CHAPTER III:

SOLUTION OF POLARIZED HOMOGENEOUS EQUATION OF TRANSFER FOR COUPLED ATMOSPHERE OCEAN SYSTEM

3.1. Discretization of equation of transfer:

In this section we shall solve the equation of transfer employing a new version of Chandrasekhar's discrete ordinate (DOM). First we rewrite (Dropping "s" in intensity) the homogeneous version of the reduced radiative transfer equations for atmosphere and ocean medium from equations (1.14.9) and (1.14.10).

$$\mu\frac{d}{dz}I_{AT}(z,\mu)+I_{AT}(z,\mu)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}PAT_J^S(z,\mu)\int_{-1}^{+1}\mathbf{P}_J^S(\mu')I_{AT}(z,\mu') \qquad (3.1.1)$$

$$\mu\frac{d}{dz}I_{OC}(z,\mu)+I_{OC}(z,\mu)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}POC_J^S(z,\mu)\int_{-1}^{+1}\mathbf{P}_J^S(\mu')I_{OC}(z,\mu') \qquad (3.1.2)$$

We shall use the set of separated but coupled non-linear integro differential equations (2.14.1 – 2.14.4) and (2.14.5 – 2.14.8) for each component of stokes vector for discretization. We shall use Chandrasekhar discretization (1960) scheme to break the continuous radiation field into **2N** quadrature directions with corresponding weights keeping optical depth dependence exact, for $\mathbf{i,k=\pm1,\pm2,\pm3,......,\pm N}$. This enable us to form a set of n equations for each $\mathbf{i}$ (negative as well as positive). Each **homogeneous version of equations** of the set (2.14.1 – 2.14.4 and 2.14.5 – 2.14.8) is replaced by N equivalent equations, separated for positive and negative quadrature directions in the following form, with corresponding Gaussian weight functions expressed and interchanging the Gaussian summation with Fourier summation.

$$w_k=\frac{1}{P'^S(\mu_k)}\int_{-1}^{+1}\frac{P(\mu_k)}{\mu-\mu_k}d\mu. \qquad (3.1.3)$$

$$\mu_i\frac{d}{dz}I_{AT}(z,\mu_i)+I_{AT}(z,\mu_i)=\frac{\omega_{AT}}{2}\sum_{J=S}^{M}PAT_J^S(z,\mu_i)\sum_{k=1}^{N}\omega_k(\mathbf{P}_J^S(\mu_kI_{AT}(z,\mu_K)+\mathbf{P}_k^S(-\mu_k)I_{AT}(z,-\mu_k)) \qquad (3.1.4)$$

$$-\mu_i\frac{d}{dz}I_{AT}(z,-\mu_i)+I_{AT}(z,-\mu_i)=\frac{\omega_{AT}}{2}\sum_{J=S}^{M}PAT_J^S(z,-\mu_i)\sum_{k=1}^{N}\omega_k(\mathbf{P}_J^S(\mu_{k)}I_{AT}(z,\mu_K)+\mathbf{P}_k^S(-\mu_k)I_{AT}(z,-\mu_k)) \qquad (3.1.5)$$

$$\mu_i \frac{d}{dz} I_{OC}(z,\mu_i) + I_{OC}(z,\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} POC_J^S(z,\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k) I_{OC}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{OC}(z,-\mu_k)) \tag{3.1.6}$$

$$-\mu_i \frac{d}{dz} I_{OC}(z,-\mu_i) + I_{OC}(z,-\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} POC_J^S(z,-\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k) I_{OC}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{OC}(z,-\mu_k) \tag{3.1.7}$$

We shall now write down the exact expression explicitly for each stokes components for both the media from equations for "**S**"**=0**

