C H A P T E R – 4

COMPUTATION IN

ASSIGNMENT PROBLEM

Page

[7]A NOTE ON HUNGARIAN METHOD FOR SOLVING ASSIGNMENT PROBLEM

1. INTRODUCTION

The linear *assignment problem* is a standard problem of assigning n jobs to n persons as one-to-one basis. Let c_{ij} be the cost of assigning *j*th job to *i*th person (i, j = 1, 2, 3, . . . , n). The objective of the *assignment problem* is to assign the jobs to the persons (one job to one person exclusively) at the least total cost or the maximum total profit. So, the *assignment problem* can be represented in the form of n×n cost matrix (c_{ij}) as given below.

Jobs

Persons

	1	2	…	j	…	n
1	c_{11}	c_{12}	…	c_{1j}	…	c_{1n}
2	c_{21}	c_{22}	…	c_{2j}	…	c_{2n}
…	…	…	…	…	…	…
i	c_{i1}	c_{i2}	…	c_{ij}	…	c_{in}
…	…	…	…	…	…	…
n	c_{n1}	c_{n2}	…	c_{nj}	…	c_{nn}

[7]Based on the communication to the Journal of Information and Optimization Sciences, 2014.

This cost matrix is also known as effectiveness matrix or payment matrix. However, without loss of generality, henceforth it refers as cost matrix throughout this paper.

Following are some of the methods may be used to solve assignment problem:

a) Enumeration method
b) Simplex method
c) Transportation method
d) Hungarian method (HM).

The optimal assignment of a linear *assignment problem* consists of choosing n entries (exactly one from each row and one from each column) from a n×n cost matrix, so that the total cost of the selection is minimum. Hungarian method (HM) [86] is the well-known, popular and widely used method to solve the linear assignment problem. This method requires a prior existence of a cost matrix (c_{ij}) and the method works on some arithmetic operations on the elements of the cost matrix. In past few years so many authors have considered the Hungarian method and modify it to reduce the execution time [71, 149]. In this paper it has been shown that if partial trial-assignment is made and if the element which is situated at the intersection of non-assigned row and column is the minimum of all uncovered elements then computational cost is remarkably reduced.

2. HUNGARIAN METHOD (HM)

The solution of the linear *assignment problem* can be made by the Hungarian or Kuhn-Munkres algorithm, originally presented by H. W. Kuhn [86] in 1955 and re-structured by

Munkres [102] in 1957. The time complexity of the Hungarian algorithm is $O(n^3)$, where n is the order of the cost matrix.

Following Kalavathy [73] and Udgata et al. [142] the *Hungarian method* (HM) is described below concisely.

Step 01: Subtract the minimum of each row of the cost matrix from all the elements of the respective rows.

Step 02: Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, we obtained the first modified cost matrix.

Step 03: Then, draw the minimum number of horizontal and vertical lines to cover all zeros in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities.

(i) If $N = n$, where n is the order of the cost matrix, then an optimal assignment can be made. So go to the Step 06 to Step 08 for making the optimal assignment.

(ii) If $N < n$, then proceed to Step 04.

Step 04: Determine the smallest element (x) in the first modified cost matrix not covered by the N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified cost matrix is obtained.

Step 05: Again repeat the Steps 03 and Step 04 until we get the case (i) of Step 03.

Step 06: Examine the rows successively until a row-wise exactly single zero element is found. Mark this zero by □ to make an assignment. Then mark a cross (×) over all zeros if lying in the column of the marked □ zero, showing that they cannot be considered for future assignment. Continue in this manner until all the rows having examined. Repeat the same procedure for the columns also.

Step 07: Repeat the Step 06 successively until one of the following situation arise.

(i) If no unmarked zero is left, then the process ends.

(ii) If there lies more than one unmarked zero in any column or row then mark one of the unmarked zeros arbitrarily and mark a cross (×) in the cells of remaining zeros in its row or column. Repeat the process until no unmarked zero is left in the cost matrix.

