More results on representation of digraphs/bigraphs using intervals or circular-arcs

By

Asim Basu

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Asim Basu
Department of Mathematics
Ananda Chandra College
Jalpaiguri

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Introduction

1.1 Basic definitions

At the outset we review some basic terminology about graphs, directed graphs and relations used throughout this dissertation.

A *graph* $G(V, E)$ consists of a set of *vertices* V (or V(G)) and a collection of *edges* E (or E(G)) which in turn consists of distinct unordered pairs of distinct elements of V. Most authours call this a *simple graph*, we call it simply a *graph*. A graph will often be denoted by an ordered pair that indicates both the vertex set and edge set: $G(V, E)$. In terms of relations, a graph G is an irreflexive symmetric relation E on V. To denote an edge we juxtapose two vertices and say the vertices are *adjacent*; that is, for $u, v \in V$, $uv \in E$ denotes u and v are adjacent or uv is an edge. A graph G' is a *subgraph* of G if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. A subgraph G' is a *generated subgraph* or an *induced subgraph* of G if $V(G') \subset V(G)$ and two vertices are adjacent in G' if and only if they are adjacent in G. The *complement* $\overline{G}(V, E)$ of a graph G has the same vertex set as G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A *stable set* (*independent set*) is a set of vertices of G such that no two vertices in the set are adjacent. A *bipartite* graph is a graph $H(V, E)$ whose vertex set V can be partitioned into two stable sets X and Y and we write $H = H(X, Y, E)$. A *complete bipartite* graph is a bipartite graph $H(X, Y, E)$ where every pair of vertices that belong to different partite sets are adjacent.

A *directed graph* or *digraph* $D(V\ E)$ is a generalization of a graph in which E, the set of *edges or arcs* consists of ordered pairs of V, and for vertices u and v, if the ordered pair $(u, v) \in E$, then we denote this by $u \rightarrow v$. We use $u \nrightarrow v$ to mean $(u,v) \notin E$. So, a digraph $D(V, E)$ is simply a binary relation E on V having no restriction; that is, the relation could be reflexive for some elements of V and not others, and $(u, v) \in E$ does not imply $(v, u) \notin E$.

We generally represent a graph with a drawing in which vertices are depicted by small circles and if two vertices are adjacent, then a line connecting them is drawn. A digraph is typically represented in the same way, but with arrows indicating the order; that is, if $(u, v) \in E$, an arrow is drawn from u to v.

The *successor* set of a vertex v in a digraph $D(V, E)$ is the set of vertices u such that $vu \in E$. The *predecessor* set of a vertex v in $D(V, E)$ is the set of vertices u such that $uv \in E$.

The *adjacency* matrix of a (di)graph on a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ (0, 1)-matrix with 1 in the (i, j) position if and only if (v_i, v_j) is an edge of the (di)graph.

For any undefined term, one is referred to Roberts [69], Golumbic [35] or West [96].

1.2 Background

The subject of intersection graphs has been an important area of study for the last few decades. An *intersection graph* is a graph G whose vertices correspond to a family of sets $\mathfrak{F}$ and two vertices are adjacent if and only if their corresponding sets intersect; $\mathfrak{F}$ is called a representation of G when G is the intersection graph of $\mathfrak{F}$. It was shown by Marczewski [58] that all graphs are intersection graphs; for each vertex of G, if we associate the set of all edges incident to the vertex, then G is the intersection graph of this family of sets. Interesting classes of graphs are evolved by restricting $\mathfrak{F}$ or by modifying the rule that determine adjacency. McKee and McMorris [60] evinces the diversity and importance of the intersection graph perspective. Probably the most well known intersection graph comes up when $\mathfrak{F}$ is restricted to intervals of $\mathbb{R}$ (or arcs of a circle) and the corresponding intersection graph is known as an *interval graph* (or *circulr-arc graph*). For detailed discussion on this topic see Golumbic [35] and Fishburn [30]. A survey by Trotter [89] summarizes a variety of results and open problems.

As an extension of the study of intersection graphs, the mathematical community picked up and started running with the idea of representing digraphs from the view point of intersection representation. Intersection digraphs of a family of ordered pair of intervals on the real line and of arcs of a circle, are called *interval digraphs* and *circular arc digraphs* respectively and have been introduced and studied by Sen, West and others. Harary et al. introduced the concept of interval bigraphs quite early, which is equivalent to the concept of interval digraphs. Then Hell and Huang [47] characterized and studied this class from the bigraph point of view. Their findings provide the background material of this thesis.

1.3 Interval graphs and related topics

A graph $G = (V, E)$ is an interval graph if and only if there is a family $\mathbb{I}$ of real intervals $\{I_v : v \in V\}$ such that $uv \in E \iff I_u \cap I_v \neq \phi$.

Interval graphs have a long and rich history. Discrete Mathematics (1985) published a special issue on interval graphs and related topics. They have been introduced independently by Benzer [2], a molecular biologist and Hajos [41], a mathematician. Benzer got the idea of interval graphs from an application point of view. Hajos, from purely mathematical consideration, asked basically, what graphs have a representation by collection of intervals of $\mathbb{R}$.

Interval graphs have found applications to a wide variety of modeling real world problem. It has been used in seriation problem by Kendall [51] and Hubert [50], in archaeology by Skrien [82, 83] and in developmental psychology by Coombs and Smith [16] to name a few. Stoffers [88] and Roberts [70, 71] used interval graphs to find a solution to general traffic phasing problem. More applications in this field can be found in [34] and [95].

1.3.1 Some characterization of interval graphs

All graphs are not interval graphs. For example, it is easy to verify that C_4 is not an interval graph. First characterization of interval graphs was due to Lekkerkerker and Boland [52].

A graph G is said to be *chordal* or *triangulated* if it has no induced $C_n, n \geq 4$. An *asteroidal triple* (AT) of a graph G is a set of three vertices such that there is a path between any pair that avoids the neighbourhood of the third. Lekkerkerker and Boland [52] characterized interval graphs as those chordal graphs that have no asteroidal triple.

Theorem 1.3.1 (Lekkerkerker and Boland [52]). *A graph G is an interval graph if and only if G is chordal and does not contain any asteroidal triple.*

In that paper, they also provided a complete set of forbidden subgraphs for interval graphs as given in the following Figure 1.1.

A *transitive orientation* F of a graph $G = (V, E)$ is an assignment of a direction, or orientation, to each edge in E such that if $xy \in F$ and $yz \in F$ then $xz \in F$. A graph is called a *comparability graph* if it has a transitive orientation. For example, the even length chordless cycles C_{2k} $(k \geq 2)$ are comparability graphs, but the odd length chordless cycles C_5, C_7, etc. are not comparability graphs. Comparability graphs are also known as *transitively orientable* (TRO) graphs.

Gilmore and Hoffman in 1964 gave another characterization of interval graph.

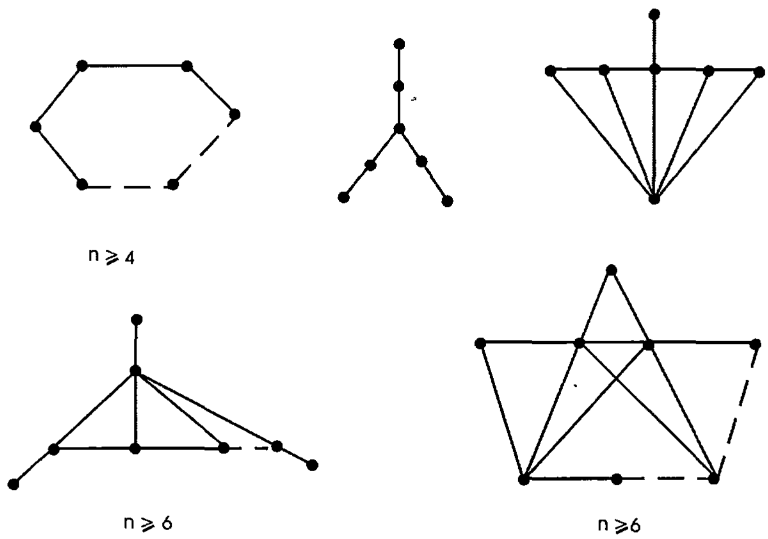


Figure 1.1: Forbidden subgraphs for interval graphs

Theorem 1.3.2 (Gilmore and Hoffman [33]**).** *A graph G is an interval graph if and only if G is chordal and its complement $\overline{G}$ is a comparability graph.*

A maximal clique of a graph is a complete subgraph which is not contained in any larger such subgraph. For a graph G, its vertex-clique incident matrix $M = (m_{ij})$ is the matrix whose rows and columns correspond to the vertices and the maximal cliques respectively of the graph such that

$$m_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ vertex belongs to } j^{th} \text{ clique} \\ 0 & \text{otherwise} \end{cases}$$

A matrix is said to have consecutive 1's property for rows if there is a permutation of its columns so that the 1's in each row appear consecutively.

Theorem 1.3.3 (Fulkerson and Gross [31]**).** *A graph is an inteval graph* *if and only if its vertex-clique incidence matrix has a consecutive* 1*'s property for rows.*

1.3.2 Indifference graphs, unit interval graphs and proper interval graphs

R. D. Luce [55] developed a model for preference motivated by the concept of psychology. He contended that, for a set of things X and the preference relation $\mathcal{R}$, one seek a function $f : X \to \mathcal{R}$ and a just-noticeable tolerance $\delta > 0$ with $x \mathcal{R} y$ (x is preferred to y) if and only if $f(x) > f(y) + \delta$; i.e., if the value placed on x is sufficiently larger than the value placed on y. This presentation led to the development of an order called semi-order. A *semi-order* $\prec$ on X is a binary relation having the properties: For $x, y, z, w \in X$, (1) $\prec$ is irreflexive, (2) $x \prec y$ *and* $z \prec w \Longrightarrow x \prec w$ *or* $z \prec y$, (3) $x \prec y$ *and* $y \prec z \Longrightarrow x \prec w$ *or* $w \prec z$. Motivated by this idea, Roberts [68] in his thesis, studied graphs with adjacency determined by the rule: vertices u and v are adjacent if and only if $|f(u) - f(v)| \leq \delta$. It is observed that a preference relation represented by f gives rise to a transitive graph. A graph G where vertices u and v are adjacent if and only if $|f(u) - f(v)| \leq \delta$ is called an *indifference graph* and f is called an *indifference representation* of G. An interval graph G that has a representation in which each interval has the same (unit) length is called a *unit interval graph.* Similarly, if G has representation in which no interval properly contains another interval, G is called a *proper interval graph.* Clearly, a unit representation is also proper. It is easy to verify that the bipartite graph $K_{1,3}$ does not have a proper interval representation. The following classical result is due to Roberts [68].

Theorem 1.3.4 (Roberts [68]). *Let $G(V, E)$ be an undirected digraph. The following conditions are equivalent:*

1) G is a unit interval graph;

2) G is a proper interval graph;

3) G is an indifference graph;

4) G is an interval graph and is $K_{1,3}$-free;

5) $\overline{G}$ is a comparability graph and every transitive orientation of $\overline{G} = (V\,\overline{E})$ is a semiorder.

1.3.3 Probe interval graphs

The interest in many classes of intersection graphs are application driven. Advancement of molecular biology, and genetics in particular, has driven scientists to find new models. In continuation of the evoluation of interval graphs to model ideas about the fine structure of gene [2], P. Zhang [98] introduced *probe interval graphs.* He attempted to aid a problem called cosmid contig mapping, a particular component of the physical mapping of DNA [98]. A *probe interval graph* is a graph $G = (V, E)$, where V can be partitioned into (P, N), such that there is an interval corresponding to each vertex and two vertices are adjacent if and only if their corresponding intervals intersect and at least one of the vertices belongs to P. Probe interval graphs generalize interval graphs (taking $N = \phi$) and provide an instance of an intersection graph with a modified adjacency rule. Now several research works are continuing on this topic and some special classes of it may be found in [8, 11, 61]. S. Ghosh et al. [32] characterized the adjacency matrix of a probe interval graph.

1.3.4 Tolerance graphs and permutation graphs

Golumbic and Monma [37] introduced tbe concept of *tolerence graphs* to generalize some of the well-known applications associated with interval graphs. Their original motivation was the need to solve scheduling problems in which resources such as rooms, vehicles, support personnel, etc may be needed on an exclusive basis, but where a measure of flexibility or tolerence would allow for sharing or relinquishing the resource if a solution is not otherwise possible. Tolerence graphs are constructed from intersecting intervals in a manner similar to interval graphs, but putting an edge between two vertices depends on measuring the size of the intersection of the two intervals. Informally, if both intervals are willing to "tolerate"or ignore the intersection, then no edge is added between their vertices in the graph. The formal definition of tolerance graphs is as follows:

A graph $G = (V, E)$ is a *tolerance graph*, if each vertex $v \in V$ can be assigned a closed interval I_v and a tolerance $t_v \in R^+$ so that $xy \in E$ if and only if $|I_x \cap I_y| \geq min\{t_x, t_y\}$. Such a collection $\langle I, t\rangle$ of intervals and tolerances is called *tolerance representation* where $I = \{I_x : x \in V\}$ and $t = \{t_x : x \in V\}$. If graph G has a tolerance representation with $t_v \leq |I_v|$ $\forall$ $v \in V$, then G is called a *bounded tolerance graph.* The topic has assumed much interest to the researchers and Golumbic and Trenk [39] has written a book on it.

A graph $G = (V, E)$ is a *permutation graph* if there is a permutation

π of $V = \{1, 2, 3, \ldots, n\}$ so that for vertices i, j we have $ij \in E$ if and only if the order of i and j are reversed in π. If a graph G is a permutation graph using π, then its complement $\overline{G}$ is also a permutation graph using the reversal of π. We will take special interest on this topic in our thesis and we will discuss it again in chapter 2.

The following theorems due to Golumbic and Monma [37] show that the class of bounded tolerance graphs is a simultaneous generalization of interval graphs and permutation graphs.

Theorem 1.3.5. *The following conditions are equivalent about a graph G:*

1) G is an interval graph;

2) G is a tolerance graph with constant tolerances;

3) G is a bounded tolerance graph with constant tolerances.

Theorem 1.3.6. *The following are equivalent statements about a graph G:*

1) G has a tolerance representation with $t_i = |I_i|$ for all $i \in V(G)$;

2) G is an interval containment graph;

3) G is a permutation graph.

1.3.5 Homogeneous graphs

In a family of intervals, a *left-end interval* is an interval whose left end-point is leftmost among all end-points of intervals in the family. Similarly, a *right-end interval* is an interval whose right end-point is the rightmost among the intervals in the family. An *end interval* is a left-end or a right-end interval.

It is well known that an interval representation of an interval graph is not unique and an interval graph may have many interval representations that differ in the order of the end points of the intervals on the real line. An excellent account of this area of study is given by P. C. Fishburn [30]. D. Skrien and J. Gimbel [84] characterized those graphs for which every vertex v, there is an interval representation such that v is an end interval. This family of graphs is defined to be *vertex-homogeneous.* S. Olariu [64] also obtained the results of Skrien and Gimbel by approaching from a different point of view.

1.3.6 Circular-arc graphs

A circular-arc graph is the intersection graph of a family of arcs on a circle. Extensive work on circular-arc graphs was done by Tucker [91, 92, 93, 94]. After a long period of thorough research by different mathematicians to find the recognition algorithm of circular-arc graph, it was R.M.McConnel [59] who could finally solve the problem in linear time. Additional references for recognition algorithms of special classes of these graphs can be found in [22, 25]. For an excellent survey of this topic see [53].

1.4 Interval bigraphs/digraphs

A bipartite graph (in short, *bigraph*) $B = (X, Y, E)$ is an *intersection bigraph* if there exists a family $\mathcal{F} = \{I_v : v \in X \cup Y\}$ of sets such that $uv \in E$ if and only if $I_u \cap I_v \neq \phi$. An intersection bigraph is an *interval bigraph* (recpectively, a *circular-arc bigraph*) if $\mathcal{F}$ is a family of intervals on the real line (respectively, arcs on a circle). Let $B = (X, Y, E)$ be a bigraph with bipartition X and Y. Then the submatrix of the adjacency matrix of B containing rows corresponding to the vertices of X and columns corresponding to the vertices of Y is the *biadjacency matrix* or the *reduced adjacency matrix* or simply the *matrix* of B. This concept was introduced in 1982 by Harary et al. [44] and later studied by Hell and Huang [47].

An *interval digraph*, analogous to the concept of interval bigraph, was introduced in 1989 by Sen et al. [75]. An interval digraph is a directed graph with an ordered pair of intervals (S_u, T_u) corresponding to each vertex u such that $u \to v$ if and only if $S_u \cap T_v \neq \phi$. We note that the models for both interval bigraphs and interval digraphs are essentially the same.

Let D be a digraph and $B(D)$ be the *associated bipartite graph* with bipartition (X, Y), obtained from D by replacing each vertex $v_i \in V(D)$ by two vertices $x_i \in X$ and $y_i \in Y$ and each arc v_iv_j of D by an edge x_iy_j. Then it is clear from the definition that D is an interval digrph if and only if $B(D)$ is an interval bigraph. Similarly, if H is a bipartite graph with bipartition (X, Y), we can orient all edges from X to Y, and observe that the resulting digraph is an interval digraph if and only if H is an interval bigraph, since for a vertex $x \in X$ only S_x matters and for a vertex $y \in Y$, it is only T_y.

1.4.1 Ferrers digraphs/ bigraphs and Ferrers dimension

Ferrers digraphs and Ferrers dimensions [14, 15, 56] play an important role in our study. This special class of digraphs was introduced independtly by Guttman [40] and Riguet [67]. Riguet defined a Ferrers digraph to be a digraph $D(V, E)$ in which $\forall x$, y, z and $t \in V$; $xy, zt \in E \Longrightarrow xt \in E$ or $zy \in E$ (inclusive). Riguet characterized Ferrers digraphs as those whose successor sets (or predecessor sets) are linearly ordered by inclusion. This suggests that by independent row and column permutation of the adjacency matrix $A(D)$, the 1's (or 0's) are clustered in the lower left sector forming a Ferrers diagram as shown in Figure 1.2.

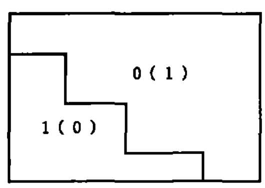


Figure 1.2: Ferrers digraph

He also showed that the above conditions are equivalent to the condition that $A(D)$ has no 2×2 permutation matrices as under:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Any of these forbidden matrices is called an *obstruction* and the zeros in an obstruction are said to form a *couple*.

In 1971, Bouchet [4] showed that any digraph D is the intersection of a family of Ferrers digraphs. This observation leads to the important notion of Ferrers dimension. The *Ferrers dimension* of a digraph D is defined to be the minimum number of Ferrers's digraphs whose intersection is D. Bouchet [4, 5] obtained several interesting results on Ferrers dimension. Cogis [13] and Doignon, Ducamp and Falmagne [23] have studied extensively the digraphs of Ferrers dimension 2 and have characterized independently this class of digraphs in different contexts. Cogis associated an undirected graph $H(D)$ with D whose vertices correspond to the 0'of the adjacency matrix of D and two such vertices are joined by an edge if and only if the corresponding 0's form a couple. He proved that D is of Ferrers dimension at most 2 if and only if $H(D)$ is bipartite. The bigraph corresponding to a Ferrers digraph is a *Ferrers bigraph/chain graph.* The *Ferrers dimension/chain dimension* of a bigraph is the Ferrers dimension of the corresponding digraph. The following theorem characterizes a bigraph of Ferrers dimension at most 2.

The following conditions are equivalent.

Theorem 1.4.1 (**Sen et al.** [75, 78],**Cogis** [13]). *The following conditions are equivalent for a bipartite graph B:*

1) B has Ferrers dimension at most 2;

2) The rows and columns of its biadjacency matrix can be permuted independently so that no 0 has a 1 both to its right and below;

3) The graph $H(B)$ of couples of B is bipartite.

In chapters 2 and 5 we will further characterize bigraphs of Ferrers dimension at most 2 in terms of permutation bigraphs and in terms of complements of two-clique circular-arc graphs.

1.4.2 Characterization of interval digraphs / bigraphs

In [78], it was proved that an interval digraph is a generalization of an interval graph. In fact, they proved the following result.

Theorem 1.4.2 (Sen, Sanyal and West [78]). *An undirected graph G is an interval graph if and only if the corresponding digraph $D(G)$ with loops at every vertex is an interval digraph.*

Let D be a digraph and its complement be $\overline{D}$. A more interesting and useful characterization of interval digraph uses its adjacency matrix.

Characterization of interval graph and digraph so far developed, involve an order of their vertices. Sanyal and Sen [73] tried to characterize them in a different way and posed the question, "Is there any ordering among the edges of a (di)graph that characterizes an interval (di)graph?" They answered this question in the affirmative. For a digraph $D(V, E)$, they introduced the notion of a *consistent ordering* of the edges of D.

The set of all edges of a digraph D is said to have a *consistent ordering* if E has a linear ordering $(<)$ such that for pq, pu, rs, $tq \in E$,

1) $pq < rs < pu \Longrightarrow ps \in E\,(q \neq u)$.

2) $pq < rs < tq \Longrightarrow rq \in E\,(p \neq t)$.

Theorem 1.4.3 (Sanyal and Sen [73]). *A digraph $D(V, E)$ is an interval digraph if and only if its edge set has a consistent ordering.*

There are quite a number of characterizations of interval bigraphs/digraphs and their subclasses. In order to get into them, we first explain the following notions.

A $(0, 1)$ matrix is said to have a *zero-partition* if the rows and columns of the matrix can be permuted independently such that each 0 can be replaced by one of $\{R, C\}$ in such a way that every R has only R's to its right and every C has only C's below it. Figure 1.3 gives an example of a matrix with a zero-partition.

$$\begin{array}{|cccc|} \hline 1 & 1 & R & R \\ 1 & 1 & 1 & R \\ C & 1 & R & R \\ C & 1 & 1 & 1 \\ C & C & 1 & R \\ \hline \end{array}$$

Figure 1.3: A matrix with a zero-partition

An alternative way of describing a zero-partition is in terms of *generalized linear ones property.* With the help of the concept of stair-partition of a matrix, we can describe this property. A *stair-partition* of a matrix is a partition of its elements into two subsets L and U by a polygonal path from the upper left to the lower right such that the set L is closed under rightward and downword movement and the set U is the complement part of L. Equivalently, U corresponds to the positions in some upper triangular matrix and L to the positions in the lower triangular matrix. This is shown in Figure 1.4.

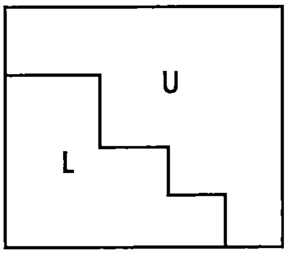


Figure 1.4: Stair partition

A (0, 1)-matrix has the *generalized linear ones property* if it has a stair partition (L, U) such that the 1's in U are consecutive and appear left-most in each row and the 1's in L are consecutive and appear top-most in each column.

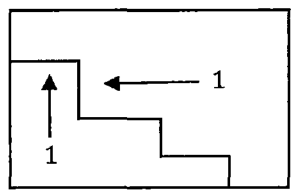


Figure 1.5: Generalized linear ones property

A complete bipartite (sub)graph is called a *biclique*. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$ be distinct induced subgraphs of a graph $G = (V, E)$. The set $\mathcal{G}$ is said to be consecutively ordered when for any $v \in V$, if $i < j < l$ and $v \in G_i \cap G_l$, then $v \in G_j$. We say $\mathcal{G}$ covers G if it forms an edge cover of G.

For a bipartite graph we have the following list of equivalent conditions characterizing an interval bigraph. The equivalences of (1), (2), (6) and (7) were obtained by Hell and Huang [47]. Das et al. [75, 76] used the digraph language to prove the equivalences of (1),(3), (4), (5) and (8).

Theorem 1.4.4. *The following statements are equivalent for a bipartite graph $H = (X, Y, E)$ with n vertices:*

1) H is an interval bigraph;

2) $\overline{H}$ is a two-clique circular arc graph in which no two arcs cover the whole circle;

3) $\overline{H}$ is the union of two disjoint Ferrers bigraphs;

4) The biadjacency matrix of H has a zero-partition;

5) There is a cover of H consisting of bicliques that can be consecutively ordered;

6) The vertices of H can be ordered $v_1 < v_2 < \ldots < v_n$ so that there do not exist $a < b < c$with v_a, v_b in the same partite set and $v_a v_c \in E$, but $v_b v_c \notin E$ (see Figure 1.6);

7) The vertices of G can be ordered $v_1 < v_2 < \ldots < v_n$ so that there do not exist $a < b < c$ with any of the four structures in Figure 1.7;

8) The biadjacency (reduced adjacency) matrix has generalized linear one's property;

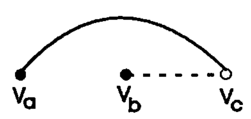


Figure 1.6: Forbidden graph of Theorem 1.4.4(6)

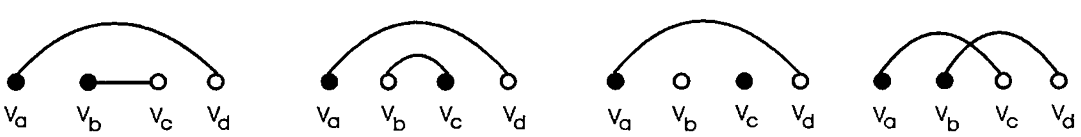


Figure 1.7: Forbidden graphs of Theorem 1.4.4(7)

Amongs the above, conditions (2) and (4) are the important ones and we will use them frequently in this thesis.

Unit interval bigraphs

Now we consider unit interval bigraphs, a special subclass of interval bigraphs. A unit interval bigraph is an interval bigraph in which all the intervals are of same length. This idea is so simple and pervasive that many researchers have converged to this notion studying from different view points. So much so that Brown in his thesis [6] has cited about 15 different but equivalent classes of structured graphs of this family. We will list only a few of them in Theorem 1.4.5. In fact, a fine account of the results of this topic of interval bigraphs and its subclasses can be found in [6].

We need to spell out some notions here.

Proper interval bigraphs are interval bigraphs having representations such that no interval properly contains another interval of the same partite set.

Indifference bigraphs are those for which there exists a real valued function on $X \cup Y$ such that for $u \in X$ and $v \in Y$, uv is an edge if and only if $|f(u) - f(v)| \leq 1$.

Analogous to Robert's work [68], these graphs have been proved to be equivalent and have been characterized in terms of monotonic consecutive arrangement [77]. A (0, 1)-matrix has a *monotone consecutive arrangement* if there exist independent row and column permutation such that the resulting matrix has the following structure: The 0's can be labeled by R and C such that every position above or to the right of an R is an R, and every position below or to the left of a C is a C.

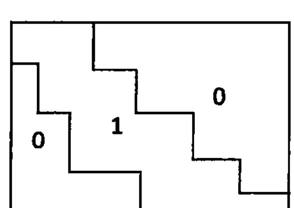


Figure 1.8: Monotone consecutive arrangement

A proper *circular-arc graph* is a circular-arc graph, which has a representation by arcs such that no arc properly contains another.

In the following, the equivalence of (1), (2), (3) and (5) is to be found in [77] while Hell and Huang [47] obtained the equivalences of (1), (4), (6) and (8). Lastly, the condition (7) was obtained by Steiner [87] who incidentally provided a linear time algorithm for recognition of these classes of bigraphs. Again, the condition (8) was obtained by Lin and West [54].

