it is of Ferrers dimension at most 2 and does not contain the tournament T_4 in Figure 3.1 as a subtournament.

Hell and Huang [47] proved that complements of interval bigraphs are precisely those two-clique circular-arc graphs which have representations such that no two arcs cover the whole circle. Brown et al [7] in their paper posed the problem of characterizing an interval tournament following this model. This problem is addressed in the last condition (9) of the main theorem 3.2.1 of this paper.

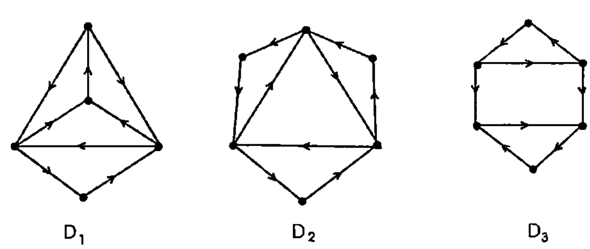

Figure 3.2: Complete list of forbidden subdigraphs for interval tournament

3.1 Preliminaries

We start with an important observation that there is only one 4-tournament which has two 3-cycles. This is given in Figure 3.3.

We first prove the following proposition which will be required to prove our main theorem.

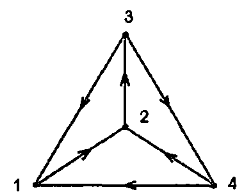

Figure 3.3: The only 4-tournament having two 3-cycles; which are (2, 3, 1) and (2, 3, 4)

Proposition 3.1.1. *If an interval* 5*-tournament has two or more* 3*-cycles, then they must have at least one vertex in common.*

Proof. The proposition is obviously true for any interval 5-tournament having exactly two 3-cycles. So we consider an interval 5-tournament T having more than two 3-cycles and suppose, if possible, the 3-cycles do not have a vertex in common. Clearly T has a 4-subtournament having two cycles (Figure 3.3) and T is obtained from this subtournament by adjoining a fifth vertex 5 to it.

In the tournament T, two possibilities arise:

I. The vertices 1, 5 and 4 form a cycle (1,5,4).

II. 1, 5 and 4 do not form a cycle.

Case I. There are four subcases regarding the edges $(2, 5)$ and $(3, 5)$.

(a) $5 \to 2, \quad 5 \to 3$

(b) $2 \to 5, \quad 3 \to 5$

(c) $5 \to 2, \quad 3 \to 5$

(d) $2 \to 5, \quad 5 \to 3$

It is now verified that in the subcases (a), (b) and (c) the bipartite graph $B(T)$ corresponding to T contains a 6-cycle. While in subcase (d),

$B(T)$ is itself a 10-cycle. From [63], it follows that none of these tournaments are interval tournaments. The bigraphs are given in Figure 3.4.

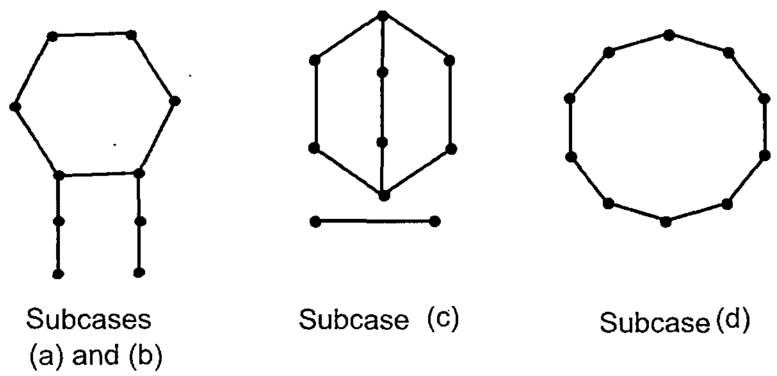

Figure 3.4: The bigraphs of 5-tournament having no common vertex of its 3-cycles

Case II. When 1, 5 and 4 do not form a cycle. In order that all 3-cycles do not have a vertex in common, either of the two possibilities must occur :

(a) there are two 3-cycles $(1, 2, 5)$ and $(3, 4, 5)$

(b) there are two 3-cycles $(3, 1, 5)$ and $(4, 2, 5)$

In each of the cases, we verify that T is isomorphic to the tournament in Case I(a). □

We now focus our attention to the digraph D_1 in Figure 3.2.