Step 08: Thus, exactly one marked □ zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked □ zeros will give the optimal assignment.

3. MODIFICATION OF HUNGARIAN METHOD (MHM)

We have modified the Step 03 of the Hungarian method stated in section 2 to reach the optimal solution of the assignment problem as follows:

(i) Minimum numbers of lines (N) are to be drawn horizontally and vertically covering all the zeros in the resulting cost matrix in such a way that it also covers the minimum of all elements other than zeros also (if possible).

(ii) If the order of the cost matrix(n) and the minimum number of lines (N) are differ by one i.e. n – N = 1 then make a partial trial-assignment according to Step 06 to

Step 08 of the Hungarian method. This will give an assignment in which (n - 1) entries will be chosen, exactly one from each row and one from each column of (n -1) rows and (n – 1) columns. There must be one row and one column is there in which no assignment was made. If the element which is situated in the intersection of such one row and one column in which no previous partial trial-assignment was made is the minimum of all uncovered elements then the previous partial trial-assignment along with this one additional assignment in the position of this smallest element of all uncovered elements will give the optimal assignment. In this case Step 04 and Step 05 of the Hungarian method are not required to execute to reach to the optimal solution. Hence, this will obviously reduced the computational cost of the Hungarian method.

(iii) If this minimum of all uncovered elements in which no partial trial-assignment was made occurs more than one places, additional assignment will also be made in each places occupied by such minimum element subject to the condition given in the Step 08.

On the other hand, if the element which is situated in the intersection of such one row and one column in which no previous assignment was made is not the minimum of all uncovered elements then the previous partial trial assignment will not give the optimal assignment. In that case the procedure of Hungarian method (HM) is to be followed.

Example 1. Let us consider the cost matrix

	S_1	S_2	S_3	S_4	S_5
P_1	30	25	33	35	36
P_2	23	29	38	23	26
P_3	30	27	22	22	22
P_4	25	31	29	27	32
P_5	27	29	30	24	32

After applying the Step 01 and Step 02 of the Hungarian method we find the first modified cost matrix as follows.

	S_1	S_2	S_3	S_4	S_5	
P_1	5	0	8	10	11	L4
P_2	0	6	15	0	3	
P_3	8	5	0	0	0	L1
P_4	0	6	4	2	7	
P_5	3	5	6	0	8	
	L2			L3		

Here n – N = 1. So we make an partial trial-assignment to the above first modified cost matrix as

	S_1	S_2	S_3	S_4	S_5
P_1	5	$\boxed{0}$	8	10	11
P_2	$\cancel{0}$	6	15	$\cancel{0}$	3
P_3	8	5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
P_4	$\boxed{0}$	6	4	2	7
P_5	3	5	6	$\boxed{0}$	8

We see that in the second row and fifth column no assignment was made. The element which is situated in the intersection of non-assigned second row and fifth column is 3 which is also the minimum of all uncovered elements. So the above partial trial-assignment along with this additional assignment $P_2 \rightarrow S_5$ will give the optimal assignment and is given by $P_1 \rightarrow S_2$, $P_2 \rightarrow S_5$, $P_3 \rightarrow S_3$, $P_4 \rightarrow S_1$, $P_5 \rightarrow S_4$ and the minimum cost is 122, as shown in the Table - 10.

4. NUMERICAL EXPERIMENTS AND COMPARISON

Let us consider the cost matrices

Example 2

	T_1	T_2	T_3	T_4	T_5
Q_1	9	3	1	13	1
Q_2	1	17	13	20	5
Q_3	0	14	8	11	4
Q_4	19	3	0	5	5
Q_5	12	8	1	6	2

Example 3

	U_1	U_2	U_3	U_4	U_5
R_1	13	8	16	18	19
R_2	9	15	24	9	12
R_3	12	9	4	4	4
R_4	6	12	10	8	13
R_5	15	17	18	12	20

We have implemented the modified Hungarian method (MHM) to the Example 1, Example 2 and Example 3. In the Table - 8, the arithmetic operations required for both the Hungarian method and the modified Hungarian method are given and it is seen that modified Hungarian method (MHM) requires less number of arithmetic operation compared to the Hungarian method (HM).