Theorem 1.4.5 ([77, 47, 54, 87]). *For a bipartite graph H, the following are equivalent:*

1) *H is a unit interval bigraph;*
2) *H is a proper interval bigraph;*
3) *H is an indifference bigraph;*
4) *$\overline{H}$ is a proper circular-arc graph;*
5) *The biadjacency matrix of H has a monotone consecutive arrangement;*
6) *H is asteroidal triple free;*
7) *H is a permutation graph;*
8) *H does not contain an induced cycle of length at least 6 or any of the graphs of Figure 1.9 as an induced subgraph.*

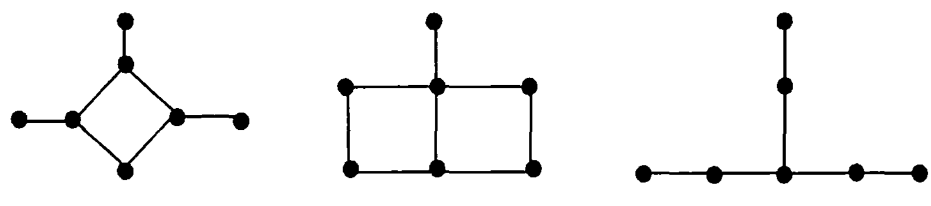


Figure 1.9: Forbidden graphs of Theorem 1.4.5(8)

1.4.3 Interior edges

Das and Sen [18] introduced the concept of *interior edges* and used this notion to find out the digraphs that are of Ferrers dimjension 2 but are not interval digraphs. Cogis [14] proved that a digraph D is of Ferrers dimension at most 2 if and only if its associated graph $H(D)$ is bipartite. Also it is known that for a digraph D having Ferrers dimension 2, the complement $\overline{D}$ is the union of two Ferrers digraphs (not necessarily disjoint). These two Ferrers digraphs are called *realization* of $\overline{D}$. Obviously realization of $\overline{D}$ is not unique. The graph $H(D)$ may have more than one connected component as well as isolated vertex(vertices) (the 0's of $A(D)$ that donot belong to any obstruction). The set of all isolated vertices is denoted by $\mathcal{I}(H)$ or $\mathcal{I}$. The graph obtained by deleting the isolated vertices from $H(D)$ is called the *bare graph* associated with D and is denoted by H^b

Since a digraph of Ferrers dimension at most 2 is equivalent to the existence of independent row and column permutation of the adjacency matrix so that the resulting matrix has no 0 with a 1 both to its right and below, it can not have an obstruction of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It was shown that an interval digraph is necessarily a digraph of Ferrers dimension at most 2 but the converse is not true. As a matter of fact, it was proved that a digraph D is an interval digraph if and only if it is the intersection of two Ferrers digraphs whose union is complete or equivalently the complement $\overline{D}$ is the union of two disjoint Ferrers digraphs. Let $H(B)$ be bipartite and (R, C) be a bicolouration of H^b. Cogis showed that there is a bicolouration (R, C) of H^b such that each of $R \cup \mathcal{I}$ and $C \cup \mathcal{I}$ is a Ferrers bigraph and their union is the complement $\overline{B}$ of B. Note that not all bicolouration of H^b have this property and a bicolouration (R, C) of H^b for which $R \cup \mathcal{I}$ and $C \cup \mathcal{I}$ are Ferrers bigraphs is called *satisfactory bicolouration.*

We know that a bigraph B is an interval bigraph if and only if its complement is the union of two disjoint Ferrers bigraphs. This means that there is a bicolouration (R, C) of H^b and a partition $\mathcal{I}$ into two disjoint subsets $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $R_1 = R \cup \mathcal{I}_1$ and $C_1 = C \cup \mathcal{I}_2$ are two disjoint Ferrers bigraphs whose union is the complement $\overline{B}$ of B. An edge I of $\mathcal{I}_1$ is said to be an *interior edge of* R_1 and is denoted by $\mathcal{I}_r$ if it has a configuration

$$\begin{pmatrix} 1 & R \\ R & \mathcal{I}_r \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} R & \mathcal{I}_r \\ 1 & R \end{pmatrix}$$

in $\overline{B}$. Similarly, an edge I of $\mathcal{I}_2$ is said to be an interior edge of C_1 and is denoted by $\mathcal{I}_c$ if it has a configuration

$$\begin{pmatrix} 1 & C \\ C & \mathcal{I}_c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C & \mathcal{I}_c \\ 1 & C \end{pmatrix}$$

With this notion of interior edges, Das and Sen [18] proved that if a digraph D of Ferrers dimension 2 is an interval digraph then for any satisfactory bicolouration of H^b, $\mathcal{I}_r \cap \mathcal{I}_c = \phi$. But the converse is not true. They gave an example of eight vertex digraph with Ferrers dimension 2 which is not an interval digraph and for which $\mathcal{I}_r \cap \mathcal{I}_c = \phi$. In [20] this idea of interior edges were used to find additional configurations of a digraph having Ferrers dimension 2 which are not interval digraphs. For further discussion see subsection 1.4.4

1.4.4 Forbidden subgraphs and forbidden substructures.

Analogue for an AT (see page 5) is an *asteroidal triple of edges*(ATE): a set of three edges such that there is a path between any two that avoids the neighbourhood of the third edge, where if $e = uv \in E$, then the neighbourhood of e is $N(e) = N(u) \cup N(v)$.

In 1997, Haiko Müller [63] gave a polynomial time $O(n^5\, m^6\, \log n)$ recognition algorithm for interval bigraphs. He introduced a family of graphs, called insects (shown in Figure 1.10) and conjectured the following characterization of interval bigraphs:

Conjecture 1.4.6. *A bipartite graph is an interval bigraph if and only if it contains no asteroidal triple of edges and no induced insects.*

This conjecture was refuted in [17, 47] who found a counter examples to this conjecture.Hell and Huang [47] obtained a class of graphs called bugs (see Figure 1.11), which are not interval bigraphs. Also they do not contain induced insects and asteroidal triple of edges and so provide counter examples

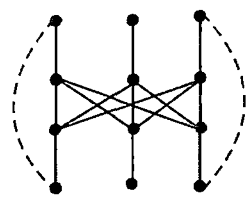


Figure 1.10: Insects

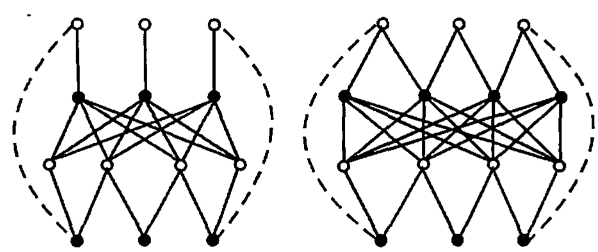
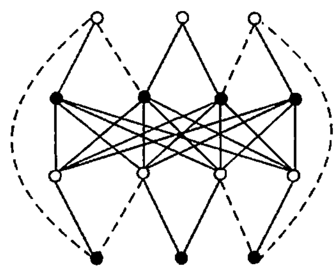


Figure 1.11: Bugs

to Müller's conjecture. They have used the term extended insect to mean either an insect or bug. Thus the list of forbidden structure for an interval bigraph has been extended. S.Das, A.Das et al. [18, 20] started from the class of bigraphs of Ferrers dimension 2 and tried to obtain a forbidden configuration characterization of interval bigraphs using biadjacency matrix. In doing so, they improved upon the results of Hell and Huang [47] by obtaining a large number of forbidden subgraphs which include the extended insects of [47]. In fact, they obtained five forbidden configurations $F^i\,(i = 0, 1, 2, 3, 4)$ of the biadjacency matrix of an interval bigraph. Specially, the class of bigraphs of configuration F^0 is given in Figure 1.12. It can be easily seen that all the graphs of the extended insects form a subclass of the class of Figure 1.12 and can be obtained by different combinations of inclusions/exclusions of the dashed edges of Figure 1.12. However, the complete list is still elusive and it remains an interesting open problem till date.



Dashed edges may or may not be present

Figure 1.12: Bigraphs of F^0

1.4.5 Related topics

Tolerance (di)graphs, bitolerance (di)graphs, homomorphic graphs

Bogart and Trenk [3] provided a different directed graph analogue of interval graphs based on work in tolerance graphs. A directed graph $D(V, E)$ is a *bounded bitolerance digraph* if each vertex $v \in V$ can be assigned a real interval $I_v = [L(v), R(v)]$, a left tolerant point $p(v) \in I_v$, and a right tolerant point $q(v) \in I_v$ so that $E = \{(x, y) : L(x) \leq q(y) \text{ and } R(x) \geq p(y)\}$. The representation $\{[L(v), R(v)], p(v), q(v) : v \in V\}$ is called a *bounded bitolerance representation* and an example is given in Figure 1.13. According to this definition, bounded bitolerance digraphs, have a loop at each vertex. They characterized bounded bitolerance digraphs in terms of two interval orders which is analogous to that of interval digraphs in terms of two Ferrers digraphs [75].

An ordered set $P = (V, \prec)$ consists of a set V together with a binary relation

$\prec$ on V that is irreflexive, transitive and therefore antisymmetric. Elements $x, y \in V$ are said to be *comparable* if $x \prec y$ or $y \prec x$; otherwise they are *incomparable*, which is denoted by $x \parallel y$. A *linear order* is one with no incomparabilities. The *dimension* of an order P is the smallest number of linear orders whose intersection is P. We construct a digraph $\widetilde{P_1} \cap \widetilde{P_2}$ from two ordered sets P_1 and P_2 that have the same ground set.

Let $P_1 = (V, \prec_1)$ and $P_2 = (V, \prec_2)$ be ordered sets. The digraph $\widetilde{P_1} \cap \widetilde{P_2} = (V, E)$ has $(x, y) \in E$ if and only if

(i) $x \prec_1 y$ or $x \|_1 y$ (ii) $x \prec_2 y$ or $x \|_2 y$.

Bogart and Trenk [3] proved the following result

Theorem 1.4.7. *A directed graph $D(V, E)$ is a bounded bitolerance digraph if and only if there exists interval orders $P_1 = (V, \prec_1)$ and $P_2 = (V, \prec_2)$ for which $\widetilde{P_1} \cap \widetilde{P_2} = D(V, E)$.*

In a bounded bitolerance representation, it is not necessary to have $p(v) \leq q(v)$. If in fact $p(v) \leq q(v)\ \forall\ v \in V$, then $D(V, E)$ is called a *totally bounded bitolerance digraph* and these digraphs were studied in [80, 81]. If a digraph $D(V, E)$ has an interval digraph representation $\{(S_v, T_v) : v \in V\}$ where $T_v \subseteq S_v\ \forall\ v \in V$, then D is called an *interval nest digraph* and this class was studied in [65]. Shull and Trenk [80] proved the following proposition.

Proposition 1.4.8. *A digraph $D(V, E)$ is a totally bounded bitolerance digraph if and only if $D(V, E)$ is an interval nest digraph.*

They have finally explored the relationship between interval digraphs and bounded bitolerance digraphs, both of which are directed graph analogues of

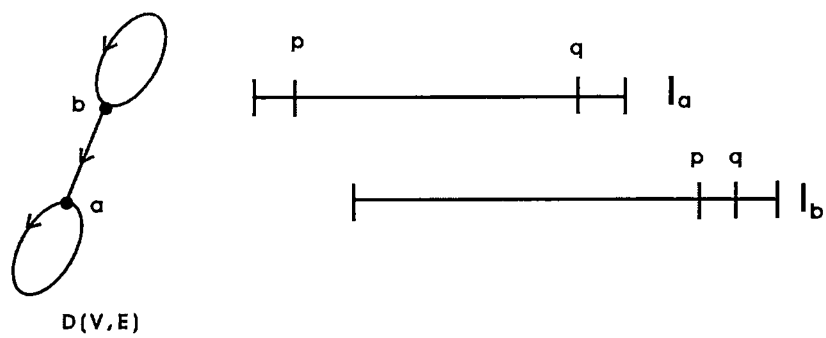


Figure 1.13: Bounded bitolerance representation

the well known interval graphs.

Theorem 1.4.9 (Shull and Trenk [81], **Sen and Sanyal** [77]**).** *The following are equivalent statements about a symmetric digraph D that has a loop at each vertex.*

1) D is a bounded bitolerance digraph;

2) D is an interval digraph;

3) The underlying simple graph is an interval graph;

4) D is a totally bounded bitolerance digraph;

5) D is an interval nest digraph.

A *homomorphism* f of a (di)graph G to a (di)graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e., $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. If there is a homomorphism of G to H, we write $G \to H$. For a fixed (di)graph H, the *homomorphism problem* HOMP(H) asks whether or not an input (di)graph G satisfies $G \to H$. The complexity of all homomorphism problems has been studied by P. Hell and J. Nestril [48]. HOMP(H) is polynomial time solvable if H is bipartite and is NP-complete otherwise. The *list homomorphism problem* for H is a generalization of the homomorphism problem for H and is defined as: Given (di)graph G, H and lists $L(v) \subseteq V(H)$, $v \in V(G)$, a *list homomorphism* of G to H with respect to the lists L is a homomorphism f of G to H, such that $f(v) \in L(V)\ \forall\, v \in V(G)$. For a fixed (di)graph H, the *list homomorphism problem* L–HOMP(H) asks whether or not an input (di)graph G, with lists L, admits a list homomorphism to H with respect to L. Note that, HOMP(H) is a restriction of L–HOMP(H), where $L(v) = V(H)$. Thus L–HOMP(H) is NP-complete when H is not bipartite.

Feder, Hell, Huang and Rafiey [29] introduced the concept of *adjusted interval digraphs* obtained by a slight change in the definition. Let $\{(S_v, T_v) : v \in V\}$ be an interval digraph representation of $D(V, E)$. If the intervals S_v, T_v, $v \in V$, can be chosen so that for each v, the intervals S_v and T_v have the same left end point, we say that D is an *adjusted interval digraph.* It is clear that an adjusted interval digraph must be reflexive. By contrast to interval digraphs, adjusted interval digraphs have a forbidden structure characterization, parallel to a characterization for undirected graphs. Feder et al. [29] related adjusted interval digraphs to a list homomorphism problem. They also observed that if H is an adjusted interval digraph, then the problem L-HOMP(H) is polynomial time solvable, and conjectured that for all other reflexive digraphs H, the problem L-HOMP(H) is NP-complete.

Interval k-graphs

Interval k-graphs, a generalization of interval bigraphs, was introduced by Brown et al. [9]. An *interval k-graph* is a k-partite graph with an interval assigned to each vertex so that two vertices are adjacent if and only if they belong to distinct partite sets and their corresponding intervals have non empty intersection. Obviously, an interval bigraph is an interval k-graph with $k = 2$.

Analogous to the idea of consecutive maximal cliques in the characterization of interval graphs, they characterized interval k-graphs in terms of consecutively ordered subgraphs. They have also proved the following result: Interval graphs $\subsetneq$ Probe interval graphs $\subsetneq$ Interval k-graphs.

Brown in his thesis [6] proved the following result.

Theorem 1.4.10. *A graph is an interval k-graph if and only if there exists a cover of complete r-partite subgraphs that can be consecutively ordered, where $1 \leq r \leq k$, for each subgraph.*

A graph G is said to be perfect if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\chi(H)$ is the chromatic number of H (i.e., the fewest number of colours needed to properly colour H) and $\omega(H)$ is the number of vertices in the largest induced clique in G. They have also proved the following results.

Theorem 1.4.11. *Interval k-graphs are perfect.*

Theorem 1.4.12. *If a graph G has an asteroidal triple of edges, then G is not an interval k-graph.*

1.5 A brief outline of the thesis

1.5.1 Permutation bigraphs

The concept of permutation graphs have been studied by B. Dushnik, E. M. Miller [26], S. Even, A. Pnueli, A. Lempel [27]. Permutation graphs can be defined in many equivalent ways. We consider the following definition.

As defined earlier, an undirected graph $G(V,\ E)$ with n-vertices is a permutation graph, if there is a bijection $L\ : V \to \{1, 2, \dots, n\}$ and a permutation π of $\{1, 2, \dots, n\}$ such that two vertices u and v are adjacent in G if and only if the orders of u and v are reversed in L and π.

Let $(P\,, <)$ be a poset. A containment representation of a poset assigns each element $x \in P$, a set S_x such that $x < y$ if and only if $S_x \subset S_y$. It is well known that interval containment posets are precisely the posets of dimension 2 [57]. Furthermore, the Ferrers dimension of the comparability digraph (a digraph that is reflexive, antisymmetric and transitive) of a poset equals the order dimension of the poset [4, 23, 75]. Sen et al. proved that interval containment digraphs are precisely the digraphs of Ferrers dimension 2. From these results, it implies that a poset P is an interval containment poset if and only if its comparability digraph is an interval containment digraph.

Permutation graphs have several known characteristics and we state a few of them.

Theorem 1.5.1. *Let $G(V,\ E)$ be an undirected graph. The following conditions are equivalent:*

1) G is a permutation graph;

2) Both G and its complement $\overline{G}$ are transitively orientable;

3) G is a containment graph of intervals on the real line;

4) G is a comparability graph of poset dimension at most 2.

In this thesis, we generalize the concept of permutation graph and introduce the concept of permutation bigraph. In chapter 2 we show that this class is equivalent to the bigraph of Ferrers dimension at most 2. Then we characterize interval bigraphs and indifference bigraphs in terms of their permutation labellings.

1.5.2 Interval tournaments

A touranment is a complete oriented digraph. A tournament that is an interval digraph is an *interval tournament.* Interval tournaments were introduced and characterized by Brown, Busch and Lundgren [7]. They have characterized them by a complete list of seven forbidden sub-tournaments. They have also proved that a tournament with n-vertices is an interval tournament if and only if it has a transitive sub-tournament having $(n-1)$-vertices. In chapter 3 we provide an alternative proof of the above theorem. This approach helps us to obtain further characterizations of interval tournaments. One of these characterizations is that a tournament is an interval tournament if and only if all of its 3-cycles have a common vertex. In continuation we have obtained three forbidden subdigraphs of an interval tournament. Lastly, we characterize the complement of an interval tournament in terms of two-clique circular-arc graphs.

1.5.3 Homogeneous interval bigraphs

Motivated by the idea of homogeneous graphs (subsection 1.3.5) we have developed in chapter 4 the concept of *homogeneous interval bigraphs* and characterize them in terms of forbidden bigraphs. An interval bigraph B is said to be *vertex-homogeneous* if for every vertex v of B, there exists an interval representation of B for which the interval representing v is an end interval. Analogously, an interval bigraph B is said to be *edge-homogeneous* if for every edge $e = uv$ of the bigraph B, there exists an interval representation of B for which the interval representing both u and v have the same end.

In chapter 4, we characterize both vertex-homogeneous and edge-homogeneous interval bigraphs in terms of forbidden subbigraphs.

1.5.4 Circular-arc bigraphs and its subclasses

Circular-arc bigraphs are the intersection bigraphs of a family of arcs on a circle. The digraph version of this class was introduced by Sen et al. [76].

Tucker [91] introduced the concept of quasi circular one's property of a $(0,1)$-matrix and used it to characterize a circular-arc graph. Sen et al. [76] extended this concept of Tucker and introduced *generalized circular ones property*: Given a $(0,1)$-matrix A and a stair partion (L, U) (see Figure 1.4), let $V_i[W_j]$ be the 1's in row i [column j] that begin at the stair and continue rightward [downward](around if possible) until a first 0 is reached. A has a generalized circular one's property if V_i's and W_j's together cover all the 1's of A. They characterized circular-arc digraphs/bigraphs in terms of this property.

In the final chapter of this thesis (Chapter 5), we obtain two new characterizations of circular-arc bigraphs. One of them is the representation of a circular arc bigraph as an intersection of two-clique circular graphs while another one represents the same as a union of interval bigraph and a Ferrers bigraph. Finally, we introduce the notions of proper and unit circular arc bigraphs, characterize them and show that, as in the case of circular-arc graphs, unit circular arc bigraphs from a proper subclass of the class of proper circular arc bigraphs.

Chapter 2

Permutation Bigraphs: An Analogue of Permutation Graphs[1]

2.1 Introduction

An undirected graph G on n vertices is a *permutation graph* if there are two orderings of $V(G)$ such that vertices are adjacent if and only if they appear in opposite order in the two orderings. The class of permutation graphs is well studied; MathSciNet lists more than 100 papers. The definition can be restated in several well-known equivalent ways.

Theorem 2.1.1. [26, 27, 38] *The following conditions are equivalent for a graph G:*

(a) G is a permutation graph;

(b) Both G and its complement $\overline{G}$ are transitively orientable;

(c) G is the containment graph of a family of intervals in $\mathbb{R}$;

[1]A part of this chapter is co-authored with D.B.West and is communicated to J.Graph Theory for possible publication

(d) G is the comparability graph of a poset of dimension at most 2.

In this chapter, we introduce two bipartite analogues of this family, one of which we call "permutation bigraphs". We do *not* mean "bipartite permutation graphs", which are those graphs that are bipartite and are permutation graphs (discussed in [86, 87]).

An *X, Y-bigraph* is a bipartite graph with partite sets X and Y. As defined above, two orderings of $X \cup Y$ produce a permutation graph G with vertex set $X \cup Y$. There are two natural ways to generate an X, Y-bigraph contained in G. In the first, one simply deletes the edges within X and within Y.

The second model is even more restrictive about the edges retained from G. Treat the first ordering L as a reference ordering, numbering $X \cup Y$ from 1 to $|X \cup Y|$. That is, L expresses (X, Y) as a partition of $[n]$, where $n = |X \cup Y|$ and $[n]$ denotes $\{1, \dots, n\}$. Create an edge xy for $x \in X$ and $y \in Y$ if and only if $x > y$ (as elements of $[n]$) and x occurs before y in the second ordering π. Note that x and y then form an inversion in π.

A *permutation bigraph* is an X, Y-bigraph that can be represented by vertex orderings in this second model. The graphs representable by the first model are the *bipermutation bigraphs*, where the first "bi" indicates that both orderings of the partite sets in an inversion are allowed to generate edges. Although the definition of bipermutation bigraphs may seem more natural, it turns out that the class of permutation bigraphs is better behaved.

A permutation bigraph may have many permutation representations. Indeed, when all of Y precedes all of X in L, every permutation π with X before Y yields a permutation representation of the complete bipartite graph $K_{|X|,|Y|}$. Also, interchanging and reversing L and π in a permutation representation yields another representation of the same graph.

We often specify a permutation graph by giving just one permutation; in this case the other permutation is the identity permutation on the vertex set $[n]$. To specify a permutation bigraph in this way, one must also give the partition of $[n]$ into X and Y. We may present this partition along with the permutation π of $[n]$ by putting underbars on the elements of Y and overbars on the elements of X, or simply underbars on Y.

Example 2.1.2. Let $X = \{1, 2, 4, 9, 10\}$ and $Y = \{3, 5, 6, 7, 8\}$ and consider a permutation $\pi = \{4, \underline{5}, 1, \underline{6}, \underline{7}, 9, 2, \underline{3}, 10, \underline{8}\}$. In this example the vertex 4 of X occurs before the vertex 3 of Y in π and since $4 > 3$ the pair $(4, 3)$ is an edge of the permutation bigraph $B(\pi)$. The bigraph $B(\pi)$ is given below in Figure 2.1

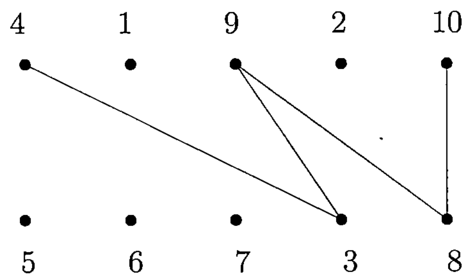


Figure 2.1: A permutation bigraph

The preference for X before Y in the inversions in π does not change which graphs are permutation bigraphs. If G is a permutation bigraph generated by L and π in this model, then reversing L and π to obtain L' and π' again generates G in the model where inversions in π' with elements of Y before those of X become edges.

Our characterizations of permutation bigraphs parallel those of permutation graphs.

Example 2.1.3. Theorem 2.1.1 characterizes permutation graphs using comparability graphs. A *poset of dimension* 2 is a partially ordered set P generated from two linear orders on its elements by putting $x < y$ if and only if x comes before y in both orders. The *comparability graph* of P is the graph whose vertices are the elements of P, with x adjacent to y if and only if $x < y$ or $y < x$ in P. Reversing one of the linear orders generating a poset of dimension 2 yields another poset of dimension 2 whose comparability graph is the complement of the first and is the permutation graph generated by the original two linear orders. Conversely, if G is a permutation graph, then the two permutations representing it give two linear orders (one reversed) that define a poset for which it is the comparability graph. $\square$

The analogous characterization of permutation bigraphs uses a generalization of linear orders. For an X,Y-bigraph B, we call the submatrix of its adjacency matrix consisting of the rows indexed by X and the columns indexed by Y the *biadjacency matrix* or simply the *reduced adjacency matrix* of B. The matrix has no 2-by-2 permutation submatrix if and only if the neighborhoods of the vertices in one (either) partite set form a chain under inclusion. A *Ferrers bigraph* is a bipartite graph satisfying this property. The property is equivalent to having independent permutations of the rows and the columns so that the positions occupied by 1s form a Ferrers diagram in the lower left. We therefore call such a matrix a *Ferrers matrix.*

Many of these concepts arose independently for directed graphs. The directed and bipartite models are essentially equivalent because the focus is on the 0,1-matrix that records adjacency. Permuting the rows or the columns in the matrix of an X, Y-bigraph does not change the graph. Allowing non-square matrices does not essentially change the class, since adding isolated vertices does not change whether a graph is a permutation bigraph, but broadening the family in this way can simplify proofs.

The *intersection* of two X, Y-bigraphs G and H with the same vertex set is the X, Y-bigraph whose matrix has 1 precisely where the matrices of G and H both have 1 (under a fixed naming of rows and columns). In Section 2.2 we characterize permutation bigraphs: a bipartite graph is a permutation bigraph if and only if it is the intersection of two Ferrers bigraphs. Furthermore, as we will show in Section 2.2 that these graphs are also the complements of two-clique circular-arc graphs, where a *circular-arc graph* is the intersection graph of a family of arcs on a circle, and a *two-clique* graph is a graph whose vertices can be covered by two complete subgraphs. We will discuss these results again in Chapter 5. Examples in Section 2.3 compare permutation bigraphs to other related classes.

The characterization using Ferrers bigraphs enables us to describe subfamilies of the permutation bigraphs that have been studied in other contexts by their permutation representations. An *interval bigraph* is an X, Y-bigraph representable by giving each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$

and $y \in Y$ are adjacent if and only if their intervals intersect. Thus an interval X, Y-bigraph arises from an interval graph with vertex set $X \cup Y$ by deleting the edges within X and within Y, just as bipermutation bigraphs arise from permutation graphs. An *indifference bigraph* is an interval bigraph having an interval representation in which all intervals have the same length.

Interval bigraphs have many known characterizations (see [75, 47], etc.). The key characterization related to permutation bigraphs is that a bipartite graph is an interval bigraph if and only if it is the intersection of two Ferrers bigraphs whose union is a complete bipartite graph. Thus every interval bigraph is a permutation bigraph. In Section 2.4 we obtain necessary and sufficient conditions on the defining permutations for a permutation bigraph to be an interval bigraph, and we similarly characterize the indifference bigraphs.

Finally, in Section 2.5 we interpret permutation bigraphs in terms of comparability graphs. Our most difficult result in this chapter is a direct proof of a characterizarion of posets whose comparability graphs are permutation graphs using permutation bigraphs. This characterization was previously expressed in [78], but there the difficult direction relied on a result of Bouchet [4] characterizing the dimension of posets. The proof here uses only a simpler and better known theorem of Cogis [13], plus our results from Section 2.2.

2.1.1 Associted permutation bigraph

Given a permutation bigraph $B = B(X, Y, E)$ defined by π describe below three other associated bigraphs such that all the four bigrphs are pairwise edge disjoint and their union is the complete bipartite graph $K_{|X|,|Y|}$.

1. Let B^*(X, Y E*) be the bigraph formed from π by subtracting each element from $n + 1$ and then reversing the order of elements. The resulting permutation is denoted by π^* and the bigraph B^* is called the conjugate bigraph of B. We will use this idea when we characterize an interval bigraph interms of its permutation bigraph.

2. Let B_1 be the bigraph whose permutationn is the reverse of the permutation π.

3. Let B_1^* be the conjugate of B_1.

Example 2.1.4. Let X $=$ $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5) = (3, 9, 8, 6, 4)$ and Y $=$ $(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4, \mathrm{y}_5) = (7, 10, 2, 5, 1)$ and let π $=$ $(\underline{7}, 3, \underline{10}, 9, \underline{2}, 8, \underline{5}, \underline{1}, 6, 4)$. The bigraph B and its three associated bigraphs are given in Figure 2.2.