3.2. Directionally Separated and Discretized equations for Atmospheric:

$$\begin{aligned} +\mu_i \frac{d}{dz} L_{AT}^S(z,+\mu_i) + L_{AT}^S(z,+\mu_i) &= \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k) [L_{AT}^S(z,\mu_k) + (-1)^J L_{AT}^S(z,-\mu_k)] \\ &+ \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) [Q_{AT}^S(z,\mu_k) + (-1)^J Q_{AT}^S(z,-\mu_k)] \\ &- \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k) [U_{AT}^S(z,\mu_k) + (-1)^J U_{AT}^S(z,-\mu_k)]. \end{aligned} \tag{3.2.1}$$

$$\begin{aligned} -\mu_i \frac{d}{dz} L_{AT}^S(z,-\mu_i) + L_{AT}^S(z,-\mu_i) &= \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k) [(-1)^J L_{AT}^S(z,\mu_k) + L_{AT}^S(z,-\mu_k)] \\ &+ \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) [(-1)^J Q_{AT}^S(z,\mu_k) + Q_{AT}^S(z,-\mu_k)] \\ &- \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k) [(-1)^J U_{AT}^S(z,\mu_k) + U_{AT}^S(z,-\mu_k)]. \end{aligned} \tag{3.2.2}$$

$$+\mu_i \frac{d}{dz} Q_{AT}^S(z,+\mu_i) + Q_{AT}^S(z,+\mu_i) = \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \gamma_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k) [L_{AT}^S(z,\mu_k) + (-1)^J L_{AT}^S(z,-\mu_k)]$$

$$+ \frac{\omega^{AT}(z)}{2} [\sum_{J=0}^{M} \alpha_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) + \sum_{J=0}^{M} \xi_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)].[Q_{AT}^S(z,\mu_k) + (-1)^J Q_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)+\frac{\omega(z)}{2}\sum_{J=0}^{M}\xi_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)][U_{AT}^S(z,\mu_k)+(-1)^J U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\in_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[V_{AT}^S(z,\mu_k)+(-1)^J V_{AT}^S(z,-\mu_k)]. \quad (3.2.3)$$

$$-\mu_i\frac{d}{dz}Q_{AT}^S(z,-\mu_i)+Q_{AT}^S(z,-\mu_i)=\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\gamma_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[(-1)^J L_{AT}^S(z,\mu_k)+L_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)].[(-1)^J Q_{AT}^S(z,\mu_k)+Q_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)][(-1)^J U_{AT}^S(z,\mu_k)+U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\in_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[(-1)^J V_{AT}^S(z,\mu_k)+V_{AT}^S(z,-\mu_k)]. \quad (3.2.4)$$

$$+\mu_i\frac{d}{dz}U_{AT}^S(z,+\mu_i)+U_{AT}^S(z,+\mu_i)=-\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\gamma_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[L_{AT}^S(z,\mu_k)+(-1)^J L_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)][Q^{ATS}(z,\mu_k)+(-1)^J Q^{ATS}(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J R_J^0(\mu_i)\sum_{k=1}^{N}R_J^0(\mu_k)][U_{AT}^S(z,\mu_k)+(-1)^J U_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\in_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu)[V_{AT}^S(z,\mu_k)+(-1)^J V_{AT}^S(z,-\mu_k)]. \quad (3.2.5)$$

$$-\mu_i \frac{d}{dz} U_{AT}^S(z,-\mu_i) + U_{AT}^S(z,-\mu_i) = -\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \gamma_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J L_{AT}^S(z,\mu_k) + L_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M} \alpha_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) + \sum_{J=0}^{M} \xi_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)][(-1)^J Q_{AT}^S(z,\mu_k) + Q_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M} \alpha_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k) + \sum_{J=0}^{M} \xi_J R_J^0(\mu_i) \sum_{k=1}^{N} R_J^0(\mu_k)][(-1)^J U_{AT}^S(z,\mu_k) + U_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu)[(-1)^J V_{AT}^S(z,\mu_k) + V_{AT}^S(z,-\mu_k)]. \tag{3.2.6}$$

$$+\mu_i \frac{d}{dz} V_{AT}^S(z,+\mu_i) + V_{AT}^S(z,+\mu_i) = -\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)[Q_{AT}^S(z,\mu_k) + (-1)^J Q_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k)[U_{AT}^S(z,\mu_k) + (-1)^J U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \delta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k)[V_{AT}^S(z,\mu_k) + (-1)^J V_{AT}^S(z,-\mu_k)]. \tag{3.2.7}$$

$$-\mu_i \frac{d}{dz} V_{AT}^S(z,-\mu_i) + V_{AT}^S(z,-\mu_i) = -\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)[(-1)^J Q_{AT}^S(z,\mu_k) + Q_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k)[(-1)^J U_{AT}^S(z,\mu_k) + U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \delta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J V_{AT}^S(z,\mu_k) + V_{AT}^S(z,-\mu_k)]. \tag{3.2.8}$$