The 3-cycles of digraph D_1 have no common vertex. So we have:

Corollary 3.1.1. *No interval tournament contains a digraph isomorphic to the digraph D_1 in Figure 3.2.*

During the proof, we observe that in every case when all the 3-cycles in a 5-tournament T do not have a common vertex, it contains a digraph isomorphic to the digraph D_1 in Figure 3.2. It is also interesting to note that the three bipartite graphs in Figure 3.4 correspond exactly to the three tournaments T_1, T_2, T_3 in Figure 3.1(See [7]).

From these, we get the following :

Corollary 3.1.2. *Let T be a 5-tournament. The following conditions are equivalent:*

1. *all the 3-cycles of T have no common vertex.*
2. *T is isomorphic to either of T_1, T_2, T_3.*
3. *T contains a subdigraph isomorphic to the digraph D_1 in Figure 3.2.*

3.2 Characterization

First we state the main theorem of this paper.

Theorem 3.2.1. *Let T be an n-tournament. Then the following conditions are equivalent:*

1) T is an interval tournament;

2) all the 3-cycles of T have a common vertex;

3) all cycles of T have a common vertex;

4) T has a transitive $(n-1)$-subtournament;

5) T has no vertex disjoint 3-cycles or the digraph D_1 in Figure 3.2;

6) T has no subdigraph isomorphic to any of D_1, D_2 or D_3 in Figure 3.2;

7) T has no subtournament T_i $(1 \leq i \leq 7)$ in Figure 3.1;

8) T is of Ferrers dimension at most 2 and T_4-free;

9) the graph complement $\overline{B(T)}$ of $B(T)$ is a two-clique circular-arc graph and $\overline{B(T_4)}$-free.

Before going into the proof of the theorem, the condition (6) of the theorem needs a little explanation. First note that all the three digraphs are interval digraphs. The equivalence of (6) and (7) means that in what ever way we add the missing arcs to these digraphs to form a tournament, they will ultimately lead to one of the forbidden tournaments T_i $(1 \leq i \leq 7)$. So, we can say that the 3 digraphs D_1, D_2, D_3 form a complete list of forbidden subdigraphs.

In order to prove the theorem we need the following lemma.

Lemma 3.2.1. *No interval tournament contains two vertex-disjoint* 3*-cycles.*

Proof We first observe that any 3-cycle (x_1, x_2, x_3) is an interval tournament having a zero-partition

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_3 & 1 & R & R \\ x_1 & C & 1 & R \\ x_2 & C & C & 1 \end{array} \qquad \ldots (1)$$

We note that this partition is not unique and two different partitions have different labellings of R's and C's; but with a given labelling of R's and C's, the zero-partition for a 3-cycle is unique.

Let, if possible $C_1 = (x_1, x_2, x_3)$ and $C_2 = (y_1, y_2, y_3)$ be two vertex-disjoint 3-cycles of an interval 6-tournament T. Then the adjacency matrix $A(T)$ of T displaying a zero-partition have two complementary submatrices of the form (1) corresponding to the two cycles. We will, in fact prove the proposition by showing that no permutation of rows and columns in $A(T)$ can display a zero-partition.

Below we list some submatrices of $A(T)$ which forbid its zero-partition.

(a) Consider a submatrix

$$\begin{array}{c|cccc} & x_i & - & - & y_j \\ \hline x_i & C & 1 & R & * \\ y_j & * & - & - & - \end{array} \qquad \ldots (F_1)$$

Since T is a tournament, either $x_i y_j$ or $y_j x_i$ must be 1. But this prevents the matrix from a zero-partition when the rows are in the given order.

(b) $$\begin{array}{c|ccc} & - & y_j & x_i \\ \hline x_i & R & * & R \\ y_j & - & R & * \end{array} \quad \ldots (F_{2a}) \quad \text{and} \quad \begin{array}{c|ccc} & - & y_j & x_i \\ \hline y_j & - & R & * \\ x_i & R & * & R \end{array} \quad \ldots (F_{2b})$$

(c) $$\begin{array}{c|cc} & - & - \\ \hline - & - & C \\ - & R & * \end{array} \qquad \ldots (F_3)$$

For the submatrix F_3 we note that $*$ lies below C and so it must be C and since it lies to the right of R, it must be R. This goes against a zero-partition.

We note that there are other forbidden submatrices of a zero-partitioned matrix, but we will need only these three submatrices to complete the proof.