It can be verified that the three associated bigraphs described above can be obtained from B by imposing different conditions on the reference labeling and the permutation π.

1. The bigraph B^*(X, Y, E*) is obtained with the conditions

   (i) $x < y$ as elements of $[n]$

   (ii) y occurs before x in π

2. B_1(X, Y, E$_1$) is the graph for which

   (i) $y < x$

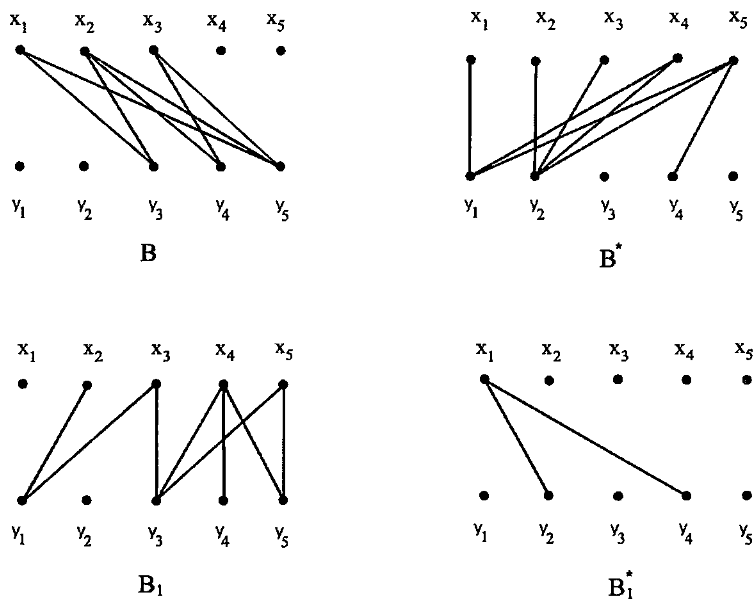


Figure 2.2: A permutation bigraph and its associates

(ii) y occurs before x in π.

3. B_1*(X, Y, E1*) is the graph with the condition

(i) $x < y$

(ii) x occurs before y in π.

Stating from B we get the three associated bigraphs B^*, B_1, ${B_1}^*$. If instead of B, we start with any one of B^*, B_1, ${B_1}^*$, then it can be seen that its three associates are the remaining three of the quadruple $\{B,\ B^*,\ B_1,\ {B_1}^*\}$.

2.2 Characterization of Permutation Bigraphs

An *interval containment bigraph* is an X, Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if the interval for y contains the interval for x. The alternative model putting x and y adjacent whenever the interval for either contains the interval for the other yields *interval bicontainment bigraphs*; the first "bi" allows both directions of containment. The ordered version corresponds to a digraph model that will be useful in Section 2.5. In [78], it was proved that a bipartite graph B is an interval containment bigraph if and only if it is the intersection of two Ferrers bigraphs; this follows easily from a useful characterization of Ferrers bigraphs.

Ferrers bigraphs (see [78, 97]) are characterized by the existence of an ordering f of the vertex set $X \cup Y$ so that xy is an edge if and only if $f(x) > f(y)$. Given a permutation of the rows and columns of the matrix A of the graph such that the 1-entries appear as a Ferrers diagram in the lower left corner, the separation between the 1-entries and 0-entries forms a "stair" that crosses each row and column once. The entry in row x and column y is below the stair if and only if the row for x is crossed after the column for y is crossed. The separation of the entries by a stair is equivalent to the existence of the function f.

To express a Ferrers X, Y-bigraph B as a permutation bigraph, let the reference ordering L consist of all of Y followed by all of X, giving X the higher numbers. Obtain π from the stair ordering f discussed above by writing the elements in decreasing order of their value under f. Now B is precisely the permutation bigraph represented by L and π.

To illustrate, consider the Ferrers bigraph (X, Y) whose reference labelling L and function f is given in Figure 2.3.

The defining permutation of the bigraph is $\pi = (7, \underline{3}, 6, 5, \underline{2}, 4, \underline{1})$. In con-

$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 4 & 1 & & \\ 5 & 1 & 1 & \\ 6 & 1 & 1 & \\ 7 & 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{c|ccc|} & 1 & 3 & 6 \\ \hline 2 & 1 & & \\ 4 & 1 & 1 & \\ 5 & 1 & 1 & \\ 7 & 1 & 1 & 1 \\ \hline \end{array}$$

(a) Labelling L (b) Function f

Figure 2.3: The two orderings of a Ferrers bigraph

tinuation of the study of bigraphs of Ferrers dimension at most 2(cf. Theo 1.4.1), below we prove that permutation bigraphs are precisely the bigraphs of Ferrers dimension at most 2.

Theorem 2.2.1. *For an X,Y-bigraph B, the following conditions are equivalent:*

(a) B is a permutation bigraph;

(b) B is an interval containment bigraph;

(c) B is the intersection of two Ferrers bigraphs.

Also, B is a bipermutation bigraph if and only if it is an interval bicontainment bigraph.

Proof. Note first that in representations that generate edges by containment, all intervals may be assumed to contain the origin. This holds because expanding two intervals outward from their centers by the same amount (added to the upper endpoints and subtracted from the lower endpoints) does not change whether one contains the other or which contains the other. After sufficient expansion, all intervals contain the origin. Hence we consider interval containment or interval bicontainment representations of this form for bipartite graphs.

Since containment is determined by the order of the endpoints, not the distance between them, we may further assume (for an n-vertex graph) that the right endpoints are $[n]$ and the left endpoints are the negatives of these points.

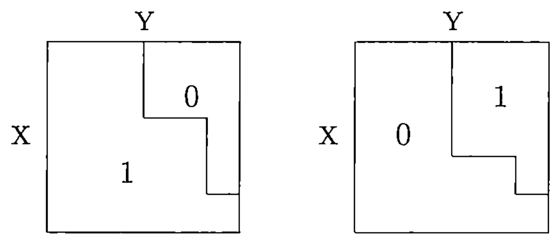
Such a set of intervals corresponds to two orderings L and π of the vertices. Define the orderings so that the left endpoint for vertex v is $-(n+1)+L(v)$ and the right endpoint is $\pi^{-1}(v)$. That is, L numbers the vertices in left-to-right order of the left endpoints, and π lists them in left-to-right order of the right endpoints. Now the interval for u contains the interval for v if and only if u and v appear in opposite order in L and π, with v appearing first in π. This proves the equivalence of (a) and (b) and proves the final statement.

For the equivalence of (b) and (c) (see [77]), consider intervals containing the origin. Let L be the left-endpoint ordering as before, and let π' be the reverse of the right-endpoint ordering. These are the stair-orderings for two Ferrers bigraphs. The interval for y contains the interval for x if and only if x follows y in each of those orderings, which is true if and only if xy is an edge in each of the two Ferrers bigraphs. □

As a supplement, we provide a direct proof of $(c) \Longrightarrow (a)$ below.

Proof. Let $B = F_1 \cap F_2$ where $F_1(X, Y, E_1)$ and $F_2(X, Y, E_2)$ are two Ferrers bigraphs.

Order the vertices of X and Y of F_1 and of F_2 so that its edges form the lower left corner of its biadjacency matrix of Figure 2.4a and the upper right corner of the biadjacency matrix of Figure 2.4b. The stair partition of the



(a) F_1 (b) F_2

Figure 2.4: B=$F_1 \cap F_2$

matrix of F_1 from the upper left corner to the lower right corner separating 1's from 0's provide us with a linear order of vertices of $\mathrm{X} \cup \mathrm{Y}$ and we treat this ordering as the reference ordering L on $\mathrm{X} \cup \mathrm{Y}$. Note that for $x \in \mathrm{X}$ and $y \in \mathrm{Y}$, $xy \in E_1$ if and only if $x > y$ in the ordering L as elemens of $[n]$. Next, the stair partition of the matrix of F_2 provide us with a linear order of its vertices and let π denote the permutation of $\mathrm{X} \cup \mathrm{Y}$ as elements of $[n]$ obtained from this order. It is clear that for $x \in \mathrm{X}$ and $y \in \mathrm{Y}$, $xy \in E_2$ if and only if x occurs before y in the permutation π.

From above it is clear that $xy \in E = E_1 \cap E_2$ if and only if (i) $x > y$ as elements of $[n]$ and (ii) x occurs before y in the permutation π. This means that the permutation π defines the bigraph B. $\square$

The above result helps us to find a permutation of the vertices of a permutation bigraph.

Consider the bigraph B of the following Figure 2.5. This bigraph is the intersection of two Ferrers bigraphs whose biadjacency matrices are given in Figure 2.6. The labeling L of $\mathrm{X} \cup \mathrm{Y}$ obtained from Figure 2.6a is

x	v_2	v_4	u_1	u_4	v_3	u_2	u_3	v_1
L	1	2	3	4	5	6	7	8

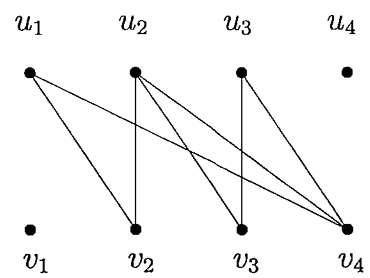


Figure 2.5: A permutation bigraph B

$$\begin{array}{c|cccc|} & v_2 & v_4 & v_3 & v_1 \\ \hline u_1 & 1 & 1 & & \\ u_4 & 1 & 1 & & \\ u_2 & 1 & 1 & 1 & \\ u_3 & 1 & 1 & 1 & \\ \hline \end{array}$$

(a) F_1

$$\begin{array}{c|cccc|} & v_1 & v_2 & v_3 & v_4 \\ \hline u_1 & 1 & 1 & 1 & 1 \\ u_2 & & 1 & 1 & 1 \\ u_3 & & & 1 & 1 \\ u_4 & & & & \\ \hline \end{array}$$

(b) F_2

Figure 2.6: Biadjacency matrices of F_1 and F_2

The permutation π obtained from Figure 2.6b is

$$\begin{aligned} \pi &= (u_1, \underline{v_1}, u_2, \underline{v_2}, u_3, \underline{v_3}, \underline{v_4}, u_4) \\ &= (3, \underline{8}, 6, \underline{1}, 7, \underline{5}, \underline{2}, 4) \end{aligned}$$

Cogis [13] obtained a simple and easily tested criterion for a bipartite graph to be the intersection of two Ferrers bigraphs. He used the language of digraphs, but the characterization describes the corresponding 0, 1-matrix.

Definition 2.2.2. (Cogis) In a 0, 1-matrix, the two zeros in a 2-by-2 permutation submatrix form an *obstruction* or *couple.* For a 0, 1-matrix D, the *associated graph* $H(D)$ has vertex set equal to the positions with 0 in D; two vertices are adjacent in $H(D)$ if and only if they form a couple in D.

Cogis proved that a bipartite graph with matrix D is the intersection of two Ferrers bigraphs if and only if $H(D)$ is bipartite. Since deleting one edge from a complete bipartite graph yields a Ferrers bigraph, every bipartite graph G is the intersection of a family of Ferrers bigraphs. The *Ferrers dimension* of a bipartite graph G (or 0, 1-matrix D) is the minimum number of Ferrers bigraphs (or Ferrers matrices) whose intersection is G (or D). Hence Theorem 2.2.1 states that the permutation bigraphs are the bipartite graphs with Ferrers dimension at most 2.

The equivalence to interval containment bigraphs yields another characterization; the proof relies on a result of Spinrad [85] described in [47].

Theorem 2.2.3. *A graph is an interval containment bigraph if and only if its complement is a two-clique circular-arc graph.*

Proof. In an interval containment representation of B where each assigned interval contains the origin, view the intervals as arcs whose union occupies half of a circle. Complement the arc for each vertex of Y to obtain a new family of arcs. The arcs for X contain the point arising from the origin, and the arcs for Y contain the opposite point on the circle, so the intersection graph G of these arcs is a two-clique circular-arc graph.

We claim that G is the complement of B. The arcs within X are pairwise intersecting, and similarly for Y. For $x \in X$ and $y \in Y$, we have $xy \in B$ if and only if the interval I_y for y contains the interval I_x for x, in which case the arc for y is disjoint from the arc for x. Otherwise, on at least one side of the origin the endpoint of I_y is closer to the origin than that of I_x, and on that side the arc for y intersects the arc for x.

Conversely, Spinrad [85] showed that if X and Y are disjoint cliques covering the vertices of a two-clique circular-arc graph, then there is a circular-arc representation and points a and b such that the arcs for X contain a, the arcs for Y contain b, and no arc contains both. Complementing the arcs for Y in this representation reverses the construction above. □

Combining Theorems 1.4.1, 2.2.1, 2.2.3, we get the following result which gives a comprehensive view of the equivalent forms of bigrphs of Ferrers dimension at most 2.

Theorem 2.2.4. *For an X, Y-bigraph B, the following conditions are equivalent:*

1) B is of Ferrers dimension at most 2;

2) The rows and columns of its biadjacency matrix can be permuted independently so that no 0 has a 1 both to its right and below;

3) The associated graph $H(B)$ of couples of B is bipartite;

4) B is a permutation bigraph;

5) B is an interval containment bigraph;

6) The graph complement of B is a 2-clique circular-arc graph.

We will however provide a direct proof of the equivalence of (1) and (6) again in Chapter 5, when we will look at the result as an extension of the characterization of an interval bigraph in terms of 2-clique circular-arc graph by Hell and Huang [47].

2.3 Examples

Example 2.3.1. *Bipartite permutation graphs form a proper subclass of permutation bigraphs.* Steiner [87] proved that the bipartite permutation graphs are precisely the indifference bigraphs, which by [75] are the intersections of two Ferrers bigraphs whose union is complete. Thus Theorem 2.2.1 implies that every bipartite permutation graph is a permutation bigraph.

The converse does not hold. Lin and West[54] gave a forbidden subgraph characterization for the indifference bigraphs (i.e., the bipartite permutation graphs) within the class of bipartite graphs with Ferrers dimension 2, using the three matrices below. From this and Theorem 2.2.1, a permutation bigraph is a bipartite permutation graph if and only if it has no induced subgraph whose matrix is one of these three. □

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Example 2.3.2. *Every bipartite permutation graph is both a permutation bigraph and a bipermutation bigraph.* Ferrers dimension 2 implies that bipartite permutation graphs are permutation bigraphs; in fact, the same permutation works in both models. Let G be a permutation graph specified by π relative to the identity permutation of $[n]$. If G is bipartite, then π has no decreasing 3-term sublist. Let X consist of all entries that begin an inversion in π, and put the remaining entries in Y. Now G is an X,Y-permutation bigraph, with Y containing the isolated vertices. No inversions occur within X or Y, and for each edge the endpoint in X occurs before the endpoint in Y.

Also, π expresses G as a bipermutation bigraph, because π has no inversions consisting of an element of Y before an element of X. □

Example 2.3.3. *Not all bipermutation bigraphs are permutation bigraphs.* To distinguish these families, consider the 6-cycle C_6. (We use C_n, P_n, K_n for the cycle, path, and complete graph with n vertices, respectively.) Applying the result of Cogis [13] with $H(C_6) = K_3$, the 6-cycle C_6 has Ferrers dimension 3. Hence C_6 is not a permutation bigraph and not a bipartite permutation graph, and C_6 is a forbidden induced subgraph for such graphs.

However, C_6 is a bipermutation bigraph. Consider the partition of [6] with $X = \{\overline{1}, \overline{4}, \overline{6}\}$ and $Y = \{\underline{2}, \underline{3}, \underline{5}\}$. Let $\pi = (\overline{4}, \underline{2}, \overline{6}, \underline{5}, \overline{1}, \underline{3}\}$. In the bipermutation model it does not matter which partite set occurs first in an inversion, so the resulting bipermutation bigraph is a 6-cycle with vertices $\underline{2}, \overline{1}, \underline{5}, \overline{6}, \underline{3}, \overline{4}$ in order.

Even without using Ferrers dimension, the definition implies directly that C_6 is not a permutation graph. Let π be a permutation of [6] expressing C_6 as a permutation graph with partite sets X and Y. Since X and Y are independent, each is in increasing order in π. If 1 and 6 are in one part, say X, then 1 precedes 6 in π, with two vertices of Y before 1 and two vertices of Y after 6; this requires $|Y| \geq 4$. Hence we may assume $6 \in X$ and $1 \in Y$.

Since 1 is the first element of Y and 6 is the last element of X in π, these two vertices have degree 3 if 6 precedes 1; hence π has the pattern $(\overline{X}, \overline{X}, \underline{1}, \overline{6}, \underline{Y}, \underline{Y})$. If $2 \in Y$, then 2 has degree 3; if $2 \in X$, then 2 has degree 1. Hence all cases are eliminated, and C_6 is not a permutation graph. □

Example 2.3.4. *Bipermutation bigraphs are edge-disjoint unions of permu-*

tation bigraphs. Let (L, π) be a representation of a bipermutation bigraph G. The same pair generates a permutation bigraph contained in G, keeping only the edges for pairs having opposite order in L and π such that the element of X comes first in the pair in π. To obtain the other edges in G, exchange L and π and generate another permutation bigraph.

It seems hard to convert this to a matrix characterization of bipermutation bigraphs. □

Example 2.3.5. *The comparability bigraph.* The *comparability digraph* $D(P)$ of a poset P is the orientation of its comparability graph obtained by putting $uv \in D(P)$ if $u \geq v$ in P; note that there is a loop at each vertex. With X and Y being two copies of the elements of P, the *comparability bigraph* $B(P)$ of P is the X,Y-bigraph B such that $xy \in E(B)$ for $x \in X$ and $y \in Y$ if and only if $x \geq y$ in P.

Note that $D(P)$ is obtained from $B(P)$ by orienting edges from X to Y and merging the two copies of each element, and $B(P)$ is obtained from $D(P)$ by splitting each vertex into a vertex of X inheriting the outgoing edges and a vertex of Y inheriting the incoming edges.

If the comparability graph $G(P)$ is a permutation graph, then there is a numbering $1, \ldots, n$ of the elements of P and a permutation π of $[n]$ such that elements are comparable in P if and only if they are equal or form an inversion in π, with the first (larger) element of each inversion above the second in P.

Given such a representation, we express $B(P)$ as a permutation bigraph. Let $X = \{\bar{i} \colon i \in [n]\}$ and $Y = \{\underline{i} \colon i \in [n]\}$, numbered as in the numbering for $G(P)$. Let the reference ordering L be $(\underline{1}, \overline{1}, \ldots, \underline{n}, \overline{n})$. Define π' from π by expanding each entry i in π into the consecutive pair $\overline{i}, \underline{i}$ in π'. Now $\overline{i}$ and $\underline{j}$ are adjacent in the resulting permutation bigraph if and only if $i = j$ or i and j are adjacent in $G(P)$ with $i > j$ in the numbering of P. This makes $\overline{i}$ and $\underline{j}$ adjacent if and only if $i \geq j$ in P.

We conclude that if $G(P)$ is a permutation graph, then $B(P)$ is a permutation bigraph. In Section 2.5 we prove the converse, thereby showing that the comparability graph of a poset is a permutation graph if and only if its comparability bigraph is a permutation bigraph. □

2.4 Interval Bigraphs and Indifference Bigraphs

Let $\mathcal{F}$ be the family of bipartite graphs with Ferrers dimension at most 2. Since interval bigraphs (and indifference bigraphs) lie in $\mathcal{F}$, one can describe these subfamilies by appropriately restricting any characterization of $\mathcal{F}$. For example, Theorem 2.2.3 characterizes $\mathcal{F}$ as the complements of two-clique circular-arc graphs, and Hell and Huang [47] proved that within this family the interval bigraphs are the graphs whose complement has a circular-arc representation with no two arcs covering the entire circle.

Since we have shown that $\mathcal{F}$ is the family of permutation bigraphs, our goal now is to characterize the permutation representations that generate interval bigraphs or indifference bigraphs. From [75], a bipartite graph is an interval bigraph if and only if it is the intersection of two Ferrers bigraphs whose union is complete. Also, the proof of Theorem 2.2.1 converts Ferrers bigraphs whose intersection is B into a permutation representation of B.

We use these two ideas to characterize the permutation representations of interval bigraphs.

We recall from Subsection 2.1.1 that for a permutation π of $[n]$, the *conjugate permutation* π^* is formed by subtracting each element from $n+1$ and then reversing the order of the elements. Given an X,Y-bigraph F, we write $\overline{F}$ for the binary complement of F; the matrices of F and $\overline{F}$ sum to the all-1 matrix. When F is a Ferrers matrix, also $\overline{F}$ is a Ferrers matrix, and the correspondence between their stair orderings suggested by the matrices below is the basis of the characterization.

$$\begin{array}{cccccc} & y_1 & y_2 & y_3 & y_4 & \\ x_1 & 1 & 0 & 0 & 0 & 2 \\ x_2 & 1 & 1 & 1 & 0 & 5 \\ x_3 & 1 & 1 & 1 & 1 & 7 \\ & 1 & 3 & 4 & 6 & \end{array} \qquad \begin{array}{cccccc} & y_4 & y_3 & y_2 & y_1 & \\ x_3 & 0 & 0 & 0 & 0 & 1 \\ x_2 & 1 & 0 & 0 & 0 & 3 \\ x_1 & 1 & 1 & 1 & 0 & 6 \\ & 2 & 4 & 5 & 7 & \end{array}$$

$$L = (\underline{1}, 2, \underline{3}, \underline{4}, 5, \underline{6}, 7) \qquad L^* = (1, \underline{2}, 3, \underline{4}, \underline{5}, 6, \underline{7})$$

$$L = (y_1, x_1, y_2, y_3, x_2, y_4, x_3) \qquad L^* = (x_3, y_4, x_2, y_3, y_2, x_1, y_1)$$

Theorem 2.4.1. *An n-vertex permutation bigraph is an interval bigraph if and only if it has a permutation representation (L, π) such that the permutation bigraph represented by the conjugate permutations (L^*, π^*) has no edges.*

Proof. Let B be a permutation bigraph and the intersection of Ferrers bigraphs F_1 and F_2 with stair orderings f_1 and f_2. As in Theorem 2.2.1, form a permutation representation of B by letting the reference ordering L put $X \cup Y$ in the same order as f_1 and π put $X \cup Y$ in the reverse order to f_2. As noted in Theorem 2.2.1, reversing this construction yields two Ferrers bigraphs whose intersection is B from any permutation representation (L, π)

of B.

When F is a Ferrers bigraph with stair-ordering f, the complementary bigraph $\overline{F}$ has x and y adjacent when $f(x) < f(y)$. To permute the matrix so that the 1s for $\overline{F}$ form a Ferrers diagram in the lower left, we reverse the rows and reverse the columns. In order to read the stair in the new matrix from upper left to lower right, we must also reverse the numbering of the elements. This is clearest when we write the stair ordering using vertex names instead of just $[n]$; the stair ordering for $\overline{F}$ is then the reverse of the stair ordering for F. When expressed using a partition of the identity permutation on $[n]$, reversal amounts to subtracting the numbers from $n+1$. Therefore, when (L, π) generates the permutation bigraph $F_1 \cap F_2$ on $[n]$, we conclude that (L^*, π^*) generates the permutation bigraph $\overline{F}_1 \cap \overline{F}_2$.

Now, recall that B is an interval bigraph if and only if F_1 and F_2 can be chosen so that $F_1 \cup F_2$ is complete, which is equivalent to $\overline{F}_1 \cap \overline{F}_2$ having no edges. The equivalent condition for the permutation representation (L, π) of B generated by F_1 and F_2 is that the permutation bigraph with the conjugate representation (L^*, π^*) has no edges. □

The matrices of indifference bigraphs were characterized (in the language of digraphs) in [77]. A *monotone consecutive arrangement* of a 0, 1-matrix consists of independent permutations of the rows and the columns and a labeling of each 0-entry as R or C such that every entry above or rightward of an R is R and every entry below or leftward of a C is C. Thus the 1-entries are consecutive in each row and in each column, and the ends of these intervals of 1-entries behave monotonically across the columns or down the rows.

Theorem 2.4.2. [77] *A bipartite graph is an indifference bigraph if and only if its matrix has a monotone consecutive arrangement.*

A monotone consecutive arrangement expresses the bipartite complement as the union of disjoint Ferrers bigraphs, so such a graph is an interval bigraph. To characterize indifference bigraphs among permutation bigraphs, we translate the conditions for a monotone consecutive arrangement into conditions on the resulty permutation representation.

Theorem 2.4.3. *A permutation bigraph B is an indifference bigraph if and only if it has a permutation representation (L, π) such that*
(1) the permutation bigraph generated by (L^*, π^*) has no edges, and
(2) each partite set appears in increasing order in π.

Proof. Let B be an X, Y-bigraph that is a permutation bigraph. By Theorem 2.2.1, B is the intersection of Ferrers bigraphs F_1 and F_2; let f_1 and f_2 be the corresponding stair orderings. As in Theorem 2.4.1, there is a permutation representation (L, π) of B in which L puts $X \cup Y$ in the same order as f_1 and π puts $X \cup Y$ in the reverse order to f_2, and (L^*, π^*) generates no edges if and only if $F_1 \cup F_2$ is complete.

If the matrix of B has a monotone consecutive arrangement, then the row and column permutations of the matrices of F_1 and F_2 exhibiting Ferrers diagrams in the lower left are reversals of each other. That is, in the stair ordering f_2, both X and Y are ordered in reverse of their ordering in the stair ordering f_1. Since f_2 is reversed to form π, the numbers assigned by L to X appear in increasing order in π, and similarly for Y.

Conversely, if a permutation representation (L, π) satisfies this increasing subsequence condition for X and Y, transforming back to Ferrers bigraphs via the stair orderings yields two Ferrers bigraphs whose row orderings (and column orderings) are reverses. If (L^*, π^*) generates no edges, then the union of these two Ferrers bigraphs is complete, and hence the orderings yield a monotone consecutive arrangement for the matrix of B. □

As mentioned earlier, Steiner [87] proved that an X, Y-bigraph is an indifference bigraph if and only if it is a bipartite permutation graph. Theorem 2.4.3 allows us to express bipartite permutation graphs as a special case of permutation bigraphs.

Corollary 2.4.1. *If B is an indifference bigraph with permutation representation (L, π) from a monotone consecutive arrangement, then (L, π) also represents B as a permutation graph.*

Proof. By Theorem 2.4.3, each partite set (X or Y) occurs in the same order in L and π. Therefore, as a permutation graph with reference order L, no edges are generated within X or within Y by π. Also, since the union of the corresponding Ferrers bigraphs F_1 and F_2 is complete, there is no $x \in X$ and $y \in Y$ with y after x in L and x after y in π (that would put (x, y) above the stair in both F_1 and F_2). Hence the only inversions in π relative to L are those that generate edges in B, and the permutation graph arising from (L, π) is B itself. □

2.5 Comparability graphs and Permutation bigraphs

As mentioned in Example 2.3.5, we prove here that if the comparability bigraph of a poset P is a permutation bigraph, then the comparability graph of P is a permutation graph. With Example 2.3.5, this completes the proof that these conditions are equivalent.

We use digraphs and posets associated with families of intervals. An *interval containment poset* is a poset P representable by giving each element an interval in $\mathbb{R}$ so that $x \leq y$ in P if and only the interval for y contains the interval for x. In Section 2.2 we defined an *interval containment bigraph* to be an X, Y-bigraph representable by assigning each vertex an interval in $\mathbb{R}$ so that vertices $x \in X$ and $y \in Y$ are adjacent if and only if the interval for y contains the interval for x. The natural digraph analogue is an *interval containment digraph*, defined to be a digraph D representable by assigning intervals S_w and T_w to each vertex w so that $uv \in E(D)$ if and only if $S_u \subseteq T_v$.

The relationship between interval containment digraphs and interval containment bigraphs is like that between the comparability digraph and comparability bigraph of a poset: splitting the vertices of an interval containment digraph D into copies in X and Y yields an interval containment bigraph, represented by assigning to the copies of w in X and Y the intervals S_w and T_w in an interval containment representation of D.