Directionally Separated and Discretized equations for Ocean:

$$+\mu_i \frac{d}{dz} L_{OC}^S(z,+\mu_i) = -L_{OC}^S(z,+\mu_i) + \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \beta'_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k)[L_{OC}^S(z,\mu_k) + (-1)^J L_{OC}^S(z,-\mu_k)]$$

$$+\frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k)[Q_{OC}^S(z,\mu_k) + (-1)^J Q_{OC}^S(z,-\mu_k)]$$

$$-\frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)[U_{OC}^S(z,\mu_k) + (-1)^J U_{OC}^S(z,-\mu_k)]. \tag{3.2.9}$$

$$-\mu_i \frac{d}{dz} L_{OC}^{S}(z,-\mu_i) = -L_{OC}^{S}(z,-\mu_i) + \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\beta'_J P_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J L_{OC}^{S}(z,\mu_k) + L_{OC}^{S}(z,-\mu_k)]$$

$$+\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\gamma'_J P_J^0(\mu_i)\sum_{k=1}^{N} w_k R_J^0(\mu_k)[(-1)^J Q_{OC}^{S}(z,\mu_k) + Q_{OC}^{S}(z,-\mu_k)]$$

$$-\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\gamma'_J P_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)[(-1)^J U_{OC}^{S}(z,\mu_k) + U_{OC}^{S}(z,-\mu_k)]. \quad (3.2.10)$$

$$+\mu_i \frac{d}{dz} Q_{OC}^{S}(z,+\mu_i) = -Q_{OC}^{S}(z,+\mu_i) + \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\gamma'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[L_{OC}^{S}(z,\mu_k) + (-1)^J L_{OC}^{S}(z,-\mu_k)]$$

$$+\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k) + \sum_{J=0}^{M}\xi'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)][Q_{OC}^{S}(z,\mu_k) + (-1)^J Q_{OC}^{S}(z,-\mu_k)]$$

$$-\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k) + \sum_{J=0}^{M}\xi'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k R_J^0(\mu_k)][U_{OC}^{S}(z,\mu_k) + (-1)^{JS}\, UOC(z,-\mu_k)]$$

$$+\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\, T_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[V_{OC}^{S}(z,\mu_k) + (-1)^J V_{OC}^{S}(z,-\mu_k)]. \quad (3.2.11)$$

$$-\mu_i \frac{d}{dz} Q_{OC}^{S}(z,-\mu_i) = -Q_{OC}^{S}(z,-\mu_i) + \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\gamma'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J L_{OC}^{S}(z,\mu_k) + L_{OC}^{S}(z,-\mu_k)]$$

$$+\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k) + \sum_{J=0}^{M}\xi'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)][(-1)^J Q^{OCS}(z,\mu_k) + Q^{OCS}(z,-\mu_k)]$$

$$-\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k) + \sum_{J=0}^{M}\xi'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k R_J^0(\mu_k)][(-1)^J U_{OC}^{S}(z,\mu_k) + U_{OC}^{S}(z,-\mu_k)]$$

$$+\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\, T_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J V_{OC}^{S}(z,\mu_k) + V_{OC}^{S}(z,-\mu_k)]. \quad (3.2.12)$$

$$+\mu_i \frac{d}{dz} U_{OC}^S(z,+\mu_i) = -U_{OC}^S(z,+\mu_i) - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\gamma'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[L_{OC}^S(z,\mu_k)+(-1)^J L_{OC}^S(z,-\mu_k)]$$