If $A(T)$ has a zero-partition, we observe that any submatrix of $A(T)$ has also a zero-partition with the same labelling of R's and C's as in $A(T)$. So we have to consider only those permutations where the orders of rows and columns of the respective submatrices of C_1 and C_2 remain unchanged. Without loss of generality, we suppose C_1 and C_2 have the zero-partition given by

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_3 & 1 & R & R \\ x_1 & C & 1 & R \\ x_2 & C & C & 1 \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline y_3 & 1 & R & R \\ y_1 & C & 1 & R \\ y_2 & C & C & 1 \end{array}$$

We give two examples below, regarding the problem of sorting out the permutations which are to be considered for our proof.

The permutation of the columns $(x_1, x_2, y_1, x_3, y_2, y_3)$ is to be considered but not the one like $(x_1, x_2, y_2, x_3, y_1, y_3)$, because in that case, the labels of R's and C's in the submatrix of C_2 get changed.

We classify all the short listed permutations according to the vertices of the last two columns of the matrix. In any permutation of the columns, we will examine, the vertices of the last two columns are either of the following:

(I) y_2, y_3 ; (II) y_3, x_3;

(III) x_2, x_3 ; (IV) x_3, y_3.

Amongst them, cases (III) and (IV) are the result of interchange of labels of the vertices of C_1 and C_2 and so we need consider the two cases (I) and

(II) only.

Case (I) Consider the following biadjacency matrix

$$\begin{array}{c|cccc|cc}
 & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\
\hline
x_3 & 1 & R & R & & & \\
x_1 & C & 1 & R & - & * & \\
x_2 & C & C & 1 & & & \\
y_3 & - & & & 1 & R & R \\
y_1 & - & & & C & 1 & R \\
y_2 & * & & & C & C & 1 \\
 & & & & & &
\end{array}$$

What ever be the permutation of the columns of the left sector, the column x_3 lies to the left of the column y_2 and until and unless the row y_2 lies above the row x_1, we get a submatrix F_1 in the matrix. So consider the submatrix when the row y_2 goes above the row x_1 in a permutation of its rows and then we have the submatrix

$$\begin{array}{c|cccc|cc}
 & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\
\hline
y_2 & - & - & - & C & C & 1 \\
x_1 & C & 1 & R & - & * & \\
 & & & & & &
\end{array}$$

which is F_3, irrespective of any permissible permutation of the columns in the left sector.

Case(II) First we consider the matrix

$$\begin{array}{c|cccc|cc}
 & x_1 & x_2 & y_1 & y_2 & y_3 & x_3 \\
\hline
x_3 & 1 & R & - & - & * & R \\
x_1 & C & 1 & - & - & - & R \\
x_2 & C & C & - & - & - & 1 \\
y_3 & & & 1 & R & R & * \\
y_1 & & & C & 1 & R & - \\
y_2 & & & C & C & 1 & - \\
 & & & & & &
\end{array}$$

Here we have the forbidden submatrix F_{2a}. This is preserved for any permissible permutation of its columns in the left sector so long as the row y_3 lies below the row x_3. In the case when the row y_3 lies above the row x_3 then we get the submatrix F_{2b}. □

To prove the theorem we need the following lemma.

Lemma 3.2.2. *No interval* 6*-tournament* T *contains the digraph* D_0 *or* D_2 *in Figure 3.5 as a subdigraph.*

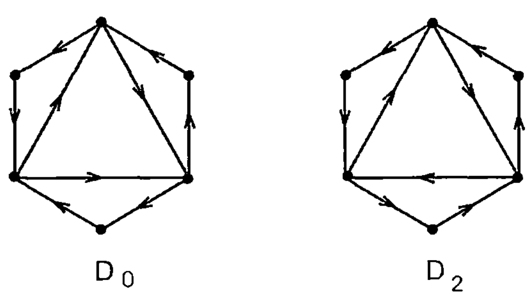

Figure 3.5: The digraphs D_0 and D_2

Proof. We first observe that the two digraphs D_0 and D_2 have the same underlying undirected graph and have orientations such that the three outer triangles are directed cycles in any order, clockwise or anticlockwise.

We first suppose that an interval 6-tournament contains the digraph D_0 in Figure 3.5 as its subdigraph.