Given an interval containment representation of a poset P, with interval I_w assigned to element w, letting $S_w = T_w = I_w$ expresses the comparability digraph as an interval containment digraph; we are given $x \leq y$ in P if and only if $I_x \subseteq I_y$, which is now equivalent to $S_x \subseteq T_y$. This proves necessity in Theorem 2.5.2. Sufficiency is not immediate, because there is no immediate way to reverse the transformation when $S_w \neq T_w$.

In [78], sufficiency was proved indirectly using the result of Bouchet [4] that the order dimension of a poset equals the Ferrers dimension of its comparability digraph. Our argument below uses only the result of Cogis, which we state first. We use the term *bipartition* for the vertex partition given by a proper 2-coloring of a bipartite graph. For a digraph D, recall the notion of the *associated graph* $H(D)$ from Definition 2.2.2. We give a more complete statement of the theorem of Cogis.

Theorem 2.5.1. (**Cogis** [13]) *Let $H(D)$ be the associated graph of a digraph D, and let H' be the subgraph of $H(D)$ consisting of the nontrivial components of $H(D)$. If $H(D)$ is bipartite, with $\mathbf{I}$ denoting its set of isolated vertices, then there is a bipartition $\{\mathbf{R}, \mathbf{C}\}$ of H' such that the positions corresponding to $\mathbf{R} \cup \mathbf{I}$ form a Ferrers digraph F_1, the positions corresponding to $\mathbf{C} \cup \mathbf{I}$ form a Ferrers digraph F_2, and $\overline{D} = F_1 \cup F_2$.*

Theorem 2.5.2. [78] *A poset is an interval containment poset if and only if its comparability digraph is an interval containment digraph.*

Proof. We have noted necessity of the condition. For sufficiency, let P be a poset whose comparability digraph D is an interval containment digraph; we prove that P is an interval containment poset. As noted in Theorem 2.2.1, we may consider an interval containment representation of D in which all intervals contain the origin. The left endpoints and right endpoints both give stair orderings to show that D has Ferrers dimension at most 2, as in the equivalence of (b) and (c) in Theorem 2.2.1.

Since D has Ferrers dimension at most 2, and complements of Ferrers digraphs are Ferrers digraphs, the 0-positions in D can be expressed as the union of two Ferrers digraphs. Since the isolated vertices of $H(D)$ form no couples, they can be included in both Ferrers digraphs. Put the nonisolated vertices into $\mathbf{R}$ or $\mathbf{C}$ when they lie in the first or second Ferrers digraph, respectively. Now $H(D)$ is bipartite, $\{\mathbf{R}, \mathbf{C}\}$ is a bipartition of the subgraph H' of nonisolated vertices, and the positions in $\mathbf{R} \cup \mathbf{I}$ and $\mathbf{C} \cup \mathbf{I}$ form Ferrers digraphs. This is the trivial direction of Cogis' result. We study the resulting coloring in more detail.

(i) If x and y are incomparable in P, then positions (x, y) and (y, x) in the matrix of D contain 0. Since D is reflexive, these positions form a couple. Hence they have distinct colors in any bicoloration of $H(D)$.

(ii) If $x < y$ in P, then $xy \in E(D)$. Hence position (x, y) is 1 and position (y, x) is 0 in the matrix. If position (y, x) forms a couple with (u, v), then positions (u, x) and (y, v) are 1. Now $u \leq x < y \leq v$, so transitivity of P requires $u < v$, and position (u, v) is 1, a contradiction. We conclude that (y, x) is an isolated vertex in $H(D)$.

Let $\mathbf{E}$ be the set of positions containing 1 in the matrix of D. We have shown that $\mathbf{E}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{I}$ partition the positions. Position (x, x) lies in E. For $x \neq y$, we have

$$(x, y) \in \mathbf{E} \Leftrightarrow (y, x) \in \mathbf{I} \text{ and } (x, y) \in \mathbf{R} \Leftrightarrow (y, x) \in \mathbf{C}. \qquad (2.1)$$

We next obtain an ordering of P such that when the rows and the columns of the matrix of D are simultaneously given this ordering, $\mathbf{C} \cup \mathbf{I}$ occupies the lower triangle of positions below the diagonal, while $\mathbf{R} \cup \mathbf{E}$ occupies the diagonal and the positions above it.

Since $\mathbf{C} \cup \mathbf{I}$ is a Ferrers digraph, there exist orderings of the rows and columns of D so that $\mathbf{C} \cup \mathbf{I}$ occupies the positions of a Ferrers diagram in the lower left. The complement $\mathbf{R} \cup \mathbf{E}$ then occupies a Ferrers diagram in the upper right.

The positions corresponding to the loops xx now occupy the positions of the 1s in a permutation matrix. Let σ be the corresponding permutation, with the loop in row i occurring in column $\sigma(i)$. Since $\mathbf{R} \cup \mathbf{E}$ is a Ferrers diagram in the upper right, every position (r, s) such that $r \leq i$ and $s \geq \sigma(i)$ lies in $\mathbf{R} \cup \mathbf{E}$. These positions are those *generated by* σ. Since they lie in $\mathbf{R} \cup \mathbf{E}$, there are at most $\binom{n}{2} + n$ of them. We show that only the identity permutation generates this few positions.

If σ is not the identity permutation, then there is some greatest i such that $\sigma(i) < i$. Let $j = \sigma(i)$ and $k = \sigma^{-1}(i)$. Let τ agree with σ except for $\tau(i) = i$ and $\tau(k) = j$; that is, j and i are switched when the word form of σ is modified to obtain the word form of τ. Every position generated by τ is also generated by σ, but position (i, j) is generated by σ and not by τ. Repeating this argument shows that every nonidentity permutation generates more positions than the identity permutation.

Since $\mathbf{R} \cup \mathbf{E}$ contains all positions generated by its loops, and (2.1) implies $|\mathbf{R} \cup \mathbf{E}| = \binom{n}{2} + n$, we conclude that the loops must appear along the diagonal. The resulting common ordering of the rows and columns is a numbering f

of the elements P by 1 through n such that $x < y$ in P implies $f(x) < f(y)$.

From this ordering, we obtain an interval containment representation of P. Let the right endpoint of the interval I_x for x be $f(x)$. For the left endpoints, we define g mapping P into $\{-1, \ldots, -n\}$. At step i, among the current minimal elements, let the one with the rightmost right endpoint be assigned $-i$ as its left endpoint. Delete this element and continue.

If $x < y$ in P, then because f is a linear extension we have $f(x) < f(y)$. Also, in the assignment of left endpoints, y cannot receive a (negative) left endpoint before x; hence $g(y) < g(x)$, and $I(x) \subseteq I(y)$.

If the representation fails, then there exist x and y incomparable in P with $I_x \subseteq I_y$. Let x be the element with smaller interval in such a pair, and let y be a minimal element among those incomparable to x whose intervals contain I_x. Since $f(x) < f(y)$, the construction procedure requires that y is not a minimal remaining element when $g(x)$ is assigned. Hence y is then above some currently minimal element z. Since x is chosen now in preference to z, we have $f(z) < f(x)$. We have $f(x) < f(z) < f(y)$, but x is incomparable to y and z.

With the given common ordering of rows and column, in which $\mathbf{C} \cup \mathbf{I}$ consists of the positions below the diagonal, $\mathbf{R} \cup \mathbf{E}$ consists of those on the diagonal and above, and the diagonal corresponds to the loops, we have obtained the submatrix below, which contradicts that $\mathbf{R} \cup \mathbf{I}$ is a Ferrers matrix. Hence we have successfully constructed a representation, and P is an interval containment poset. $\square$

$$\begin{array}{cccc} & z & x & y \\ z & 1 & R & 1 \\ x & & 1 & R \\ y & & & 1 \end{array}$$

Finally, we remark that relative to posets there is a correspondence betwee permutation graphs and permutation bigraphs. That is, the comparability graph of a poset is a permutation graph if and only if its comparability bigraph is a permutation bigraph.

Corollary 2.5.1. *For a poset P, the following conditions are equivalent.*

(a) The comparability graph of P is a permutation graph.

(b) The comparability graph of P is an interval containment graph.

(c) P is an interval containment poset.

(d) The comparability digraph of P is an interval containment digraph.

(e) The comparability bigraph of P is an interval containment bigraph.

(f) The comparability bigraph of P is a permutation bigraph.

Proof. Equivalence of (a),(b),(c) follows from Theorem 2.1.1, as elaborated in Example 2.1.3. Equivalence of (c) and (d) is Theorem 2.5.2. Equivalence of (d) and (e) is Example 2.3.5. Equivalence of (e) and (f) is Theorem 2.2.1.□

Chapter 3

Further Characterizations for Interval Tournaments[1]

A *tournament* is a complete oriented graph, that is, a digraph with no loops such that $u \to v$ iff $v \not\to u$. A tournament with n vertices is an *n-tournament.* A tournament that is an interval digraph is an *interval tournament* and an interval tournament with n-vertices is an *interval n-tournament.* Interval tournaments were introduced and characterized by Brown et. al [7]. They have characterized them by a complete list of forbidden subtournaments given in Figure 3.1. They have also proved that an n-tournament is an interval tournament if and only if it has a transitive $(n-1)$-subtournament.

A.H.Busch [10] has shown that the vertices of every loopless interval digraph can be partitioned into two acyclic digraphs from which he has shown that every interval tournament can be partitioned into two transitive tournaments. He then uses this result to provide a short proof of the result that every interval n-tournament has a transitive $(n-1)$-tournament.

As an extension of the study of interval tournaments, Drust, Das Gupta and

[1]A part of this chapter has appeared in *J. Indian Math. Soc.* (2011)**78**(1–4)15–26.

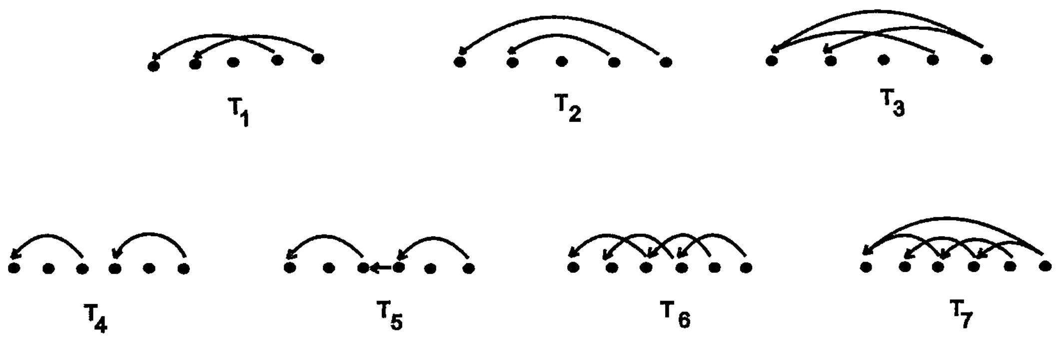


Figure 3.1: Complete list of forbidden tournaments for interval tournament. (The arcs which are not shown are from left to right)

Lundgren [24] started with an oriented digraph D which is not a tournament and explored the restrictions to be placed on D to gurantee that it is an interval digraph. In particular, they investigated a broader class of oriented graphs on n-vertices that contain a transitive $(n-2)$-tournament as a subdigraph.

It is well known [63] that a cycle of length 6 or more is not an interval bigraph. This example will be used very often in the present chapter.

The present chapter is actually an extended study of the paper by Brown et al. [7]. We first apply the zero partition of an interval digraph to obtain an alternative proof of the main theorem 3 of [7]. The importance of this alternative approach is that during the process we have obtained several additional characterizations of an interval tournament. We prove that a tournament is an interval tournament if and only if all its cycles have a common vertex. Then we show that the three digraphs D_1, D_2, D_3 in Figure 3.2 are the only three forbidden subdigraphs of an interval tournament. Next we show that a tournament T is an interval tournament if and only if it is of Ferrers dimension at most 2 and does not contain the tournament T_4 in Figure 3.1 as a subtournament.

Hell and Huang [47] proved that complements of interval bigraphs are precisely those two-clique circular-arc graphs which have representations such that no two arcs cover the whole circle. Brown et al [7] in their paper posed the problem of characterizing an interval tournament following this model. This problem is addressed in the last condition (9) of the main theorem 3.2.1 of this paper.

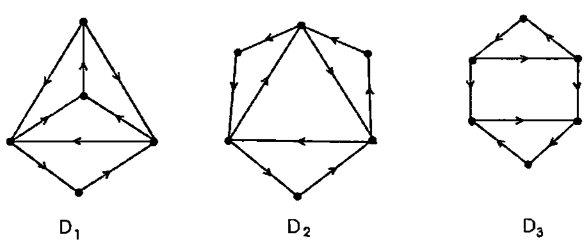


Figure 3.2: Complete list of forbidden subdigraphs for interval tournament

3.1 Preliminaries

We start with an important observation that there is only one 4-tournament which has two 3-cycles. This is given in Figure 3.3.

We first prove the following proposition which will be required to prove our main theorem.

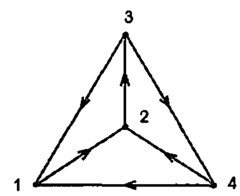


Figure 3.3: The only 4-tournament having two 3-cycles; which are (2, 3, 1) and (2, 3, 4)

Proposition 3.1.1. *If an interval* 5*-tournament has two or more* 3*-cycles, then they must have at least one vertex in common.*

Proof. The proposition is obviously true for any interval 5-tournament having exactly two 3-cycles. So we consider an interval 5-tournament T having more than two 3-cycles and suppose, if possible, the 3-cycles do not have a vertex in common. Clearly T has a 4-subtournament having two cycles (Figure 3.3) and T is obtained from this subtournament by adjoining a fifth vertex 5 to it.

In the tournament T, two possibilities arise:

I. The vertices 1, 5 and 4 form a cycle (1,5,4).

II. 1, 5 and 4 do not form a cycle.

Case I. There are four subcases regarding the edges $(2, 5)$ and $(3, 5)$.

(a) $5 \to 2, \quad 5 \to 3$

(b) $2 \to 5, \quad 3 \to 5$

(c) $5 \to 2, \quad 3 \to 5$

(d) $2 \to 5, \quad 5 \to 3$

It is now verified that in the subcases (a), (b) and (c) the bipartite graph $B(T)$ corresponding to T contains a 6-cycle. While in subcase (d),

$B(T)$ is itself a 10-cycle. From [63], it follows that none of these tournaments are interval tournaments. The bigraphs are given in Figure 3.4.

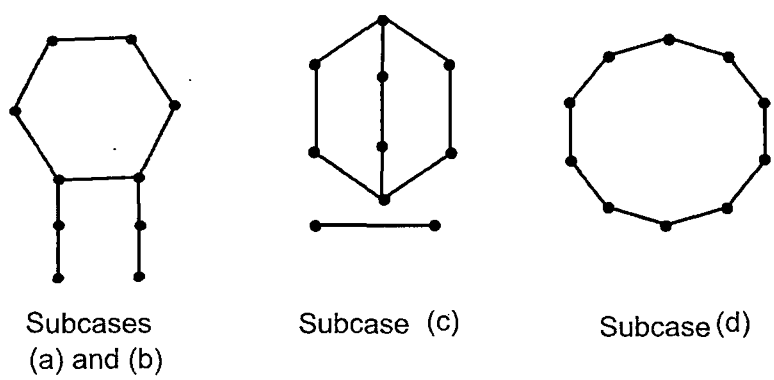


Figure 3.4: The bigraphs of 5-tournament having no common vertex of its 3-cycles

Case II. When 1, 5 and 4 do not form a cycle. In order that all 3-cycles do not have a vertex in common, either of the two possibilities must occur :

(a) there are two 3-cycles $(1, 2, 5)$ and $(3, 4, 5)$

(b) there are two 3-cycles $(3, 1, 5)$ and $(4, 2, 5)$

In each of the cases, we verify that T is isomorphic to the tournament in Case I(a). □

We now focus our attention to the digraph D_1 in Figure 3.2.

The 3-cycles of digraph D_1 have no common vertex. So we have:

Corollary 3.1.1. *No interval tournament contains a digraph isomorphic to the digraph D_1 in Figure 3.2.*

During the proof, we observe that in every case when all the 3-cycles in a 5-tournament T do not have a common vertex, it contains a digraph isomorphic to the digraph D_1 in Figure 3.2. It is also interesting to note that the three bipartite graphs in Figure 3.4 correspond exactly to the three tournaments T_1, T_2, T_3 in Figure 3.1(See [7]).

From these, we get the following :

Corollary 3.1.2. *Let T be a 5-tournament. The following conditions are equivalent:*

1. *all the 3-cycles of T have no common vertex.*
2. *T is isomorphic to either of T_1, T_2, T_3.*
3. *T contains a subdigraph isomorphic to the digraph D_1 in Figure 3.2.*

3.2 Characterization

First we state the main theorem of this paper.

Theorem 3.2.1. *Let T be an n-tournament. Then the following conditions are equivalent:*

1) T is an interval tournament;

2) all the 3-cycles of T have a common vertex;

3) all cycles of T have a common vertex;

4) T has a transitive $(n-1)$-subtournament;

5) T has no vertex disjoint 3-cycles or the digraph D_1 in Figure 3.2;

6) T has no subdigraph isomorphic to any of D_1, D_2 or D_3 in Figure 3.2;

7) T has no subtournament T_i $(1 \leq i \leq 7)$ in Figure 3.1;

8) T is of Ferrers dimension at most 2 and T_4-free;

9) the graph complement $\overline{B(T)}$ of $B(T)$ is a two-clique circular-arc graph and $\overline{B(T_4)}$-free.

Before going into the proof of the theorem, the condition (6) of the theorem needs a little explanation. First note that all the three digraphs are interval digraphs. The equivalence of (6) and (7) means that in what ever way we add the missing arcs to these digraphs to form a tournament, they will ultimately lead to one of the forbidden tournaments T_i $(1 \leq i \leq 7)$. So, we can say that the 3 digraphs D_1, D_2, D_3 form a complete list of forbidden subdigraphs.

In order to prove the theorem we need the following lemma.

Lemma 3.2.1. *No interval tournament contains two vertex-disjoint 3-cycles.*

Proof We first observe that any 3-cycle (x_1, x_2, x_3) is an interval tournament having a zero-partition

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_3 & 1 & R & R \\ x_1 & C & 1 & R \\ x_2 & C & C & 1 \end{array} \qquad \ldots (1)$$

We note that this partition is not unique and two different partitions have different labellings of R's and C's; but with a given labelling of R's and C's, the zero-partition for a 3-cycle is unique.

Let, if possible $C_1 = (x_1, x_2, x_3)$ and $C_2 = (y_1, y_2, y_3)$ be two vertex-disjoint 3-cycles of an interval 6-tournament T. Then the adjacency matrix $A(T)$ of T displaying a zero-partition have two complementary submatrices of the form (1) corresponding to the two cycles. We will, in fact prove the proposition by showing that no permutation of rows and columns in $A(T)$ can display a zero-partition.

Below we list some submatrices of $A(T)$ which forbid its zero-partition.

(a) Consider a submatrix

$$\begin{array}{c|cccc} & x_i & - & - & y_j \\ \hline x_i & C & 1 & R & * \\ y_j & * & - & - & - \end{array} \qquad \ldots (F_1)$$

Since T is a tournament, either $x_i y_j$ or $y_j x_i$ must be 1. But this prevents the matrix from a zero-partition when the rows are in the given order.

(b) $$\begin{array}{c|ccc} & - & y_j & x_i \\ \hline x_i & R & * & R \\ y_j & - & R & * \end{array} \qquad \ldots (F_{2a}) \quad \text{and} \quad \begin{array}{c|ccc} & - & y_j & x_i \\ \hline y_j & - & R & * \\ x_i & R & * & R \end{array} \qquad \ldots (F_{2b})$$

(c) $$\begin{array}{c|cc} & - & - \\ \hline - & - & C \\ - & R & * \end{array} \qquad \ldots (F_3)$$

For the submatrix F_3 we note that $*$ lies below C and so it must be C and since it lies to the right of R, it must be R. This goes against a zero-partition.

We note that there are other forbidden submatrices of a zero-partitioned matrix, but we will need only these three submatrices to complete the proof.

If $A(T)$ has a zero-partition, we observe that any submatrix of $A(T)$ has also a zero-partition with the same labelling of R's and C's as in $A(T)$. So we have to consider only those permutations where the orders of rows and columns of the respective submatrices of C_1 and C_2 remain unchanged. Without loss of generality, we suppose C_1 and C_2 have the zero-partition given by

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_3 & 1 & R & R \\ x_1 & C & 1 & R \\ x_2 & C & C & 1 \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline y_3 & 1 & R & R \\ y_1 & C & 1 & R \\ y_2 & C & C & 1 \end{array}$$

We give two examples below, regarding the problem of sorting out the permutations which are to be considered for our proof.
The permutation of the columns $(x_1, x_2, y_1, x_3, y_2, y_3)$ is to be considered but not the one like $(x_1, x_2, y_2, x_3, y_1, y_3)$, because in that case, the labels of R's and C's in the submatrix of C_2 get changed.

We classify all the short listed permutations according to the vertices of the last two columns of the matrix. In any permutation of the columns, we will examine, the vertices of the last two columns are either of the following:

(I)	y_2, y_3 ;	(II)	y_3, x_3;
(III)	x_2, x_3 ;	(IV)	x_3, y_3.

Amongst them, cases (III) and (IV) are the result of interchange of labels of the vertices of C_1 and C_2 and so we need consider the two cases (I) and

(II) only.

Case (I) Consider the following biadjacency matrix

$$\begin{array}{c|cccc|cc}
 & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ \hline
x_3 & 1 & R & R & & & \\
x_1 & C & 1 & R & - & * & \\
x_2 & C & C & 1 & & & \\
y_3 & - & & & 1 & R & R \\
y_1 & - & & & C & 1 & R \\
y_2 & * & & & C & C & 1 \\
 & & & & & &
\end{array}$$

What ever be the permutation of the columns of the left sector, the column x_3 lies to the left of the column y_2 and until and unless the row y_2 lies above the row x_1, we get a submatrix F_1 in the matrix. So consider the submatrix when the row y_2 goes above the row x_1 in a permutation of its rows and then we have the submatrix

$$\begin{array}{c|cccc|cc}
 & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ \hline
y_2 & - & - & - & C & C & 1 \\
x_1 & C & 1 & R & - & * & \\
 & & & & & &
\end{array}$$

which is F_3, irrespective of any permissible permutation of the columns in the left sector.

Case(II) First we consider the matrix

$$\begin{array}{c|cccc|cc}
 & x_1 & x_2 & y_1 & y_2 & y_3 & x_3 \\ \hline
x_3 & 1 & R & - & - & * & R \\
x_1 & C & 1 & - & - & - & R \\
x_2 & C & C & - & - & - & 1 \\
y_3 & & & 1 & R & R & * \\
y_1 & & & C & 1 & R & - \\
y_2 & & & C & C & 1 & - \\
 & & & & & &
\end{array}$$

Here we have the forbidden submatrix F_{2a}. This is preserved for any permissible permutation of its columns in the left sector so long as the row y_3 lies below the row x_3. In the case when the row y_3 lies above the row x_3 then we get the submatrix F_{2b}. □

To prove the theorem we need the following lemma.

Lemma 3.2.2. *No interval* 6*-tournament* T *contains the digraph* D_0 *or* D_2 *in Figure 3.5 as a subdigraph.*

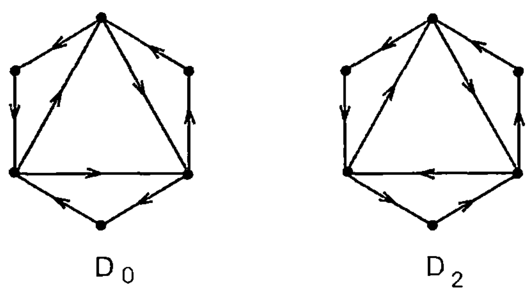


Figure 3.5: The digraphs D_0 and D_2

Proof. We first observe that the two digraphs D_0 and D_2 have the same underlying undirected graph and have orientations such that the three outer triangles are directed cycles in any order, clockwise or anticlockwise.

We first suppose that an interval 6-tournament contains the digraph D_0 in Figure 3.5 as its subdigraph.

In the tournament either $q \to b$ or $b \to q$. But none is possible, because if $q \to b$, then $T - \{x\}$ is the digraph D_1 in Figure 3.2 and if $b \to q$ then

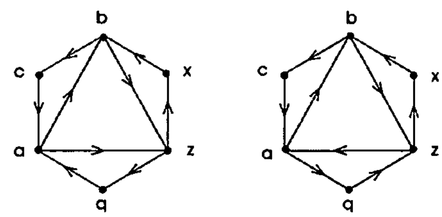


Figure 6(a) Figure 6(b)

Figure 3.6: The digraphs D_0 and D_2 with labelled vertices

so is $T - \{c\}$. Next suppose that an interval 6-tournament contains D_2 in Figure 3.5 as its subgraph.

We must have $c \to q$, because otherwise (a, q, c) and (x, b, z) are two vertex disjoint three cycles. Similarly $q \to x$ and $x \to c$. But then the cycles (c, q, x), (a, b, z) are two disjoint 3-cycles. □

Proof of Theorem 3.1. (1) $\Rightarrow$ (2). If the tournament is transitive then there is nothing to prove. So we suppose it is not transitive. Let the result be true for any interval n-tournament $(n \geq 5)$. Let $T(V, E)$ be an interval $(n+1)$-tournament and let $x \in V$. Let $T^{'} = T - \{x\}$. Then by hypothesis all the 3-cycles pass through a common vertex, say a. Let $T^{''} = T - \{a\}$. Clearly all 3-cycles of T'' pass through x. Here we note that all the 3-cycles of T contains either x or a or both. Let (a, b, c) and (x, y, z) be two 3-cycles of $T^{'}$ and $T^{''}$ respectively, $x \neq a, b, c$. Let $A_1 = \{(a, b, c), (x, y, z)\}$

These two 3-cycles can not be vertex disjoint in T by Lemma 3.2.1. Then either of y and z must be b or c. We prove the theorem for $y = b$. (All other cases follow similarly)

Then $A_1 = \{(a, b, c), (x, b, z)\}$

Now append to A_1 another (a, p, q) from $T^{'}$ and let

$A_2 = \{(a,b,c),(x,b,z),(a,p,q)\}$, $x \neq a,b,c,p,q$. Clearly $q \neq b$ and if $p = b$ then b is the common vertex in the 3-cycles of A_2. So we consider $p \neq b$. In this case we must have $p = z$ or $q = z$. If $p = z$, then $A_2 = \{(a,b,c),(x,b,z),(a,z,q)\}$ and the digraph comprising of the 3-cycles of A_2 is the digraph D_0 in Figure 3.5 and hence the tournament can not be an interval tournament. Similarly, if $q = z$ then the digraph comprising the 3-cycles of A_2 is the digraph D_2 in Figure 3.5 and hence T cannot be an interval tournament. So there must be a comon vertex in the 3-cycles of A_2. So $A_2 = \{(a,b,c),(a,b,q),(x,b,z)\}$. Now we append another 3-cycle (x,u,v) of T'' and let

$$A_3 = \{(a,b,c),(a,b,q),(x,b,z),(x,u,v)\}, x \neq a,b,c,q; v \neq b.$$

If $u = b$ then b is the common vertex. So we suppose $u \neq b$. In order to prevent the occurrence of two disjoint 3-cycles in T we must have $(u,v) = (c,q)$ or (q,c). In any case, we verify that the digraph $T - \{z\}$ contains the digraph D_1.