In the Table – 9, the number of lines required to be drawn to find out the optimal solution for both the Hungarian method and modified Hungarian method is shown. From the Table – 9, it is seen that the number of lines required to be drawn for the modified Hungarian method is less than the Hungarian method.

In the Table - 10, optimal assignment and optimal cost in both the Hungarian method and modified Hungarian method is shown. For all these three examples optimal assignment and optimal cost is coincides with the Hungarian method.

Table - 8: Comparison Table - arithmetic operations

Example	Hungarian method	Modified Hungarian method
Example 1	63	50
Example 2	61	50
Example 3	63	50

Table - 9: Comparison Table - number of lines drawn

Example	Hungarian method	Modified Hungarian method
Example 1	9	4
Example 2	9	4
Example 3	9	4

Table - 10: Optimal Solution

Example	**Optimal assignment in Hungarian method**	**Optimal cost in Hungarian method**	**Optimal assignment in modified Hungarian method**	**Optimal cost in modified Hungarian method**
Example 1	$P_1 \rightarrow S_2$, $P_2 \rightarrow S_5$, $P_3 \rightarrow S_3$, $P_4 \rightarrow S_1$, $P_5 \rightarrow S_4$	122	$P_1 \rightarrow S_2$, $P_2 \rightarrow S_5$, $P_3 \rightarrow S_3$, $P_4 \rightarrow S_1$, $P_5 \rightarrow S_4$	122
Example 2	$Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_1$, $Q_3 \rightarrow T_5$, $Q_4 \rightarrow T_3$, $Q_5 \rightarrow T_4$ OR $Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_1$, $Q_3 \rightarrow T_5$, $Q_4 \rightarrow T_4$, $Q_5 \rightarrow T_3$ OR $Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_5$, $Q_3 \rightarrow T_1$,	14	$Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_1$,$Q_3 \rightarrow T_5$, $Q_4 \rightarrow T_3$, $Q_5 \rightarrow T_4$ OR $Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_1$, $Q_3 \rightarrow T_5$, $Q_4 \rightarrow T_4$, $Q_5 \rightarrow T_3$ OR $Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_5$, $Q_3 \rightarrow T_1$,	14

	$Q_4 \rightarrow T_3$, $Q_5 \rightarrow T_4$ OR $Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_5$, $Q_3 \rightarrow T_1$, $Q_4 \rightarrow T_4$, $Q_5 \rightarrow T_3$		$Q_4 \rightarrow T_3$, $Q_5 \rightarrow T_4$ OR $Q_1 \rightarrow T_2$, $Q_2 \rightarrow T_5$, $Q_3 \rightarrow T_1$, $Q_4 \rightarrow T_4$, $Q_5 \rightarrow T_3$	
Example 3	$R_1 \rightarrow U_2$, $R_2 \rightarrow U_5$, $R_3 \rightarrow$ U_3, $R_4 \rightarrow U_1$, $R_5 \rightarrow U_4$	42	$R_1 \rightarrow U_2$, $R_2 \rightarrow U_5$, $R_3 \rightarrow$ U_3, $R_4 \rightarrow U_1$, $R_5 \rightarrow U_4$	42

5. CONCLUSION

In this paper it is shown that the computational cost of the Hungarian method is reduced when the order of the cost matrix (n) and the minimum number of horizontal and vertical lines (N) is differ by one. In the future course of study we shall also give more emphasis on the reduction of the computational cost of the Hungarian method when the order of the cost matrix (n) and the minimum number of horizontal and vertical lines (N) is differ by more than one.