In the tournament either $q \to b$ or $b \to q$. But none is possible, because if $q \to b$, then $T - \{x\}$ is the digraph D_1 in Figure 3.2 and if $b \to q$ then

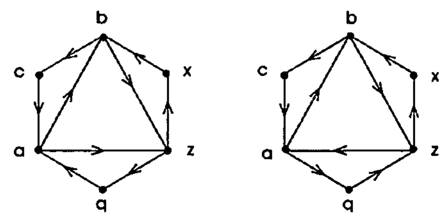

Figure 3.6: The digraphs D_0 and D_2 with labelled vertices

so is $T - \{c\}$. Next suppose that an interval 6-tournament contains D_2 in Figure 3.5 as its subgraph.

We must have $c \to q$, because otherwise (a, q, c) and (x, b, z) are two vertex disjoint three cycles. Similarly $q \to x$ and $x \to c$. But then the cycles (c, q, x), (a, b, z) are two disjoint 3-cycles. □

Proof of Theorem 3.1. (1) $\Rightarrow$ (2). If the tournament is transitive then there is nothing to prove. So we suppose it is not transitive. Let the result be true for any interval n-tournament ($n \geq 5$). Let $T(V, E)$ be an interval $(n+1)$-tournament and let $x \in V$. Let $T^{'} = T - \{x\}$. Then by hypothesis all the 3-cycles pass through a common vertex, say a. Let $T^{''} = T - \{a\}$. Clearly all 3-cycles of T'' pass through x. Here we note that all the 3-cycles of T contains either x or a or both. Let (a, b, c) and (x, y, z) be two 3-cycles of $T^{'}$ and $T^{''}$ respectively, $x \neq a, b, c$. Let $A_1 = \{(a, b, c), (x, y, z)\}$

These two 3-cycles can not be vertex disjoint in T by Lemma 3.2.1. Then either of y and z must be b or c. We prove the theorem for $y = b$. (All other cases follow similarly)

Then $A_1 = \{(a, b, c), (x, b, z)\}$

Now append to A_1 another (a, p, q) from $T^{'}$ and let

$A_2 = \{(a,b,c),(x,b,z),(a,p,q)\}$, $x \neq a,b,c,p,q$. Clearly $q \neq b$ and if $p = b$ then b is the common vertex in the 3-cycles of A_2. So we consider $p \neq b$. In this case we must have $p = z$ or $q = z$. If $p = z$, then $A_2 = \{(a,b,c),(x,b,z),(a,z,q)\}$ and the digraph comprising of the 3-cycles of A_2 is the digraph D_0 in Figure 3.5 and hence the tournament can not be an interval tournament. Similarly, if $q = z$ then the digraph comprising the 3-cycles of A_2 is the digraph D_2 in Figure 3.5 and hence T cannot be an interval tournament. So there must be a comon vertex in the 3-cycles of A_2. So $A_2 = \{(a,b,c),(a,b,q),(x,b,z)\}$. Now we append another 3-cycle (x,u,v) of T'' and let

$$A_3 = \{(a,b,c),(a,b,q),(x,b,z),(x,u,v)\}, x \neq a,b,c,q; v \neq b.$$

If $u = b$ then b is the common vertex. So we suppose $u \neq b$. In order to prevent the occurrence of two disjoint 3-cycles in T we must have $(u,v) = (c,q)$ or (q,c). In any case, we verify that the digraph $T - \{z\}$ contains the digraph D_1.

Now we are left with the cycles of the form (a,x,w) or (x,a,w) in T and if we append a cycle of this form to A_3 we verify that this cycle along with the cycles $(a,b,c),(x,b,z)$ gives us the digraph D_0 in Figure 3.5. □

(2) ⇒ (3). If possible let an interval tournament T have a k-cycle ($k > 3$) which does not pass through the common vertex, say v, of all the 3-cycles of the tournament. Consider the induced subtournament of this k-cycle. This subtournament will have a 3-cycle which does not pass through v, a contradiction. □

(3) ⇒ (4). Let $v \in V(T)$ be common to all the cycles of T. Then $T - \{v\}$ is

acyclic and so is transitive. □

(4) ⇒ (5). Brown et al. [7] has proved that every n-tournament T contains one of the following structures:

(1) Two disjoint 3-cycles;

(2) A subtournament isomorphic to T_1, T_2 or T_3 in Figure 3.1;

(3) A transitive $(n-1)$-subtournament.