Now we are left with the cycles of the form (a,x,w) or (x,a,w) in T and if we append a cycle of this form to A_3 we verify that this cycle along with the cycles $(a,b,c),(x,b,z)$ gives us the digraph D_0 in Figure 3.5. □

(2) $\Rightarrow$ (3). If possible let an interval tournament T have a k-cycle ($k > 3$) which does not pass through the common vertex, say v, of all the 3-cycles of the tournament. Consider the induced subtournament of this k-cycle. This subtournament will have a 3-cycle which does not pass through v, a contradiction. □

(3) $\Rightarrow$ (4). Let $v \in V(T)$ be common to all the cycles of T. Then $T - \{v\}$ is

acyclic and so is transitive. □

(4) $\Rightarrow$ (5). Brown et al. [7] has proved that every n-tournament T contains one of the following structures:

(1) Two disjoint 3-cycles;

(2) A subtournament isomorphic to T_1, T_2 or T_3 in Figure 3.1;

(3) A transitive $(n-1)$-subtournament.

From the above and Corollary 3.1.2 the result follows. □

(5) $\Rightarrow$ (6). Since D_3 has two vertex disjoint 3-cycles, T can not contain D_3 as its subdigraph. Also Lemma 3.2.2 shows that T can not contain D_2 as its subdigraph. □

(6) $\Rightarrow$ (7). From Corollary 3.1.2, it follows that the digraph D_1 is a subdigraph of each of T_1, T_2 and T_3. We prove the result by showing T_6 and T_7 contain D_2 as its subdigraph while T_4 and T_5 contain D_3 as a subdigraph.

With the labellings of the tournaments T_6 and T_7 as in Figure 3.1, we see that the union of the 3-cycles (v_3, v_1, v_5), (v_3, v_4, v_2) and (v_5, v_6, v_4) yields the digraph D_2 as a subdigraph of T_6 while the 3-cycles (v_1, v_2, v_3), (v_1, v_5, v_6) and (v_3, v_4, v_5) give us the same digraph D_2 as a subdigraph of T_7. Next we easily verify that the union of the two disjoint 3-cycles (v_1, v_2, v_3) and (v_4, v_5, v_6) along with the edges (v_1, v_5) and (v_2, v_6) constitute the digraph D_3 as a subdigraph of both T_4 and T_5. □

(7) $\Rightarrow$ (1). It has been proved in [7]. We rely heavily on the following result from lemma 5 of [7] and provide an easier proof for the sake of completeness.

Lemma 3.2.3. [7] Let $X = \{x_0, x_1, x_2\}$ be the vertices of a 3-cycle in a 6-tournament that does not contain a copy of D_1 (and so T_1, T_2, T_3). Let $u \to v$

be an arc of T not incident to X. Then $|N^+(v) \cap X| \geq 2 \Rightarrow |N^+(u) \cap X| \geq 2$ where $N^+(v)$ is the successor set of v.

Let $Y = \{y_0, y_1, y_2\}$ be another 3-cycle of T disjoint from X. Then from the lemma and $y_0 \to y_1 \to y_2 \to y_0$ implies that $2 \leq |N^+(y_i) \cap X| \leq 3$ $(i = 0, 1, 2)$.[2] Let $p_i = |N^+(y_i) \cap X|$ $(i = 0, 1, 2)$ and since $2 \leq p_i \leq 3$, there are four distinct possibilities for the sequence (p_0, p_1, p_2):

(a) $(3, 3, 3)$ (b) $(2, 3, 3)$

(c) $(2, 2, 3)$ (d) $(2, 2, 2)$

Because $\{y_0, y_1, y_2\}$ is a cycle, any permutation of (p_0, p_1, p_2) will yield the same digraph.

It is now a matter of easy verification that the four cases give us exactly the four tournaments T_4, T_5, T_6, T_7 respectively. □

(7) $\Leftrightarrow$ (8). The equivalence of (1) and (7) means that T_i $(1 \leq i \leq 7)$ are the only forbidden tournaments of an interval tournament. All the bipartite graphs $B(T_i)$ associated with T_i, except $B(T_2)$ and $B(T_4)$, have an *asteroidal triple of edges* (see Figure 3.1, Figure 3.3 and Figure 3.4 of [7]). Also $B(T_2)$ is not *bichordal*[3]. It is proved in [19] that a bigraph of Ferrers dimension ≤ 2 is bichordal and ATE free (see also [63]). So it follows that any bigraph which is not bichordal nor containing an ATE is of Ferrers dimension > 2. So all the digraphs T_i $(i \neq 4)$ are of Ferrers dimension > 2. T_4 is the only digraph in this set which is of Ferrers dimension 2. Since an interval tournament is

[2] In case $|N^+(y_i) \cap X| < 2$ $(i = 0, 1, 2)$, interchanging X and Y will serve our purpose

[3] A bigraph is called chordal bipartite or simply bichordal, if every cycle of length ≥ 6 has a chord.

necessarily of Ferrers dimension at most 2, the result follows. □

(8) $\Leftrightarrow$ (9). In [47] it was proved that a bigraph is an interval bigraph if and only if its graph complement is a two-clique circular-arc graph such that no two arcs cover the whole circle. Extending this result it is proved in 2 that bigraphs of Ferrers dimension at most 2 are precisely the complements of two-clique circular-arc graphs. (8) $\Leftrightarrow$ (9) follows as an immediate consequence of this result. □

Chapter 4

Homogeneous Interval Bigraphs: An Alternative Approach[1]

4.1 Introduction

A bipartite graph $B(X, Y, E)$ is an interval bigraph if there exists a family $\mathcal{F} = \{I_v : v \in X \cup Y\}$ of sets, where I_v is an interval on the real line, such that $uv \in E$ if and only if $I_u \cap I_v \neq \phi$.

It is well known that an interval representation of an interval graph is not unique and an interval graph may have many interval representations that differ in the order of the end points of the intervals on the line. An excellent account of this area of study is given in Fishburn [30]. Skrien and Gimbel [84] in particular, studied the problem of homogeneous representation of an interval graph. The problem is to characterize those interval graphs such that for every vertex v, there is an interval representation of the graph where the interval representing v is an end interval (leftmost or rightmost). S.Olariu

[1]A shorter version of this chapter ([79]) was earlier published in *Elec. notes in Disc. Math.* The present chapter however is a thoroughly different version of that publication.

[64] obtained the result of [84] by approaching the problem from a different angle. He first showed that the property of homogeneous representation of an interval graph is rooted at the existence of transitive orientations featuring a special property and used this idea to get the desired result.
Motivated by this problem on interval graph, we study in this chapter the corresponding problem on interval bigraph.

In a family of intervals, a *left-end interval* is an interval whose left endpoint is leftmost among all endpoints of intervals in the family. Similarly, a *right-end interval* is an interval whose right endpoint is rightmost. An *end interval* is a left-end or right-end interval. An interval bigraph is said to be *vertex-homogeneous* if for every vertex v of B there is an interval representation such that the interval corresponding to v is an end interval. Analogously, an interval bigraph B is said to be *edge-homogeneous* if every edge $e = uv$, there is an interval representation such that the intervals corresponding to both u and v are intervals of the same end.

As mentioned in the introduction, the forbidden subgraph characterization of an interval bigraph remains still elusive. In this chapter, we provide the forbidden induced subbigraph characterization for vertex-homogeneous and edge-homogeneous interval bigraphs.

We recall (Theorem 1.4.4) that a bigraph is an interval bigraph if and only if its biadjacency matrix has a zero-partition.

Given an interval representation of an interval bigraph, permute the biadjacency matrix by listing the rows in order of left-end points of intervals in U and the columns in order of the left-end points of intervals in V. The total order of the left end-points of the intervals of the union of these two families generates a stair path [76] from upper left to to the lower right corner of the matrix and a 0 in upper right sector of the matrix has all 0's to its right and a 0 in the lower left sector has all 0's below it. The 0's of the upper right sector are all labeled R and the 0's in the lower left sector are labelled C. Thus we obtain a zero-partition. This process is reversible and therefore we can say that an interval bigraph is vertex-homogeneous if and only if there is a permutation of rows and columns of its biadjacency matrix such that any vertex can occupy the first row or first column to have a zero-partition. Similarly for edge-homogeneity, the end vertices of any edge can occupy the first row and the first column of its biadjacency matrix and permutation of the remaining rows and columns will lead to a zero-partition.

By Theorem 1.4.4, the Ferrers dimension of an interval digaph is at most 2. The converse is false. It was shown in [75] that there exists a bigraph of Ferrers dimension 2 which is not an interval bigraph. Bigraphs of Ferrers dimension at most 2 were characterized by Cogis [13]. We discussed it briefly in the introduction and we explain it in more details below.

For this, he used the idea of a 2×2 permutation submatrix or a *couple*. A couple in a binary matrix is a pair of zeros lying at opposite corners of a 2×2 permutation submatrix. Cogis then defined an undirected graph $H(B)$, the graph of couples in B whose vertices correspond to the 0's of the biadjacency matrix of B with two such vertices joined by an edge if they form a couple. He proved that a finite bigraph B has Ferrers dimension at most 2 if and only if $H(B)$ is bipartite. This result has a pivotal role in the development of this chapter. The zero(s) that do not form a couple is (are) called *isolated vertex (vertices).* Let $\mathbf{I}$ denote the set of isolated vertices of $H(B)$. The graph obtained by deleting $\mathbf{I}$ from $H(B)$ is denoted by H^b and is called the bare graph of $H(B)$. Let $(\mathbf{R}, \mathbf{C})$ be a bicolouration of H^b. Cogis [13] showed that there is a bicolouration $(\mathbf{R}, \mathbf{C})$ of H^b such that each of $\mathbf{R} \cup \mathbf{I}$ and $\mathbf{C} \cup \mathbf{I}$ is a Ferrers bigraph and their union is the complement $\overline{B}$ of B. Note that not all bicolourations of H^b have this property and a bicolouration (R, C) of H^b for which $\mathbf{R} \cup \mathbf{I}$ and $\mathbf{C} \cup \mathbf{I}$ are Ferrers bigraph is called a *satisfactory bicolouration.*

We know that a bigraph B is an interval bigrph if and only if its complement is the union of two disjoint Ferrers bigraphs. This means that there is a bicolouration $(\mathbf{R}, \mathbf{C})$ of H^b and a partition of $\mathbf{I}$ into two disjoint subsets $\mathbf{I}_1$ and $\mathbf{I}_2$ ($\mathbf{I}_1 \cap \mathbf{I}_2 = \phi$) such that $\mathbf{R} \cup \mathbf{I}_1$ and $\mathbf{C} \cup \mathbf{I}_2$ are two disjoint Ferrers bigraphs whose union is the complement $\overline{B}$ of B.

Given a bigraph of Ferrers dimension at most 2 and a realization of $\overline{B}$ as the union of two Ferrers bigraphs, the notion of *interior edges* of these two Ferrers bigraphs was introduced in [18].

Below we explain this notion in the specific context of an interval bigraph. Let $\mathbf{R}_1 = \mathbf{R} \cup \mathbf{I}_1$ and $\mathbf{C}_1 = \mathbf{C} \cup \mathbf{I}_2$ be two disjoint Ferrers bigraphs whose union is $\overline{B}$. An edge I of $\mathbf{I}_1$ (respectively I of $\mathbf{I}_2$) is said to be an *interior edge* of $\mathbf{R}_1$ if in the biadjacency matrix of $\overline{B}$ it belongs to a 2×2 configuration, where the opposite corner is 0 and the other diagonal is in R (respectively the other diagonal is in C). It is denoted by I_r (respectively I_c). This means that in a zero-partitioned matrices of B, an I_r belongs to a submatrix of the form:

$$\begin{pmatrix} 1 & R \\ R & I_r \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} R & I_r \\ 1 & R \end{pmatrix}$$

and an I_c belongs to a submatrix of the form

$$\begin{pmatrix} 1 & C \\ C & I_c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C & 1 \\ I_c & C \end{pmatrix}$$

The set of all interior edges of R_1 is denoted by $\mathbf{I_r}$ and of C_1 by $\mathbf{I_c}$. The following proposition was proved in [18].

Proposition 4.1.1. *Let B be an interval bigraph and let $(\mathbf{R_1}, \mathbf{C_1})$ be a realization of $\overline{B}$ into two Ferrers bigraphs, where $\mathbf{R_1} = \mathbf{R} \cup \mathbf{I_1}$ and $\mathbf{C_1} = \mathbf{C} \cup \mathbf{I_2}$. The two bigraphs $H_1 = \mathbf{R} \cup \mathbf{I_r}$ and $H_2 = \mathbf{C} \cup \mathbf{I_c}$ are Ferrers bigraphs.*

4.2 Preliminaries

Let $\{S_i : u_i \in U\}$ and $\{T_i : v_i \in V\}$ be an interval representation of interval bigraph $B(U, V; E)$, where $S_i = [a_i, b_i]$ and $T_i = [c_i, d_i]$ are the intervals on the real line corresponding to the vertices u_i and v_i respectively. A source interval $\{S_i : u_i \in U\}$ is a *left-end interval* if $a_i \leq a_j$ and $a_i \leq c_j$, for all j and a *right-end interval* if $b_j \leq b_i$ and $d_j \leq b_i$, for all j. It is an *end interval* if it is either a left-end interval or a right-end interval.

Similarly an end interval for a sink interval can be defined.

Given a vertex v of an interval bigraph B, we say that B is *v-homogeneous* if there is an interval representation of B for which the interval representing v is an end interval. Analogously, given an edge $e = uv$ of an interval bigraph

B, the bigraph is said to be *e-homogeneous* if both of u and v have the end intervals and have the same end points in an inteval representation.

An interval bigraph B is said to be *homogeneously representable with respect to its vertices* or simply *vertex-homogeneous* if for every vertex v of the bigraph there exists an interval representation of B in which the interval representing v is an end interval. If an interval bigraph $B(U, V, E)$ is not vertex-homogeneous then there exists a vertex $x \in U \cup V$ such that x can not be an end interval in any representation of B. The vertex x is called the *critical vertex* of B. An interval bigraph B is said to be *homogeneously representable with respect to its edges* or simply *edge-homogeneous* if for every edge $e = uv$ of the bigraph there exists an interval representation of B in which the intervals representing u and v have the intervals at the same end.

Let P_n be an n-vertex path. We prove the folling propositions.

Proposition 4.2.1. *P_5 is vertex-homogeneous as well as edge-homogeneous.*

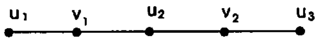


Figure 4.1: P_5

Proof. The bigraph P_5 as shown in Figure 4.1 is both vertex-homogeneous and edge-homogeneous as can be seen from its following interval representation in Figure 4.2.

Proposition 4.2.2. *P_6 is vertex-homogeneous but not edge-homogeneous.*

Proof. The bigraph P_6, which we shall denote by B_5, has interval representation of Figure 4.4, which shows that any vertex can be an end interval.

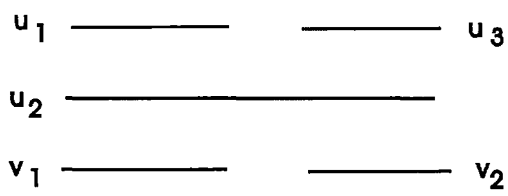


Figure 4.2: Interval representation of P_5

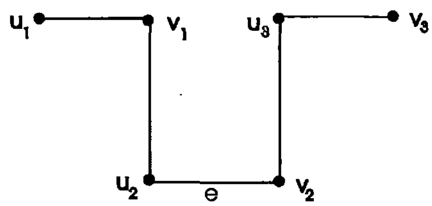


Figure 4.3: $P_6(B_5)$

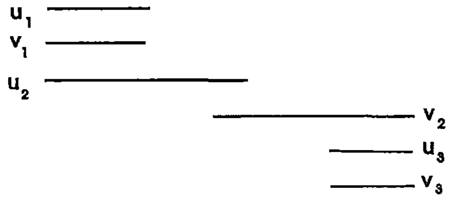


Figure 4.4: Interval representation of P_6

But it is not edge-homogeneous. If B_5 is edge-homogeneous, we may consider an interval representation assigning left-end intervals to u_2 and v_2, being the end vertices of the edge $e = u_2v_2$. Since u_1 is nonneighbour of v_2, $S(u_1)$ must be completely to the right of $T(v_2)$. $T(v_1)$ intersects both $S(u_2)$ and $S(u_1)$. Now u_3 is a neighbour of v_2 but not of v_1 and thus $S(u_3)$ is completely to the left of $T(v_1)$ and thus $S(u_3) \subset S(u_2)$. Now v_3 is a neighbour of u_3 but not u_2, which is impossible. □

Note: Edge-homogeneity of a bigraph implies its vertex-homogeneity but the converse is not true.

Proposition 4.2.3. *The interval bigraphs $B_1(P_7)$, B_2, B_3, B_4 of Figure 4.5 are not vertex-homogeneous, where B_3 and B_4 are the bigraphs obtained by omitting or including the dashed edge in Figure 4.5.*

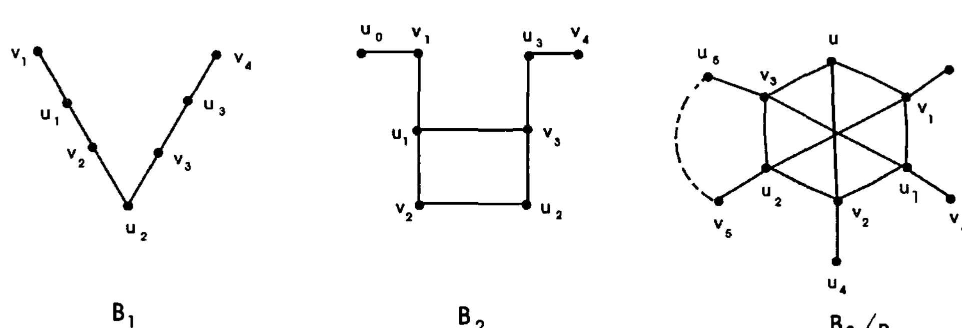


Figure 4.5: Forbidden vertex-homogeneous bigraphs

Proof. If we permute the rows and columns of B_1, B_2, B_3/B_4 according to the orders $(u_1, u_2, u_3; v_1, v_2, v_3, v_4)$, $(u_3, u_1, u_2, u_0; v_3, v_4, v_2, v_1)$,

$(u_1, u, u_2, u_3, u_4, u_5; v_4, v_1, v_2, v_3, v_5)$ respectively, zero-partition of the respective bigraphs are obtained. For B_1, we consider an interval representation assigning left-end interval to u_2. Since v_1, v_4 are nonneighbours of u_2, their intervals lie completely to the right of $S(u_2)$. By symmetry, we may assume that the left endpoint of $T(v_1) \cup T(v_4)$ belongs to $T(v_1)$. Since $S(u_3)$ intersects $T(v_4)$ but not $T(v_1)$, it begins to the right of $T(v_1)$. Now $T(v_3)$ extends from $S(u_2)$ to $S(u_3)$ and contains $T(v_1)$. This is impossible, since u_1 is a neighbour of v_1 but not v_3. Thus B_1 is not vertex-homogeneous. For the study of B_2, we assume that $S(u_2)$ is leftmost and $T(v_1)$ and $T(v_4)$ are to its right. If the left endpoint of $T(v_1) \cup T(v_4)$ belongs to $T(v_4)$, then the argument as for B_1, implies that $T(v_2)$ contains $T(v_4)$, which is impossible since u_3 is a neighbour of v_4 but not v_2. Otherwise, T_3 contains T_1, which is impossible since u_0 is a neighbour of v_1 but not v_3.

Similarly we can prove that B_3 and B_4 are not vertex-homogeneous by taking the interval for u to be the left-end interval and arriving at a contradiction. But below we use the zero-partition of their biadjacency matrices to prove the same.

In case of B_3/B_4, $H(B_3)/H(B_4)$ has two non trivial components (R, C) and (R_1, C_1) and the remaining 0's are isolated vertices as shown in the following matrix :

	v_4	v_1	v_2	v_3	v_5
u_1	1	1	1	1	R
u	I	1	1	1	R
u_2	C	1	1	1	1
u_3	I	1	R_1	R	R
u_4	I	C_1	1	R	R
u_5	C	C	C	1	–

We show that the vertex u is the critical vertex i.e. zero-partition is not obtainable by placing u in the top row of the biadjacency matrix. If we bring u to the first row then for a zero partition uv_4 and uv_5 must have the same colour R. Now the sub matrix

$$\begin{array}{c|cc} & v_4 & v_1 \\ \hline u_2 & \mathrm{C} & 1 \\ u_4 & \mathrm{I} & \mathrm{C} \end{array}$$

demands u_4v_4 must have the colour C. But

$$\begin{array}{c|cc} & v_4 & v_3 \\ \hline u & \mathrm{R} & 1 \\ u_4 & \mathrm{I} & \mathrm{R} \end{array}$$

demands that the vertex u_4v_4 has the colour R. This is a contradiction. The same argument holds for the zero-partition obtained by interchange of colours R_1, C_1.

It can be verified that B_1, B_2, B_3/B_4 are vertex-minimal graphs that are not vertex-homogeneous. □

Proposition 4.2.4. *Let an interval bigraph $B(U, V, E)$ be not vertex-homogeneous and let u be a critical vertex of B, then the vertex u must have at least two neighbours.*

Proof. We suppose that B is the minimal bigraph that is not vertex-homogeneous and let $u \in U$ be the critical vertex of B. Let if possible u has only one neighbour in B. Then in the biadjacency matrix of B, there is a vertex $v_1 \in V$ for which $(uv_1 = 1$ and $(uv_j = 0, j \neq 1$.

Let $B^* = B \setminus \{u\}$. Then B^* is vertex-homogeneous and so v_1-homogeneous. Then the rows and columns of the biadjacency matrix B^* can be arranged so that v_1 is the first column in the matrix. Adjoining the vertex u to B^* so that u is the first row, we easily see that the bigraph B has zero-partition – a contradiction. □

4.3 Vertex-homogeneous Representation

Below we present two important observations.

Observation 1. Let the biadjacency matrix of $B = (U, V)$ where U is the set of rows and V is the set of the columns, have a zero-partition. If we delete some of the columns, then the reduced matrix (U, V_1) where V_1 is the set of columns that remain, will also have a zero-partition with the same permutation of its rows and columns as in biadjacency of B. Similarly, (U, V_2), where $V_2 = V \setminus V_1$ will also have a zero-partition with the same permutation as in the biadjacency matrix of B.

Observation 2. Consider two block matrices (U, V_1) and (U, V_2) where U is the set of rows and V_i's $(i = 1, 2)$ are the set of columns of the matrices, each having zero-partion with the same permutation of its rows, being placed side by side such that (U, V_1) block is to the left of (U, V_2) block. Let A denote then the augmented matrix (i.e. taking all the columns with their same relative positions). A may or may not have a zerro-partition. We are interested in the case when A does not have it. In that case, it is not difficult to show that only two possibilities arise. They are given below:

Case I. There is an R in (U, V_1) block having a 1 below this R finds a 1 to its right in (U, V_2) block.

Case II. An R in (U, V_1) may find a 0 to its right in (U, V_2) such that there is a C above this 0 in (U, V_2) block and as such this 0 can be coloured neither R nor C in the augmented matrix.

Now we prove the main theorem of this chapter.

Theorem 4.3.1. *An interval bigraph is a vertex-homogeneous bigraph if and only if it does not contain the induced forbidden subgraphs* B_i, $(i = 1, 2, 3)$ *of Figure 4.5.*

Proof. The necessary part is proved in Proposition 4.2.3. Below we prove the sufficiency.

Let $B(U, V, E)$ be a minimal interval bigraph which is not vertex-homogeneous and let $u \in U$ be the *critical vertex* of the bigraph (meaning that B becomes vertex-homogeneous if we delete u from B). Then it follows that u has at least two neighbours and that for any zero-partition, u can not occupy the first row. Obviously u has some 0's which are not all R's. Now we bring this vertex u to the first row and place the columns of $N(u)$ (the neighbours of u) at the left, keeping the same relative orders as they were in the zero-partition and the columns of $\overline{N(u)}$ to the right of the columns of $N(u)$, also keeping the same relative order as they were in the zero-partition in $Adj(B)$ (which is the biadjacency matrix of B).

It is to be noted that, by doing so, each of the block matrices $(U \setminus \{u\}, N(u))$ and $(U \setminus \{u\}, \overline{N(u)})$ have zero-partitions and in addition the 0's in each block have the same colours as they were in the zero-partition in $Adj(B)$.

Only two possible induced submatrices are there:

Case(I)

	$N(u)$: v_2	$\overline{N(u)}$: v_3
u	1	0
u_1	R	1
u_2	1	–

Case(II)

	$N(u)$: v_1	$\overline{N(u)}$: v_2
u	1	0
u_1	–	C
u_2	R	0

where – = 0 or 1.

Case I. By Proposition 4.2.4, u must have two neighbours and so we must have structure of the form:

	$N(u)$: v_1	$N(u)$: v_2	$\overline{N(u)}$: v_3
u	1	1	0
u_1	–	R	1
u_2	–	1	–

We divide the proof into two subcases:

(IA) $u_1v_1 = 1$ **(IB)** $u_1v_1 = 0$

Subcase(IA)

The pair of edges (u_2v_1, u_2v_3) have four possibilities:

(i) $(0,0)$ (*ii*) $(0,1)$ (*iii*) $(1,0)$ (*iv*) $(1,1)$

SubcaseIA(ii). The bigraph has a 6-cycle.

SubcaseIA(i).

	$N(u)$: v_1	$N(u)$: v_2	$\overline{N(u)}$: v_3
u	1	1	0
u_1	1	R	1
u_2	0	1	0

Now interchange of u_1 and u_2 and also of v_1 and v_2 will lead to a zero-partition. To prevent it, we introduce a column

$$v_4{}^t = \begin{array}{c} \begin{array}{ccc} u & u_1 & u_2 \end{array} \\ \begin{pmatrix} 0 & - & 1 \end{pmatrix} \end{array}$$

and the matrix takes the form:

$$
\begin{array}{c|cc|cc}
 & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\
 & v_1 & v_2 & v_3 & v_4 \\
\hline
u & 1 & 1 & 0 & 0 \\
u_1 & 1 & R & 1 & - \\
u_2 & 0 & 1 & 0 & 1
\end{array}
$$

Now $u_1v_4 = 1$ leads to 6-cycle and $u_1v_4 = 0$ leads to B_1.

SubcaseIA(iii).

$$
\begin{array}{c|cc|c}
 & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\
 & v_1 & v_2 & v_3 \\
\hline
u & 1 & 1 & 0 \\
u_1 & 1 & R & 1 \\
u_2 & 1 & 1 & 0
\end{array}
$$

Interchange of rows u_1 and u_2 gives us a zero-partition and so we introduce a column

$$
{v_4}^t = \begin{array}{c} \begin{array}{ccc} u & u_1 & u_2 \end{array} \\ \left(\begin{array}{ccc} 0 & - & 1 \end{array} \right) \end{array}
$$

Two cases may arise.

SubcaseIA(iii)(a).

$$
{v_4}^t = \begin{array}{c} \begin{array}{ccc} u & u_1 & u_2 \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & 1 \end{array} \right) \end{array}
$$

and get the matrix

$$
\begin{array}{c|cc|cc}
 & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\
 & v_1 & v_2 & v_3 & v_4 \\
\hline
u & 1 & 1 & 0 & 0 \\
u_1 & 1 & \mathrm{R} & 1 & 0 \\
u_2 & 1 & 1 & 0 & 1
\end{array}
$$

To prevent the zero-partition, we introduce a row u_3 suitably.

First we observe that if a row or column of the biadjacency matrix contain all 1's, then we can ignore this vertex, because the presence or otherwise of this vertex does not affect the property of homogeneity of the graph. So we take $u_3v_1 = 0$.

In order that we can not move the u_3 row above u_1 or u_2, we take either (or both) of u_3v_3 and u_3v_4 as equal to 1.

So the matrix becomes:

$$
\begin{array}{c|cc|cc}
 & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\
 & v_1 & v_2 & v_3 & v_4 \\
\hline
u & 1 & 1 & 0 & 0 \\
u_1 & 1 & \mathrm{R} & 1 & 0 \\
u_2 & 1 & 1 & 0 & 1 \\
u_3 & 0 & - & * & *
\end{array}
$$

where at least one of the $*$ is 1.

We first consider the case

$$
\begin{array}{c|cc|cc}
 & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\
 & v_1 & v_2 & v_3 & v_4 \\
\hline
u & 1 & 1 & 0 & 0 \\
u_1 & 1 & \mathrm{R} & 1 & 0 \\
u_2 & 1 & 1 & 0 & 1 \\
u_3 & 0 & - & 0 & 1
\end{array}
$$

We verify that if $u_3v_2 = 0$ then the bigraph is B_2 and otherwise the graph contains B_1.

We can again verify that all other possibilities of the three positions in u_3-row lead either to a 6-cycle or a zero-partition.

Since the vertex u is a critical vertex and the u-row has two zeros we need not extend this matrix further from this to prevent zero-partition. All possibilities exhausted and we will get nothing new.

SubcaseIA(iii)(b).

$$v_4{}^t = \begin{array}{c} \begin{array}{ccc} u & u_1 & u_2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \end{array}$$

and get the matrix

$$\begin{array}{c|cc|cc} & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\ & v_1 & v_2 & v_3 & v_4 \\ \hline u & 1 & 1 & 0 & 0 \\ u_2 & 1 & 1 & 0 & 1 \\ u_1 & 1 & R & 1 & 1 \end{array}$$

To prevent a zero-partition with u in the first row, we add a row u_3, where $u_3v_1 = 0$, $u_3v_2 = 1$ to get the matrix

$$\begin{array}{c|cc|cc} & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\ & v_1 & v_2 & v_3 & v_4 \\ \hline u & 1 & 1 & 0 & 0 \\ u_2 & 1 & 1 & 0 & 1 \\ u_1 & 1 & R & 1 & 1 \\ u_3 & 0 & 1 & - & - \end{array}$$

For all the possibilities of the dashes in the u_3v_3 and u_3v_4 positions excepting when both of them are 0's, the graph contains a 6-cycle. In the latter case the matrix has a zero-partition. Since all possibilities have been explored, we get nothing new by adding an extra row or column to the matrix and so we do not obtain any new forbidden subgraph from this case.

SubcaseIA(iv).

$$
\begin{array}{c|cc|c}
 & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\
 & v_1 & v_2 & v_3 \\ \hline
u & 1 & 1 & 0 \\
u_1 & 1 & \mathrm{R} & 1 \\
u_2 & 1 & 1 & 1
\end{array}
$$

Here u_2 row and v_1 column has all its positions 1 and so we add a new row u_3 and a new column v_4 such that $u_3v_1 = 0$ and $u_2v_4 = 0$ and get the matrix

$$
\begin{array}{c|cc|cc}
 & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\
 & v_1 & v_2 & v_3 & v_4 \\ \hline
u & 1 & 1 & 0 & 0 \\
u_1 & 1 & \mathrm{R} & 1 & - \\
u_2 & 1 & 1 & 1 & 0 \\
u_3 & 0 & - & - & -
\end{array}
$$

If $u_1v_4 = 1$ then the matrix $(u, u_1, u_2;\ v_1, v_2, v_4)$ is the subcase I(A)(iii). So we suppose $u_1v_4 = 0$ and $u_3v_4 = 1$ to get the matrix

$$
\begin{array}{c|cc|cc}
 & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\
 & v_1 & v_2 & v_3 & v_4 \\ \hline
u & 1 & 1 & 0 & 0 \\
u_1 & 1 & \mathrm{R} & 1 & 0 \\
u_2 & 1 & 1 & 1 & 0 \\
u_3 & 0 & - & - & 1
\end{array}
$$

One of u_3v_2 and u_3v_3 must be 1 (otherwise the graph is disconnected). If both of them are 1, then the graph contains a 6-cycle, otherwise in either case the graph contains B_1.

[When $u_3v_2 = 1$, $u_3v_3 = 0$ then u is the critical vertex where as when $u_3v_2 = 0$, $u_3v_3 = 1$, then u_1 is the critical vertex. In the latter case the matrix has a zero-partition with u in the first row but not u_1 in the first row and so we shall stop here.]

CaseIB. Here the matrix is:

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\ & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 \\ u_1 & 0 & \mathrm{R} & 1 \\ u_2 & - & 1 & - \end{array}$$

The four possibilities of the pair of positions (u_2v_1, u_2v_3) are:

(i) $(0,0)$ (ii) $(0,1)$ (iii) $(1,0)$ (iv) $(1,1)$

SubcaseIB(i)

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\ & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 \\ u_1 & 0 & \mathrm{R} & 1 \\ u_2 & 0 & 1 & 0 \end{array}$$

To prevent the zero-partition we add a column

$$v_4{}^t = \begin{array}{c} \begin{array}{ccc} u & u_1 & u_2 \end{array} \\ \begin{pmatrix} 0 & - & 1 \end{pmatrix} \end{array}$$

If $u_1v_4 = 1$ then the bigraph is B_1 with u_2 as the critical vertex. So we suppose $u_1v_4 = 0$ and the zero-partition of the matrix is

$$\begin{array}{c|cc|cc} & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\ & v_1 & v_2 & v_4 & v_3 \\ \hline u & 1 & 1 & 0 & 0 \\ u_2 & 0 & 1 & 1 & 0 \\ u_1 & 0 & 0 & 0 & 1 \end{array}$$

Since all possibilities for the column v_4 has been explored, we proceed by adding a row u_3 to the matrix and try to prevent a zero-partition (with u in the first row).

We observe that if $u_3v_3 = 0$, then the graph gets disconnected and so we suppose $u_3v_3 = 1$ with at least one of the three other positions in the u_3 row to be 1. So the matrix is

$$\begin{array}{c|cc|cc} & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\ & v_1 & v_2 & v_4 & v_3 \\ \hline u & 1 & 1 & 0 & 0 \\ u_2 & 0 & 1 & 1 & 0 \\ u_1 & 0 & 0 & 0 & 1 \\ u_3 & - & - & - & 1 \end{array}$$

First we suppose that $u_3v_1 = 0$. In this case we verify that for different possibilities of the pair (u_3v_2, u_3v_4) excepting when it is $(1,1)$, is either B_1 or B_2 contains a 6-cycle. In the last case the u_3 row has all its positions 1 and the matrix has a zero-partition, so no forbidden graph is coming out of this case.

Next we consider the case when the pair (u_3v_2, u_3v_4) has the value $(0,1)$, the graph is B_1 with u_2 or v_4 as the critical verterx; when the pair is $(1,1)$ then the graph is B_2 when again the critical vertices are u_2 or v_4. In other cases the matrix has a zero-partition and so no forbidden graph arises from these cases.

SubcaseIB(ii). The matrix is

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\ & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 \\ u_1 & 0 & 0 & 1 \\ u_2 & 0 & 1 & 1 \end{array}$$

Interchanging u_1 and u_2 we get a zero-partition and to prevent it we add a row u_3

$$u_3 = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & - & - \end{pmatrix} \end{array}$$

If $(u_3v_2, u_3v_3) = (0,1)$, then the graph has a 6-cycle while If $(u_3v_2, u_3v_3) = (0,0)$, then the graph is B_1 with v_2 as the critical vertex. If lastly $(u_3v_2, u_3v_3) = (1,1)$ then the matrix has a zero-partition given by

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\ & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 \\ u_3 & 1 & 1 & 1 \\ u_2 & 0 & 1 & 1 \\ u_1 & 0 & 0 & 1 \end{array}$$

To prevent the zero-partition, we add a column v_4

$$v_4{}^t = \begin{array}{c} \begin{array}{cccc} u & u_3 & u_2 & u_1 \end{array} \\ \begin{pmatrix} 0 & - & - & - \end{pmatrix} \end{array}$$

and get the matrix

	$N(u)$		$\overline{N(u)}$	
	v_1	v_2	v_3	v_4
u	1	1	0	0
u_3	1	1	1	–
u_2	0	1	1	–
u_1	0	0	1	–

Clearly any value of the triple (u_3v_4, u_2v_4, u_1v_4) the matrix has a zero-partition with u in the first row . (Note that the graph contains B_1 when $(u_2v_4, u_1v_4) = (o, 1)$). Since all possibilities have been explored, the matrix does not generate any new forbidden graph.

SubcaseIB(iii).

	$N(u)$		$\overline{N(u)}$
	v_1	v_2	v_3
u	1	1	0
u_1	0	R	1
u_2	1	1	0

Here the two columns v_1 and v_2 are identical. To make them different, we add a row u_3 and suppose without loss of generality, $u_3v_1 = 1$, $u_3v_2 = 0$. So we get

	$N(u)$		$\overline{N(u)}$
	v_1	v_2	v_3
u	1	1	0
u_1	0	R	1
u_2	1	1	0
u_3	1	0	–

Now $u_3v_3 = 1$ leads us to subcase IA(iii) where as $u_3v_3 = 0$ returns back to subcase IB(i).

SubcaseIB(iv).

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\ & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 \\ u_1 & 0 & \mathrm{R} & 1 \\ u_2 & 1 & 1 & 1 \end{array}$$

Here again the two columns v_1 and v_2 are identical. So as earlier we add a row u_3 to obtain

$$\begin{array}{c|cc|c} & \multicolumn{2}{c}{N(u)} & \overline{N(u)} \\ & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 \\ u_1 & 0 & \mathrm{R} & 1 \\ u_2 & 1 & 1 & 1 \\ u_3 & 1 & 0 & - \end{array}$$

If $u_3v_3 = 1$, then we go back subcase IA(iv) where as $u_3v_3 = 0$ returns to subcase IB(i).

Case(II).

$$\begin{array}{c|c|c} & N(u) & \overline{N(u)} \\ & v_1 & v_2 \\ \hline u & 1 & 0 \\ u_1 & - & \mathrm{C} \\ u_2 & \mathrm{R} & 0 \end{array}$$

Since u_1v_2 is C and u_2v_1 is R, the matrix can be expanded in the following form:

$$\begin{array}{c|cc|cc} & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\ & v_4 & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 & 0 \\ u_4 & - & - & 1 & \mathrm{R} \\ u_1 & - & - & \mathrm{C} & 1 \\ u_2 & 1 & \mathrm{R} & 0 & - \\ u_3 & \mathrm{C} & 1 & 0 & - \end{array}$$

Note that the bigraph induced by $N(u)$ (or $\overline{N(u)}$) and $U \setminus \{u\}$ is an interval bigraph. If any of the positions u_4v_4, u_4v_1, u_1v_4, u_1v_1 be 0, they must be R, since these positions have a 1 below them, and in that case we return to Case I. So, we consider the case when $u_4v_4 = u_4v_1 = u_1v_4 = u_1v_1 = 1$. With the same reasoning, $u_2v_3 = 0$ and the matrix becomes

$$\begin{array}{c|cc|cc} & \multicolumn{2}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\ & v_4 & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 0 & 0 \\ u_4 & 1 & 1 & 1 & \mathrm{R} \\ u_1 & 1 & 1 & \mathrm{C} & 1 \\ u_2 & 1 & \mathrm{R} & 0 & 0 \\ u_3 & \mathrm{C} & 1 & 0 & - \end{array}$$

At this point, the rows u_1 and u_2 can still be interchanged and to prevent it, we have to introduce column v_5 given by

$$v_5{}^t = \begin{array}{c} \begin{matrix} u & u_4 & u_1 & u_2 & u_3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & \mathrm{C} & \mathrm{C} \end{pmatrix} \end{array}$$

and the induced sub matrix becomes

$$\begin{array}{c|ccc|cc} & \multicolumn{3}{c}{N(u)} & \multicolumn{2}{c}{\overline{N(u)}} \\ & v_5 & v_4 & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 1 & 0 & 0 \\ u_4 & 1 & 1 & 1 & 1 & \mathrm{R} \\ u_1 & 1 & 1 & 1 & \mathrm{C} & 1 \\ u_2 & \mathrm{C} & 1 & \mathrm{R} & 0 & 0 \\ u_3 & \mathrm{C} & \mathrm{C} & 1 & 0 & - \end{array}$$

Again a zero-partition of the above matrix can be obtained by interchanging the v_5 and v_4 columns, changing the label C of u_2v_5 by R and bringing the u_2 row to a position above u_1. To prevent it, we introduce a row

$$u_5 = \begin{array}{c} \begin{matrix} v_5 & v_4 & v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(Note that by introducing this column, the C's of u_2v_5 and u_3v_5 actually form couples with u_5v_4 and u_5v_1 respectively.)

In this case the induced submatrix takes the form

	$N(u)$			$\overline{N(u)}$	
	v_5	v_4	v_1	v_2	v_3
u	1	1	1	0	0
u_4	1	1	1	1	0
u_5	1	0	0	0	0
u_1	1	1	1	0	1
u_2	C	1	0	0	0
u_3	C	C	1	0	–

which is B_3/B_4.

Note: It may be noted that, we can also prevent the C of u_2v_5 from labeling R by introducing a row u_6 and a column v_6 as shown below:

$$u_5 = \begin{array}{c} \begin{matrix} v_5 & v_4 & v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$v_6{}^t = \begin{array}{c} \begin{matrix} u & u_4 & u_6 & u_1 & u_2 & u_3 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

and we get the matrix :

	$N(u)$				$\overline{N(u)}$	
	v_5	v_4	v_6	v_1	v_2	v_3
u	1	1	1	1	0	0
u_4	1	1	1	1	1	0
u_6	1	1	0	0	0	0
u_1	1	1	1	1	0	1
u_2	1	C	1	0	0	0
u_3	C	C	1	1	0	–

But in such a case we get an induced submatrix $B_1 = (u_2, u_6, u_3; v_5, v_6, v_4, v_1)$ where the critical vertex is changed to u_6 in place of the vertex u. So we ignore this case. □

4.4 Edge-homogeneous Representation

Theorem 4.4.1. *An interval bigraph $B(U, V; E)$ is edge-homogeneous if and only if it does not contain B_5 of Figure 4.3 or B_6 of Figure 4.6 as its induced subbigraphs.*

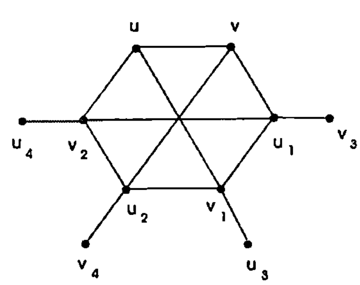


Figure 4.6: B_6

Proof. Necessary. We have already shown in Proposition 4.2.2 that B_5 is not edge-homogeneous. Now the bigraph B_6 is a subgraph of B_3. Since B_3 is a minimal bigraph which is not vertex homogeneous, it follows that B_6 is vertex homogeneous. We show that the edge $e = uv$ prevents the bigraph from being edge-homogeneous (that is, no zero-partition of the biadjacency matrix can have the u row as the first row and v column as the first column simultaneously). A zero-partition of the matrix with u in the first row is

	v_1	v_2	v	v_3	v_4
u	1	1	1	I	I
u_1	1	1	1	1	R
u_3	1	R_1	I	I	I
u_2	1	1	1	C	1
u_4	C_1	1	I	I	I

$H(B)$ has two non-trivial components (u_1v_4, u_2v_3) and (u_3v_2, u_4v_1) and all other 0's are isolated vertices. If we now bring the vertex v to the first column, and try to obtain a zero-partition of the matrix, we see that the I's of (uv_3) and (uv_4)must be coloured R and the I's of (u_3v) and (u_4v) must be coloured C. Thus the matrix becomes

	v	v_1	v_2	v_3	v_4
u	1	1	1	R	R
u_1	1	1	1	1	R
u_2	1	1	1	C	1
u_3	C	1	R	I	I
u_4	C	C	1	I	I

In the above matrix we observe the submatrix

	v_2	v_3
u	1	R
u_3	R	I

demands the I of (u_3v_3) must be coloured R and the matrix

	v	v_3
u_2	1	C
u_3	C	I

demands that (u_3v_3) be coloured C. Thus (u_3v_3) can not be given colour R or C exclusively.

The same argument holds for a zero-partition of the matrix obtained by the interchange of colours of the component (u_3v_2, u_2v_1).

Sufficient. Let B be a minimal bigraph which is not edge-homogeneous (that is a bigraph obtained by eliminating any edge from B becomes edge-homogeneous. Again since any edge-homogeneous graph is vertex-homogeneous and the bigraph B is minimal edge-homogeneous, B is vertex-homogeneous.

Let $e = uv$ be the critical edge which prevents B from being edge-homogeneous. B is u-homogeneous and v-homogeneous separately, so there is zero-partition of the matrix of B with u in the first row, but certainly fails when we bring the vertex v in the first column. This means that there is a R in the first column (v-column) in the above matrix which has a 1 both to its right and below.

$$\begin{array}{c|cc|c} & v & v_1 & v_2 \\ \hline u & 1 & 1 & 0 \\ u_1 & \mathrm{R} & 1 & - \\ u_2 & 1 & - & - \end{array}$$

Case I If R belongs to a non-trivial component then

$$\begin{array}{c|cc|c} & v & v_1 & v_2 \\ \hline u & 1 & 1 & 0 \\ u_1 & \mathrm{R} & 1 & - \\ u_2 & 1 & \mathrm{C} & - \end{array}$$

It is possible that the interchange of u_1 and u_2 rows yield a zero-partition and in order to prevent it, the v_2 column should be

$$v_2 = \begin{array}{c} u \\ u_1 \\ u_2 \end{array} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the submatrix becomes

$$\begin{array}{c|cc|c} & v & v_1 & v_2 \\ \hline u & 1 & 1 & 0 \\ u_1 & \mathrm{R} & 1 & 0 \\ u_2 & 1 & \mathrm{C} & 1 \end{array}$$

This is B_5.

Case II R is an isolated vertex (so it is an I and does not belong to a couple). So the initial structure is

$$\begin{array}{c|cc} & v & v_1 \\ \hline u & 1 & 1 \\ u_1 & \mathrm{I} & 1 \\ u_2 & 1 & 1 \end{array}$$

In order to prevent the bigraph from being edge-homogeneous, the u_2 row cannot be moved above u_1.(Note that if we delete the column v from the matrix, the matrix is a zero-partition). So to prevent u_2 row from moving above the u_1 row, the two possibilities arise:

Case IIa

$$\begin{array}{c|ccc} & v & v_1 & v_3 \\ \hline u & 1 & 1 & - \\ u_1 & \mathrm{I} & 1 & \mathrm{R} \\ u_2 & 1 & 1 & \mathrm{C} \end{array}$$

Case IIb

$$\begin{array}{c|cccc} & v & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & - & - \\ u_1 & \mathrm{I} & 1 & \mathrm{R} & 0 \\ u_2 & 1 & 1 & - & \mathrm{C} \end{array}$$

We show below that the case IIa leads to a contradiction and the case IIb leads upto the bigraph B_6.

Case IIa

The R of (u_1, v_3)gives us a couple

	v_2	v_3
u_1	1	R
u_3	C	1

The u_3 row must be above the u_2 row (because there can be no 1 below the C of v_3 column). So we have

	v	v_1	v_2	v_3
u	1	1	–	–
u_1	I	1	1	R
u_3	–	–	C	1
u_2	1	1	0	C

From the above structure we see that the 0 of u_2v_2 form a couple with the I of u_1v, which is not possible.

Case IIb.

	v	v_1	v_2	v_3
u	1	1	–	–
u_1	I	1	R	0
u_2	1	1	–	C

The C of u_2v_3 gives us

	v	v_1	v_2	v_3	v_4
u	1	1	–	–	–
u_3	–	–	–	1	R
u_1	I	1	R	0	0
u_2	1	1	–	C	1

The R of u_1v_2 gives us

$$\begin{array}{c|ccccc} & v & v_1 & v_2 & v_3 & v_4 \\ \hline u & 1 & 1 & - & - & - \\ u_3 & - & - & - & 1 & \mathrm{R} \\ u_1 & \mathrm{I} & 1 & \mathrm{R} & 0 & 0 \\ u_2 & 1 & 1 & - & \mathrm{C} & 1 \\ u_4 & - & \mathrm{C} & 1 & 0 & - \end{array}$$

Since the matrix for $B \setminus \{v\}$ has a zero-partition, $u_3v_1 = u_3v_2 = u_2v_2 = 1$. So the matrix becomes

$$\begin{array}{c|ccccc} & v & v_1 & v_2 & v_3 & v_4 \\ \hline u & 1 & 1 & - & - & - \\ u_3 & - & 1 & 1 & 1 & \mathrm{R} \\ u_1 & \mathrm{I} & 1 & \mathrm{R} & 0 & 0 \\ u_2 & 1 & 1 & 1 & \mathrm{C} & 1 \\ u_4 & - & \mathrm{C} & 1 & 0 & - \end{array}$$

Since $u_1v = \mathbf{I}$, u_4v should be 0, else we get a couple comprising of $u_1v, u_1v_1, u_4v, u_4v_1$. Now interchange of u_1 and u_2 give an awkward situation in which the 0 of u_1v_3 is forced in the intersection of R's and C's. This situation can only be avoided if we can interchange v_2 and v_3. Thus to prevent this interchange we take $uv_2 = 1$ and $uv_3 = 0$ and so $uv_4 = 0$ and so the matrix becomes

$$\begin{array}{c|ccccc} & v & v_1 & v_2 & v_3 & v_4 \\ \hline u & 1 & 1 & 1 & 0 & 0 \\ u_3 & - & 1 & 1 & 1 & \mathrm{R} \\ u_1 & \mathrm{I} & 1 & \mathrm{R} & 0 & 0 \\ u_2 & 1 & 1 & 1 & \mathrm{C} & 1 \\ u_4 & 0 & \mathrm{C} & 1 & 0 & - \end{array}$$

B being an interval bigraph, we would place v to the left of v_3 and this implies that $u_3v = 1$. Lastly we see that if $u_4v_4 = 1$, the bigraph contains B_5 and if u_4v_4=0, the bigraph is B_6. □

Chapter 5

Circular-arc bigraphs and its subclasses[1]

5.1 Introduction

A circular-arc bigraph is the intersection bigraph of a family of arcs on a circle (see page 11).

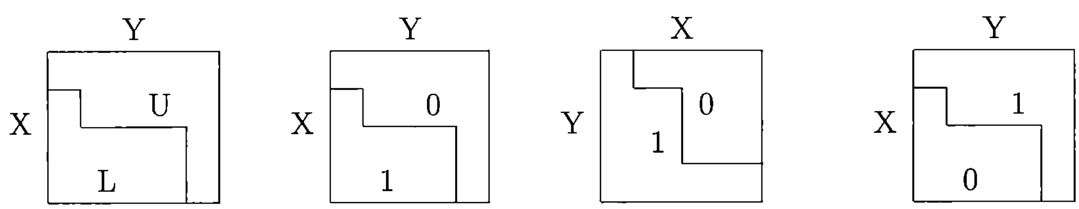
A graph whose vertex set can be partitioned into two cliques is a *two-clique graph.* In [46], Hell and Huang obtained a characterization of two-clique circular-arc graph. Given a graph G, they defined an *auxiliary graph* G^* as follows: The vertex set $V(G^*)$ of the graph G^* is the edge set $E(G)$ and two vertices of G^* are adjacent if the vertices of the corresponding edges of G induce a chordless four-cycle. They proved that a two clique graph G is a circular-arc graph if and only if its auxiliary graph G^* is bipartite.

Two-clique circular arc graphs have arisen as an important subclass of circular arc graph. Tucker [94] first observed that if H is a two-clique circular-

[1]A part of this chapter has been accepted for publication in J.Graph Theory with suggestions for a few minor changes by the referees.

arc graph, then in any representation of H by circular arcs, there exist two points p, q on the circle such that each arc contains at least one of them. Spinrad [85] modified this result and proved that for a fixed pair U, U' of vertex disjoint cliques that cover the circular arc graph H, one can choose two points p, q on the circle and a representation of H such that each arc representing a vertex in U contains p but not q and each arc representing a vertex in U' contains q but not p. Based on this observation, Spinrad [85] characterized two-clique circular arc graphs in terms of the dimension of an associated poset. Trotter and Moore [90] characterized two-clique circular arc graphs in terms of several infinite families of forbidden subgraphs. Feder et al. [28] proved in a compact form that a two-clique graph is a circular arc graph if and only if its complement contains no induced cycle of length at least six and no edge-asteroid. An *edge-asteroid* is a set of edges $e_0, e_1, \ldots, e_{2k}$ such that, for each $i = 0, 1, 2, \ldots, 2k$, there is a path joining e_i and e_{i+1} and containing both e_i and e_{i+1}, that avoids the neighbourhoods of e_{i+k+1}; the subscript addition is modulo $2k+1$. Note that there is a technical difference between asteroidal triple of edges (see page 23) and edge-asteroid of size three: An edge-asteroid e_0, e_1, e_2 is an asteroidal triple of edges, with the additional property that the paths joining e_i and e_{i+1}, and avoiding the neighbourhoods of e_{i+2}, can be chosen to include the edges e_i and e_{i+1}. Thus a six-cycle contains an asteroidal triple of edges but no edge-asteroid of three edges.

As mentioned earlier (see page 16) a *stair partition* of a matrix is a partition of its positions into two sets (L, U) by a polygonal path from the upper left corner to the lower right corner of the matrix such that the set L (respectively, U) is closed under leftward or downward (respectivly, rightward or upward) movements. Given the partite sets, a Ferrers bigraph F uniquely corresponds to a stair partition (L, U) of its biadjacency matrix in which all the entries in L are 1 and those in U are 0. In this case, the stair partition (L, U) is said to *correspond* with the Ferrers bigraph F and vice-versa. Again note that if (L, U) is the stair partition corresponding to a Ferrers bigraph F, then with the same partition of the transpose of the biadjacency matrix of $\overline{F}$, all entries of the matrix in L are 0 and those in U are 1.



(a) A stair partition (L, U)   (b) Biadjacency matrix of F (c) Biadjacency matrices of $\overline{F}$ and its transpose

Figure 5.1: Stair partitions of (Ferrers) bigraph.

Every bigraph B is the intersection of a finite number of Ferrers bigraphs and the minimum number of Ferrers bigraphs whose intersection is the bigraph B is its *Ferrer dimension.* The following is an important characterization of bigraphs of Ferrers dimension at most 2. A 2×2 permutation matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ is a couple in a binary matrix. Cogis then defined an undirected graph $H(B)$, the graph associated to B whose vertices correspond to the 0's of its biadjacency matrix with two such vertices joined by an edge if and only if the corresponding 0's belong to a couple. He proved that a bigraph B is of Ferrers dimension at most 2 if and only if $H(B)$ is bipartite (see Theorem 1.4.1).

In [47], Hell and Huang proved that complements of interval bigraphs are precisely those circular-arc graphs clique covering number 2 which admit a representation such that no two arcs cover the whole circle. In [76], it was shown that the bigraph complement of a bigraph of Ferrers dimention at most 2 is a circular-arc bigraph.

As an extension of these results, it was proved in Theorem 2.2.4 that bigraphs of Ferrers dimension at most 2 are precisely the complements of 2-clique circular-arc graphs. Below we give an alternative proof of the above result, coming as an immediate consequence of the results of Cogis [13] and Hell and Huang [47].

Theorem. *A bigraph G is of Ferrers dimension at most 2 if and only if its graph complement G' is a 2-clique circular-arc graph.*

Proof. Let $G = (V, E)$ be a two-clique graph and let X, Y be two disjoint cliques of the graph covering its vertices. From the definition of the auxiliary graph G^* of G, it follows that no two edges of clique X (and Y) are adjacent in G^* and two edges e and e' in G are adjacent in G^* if and only if each edge has one end point in X and the other end point in Y and their corresponding vertices form a couple in the adjacency matrix of G. Clearly, G' is a bipartite graph and it easily follows that $H(G') = G^*$, where $H(G')$ is the associated graph of G' as defined by Cogis in [13]. Now the following implications prove the theorem.

G is a 2-clique circular-arc graph $\Longleftrightarrow$ G^* is bipartite $\Longleftrightarrow$ $H(G')$ is bipartite $\Longleftrightarrow$ G' is of Ferrers dimension at most 2. $\square$

We now provide an alternative proof of an interval bigraph characterization of Hell and Huang [47] as an application of the above theorem.