From the above and Corollary 3.1.2 the result follows. □

(5) ⇒ (6). Since D_3 has two vertex disjoint 3-cycles, T can not contain D_3 as its subdigraph. Also Lemma 3.2.2 shows that T can not contain D_2 as its subdigraph. □

(6) ⇒ (7). From Corollary 3.1.2, it follows that the digraph D_1 is a subdigraph of each of T_1, T_2 and T_3. We prove the result by showing T_6 and T_7 contain D_2 as its subdigraph while T_4 and T_5 contain D_3 as a subdigraph.

With the labellings of the tournaments T_6 and T_7 as in Figure 3.1, we see that the union of the 3-cycles (v_3, v_1, v_5), (v_3, v_4, v_2) and (v_5, v_6, v_4) yields the digraph D_2 as a subdigraph of T_6 while the 3-cycles (v_1, v_2, v_3), (v_1, v_5, v_6) and (v_3, v_4, v_5) give us the same digraph D_2 as a subdigraph of T_7. Next we easily verify that the union of the two disjoint 3-cycles (v_1, v_2, v_3) and (v_4, v_5, v_6) along with the edges (v_1, v_5) and (v_2, v_6) constitute the digraph D_3 as a subdigraph of both T_4 and T_5. □

(7) ⇒ (1). It has been proved in [7]. We rely heavily on the following result from lemma 5 of [7] and provide an easier proof for the sake of completeness.

Lemma 3.2.3. [7] Let $X = \{x_0, x_1, x_2\}$ be the vertices of a 3-cycle in a 6-tournament that does not contain a copy of D_1 (and so T_1, T_2, T_3). Let $u \to v$

be an arc of T not incident to X. Then $|N^+(v) \cap X| \geq 2 \Rightarrow |N^+(u) \cap X| \geq 2$ where $N^+(v)$ is the successor set of v.

Let $Y = \{y_0, y_1, y_2\}$ be another 3-cycle of T disjoint from X. Then from the lemma and $y_0 \to y_1 \to y_2 \to y_0$ implies that $2 \leq |N^+(y_i) \cap X| \leq 3$ $(i = 0, 1, 2)$.[2] Let $p_i = |N^+(y_i) \cap X|$ $(i = 0, 1, 2)$ and since $2 \leq p_i \leq 3$, there are four distinct possibilities for the sequence (p_0, p_1, p_2):

(a) $(3, 3, 3)$ (b) $(2, 3, 3)$

(c) $(2, 2, 3)$ (d) $(2, 2, 2)$

Because $\{y_0, y_1, y_2\}$ is a cycle, any permutation of (p_0, p_1, p_2) will yield the same digraph.

It is now a matter of easy verification that the four cases give us exactly the four tournaments T_4, T_5, T_6, T_7 respectively. □

(7) $\Leftrightarrow$ (8). The equivalence of (1) and (7) means that T_i $(1 \leq i \leq 7)$ are the only forbidden tournaments of an interval tournament. All the bipartite graphs $B(T_i)$ associated with T_i, except $B(T_2)$ and $B(T_4)$, have an *asteroidal triple of edges* (see Figure 3.1, Figure 3.3 and Figure 3.4 of [7]). Also $B(T_2)$ is not *bichordal*[3]. It is proved in [19] that a bigraph of Ferrers dimension ≤ 2 is bichordal and ATE free (see also [63]). So it follows that any bigraph which is not bichordal nor containing an ATE is of Ferrers dimension > 2. So all the digraphs T_i $(i \neq 4)$ are of Ferrers dimension > 2. T_4 is the only digraph in this set which is of Ferrers dimension 2. Since an interval tournament is

[2]In case $|N^+(y_i) \cap X| < 2$ $(i = 0, 1, 2)$, interchanging X and Y will serve our purpose

[3]A bigraph is called chordal bipartite or simply bichordal, if every cycle of length ≥ 6 has a chord.

necessarily of Ferrers dimension at most 2, the result follows. □

(8) $\Leftrightarrow$ (9). In [47] it was proved that a bigraph is an interval bigraph if and only if its graph complement is a two-clique circular-arc graph such that no two arcs cover the whole circle. Extending this result it is proved in 2 that bigraphs of Ferrers dimension at most 2 are precisely the complements of two-clique circular-arc graphs. (8) $\Leftrightarrow$ (9) follows as an immediate consequence of this result. □