Theorem 5.1.1 (Hell and Huang [47]). *Let H be a bipartite graph. Then H is an interval bigraph if and only if its graph complement H' is a two-clique circular-arc graph in which no two arcs cover the whole circle.*

Proof. Let H be an interval bigraph. Then H is of Ferrers dimension at most 2 and H' is a two-clique circular-arc graph. Now we know that a bigraph B is an interval bigraph if and only if its biadjacency matrix has a zero-partition 1.4.4

Now consider the bipartite complement $\overline{H}$ of H and reverse the order of rows and columns of its biadjacency matrix. Let the orders of the vertices of X and Y in the revised arrangement be $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$. Labeling the 1's of this matrix again by R's and C's, we see that 1's of the matrix have a partition (R,C) so that any position to the left of an R is R and any position above a C is C. Now in the adjacency matrix of the complement H' of H with the permutation of its vertices as $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$, it is easy to observe that it has a quasi-circular 1's property. Since R's and C's are disjoint, the set of 1's belonging to the set of u_i's is disjoint from the set of 1's belonging to the set of v_j's. In other words, no 1 can belong to both u_i and v_j. We have observed that in a circular-arc representation, two arcs A_i and A_j cover the whole circle if and only if the 1 in the (i, j) position belongs to two sets u_i and v_j. Consequently, it follows that a bigraph is an interval bigraph if and only if its complement is a two-clique circular-arc in which no two arcs cover the whole circle. $\square$

Let $B = (U, V, E)$ be a bipartite graph and let $\hat{B}$ be the graph obtained from B by adjoining cliques to the partite sets. The graph $\hat{B}$ is *the associated 2-clique graph of* B. We first observe that a bigraph B is a Ferrers bigraph if and only if $\hat{B}$ is a 2-clique interval graph [12]. It is also not difficult to prove that a bigraph is an interval bigraph if and only if its associated 2-clique graph is the intersection of two 2-clique interval graphs whose union is complete. These results motivate us to characterize a circular-arc bigraph. We show that a bigraph is a circular-arc bigraph if and only if its associated 2-clique graph is the intersection of two 2-clique circular-arc graphs with some special properties.

For the second characterization, we observe that if we cut the circle at an arbitrary point, the resulting bigraph becomes an interval bigraph. Dwelling on this idea we show how a circular arc bigraph can be characterized as the union of an interval bigraph and a related Ferrers bigraph.

A proper interval graph is an interval graph in which no interval properly contains another interval. For a unit interval graph, all the intervals are of unit (same) length. Analogously, a proper circular-arc (PCA) graph and a unit circular-arc (UCA) graph is defined. Tucker [25, 91, 92, 94] introduced and extensively studied circular-arc graphs, PCA graphs and UCA graphs. While it is known that an interval graph is proper if and only if it is a unit interval graph [68], the same is not true for PCA graphs and UCA graphs. Tucker [?] showed that UCA graphs form a proper subclass of PCA graphs.

We introduce in this chapter the notions of proper circular-arc bigraphs and unit circular-arc bigraphs. A circular-arc bigraph $B = (U, V, E)$ is a *proper circular-arc bigraph (PCA-bigraph)* if there is a circular-arc representation of B such that no arc is properly contained in another arc of the same partite set. A *unit circular-arc bigraph (UCA-bigraph)* has a circular-arc representation where each arc is of same length. Proper interval bigraphs and unit interval bigraphs have been defined in a similar fashion [77]. They have been shown to be equivalent and have been characterized by the property of monotone consecutive arrangement of their biadjacency matrices.

A symmetric $(0, 1)$-matrix is said to have *circular compatibility* of 1's if the 1's in each column are circular and if after inverting and / or cyclically permuting the order of the rows and corresponding columns, either column 1 or 2 has all 1's or starting from row 2 and going down one finds the first 0 of column 1 in the same row or before the first 0 of column 2. In Section 4 we introduce the notion of monotone circular arrangement of a matrix, which is a generalization of the notions of MCA and also of circular compatibility of 1's in a symmetric matrix [91]. We then characterize biadjacency matrix of a PCA-bigraph in terms of this property.

The Harary graph $H_{2r,n}$ for $n > 2r$ is a graph with n vertices which are all placed on a circle where each vertex is adjacent to nearest r vertices in both directions. In the last section of the paper we first charaterize a UCA-graph in terms of Harary graphs. Then we extend the idea of Harary graphs to bigraphs and obtain an analogous characterization of UCA-bigraphs. Finally we show that, as in the case of circular-arc graphs, UCA-bigraphs form a proper subclass of PCA-bigraphs.

5.2 Circular-arc bigraphs

Let B be a bigraph with biadjacency matrix A. The matrix A is said to satisfy *generalized linear 1's property* if it has a stair partition (L, U) such that the 1's in U are consecutive and appear leftmost in each row, and the 1's in L are consecutive and appear topmost in each column. It follows from Theorem 4 of [75] that B is an interval bigraph if and only if the rows and columns of A can be permuted independently so that the resulting matrix has the generalized linear 1's property.

Further we recall the notion of quasi-circular ones property of a matrix and the characterization of a circular-arc graph in terms of this property by Tucker [91]. Let M be a symmetric $(0, 1)$ matrix with 1's in the main diagonal. Let V_i be the circular set of 1's in the row i starting at the main diagonal and going right (and around) as far as possible until a 0 is reached. Let W_j be the analogous set of 1's in column j starting at the main diagonal and going down (and around). Then M is said to have a *quasi-circular 1's property* if V_i's and W_j's cover all the 1's in M. Tucker [91] proved that G is a circular-arc graph (with loop at every vertex) if and only if its vertices can be indexed so that its adjacency matrix has the quasi-circular 1's property. Now suppose that C is a circular-arc graph and A is its adjacency matrix with the quasi-circular 1's property. A circular-arc representation of C is constructed [91] in the following way. We set the anticlockwise end point of the arc of a vertex by the row number (which is as well as the column number) of the vertex in A. The clockwise end point of the vertex is determined by the column number corresponding to the vertex for which the last consecutive 1 (after the main diagonal) occurs in the row of the former vertex (even if the stretch of 1's goes around).

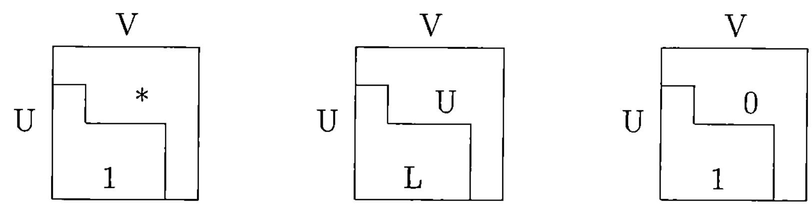
The above characterizations of interval bigraphs and circular-arc graphs are extended to the class of circular-arc bigraphs in the following way: Given a biadjacency matrix A of a bigraph B and stair partition (L, U) of A, let V_i (respectively, W_j) be the 1's in row i (respectively, in column j) that begins at the stair and continues rightward (respectively, downward) (and arround if necessary) until the first 0 is reached. Then A is said to have the *generalized circular* 1*'s property* if it has a stair partition (L, U) such that the V_i's and W_j's together cover all 1's in A. According to a result of [76] we have that a bigraph B is a circular-arc bigraph if and only if its biadjacency matrix A has the generalized circular 1's property.

Let C be a 2-clique circular-arc graph with clique partitions U and V. Let $D(C)$ be the circular-arc bigraph with partite sets U and V obtained from C by removing the edges of the cliques U and V.

Proposition 5.2.1. *Let C be a 2-clique circular-arc graph and A be the biadjacency matrix of $D(C)$. Then rows and columns of A can be (independently) permuted in such a way that there exists a stair partition (L, U) of A in which L is either empty or all entries of L are* 1 *and A satisfies the generalized circular* 1*'s property with respect to (L, U).*

Proof. Let C be a 2-clique circular-arc graph. Then by Theorem 2.2.4, the graph complement of C, say C' is a bigraph of Ferrers dimension at most 2. Let A' be the biadjacency matrix of C'. Again by Theorem 2.2.4, rows and columns of A' can be (independently) permuted in such a way that no 0 has 1's both below and to its right. Denote this matrix by A'_1. Let A be the matrix obtained by interchanging 0's and 1's in A'_1 and reversing the orders of vertices in rows and columns. Then A is the biadjacency matrix of $D(C)$ and no 1 of A has 0's both above and to its left. Then it follows from the proof of Theorem 7 of [76] that A satisfies the generalized circular 1's property with respect to this stair partition $(\mathsf{L}_0, \mathsf{U}_0)$, where $\mathsf{L}_0 = \emptyset$ and U is the entire matrix A. Now consider any stair partition $(\mathsf{L}_1, \mathsf{U}_1)$ of A (keeping the same arrangement of vertices in rows and columns) in which all entries of L_1 are 1 ($\mathsf{L}_0 \subseteq \mathsf{L}_1$ and $\mathsf{U}_1 \subseteq \mathsf{U}_0$). Then also A satisfies the generalized circular 1's property with respect to $(\mathsf{L}_1, \mathsf{U}_1)$ as A satisfies the same with respect to $(\mathsf{L}_0, \mathsf{U}_0)$. □

Definition 5.2.2. *Let C be a 2-clique circular-arc graph and A be the biadjacency matrix of $D(C)$. A stair partition* (L, U) *of A is said to be* fundamental *if all entries of* L *are* 1 (L may be empty) *and A has the generalized circular* 1*'s property with respect to* (L, U)*. Let* (L, U) *be a fundamental stair partition of A and A_1 be the matrix obtained from A by replacing all* 1*'s in U by* 0*. Then A_1 becomes the biadjacency matrix of a Ferrers bigraph, say, F. Then F is called a* fundamental *Ferrers bigraph to C. The adjacency matrix of C with the above arrangement of vertices is called a* fundamental matrix *of C with respect to F.*

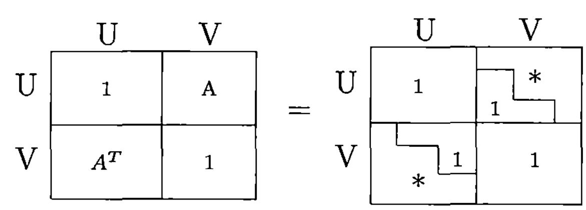


(a) Biadjacency matrix A of $D(C)$

(b) A fundamental stair partition of A

(c) A fundamental Ferrers bigraph F to C

Figure 5.2: Fundamental stair partition



(a) (b)

Figure 5.3: A fundamental matrix of a 2-clique circular-arc graph C with respect to F

Corollary 5.2.1. *For any* 2*-clique circular-arc graph, there is a Ferrers bigraph which is fundamental to it.*

Proof. Follows from Proposition 5.2.1 and Definition 5.2.2. □

Definition 5.2.3. *Two* 2*-clique circular-arc graphs C_1 and C_2 (with the same clique partitions, say, U and V) are called* supplementary *if there exists a Ferrers bigraph F such that*

(1) *F and $\overline{F}$ are fundamental to C_1 and C_2 respectively;*

(2) *If in a row of $A(C_1)$ (respectively, $A(C_2)$) a* 1 *appears in its first column which has a* 0 *above it (in any one of the rows above this row), then the corresponding row in $A(C_2)$ (respectively in $A(C_1)$) does not contain any* 0*, where $A(C_1)$ and $A(C_2)$ are fundamental matrices of C_1 and C_2 respectively.*

Remark 5.2.4. *It is important to note the following:*

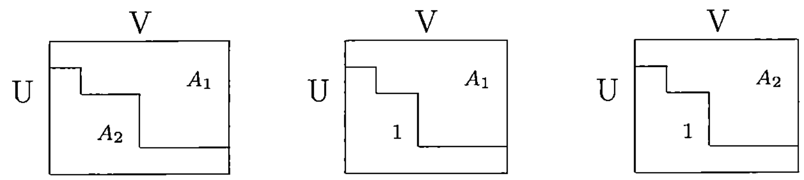
1. *If the arrangement of vertices in $A(C_1)$ is $(u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n)$, then the same in $A(C_2)$ is $(v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m)$, where $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$.*

2. *The interval graphs $\widehat{F}$ and $\widehat{\overline{F}}$ are subgraphs of C_1 and C_2 respectively and since $\widehat{F} \cup \widehat{\overline{F}}$ is a complete graph, we have $C_1 \cup C_2$ is complete.*

Theorem 5.2.5. *A bipartite graph B is a circular-arc bigraph if and only if its associated 2-clique graph $\widehat{B}$ can be expressed as $\widehat{B} = C_1 \cap C_2$, where C_1 and C_2 are two supplementary 2-clique circular-arc graphs.*

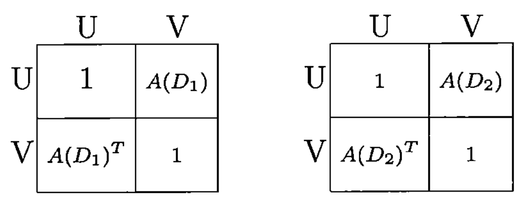
Proof. *Necessity:* Let B be a circular-arc bigraph with partite sets U and V. Let A be the biadjacency matrix of B. Then there is a stair partition of B with respect to which A has the generalized circular 1's property. Let D_1 (respectively, D_2) be the bipartite graph corresponding to the biadjacency matrix $A(D_1)$ (respectively, $A(D_2)$) obtained by putting 1 to every entry below (respectively, above) the stair. Let C_1 (respectively, C_2) be the graph obtained from D_1 (respectively, D_2) by adjoining cliques to the partite sets. The adjacency matrices of C_1 and C_2 thus obtained are denoted by $A(C_1)$ and $A(C_2)$ respectively.

Since A has the generalized circular 1's property with respect to the stair mentioned above, the adjacency matrices of C_1 and C_2 satisfy quasi-circular ones property and so they are 2-clique circular-arc graphs. Also if F is the Ferrers bigraph corresponding to the stair, then F and $\overline{F}$ are fundamental to



(a) A (b) $A(D_1)$ (c) $A(D_2)$

Figure 5.4: Biadjacency matrix of a circular-arc bigraph and its stair partition



(a) $A(C_1)$ (b) $A(C_2)$

Figure 5.5: Biadjacency matrices by adjoining cliques to its partite sets

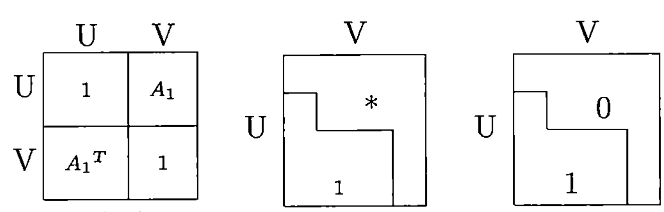
C_1 and C_2 respectively. Further $A(C_1)$ and $A(C_2)$ are fundamental matrices of C_1 and C_2 respectively.

Now since all the first column elements at each row of $A(C_1)$ corresponding to any $u \in U$ are 1, there is no such 1 in the first column of the row corresponding to any $u_i \in U$ has a 0 above it. Suppose there is a 1 in the first column of the row corresponding to the vertex $v_j \in V$ in $A(C_1)$ which has a 0 above it. Then in the biadjacency matrix A of B, the column corresponding to v_j starts with 1 which has a 0 (at some column) left to it and above the stair, as in the biadjacency matrix of D_1 all the entries below the stair are 1. Also since A has the generalized circular 1's property, the above mentioned 1 in the column corresponding to v_j is a continuation of the downward stretch of 1's which begins just below the stair, continues up to the end and goes arround. Thus the row corresponding to v_j in $A(C_2)$ consists of 1 only. Indeed, the entries left to the stair (corresponding to $\overline{F}$)

of the row are all 1 as $\overline{F}$ is fundamental to C_2. Next the entries right to the stair are the entries corresponding to the downward stretch of 1's below the stair in the corresponding column of A which are all 1.

Similar arguments hold to prove the other part of condition (2) of Definition 5.2.3. Therefore C_1 and C_2 are two supplementary 2-clique circular-arc graphs. Also by the above construction $B = D_1 \cap D_2$ and hence $\widehat{B} = C_1 \cap C_2$.

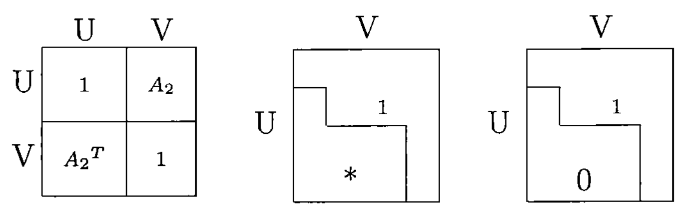
Sufficiency: Let C_1 and C_2 be two supplementary 2-clique circular-arc graphs with clique partitions U and V and F be the Ferrers bigraph such that F and $\overline{F}$ are fundamental to C_1 and C_2 respectively. Let $D_1 = D(C_1)$ and $D_2 = D(C_2)$. Let $A(C_1)$ and $A(C_2)$ be fundamental matrices of C_1 and C_2 with respect to F and $\overline{F}$ respectively. Suppose A_1 is the $U \times V$ submatrix of $A(C_1)$ and A_2 is the $V \times U$ submatrix of $A(C_2)$.



(a) $A(C_1)$ (b) A_1 (c) Biadjacency matrix of F

Figure 5.6: Supplementary 2-clique circular-arc graph and its fundamental matrix with respect to C_1

Then by Remark 5.2.4(1), A_1 and A_2^T are the biadjacency matrices of D_1 and D_2 respectively with same arrangement of vertices in rows and columns. Also A_1 and A_2^T satisfy the generalized circular 1's property with respect to the stair partition, say, (L, U) corresponding to F. $A_1 = (x_{ij})_{m\times n}$ and $A_2^T = (y_{ij})_{m\times n}$ (where $m = |U|$ and $n = |V|$). Consider the matrix $A =$



(a) $A(C_2)$ (b) A_2^T (c) Biadjacency matrix of $\overline{F}^T$

Figure 5.7: Supplementary 2-clique circular-arc graph and its fundamental matrix with respect to C_2

$(x_{ij}y_{ij})_{m\times n}$. Then A is the biadjacency matrix of $D_1 \cap D_2$. Now consider the stair partition (L, U) in A. We claim that A satisfies the generalized circular 1's property with respect to this stair partition.

Since A_1 and A_2 have the generalized circular 1's property with respect to the stair partition (L, U), in each row of A, the continuous stretch of 1's that begins at the right of the stair continues rightward up to the last column unless the first 0 is reached. Also in each column of A, the continuous stretch of 1's that begins jst below the stair continues downward up to the last row unless the first 0 is reached.

Suppose there is a 1 ($= x_{ij}$, say) in the first row of A which is above the stair and has a 0 left to it (in any one of the columns above the stair and left to the jth column). Let the jth column corresponds to the vertex $v_j \in V$. Now in the submatrix $A_1^T = (\bar{x}_{ij})$ of $A(C_1)$ (where $\bar{x}_{ij} = x_{ji}$), there is a 1 ($= \bar{x}_{j1}$) in the first column of a row so that there is a 0 above it (in some row above the this row). Then by condition (2) of Definition 5.2.3 the corresponding row (the row corresponding to the vertex $v_j \in V$) in $A(C_2)$ does not contain any 0. This implies the column corresponding to v_j in the submatrix A_2^T of

$A(C_2)$ does not contain any 0. Moreover A_1 has the generalized circular 1's property with respect to the stair. So in the jth column of A, the stretch of 1's that begins just below the stair continues downward and arround unless the first 0 is reached.

Similar arguments hold for a row of A which has a 1 in the first column below the stair such that the column has a 0 in some row below the stair and above that 1. Thus A satisfies the generalized circular 1's property with respect to the stair (L, U). Consequently $D_1 \cap D_2$ is a circular-arc bigraph. Finally if B is a bipartite graph such that $\widehat{B} = C_1 \cap C_2$, then $B = D_1 \cap D_2$ and hence a circular-arc bigraph, as required. $\square$

Remark 5.2.6. *In the trivial case, where F is the null bigraph, C_2 is complete and consequently $\widehat{B} = C_1$, i.e., a 2-clique circular-arc graph.*

Example 5.2.7. Consider the circular-arc bigraph B whose biadjacency matrix is given by

vertices	v_1	v_2	v_3	v_4	v_5	v_6	arcs
u_1	1	1	1	0	1	1	$[1,7]$
u_2	1	1	0	0	1	0	$[2,4]$
u_3	1	1	1	1	1	1	$[5,12]$
u_4	0	1	0	0	0	0	$[6,6]$
u_5	0	1	0	1	1	0	$[8,10]$
u_6	1	1	1	0	1	1	$[11,7]$
arcs	$[3,5]$	$[4,12]$	$[7,7]$	$[9,9]$	$[10,5]$	$[12,1]$	

Figure 5.8: A circular-arc bigraph and its biadjacency matrix

Let F be the Ferrers bigraph with the following biadjacency matrix:

Then $\widehat{B} = C_1 \cap C_2$, where C_1 and C_2 are two supplementary 2-clique

	v_1	v_2	v_3	v_4	v_5	v_6
u_1	0	0	0	0	0	0
u_2	0	0	0	0	0	0
u_3	1	1	0	0	0	0
u_4	1	1	0	0	0	0
u_5	1	1	1	0	0	0
u_6	1	1	1	1	1	0

Figure 5.9: A Ferrers bigraph and its biadjacency matrix

circular-arc graphs, where $A(C_1)$ and $A(C_2)$ are given by

	u_1	u_2	u_3	u_4	u_5	u_6	v_1	v_2	v_3	v_4	v_5	v_6	arcs
u_1	1	1	1	1	1	1	1	1	1	0	1	1	$[1,9]$
u_2	1	1	1	1	1	1	1	1	0	0	1	0	$[2,8]$
u_3	1	1	1	1	1	1	1	1	1	1	1	1	$[3,12]$
u_4	1	1	1	1	1	1	1	1	0	0	0	0	$[4,8]$
u_5	1	1	1	1	1	1	1	1	1	1	1	0	$[5,11]$
u_6	1	1	1	1	1	1	1	1	1	1	1	1	$[6,12]$
v_1	1	1	1	1	1	1	1	1	1	1	1	1	$[7,12]$
v_2	1	1	1	1	1	1	1	1	1	1	1	1	$[8,12]$
v_3	1	0	1	0	1	1	1	1	1	1	1	1	$[9,1]$
v_4	0	0	1	0	1	1	1	1	1	1	1	1	$[10,12]$
v_5	1	1	1	0	1	1	1	1	1	1	1	1	$[11,3]$
v_6	1	0	1	0	0	1	1	1	1	1	1	1	$[12,1]$

Note that F and $\overline{F}$ are fundamental to C_1 and C_2 respectively and $A(C_1)$ and $A(C_2)$ are fundamental matrices of C_1 and C_2 with respect to F and $\overline{F}$ respectively. In each of the rows corresponding to v_5 and v_6 of $A(C_1)$, there is a 1 in the first column which have a 0 above it. Both the rows corresponding to v_5 and v_6 of $A(C_2)$ do not contain any 0. Similarly, the row corresponding to u_6 of $A(C_2)$ contains a 1 in the first column which has a 0 above it and the row corresponding to u_6 of $A(C_1)$ does not contain any 0.

	v_1	v_2	v_3	v_4	v_5	v_6	u_1	u_2	u_3	u_4	u_5	u_6	arcs
v_1	1	1	1	1	1	1	1	1	1	0	0	1	[1,9]
v_2	1	1	1	1	1	1	1	1	1	1	1	1	[2,12]
v_3	1	1	1	1	1	1	1	1	1	1	0	1	[3,10]
v_4	1	1	1	1	1	1	1	1	1	1	1	0	[4,11]
v_5	1	1	1	1	1	1	1	1	1	1	1	1	[5,12]
v_6	1	1	1	1	1	1	1	1	1	1	1	1	[6,12]
u_1	1	1	1	1	1	1	1	1	1	1	1	1	[7,12]
u_2	1	1	1	1	1	1	1	1	1	1	1	1	[8,12]
u_3	1	1	1	1	1	1	1	1	1	1	1	1	[9,12]
u_4	0	1	1	1	1	1	1	1	1	1	1	1	[10,12]
u_5	0	1	0	1	1	1	1	1	1	1	1	1	[11,12]
u_6	1	1	1	0	1	1	1	1	1	1	1	1	[12,3]

Now we obtain another characterization of a circular-arc bigraph in terms of an interval bigraph and a Ferrers bigraph. Let $I = (U, V, E)$ be an interval bigraph. Let $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the orders of vertices in U and V respectively such that with these orders the biadjacency matrix of I has the generalized linear ones property with respect to a stair partition $(\mathsf{L}, \mathsf{U}) = P$, say. Let $U_1 \subseteq U$ be the subset of U such that 1's of the rows corresponding to any of its vertices (after starting from the right of the stair partition) end up at the last column of the matrix. Similarly consider the subset $V_1 \subseteq V$. It is easy to verify that the set $\{uv \mid u \in U_1,\ v \in V_1\}$ form a biclique of the bigraph I and that there is an interval representation of I such that the right end points of the intervals corresponding to these vertices are same, which is the right-most end point of the interval representation of I. This biclique will be referred to as the *right-most biclique* of I (with respect to the stair partition P).

Now we construct a Ferrers bigraph F whose partite sets are subsets of

U and V in the following manner. Let $U_2 \subseteq U_1$. Arrange the vertices of U_2 (in any order) and then order the remaining vertices of U in the increasing order of the left end points of their intervals. Let $V_2 \subseteq V_1$ and order the vertices of V similarly. With these orders construct a Ferrers bigraph so that edges of the biclique $\{uv \mid u \in U_2,\ v \in V_2\}$ occupy the top left corner of a Ferrers diagram and the neighbors of U (and of V) form a nested sequence (in decreasing order). The Ferrers bigraph so obtained will be referred to as a *neighboring Ferrers bigraph of the interval bigraph* (with respect to the stair partition P). Note that this Ferrers bigraph is, in general, not a subgraph of I.

Theorem 5.2.8. *A bipartite graph B is a circular-arc bigraph if and only if $B = I \cup F$, where I is an interval bigraph and F is a neighboring Ferrers bigraph of I (with respect to a stair partition P).*

Proof. Sufficiency: Let $I = (U, V, E)$ be an interval bigraph. Then the biadjacency matrix A of I has the generalized linear ones property with respect to a stair partition P. Let F be a neighboring Ferrers bigraph of I with respect to P. If we add to A all the 1's corresponding to the edges of F, it can be easily verified that the resulting matrix has the generalized circular ones property.

Necessity: Let $B = (U, V, E)$ be a circular-arc bigraph. Then there is a stair partition P of its biadjacency matrix with respect to which it has the generalized circular ones property. Consider those arcs of $U \cup V$ which cross the last end point p in the circular list of the arcs of the circle (and moves clockwise). If we truncate these arcs at the point p, we get an interval bigraph $I = (U, V, E_1)$. Let $R = (U', V', E')$ denote the right most biclique of I. Let $u \in U'$. Define

$$\gamma(u) = \max\{\text{anticlockwise end point of } v \mid uv \in E \setminus E_1\}.$$

Arrange the vertices of U' in the decreasing order of $\gamma(u)$. We arrange the vertices of V' similarly. Then it is easy to verify that the edges in $E \setminus E_1$ along with those of the right most biclique form a neighboring Ferrers bigraph, say, F of I and $B = I \cup F$. □

5.3 Proper circular-arc bigraphs

We recall from the introduction that a proper circular-arc bigraph is a circular-arc bigraph such that in a circular-arc representation no two arcs of the same partite set are contained in one another.

Example 5.3.1. *Consider the bigraph whose biadjacency matrix is given by*

$$B_1 = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline u & 1 & 1 & 1 \\ u_1 & 1 & 0 & 0 \\ u_2 & 0 & 1 & 0 \\ u_3 & 0 & 0 & 1 \end{array}$$

Consider the bigraph B' obtained by deleting the vertex u. This consists of three mutually disjoint edges u_1v_1, u_2v_2, u_3v_3. It has a proper circular arc representation where we assume that the arcs of u_1, u_2, u_3 move clockwise in any order. Since the arc of u intersects all the arcs of v_1, v_2, v_3, it follows that the arc of u must properly contain one of the arcs of u_1, u_2 and u_3.

Definition 5.3.2. *A sequence of points $(x^1, x^2, ..., x^n)$ for $n \geq 3$ on a circle ordered in a clockwise sense will be called a circular list. Naturally, the list remains invariant under any circular permutation of the sequence.*

Here we consider all the end points of the arcs are distinct. If $[l_i, r_i]$ and $[l_j, r_j]$ are the two intersecting arcs of a proper circular arc bigraph, and they do not cover the entire circle then the circular list of the four end points (moving clockwise) is $[l_i, l_j, r_i, r_j]$ or $[l_j, l_i, r_j, r_i]$. If however these two arcs cover the whole circle, then the list is $[l_i, r_j, l_j, r_i]$. (Note that in this case relation between these two arcs is a symmetric one.)

If, however, we withdraw the restriction that all the end points of the arcs in the representation are distinct, then the proper circular-arc bigraph will be said to be *weak*. It can be easily seen that the two models are equivalent (cf. [78])

Definition 5.3.3. *An $m \times n$ $(0, 1)$ matrix A (with no nonzero rows/columns) having circular ones property for rows has proper circular-arc arrangement if it satisfies the following:*

If $[a_i, b_i]$ is the circular stretch of 1's of the vertex u_i in row i and if the arcs for u_i and u_j have nonempty intersection then for the circular list of the four end points is $[a_i, a_j, b_i, b_j]$ for some (i, j), when the two arcs overlap at one end and $[a_i, b_j, a_j, b_i]$ when they overlap at two ends (with ties broken arbitrarily).

The following definition generalizes the above notion of monotone consecutive arrangements of a matrix [77] and also of circular compability of 1's of a symmetric $(0, 1)$-matrix [91].

Definition 5.3.4. *Let A be an $m \times n$ $(0, 1)$ matrix (with no non-zero rows/columns) having circular ones property for rows. If $[a_i, b_i]$ is the circular stretch of 1's in row i, let $\lambda_i = a_i$, $(i = 1, 2, \ldots, m)$,*

$$\mu_i = \begin{cases} b_i, & a_i \leq b_i \\ n + b_i, & b_i < a_i \end{cases}$$

Then A has a monotone circular arrangement *if there exists a linear order $u_1, u_2, \ldots, u_m$ of U such that $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and $\{\mu_1, \mu_2, \ldots, \mu_m\}$ are non-decreasing sequences.*

Clearly $\{a_1, a_2, \ldots, a_m\}$ is non-decreasing and $\{b_1, b_2, \ldots, b_m\}$ is a cyclic permutation of non decreasing sequence of elements taken from $\{1, 2, \ldots, n\}$. Note however that the other direction is not true.

For the bigraph B_2 of example 5.3.1, we see that $\{a_i\} = \{1, 1, 2, 3\}$ and $\{b_i\} = \{3, 1, 2, 3\}$, $\{\mu_i\} = \{3, 4, 5, 3\}$. So $\{\mu_i\}$ fails to be a nondecreasing sequence and the matrix has no monotone circular arrangement.

It is known that in a matrix having monotone consecutive arrangement 1's appear consecutively in both rows and columns and analogously it is easy to see that in a matrix having monotone circular arrangement, 1's occur circularly in both rows and columns. But while there are matrices where 1's appear consecutively in both rows and columns, but has no MCA , it is not known whether the result mimics in the case of matrices where 1's appear circularly in both rows and columns.

Example 5.3.5. The following is an example of a matrix having monotone circular arrangement.

$$\begin{array}{c|ccccccc|}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
u_1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
u_2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
u_3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
u_4 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
u_5 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
u_6 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
\end{array}$$

Figure 5.10: A matrix and its monotone circular arrangement

We note that the matrix of Example 5.3.5, also satisfies the proper circular arrangement. Actually in the next theorem, we show that the two concepts are equivalent and characterize a proper circular-arc bigraph.

Theorem 5.3.6. *Let $B = (U, V, E)$ be an $m \times n$ bigraph. Then the following conditions are equivalent:*

1) B is a proper circular-arc bigraph;

2) The biadjacency matrix $A(B)$ of B has proper circular arrangement;

(3) The biadjacency matrix $A(B)$ of B has monotone circular arrangement.

Proof. **(1) $\Longrightarrow$ (2):** The proof is straightforward and so is omitted.

(2) $\Longrightarrow$ (3): Let $A(B)$ have a proper circular arc arrangement. Let the arc representation of a vertex $u_i \in U$ in the matrix be $[a_i, b_i]$. Let $u_1 \in U$

corresponding to first row of the matrix. By rotating the columns of the matrix, we bring a_1 to the first column so that $a_1 = 1$. By construction in the earlier part, it is clear that $a_1 \leq a_2 \leq \ldots \leq a_n$. For two consecutive rows u_i and u_{i+1} the circular list of the four end points is $(a_i, a_{i+1}, b_i, b_{i+1})$ when the two ends do not cover the whole circle and $(a_i, b_{i+1}, a_{i+1}, b_i)$ when they cover the whole circle.

Let $\lambda_i = a_i$, $(i = 1, 2, \ldots, n)$

$$\mu_i = \begin{cases} b_i, & a_i \leq b_i \\ n + b_i, & b_i < a_i \end{cases}$$

Then for either of the two cases, we see that $\lambda_i \leq \lambda_{i+1}$, $\mu_i \leq \mu_{i+1}$ for $i = 1, 2, \ldots, n - 1$. Clearly $\{\lambda_i\}$ and $\{\mu_i\}$ are two nondecreasing sequence.

(3) $\Longrightarrow$ (1): We have to show that B has a proper circular arc representation in which all the end points are distinct. If λ_i's and μ_i's are all distinct then a_i's and b_i's are all distinct and we have nothing to prove.

Consider the case when the two arcs for u_i and u_{i+1} do not cover the whole circle and $\lambda_i = \lambda_{i+1}$ so that $a_i = a_{i+1}$. Hence we made the end point of a_i to a (sufficient small) arc distance in the anticlockwise direction (which is less than the smallest of the distances between any two consecutive end points of this arcs). By so moving, the intersection relation between the vertices μ_i and μ_{i+1} remains unchanged. Again when $\mu_i = \mu_{i+1}$, then $b_i = b_{i+1}$. We shift the point b_{i+1} to a sufficiently small arc distance in the clockwise sense. Similar shiftings are made when $a_{i+1} = b_i$. The same reasonings are applied when the two arcs cover the whole circle. □

5.4 Unit circular-arc graphs and bigraphs

A circular-arc graph $G = (V, E)$ is a *unit circular-arc graph* if G has a circular-arc representation such that every arc is of same length.

Example 5.4.1. *Place n vertices around a circle (equally spaced). Form the graph $H_{2r,n}$ by making each vertex adjacent to nearest r vertices in each direction around the circle. Then it is easy to verify that $H_{2r,n}$ is a unit circular-arc graph with a unit circular-arc representation*

$$S = \{[i, j(i)] \mid i = 1, 2, \ldots, n\}, \text{ where } j(i) \equiv (i + r) mod(n) \text{ and } 1 \leqslant j(i) \leqslant n.$$

Note that $H_{2r,n}$ is $2r$-regular and it is complete if and only if $n = 2r + 1$.

In fact, we note that if G is any unit circular-arc graph with the unit circular-arc representation S, then G is complete if and only if $n \leqslant 2r + 1$. In [92], a graph is called *(open) reduced* if it contains no pair of vertices with the same (open) neighbors. Following this we define the concept below.

Definition 5.4.2. *Let $G = (V, E)$ be a graph. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a graph obtained from G by merging the vertices of G having the same set of (open) neighbors. Then $\tilde{G}$ is said to be the* reduced graph *of G.*

Lemma 5.4.1. *Let G be a graph. Then $\tilde{G}$ is a unit circular-arc graph if and only if $\tilde{G}$ is an induced subgraph of $H_{2r,n}$ for some $n, r \in \mathbb{N}$ such that $n > 2r$.*

Proof. Sufficiency: Since every induced subgraph of a unit circular-arc graph is also so, the given condition is sufficient.

Necessity: Let G be a graph such that $\tilde{G}$ is a unit circular-arc graph. Without loss of generality we may assume that there is a unit circular-arc representation of $\tilde{G}$ such that end points of each arc and the uniform length of each arc are positive integer valued. Moreover we may consider the minimum of the anti-clockwise end points is 1 and the maximum of the anti-clockwise end points is n. Suppose the uniform length be r.

Now since $\tilde{G}$ is reduced, no two anti-clockwise end points are same for otherwise, the corresponding vertices would have the same set of neighbors. Thus the set of circular arcs in the above unit circular-arc representation of $\tilde{G}$ is a subset of S, defined in Example 5.4.1. Now if $n > 2r + 1$, then the circular-arc graph with the arc representation S is $H_{2r,n}$. But this implies $\tilde{G}$ is an induced subgraph of $H_{2r,n}$ as the set of arcs in the representation of $\tilde{G}$ is a subset of S. Otherwise $n \leqslant 2r+1$. In this case the circular-arc graph with the representation S is complete. Then $\tilde{G}$ is also complete and hence the trivial graph with one vertex as $\tilde{G}$ is reduced. Thus $\tilde{G}$ may be considered as an induced subgraph of $H_{2r,n}$ (for any $n, r \in \mathbb{N}$ with $n > 2r$). This completes the proof. □

Theorem 5.4.3. *A graph G is a unit circular-arc graph if and only if $\tilde{G}$ is an induced subgraph of $H_{2r,n}$ for some $n, r \in \mathbb{N}$ such that $n > 2r$.*

Proof. Immediate, from the above lemma and the obvious fact that that G is a unit circular-arc graph if and only if $\tilde{G}$ is also so. □

Example 5.4.4. *Consider the Harary graph $H_{2r,n} = (V, E)$. Define a bigraph $HB_{2r,n} = (V_1, V_2, E_1)$ with $V_1 = V_2 = V$ and for any $u \in V_1$ and $v \in V_2$,*

$uv \in E_1$ if and only if either $u = v$ or $uv \in E$. Then the biadjacency matrix of $HB_{2r,n}$ is same as the augmented adjacency matrix of $H_{2r,n}$. Hence $HB_{2r,n}$ is a unit circular-arc bigraph with the same circular-arc representation of that of $H_{2r,n}$.

Definition 5.4.5. *Let $B = (U, V, E)$ be a bigraph. Then the bigraph $\tilde{B} = (\tilde{U}, \tilde{V}, \tilde{E})$ obtained by merging the vertices of B having the same set of (open) neighbors is called the* reduced bigraph *of B.*

Theorem 5.4.6. *A bigraph B is a unit circular-arc bigraph if and only if $\tilde{B}$ is an induced subbigraph of $HB_{2r,n}$ for some $n, r \in \mathbb{N}$ with $n > 2r$.*

Proof. The proof is similar to that of Theorem 5.4.3 and hence omitted. □

Example 5.4.7. *Consider any PCA-graph G which is not a UCA-graph [92]. Construct a bigraph B whose biadjacency matrix is the augmented adjacency matrix of G. If B is a unit circular-arc bigraph, then by the above theorem, it is an induced subbigraph of $HB_{2r,n}$ for some $n, r \in \mathbb{N}$ with $n > 2r$, as B is reduced. But then it follows from the construction of B from G and that of $HB_{2r,n}$ from $H_{2r,n}$ that G is an induced subgraph of $H_{2r,n}$. This implies G is a unit circular-arc graph which is a contradiction. Thus B is a proper circular-arc bigraph which is not a unit circular-arc bigraph.*

Conclusion

In the concluding part of this thesis, we list below some of the open problems of this area of study which remain unresolved and also those which arise as a consequence of the present work and should be a motivation for further research.

1) The problem of finding out the forbidden graphs of an inverval graph and the determination of its recognition algorithm have been thoroughly resolved. But the complete list of forbidden bigraphs or forbidden structure for an interval bigraph and its recognition algorithm is not yet obtained. This is the most formidable open problem in the study of interval bigraph.

2) Interval bigraph is a generalization of interval graph (see [78]). One may try to find similar relationship between:

   i) circular-arc bigraph and circular-arc graph;

   ii) interval containment bigraph and interval containment graph;

   iii) overlap bigraph and overlap graph.

3) In chapter 2 we have referred to the class of graphs called bipermutation bigraph which is an extended concept of permutation bigraph and comes more naturally from the permutation graph. It has been noted that bipermutation bigraphs are edge-disjoint unions of permutation bigraphs and it seems hard to convert this result to a matrix characterization of this class of graphs.

   One can also extend the concept of circular permutation graphs [72] to circular permutation bigraphs, characterize them and obtain a recognition algorithm.

4) Interval tournaments have been characterized by forbidden structure and other ways. Characterization of circular-arc tournaments and its recognition algorithm remains unsolved.

5) In chapter 4 we have characterized a vertex (edge) homogeneous interval bigraphs in terms of forbidden subgraphs. But the problem of finding a recognition algorithm of these graphs has not been solved.

6) The recognition problems of circular-arc graphs and proper circular-arc graphs have been solved in linear time after a long period of thorough research [22, 59]. A quadratic time algorithm for recognizing UCA-graphs is given in [25]. But the problem of recognition algorithm of circular-arc bigraphs along with its complexity question is yet to be solved. Nevertheless, we hope for an early solution to the problem of determining the complexity and the recognition algorithm of the two subclasses, PCB and UCB of circular-arc bigraphs with the help of the

characterization of the corresponding bigraphs given in chapter 5.

Bibliography

[1] L.W.Beineke and C.M.Zamfirescu, Connection digraphs and second order line graphs, *Disc. Math.,* **39** (1982) 237–163.

[2] S.Benzer, On the topology of the genetic fine structure,*Proc. Natl. Acad. Sci.,* USA **45** (1959) 1607–1620.

[3] K.P.Bogart and A.N.Trenk, Bounded bitolerance digraphs, *Discrete Math.,* **215** (2000) 13–20.

[4] A.Bouchet, Etude combinatoire des ordonnes finis–application, Thesis universite Scientifique et Medicale de Grenoble, France (1971).

[5] A.Bouchet, Codages et dimension de relations binaries, In ordess : Description et Roles (M.Pouzet, ed.) (P roc. Lyon 1982), Annals *Disc. Math.,* (1984) 387–396.

[6] D.E.Brown, Variations on interval graphs, Ph.D Thesis, University of Colorado at Denver, U.S., (2004).

[7] D.E.Brown, A.H.Busch and J.R.Lundgren , Interval Tournaments, *J.Graph Theory,* **56** (2007) 72–81.

[8] D.E.Brown, J.R.Lundgren, Bipartite probe interval graphs, circular arc graphs, and interval point bigraphs, *Australas J.Combin.,* **35** (2006) 221–236.

[9] D.E.Brown, J.R.Lundgren and S.C.Flink, Interval k-graphs,*Congress Numerantium,* **156** (2002) 5-16

[10] A.H.Busch, A short proof for interval tournaments, (2008) academic.udayton.edu.

[11] G.J.Chang, T.kloks, S.L.Peng, Probe interval bigraphs, *Electron. Notes Discrete Math.,* **19** (2005) 195–324.

[12] S. Chatterjee and S. Ghosh, Ferrers dimension and boxicity, *Disc. Math.,* **310** (2010) 2443–2447.

[13] O.Cogis, A characterization of digraphs with Ferrers dimension 2, *Rapport de Recherche.* 19, G.R.CNRS, 22 Paris (1979).

[14] O.Cogis, Ferrers digraphs and threshold graphs, *Discrete Math.,* **38** (1982) 33–46.

[15] O.Cogis, On the Ferrers dimension of a digraph, *Disc. Math.,* **38** (1982) 47–52.

[16] C.Coombs and J.Smith, On the direction of structures in attitudes and developmental process, *Psych. Rev.,* **80** (1973) 337–351.

[17] A.K.Das, More results on interval digraphs, Base interval digraphs and containment digraphs, Ph.D. Thesis, University of North Bengal, India (2000) .

[18] S.Das and M.Sen, An interval digraph in relation to its associated bipartite graph,*Dist. Math.* **122** (1993) 113-136.

[19] A.K.Das and M.K.Sen, Bigraphs/digraphs of Ferrers dimension 2 and Asteroidal triple of edges, *Discrete Mathematics,* **295** (2005) 191-195.

[20] S.Das, M.Sen, A.K.Das, D.B.West, Forbidden Substructure Characterization for Interval digraphs/ bigraphs, *2nd Canadian Discrete and Algorithmic Mathematics Conference, Monstreal, Canada,* (2009).

[21] S.Das, P.Talukdar and M.Sen, Homogeneously Representable Interval Bigraphs, *Electronic notes in Dist. Math.,* **15** (2003) 77-80.

[22] X.Deng, P.Hell and J.Huang, Linear-Time Representation Algorithms for Proper Circular-Arc Graphs and Proper Interval Graphs, *SIAM Journal on Computing,* **25** (1996) 390-403.

[23] J.P.Doignon, A.Ducamp and J.C.Falmagne, On realizable biorders and the biorder dimension of a relation,*J.Math. Psych.,* **28** (1984) 73–109.

[24] E.L.O.Drust, S.Dasgupta and J.R.Lundgren, A class of interval digraphs, *Congr. Numerantium,* **192** (2008) 97–113.

[25] G.Durán, A.Gravano, R.M.McConnell, J.P.Spinrad, A.Tucker, Polynomial time recognition of unit circular-arc graphs, *J.Algorithms,* **58**(1) (2006) 76–78.

[26] B.Dushnik and E.M.Miller, Partially ordered set, *Amer. J.Math.,* **63** (1941) 600–610.

[27] S.Even, A.Pnueli and A.Lempel, Permutation graphs and transitive graphs, *J.Assoc.Comput.Math.,* **19** (1972) 400–410.

[28] T.Feder, P.Hell, and J.Huang, List homomorphism and circular arc graphs, *Combinatorica,* **19** (1999) 487–505.

[29] T.Feder, P.Hell, J.Huang and A.Rafiey, Adjusted interval digraphs, *Electronic notes in Discrete Mathematics,* **32** (2009) 83-91.

[30] P.C.Fishburn, *Interval order and interval graphs,* John Wiely and Sons, N.Y.(1985).

[31] D.R.Fulkerson and O.A.Gross, Incidence matrices and interval graphs, *Pacific J. Math.,***15** (1965) 835–855.

[32] S.Ghosh, M.Podder, M.K.Sen, Adjacency matrices of probe interval graphs, *Discrete Applied Mathematics,* **158** (2010) 2004–2013.

[33] P.C.Gilmore and A.J.Hoffman, A characterization of comparability graphs and of interval graphs, *Canad. J.Math.,* **16** (1964) 539–548.

[34] P.W.Goldberg, M.C.Golumbic, H.Kaplan and R.Shamir, Four strikes against physical mapping of DNA, *Computer Bio.,* **2** (1995) 139–152.

[35] M.C.Golumbic, *Algorithmic graph theory and perfect graphs,* Annals of Discrete Math. Elsevier Science, **57** (2004)

[36] M.C.Golumbic and C.F.Goss, Perfect elimination and chordal bipartite graphs, *J.Graph Theory,* **2** (1978) 155–163.

[37] M.C.Golumbic and C.L.Monma, A generalization of interval graphs with tolerances, *Congress. Number.,* **35** (1982) 321–331.

[38] M.C.Golumbic and E. R. Scheinerman, Containment graphs, posets, and related classes of graphs, *Ann. N. Y. Acad. Sci.,* **555** (1989) 192–204.

[39] M.C.Golumbic and A.N.Trenk, *Tolerance Graphs,* Cambridge Univ. Press, UK.(2004).

[40] L.Guttman, A basis for scaling quantitative data. *Amer. Social Rev.,* **9** (1944) 139–150.

[41] G.Hajos, Uver cine art von graphen, *Interval. Math. Nachr.,* 2, (1957) 65.

[42] P.L.Hammer, F.Maffray and M.Preissman, A characterization of chordal bipartite graphs, *Rector Res. Report,* 16–89 Retgers University N.Y. (1989).

[43] P. L. Hammer, K. N. Peled and X. Sun, Difference graphs, *Disc. App. Math.,*, **28** (1990) 35–44.

[44] F.Harary, J.A.Kabell and F.R.McMorris, Bipartite intersection graphs, *Comment. Math. Universities Carolinac,* **23** (1982) 739-745.

[45] F.Harary, J.A.Kabell and F.R.McMorris, Interval acyclic digraphs, *ARC Combinatorics,* **29A** (1990) 59-64.

[46] P. Hell and J. Huang, Two remarks on circular arc graphs, *Graphs and Combinatorics,* **13** (1997) 65–72.

[47] P.Hell and J.Huang, Interval bigraphs and circular arc graphs, *Journal of Graph Theory,* **46** (2004) 313–327.

[48] P.Hell and J.Nešetřil, On the complexity of H-colouring, *J.Combin Theory B,* **48** (1990) 92–110.

[49] J. Huang, Representation characterizations of chordal bipartite graphs, *J. Combinatorial Theory, Ser. B.,* **96** (2006) 673–683.

[50] L.Hubert, Some applications of graph theory and related nonmetric techniques to problems of approximate seriation : The case of symmetry proximity measure,*Brit.J. Math. Stat Phych.,* **27** (1974) 133–153.

[51] D.G.Kendall, Seriation from abundance matrices in F.R.Hodson et al. Eds., Mathematics in archaeological and historical science, Edinburgh Univ. Press. Edinburg, (1971).

[52] C.G.Lekkerkerker and J.Ch.Boland, Representation of a finite graph by a set of intervals on the real line. *Fund.Math.,* **51** (1962) 45–64.

[53] M.C.Lin and J.L.Szwarcfiter, Characterizations and recognition of circular-arc graphs and subclasses: A survey, *Disc. Math.,* **309** (2009) 5618–5635.

[54] I.J.Lin and D.B.West, Interval digraphs that are indifference digraphs, *Graph theory, Combinatorics and Algorithms,* (Proc. 7th Intl. conf. Graph Th., Kalamazoo,1992) (Y.Alavi and A.Schwenk, eds.), Wiley (1995) 751–765.

[55] R.D.Luce, Semiorders and a theory of utility discrimination, *Econometrica,* **24** (1956) 178–191.

[56] N.V.R.Mahadev and U.N.Peled, Threshold graphs and related topics , Elsevier, NY.(1995).

[57] T.Madej and D.B.West, Interval inclusion number of a partially ordered set, *Discrete Math.,* **88** (1991) 259–277.

[58] E.(Szpilrajn-)Marczewski, Sur deux propriétés des classes dénsembles, *Fund. Math.,* **33** (1945) 303–307.

[59] R.M.McConnell, Linear-time recognition of circular-arc graphs, *Algorithmica,* **37**(2) (2003) 93–147.

[60] T.McKee and F.R.McMorris, Topics in intersection graph theory, *SIAM Publications,* Philadelphia.(1999).

[61] F.McMorris, C.Wang, P.Zhang, On probe interval graphs, *Discrete Appl. Math.,* **88** (1998) 315–324.

[62] J.Moon, Topics on tournaments, Holt, Rinehart and Winston, Inc. New York. (1986).

[63] H.Müller, Recognizing interval digraphs and interval bigraphs in polynomial time, *Discrete Applied Math.,* **78**(1997) 189-205.Erratum, http://www.comp.lccds.ac.uk/hm/pupl.html.

[64] S.Olariu, On the Homogeneous Representation of Interval graphs,*J.Graph Theory,* **15** (1991) 65–80.

[65] E.Prisner, Algorithms for interval catch digraphs, *Discrete Applied Math.* **51** (1994) 147–157.

[66] E.Prisner, A journey through intersection graph county, web manuscript,http://www.math.uni_hamburg.dc

[67] J.Riguet, Les relation des Ferrers, *C.R.Acad. Sci. Paris*, **232** (1951) 1729–1730.

[68] F.S.Roberts, *Indifference graphs, in Proof Techniques in Graph Theory,* (ed. F.Harary) 139–146. Academic Press, N.Y., (1969).

[69] F.S.Roberts, *Discrete Mathematical Models with Application to Social, Biological and Environmental Problems,* Prentice-Hall, N.J., (1976).

[70] F.S.Roberts, *Graph Theory and its Application to the problems of society,* NSFCBMS monograph # 29 SIAM Publications, Philadephia. (1978)

[71] F.S.Roberts, On the mobile frequency assignment problem and the traffic light phasing problem, *Ann. N.Y. Acad. Sci.,* **319** (1979) 446–483.

[72] D.Rotem and J.Urrutia, Circular permutation graphs, *Networks*, **12** (1982) 429–437.

[73] B.K.Sanyal and M.Sen, New characterization of digraphs represented by intervals, *J.Graph Theory,* **22**(4)(1996) 297–303.

[74] M. Sen, Some topics related to interval digraphs / bigraphs, in *Graph Theory, Research Directions,* P. Panigrahi and S. B. Rao, eds. 89–99, Narosa, New Delhi. (2011).

[75] M.Sen, S.Das, A.B.Roy and D.B.West, Interval digraphs : an analogue of interval graphs, *J.Graph Theory,* **13** (1989a) 189–202.

[76] M.Sen, S.Das and D.B.West, Circular arc digraphs : a characterization, *J.Graph Theory,* **13** (1989b) 581–592.

[77] M.Sen and B.K.Sanyal, Indifference digraphs : a generalization of indifference graphs and semiorders, *SIAM J. Discrete Math.,* (1994) 157–165.

[78] M.Sen, B.K.Sanyal and D.B.West, Representating digraphs using intervals and circular arcs, *Discrete Math.,* **147** (1995) 235–245.

[79] M.Sen, P.Talukdar and S.Das, Chronological ordering of interval digraphs, *Elec. notes in Disc. Math.,***15** (2003) 184–187.

[80] Randy Shull and A.N.Trenk, Cliques that are tolerance digraphs, *Discrete Applied Mathematics.* **80** (1997) 119–134.

[81] Randy Shull and A.N.Trenk, Interval digraphs and bounded bitolerance digraphs, *Congress. Numer.,* **151** (2001) 111–127.

[82] D.Skrien, Interval graphs, chronological ordering and related matters, Ph.D. Thesis, Washinton Univ. (1980).

[83] D.Skrien, Chronological ordering of interval graphs, *Disc. Appl. Math.,* **8** (1984) 69–83.

[84] D.Skrien and J.Gimbel, Homogeneously representable interval graphs, *Discrete Math.,* **55** (1985) 235–245.

[85] J.Spinrad, Circular arc graphs with clique cover number two, *J. Combin Theory B,* (1988) 300–306.

[86] J. Spinrad, A. Brandstadt, and L. Stewart, Bipartite permutation graphs, *Disc. App. Math.,* **18** (1987) 279–292.

[87] G.Steiner, The recognition of indifference digraphs and generalized semiorders, *J.Graph Theory,* **21**(2) (1996) 235–241.

[88] K.E.Stoffers, Scheduling of traffic lights – a new approach, Transport Res 2, (1968) 199–234.

[89] W.T.Trotter, New perspectives on Interval orders and Interval graphs, *Surveys in combinatories,* Ed. R.A.Bailey, Cambridge Univ. Press. (1997) 237–286.

[90] W.T.Trotter and J.I.Moore, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, *Discrete Math.,* **16** (1976) 361–381.

[91] A.Tucker, Matrix characterization of circular-arc graphs, *Pacific J. Math.,* **39**(2) (1971) 535–545.

[92] A.Tucker, Structure theorems for some circular-arc graphs, *Discrete Math.,* **7** (1974) 167–195.

[93] A.Tucker, Theory and Application of Graphs, Lecture Notes in Math., vol. **642**(1978) 580–589. Springer-Verlag, Berlin.

[94] A.Tucker, An efficient test for circular arc graphs, *SIAM J. Comput.,* **9** (1980) 1–24.

[95] M.S.Waterman, *Introduction to computational Biology,* Chapman and Hall, London. (1995).

[96] D.B.West, *Introduction to Graph Theory,* Prentice–Hall, New Delhi. (2003).

[97] D.B.West, Short proof for interval digraphs, *Discrete Math.,* **178** (1998) 287–292.

[98] P.Zhang, Probe interval graph and its applications to physical mapping of dna, Manuscript. (1994).