$$-\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k R_J^0(\mu_k)+\sum_{J=0}^{M}\xi'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)][Q_{OC}^S(z,\mu_k)+(-1)^J Q_{OC}^S(z,-\mu_k)]$$
$$+\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)+\sum_{J=0}^{M}\xi'_J R_J^0(\mu_i)\sum_{k=1}^{N} R_J^0(\mu_k)][U_{OC}^S(z,\mu_k)+(-1)^J U_{OC}^S(z,-\mu_k)]$$
$$-\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\ R_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu)[V_{OC}^S(z,\mu_k)+(-1)^J V_{OC}^S(z,-\mu_k)]. \qquad (3.2.13)$$

$$-\mu_i \frac{d}{dz} U_{OC}^S(z,-\mu_i) = -U_{OC}^S(z,-\mu_i) - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\gamma'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J L_{OC}^S(z,\mu_k)+L_{OC}^S(z,-\mu_k)]$$

$$-\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k R_J^0(\mu_k)+\sum_{J=0}^{M}\xi'_J R_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)][(-1)^J Q_{OC}^S(z,\mu_k)+Q_{OC}^S(z,-\mu_k)]$$
$$+\frac{\omega^{OC}(z)}{2}[\sum_{J=0}^{M}\alpha'_J T_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)+\sum_{J=0}^{M}\xi'_J R_J^0(\mu_i)\sum_{k=1}^{N} R_J^0(\mu_k)][(-1)^J U_{OC}^S(z,\mu_k)+U_{OC}^S(z,-\mu_k)]$$
$$-\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\ R_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu)[(-1)^J V_{OC}^S(z,\mu_k)+V_{OC}^S(z,-\mu_k)]. \qquad (3.2.14)$$

$$+\mu_i \frac{d}{dz} V_{OC}^S(z,+\mu_i) = -V_{OC}^S(z,+\mu_i) - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\ P_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)[Q_{OC}^S(z,\mu_k)+(-1)^J Q_{OC}^S(z,-\mu_k)]$$
$$+\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\ P_J^0(\mu_i)\sum_{k=1}^{N} w_k R_J^{0S}(\mu_k)[UOC(z,\mu_k)+(-1)^J U_{OC}^S(z,-\mu_k)]$$
$$+\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\delta'_J P_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[V_{OC}^S(z,\mu_k)+(-1)^J V_{OC}^S(z,-\mu_k)]. \qquad (3.2.15)$$

$$-\mu_i \frac{d}{dz} V_{OC}^S(z,-\mu_i) = -V_{OC}^S(z,-\mu_i) - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\ P_J^0(\mu_i)\sum_{k=1}^{N} w_k T_J^0(\mu_k)[(-1)^J Q_{OC}^S(z,\mu_k)+Q_{OC}^S(z,-\mu_k)]$$
$$+\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\in'_J\ P_J^0(\pm\mu_i)\sum_{k=1}^{N} w_k R_J^0(\mu_k)[(-1)^{JS} UOC(z,\mu_k)+U_{OC}^S(z,-\mu_k)]$$
$$+\frac{\omega^{OC}(z)}{2}\sum_{J=0}^{M}\delta'_J P_J^0(\mu_i)\sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J V_{OC}^S(z,\mu_k)+V_{OC}^S(z,-\mu_k)]. \qquad (3.2.16)$$

3.3. Eigen functions and Eigen values:

Hence forth we shall suppress the "**s**" dependence. We shall follow a slightly different approach using compact eigenfunction as follows

$$I_{AT/OC}(z,\pm\mu_i) = \mathbf{H}^{AT/OC}(\gamma,\pm\mu_i)\exp(-\frac{z}{\gamma}) \quad (3.3.1)$$

Using ansatz (3.3.1) in the set (3.2.1 – 3.2.16) we can arrive at the following set of equations for atmosphere and ocean respectively

$$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{AT}(\gamma,\mu_i) = \frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{AT}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{AT}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{AT}(\gamma,-\mu_i)\right] \quad (3.3.2)$$

Or $$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{AT}(\gamma,\mu_i) = \frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{AT}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k} \quad (3.3.3)$$

$$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{AT}(\gamma,-\mu_i) = \frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{AT}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{AT}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{AT}(\gamma,-\mu_i)\right] \quad (3.3.4)$$

Or $$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{AT}(\gamma,-\mu_i) = \frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{AT}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k} \quad (3.3.5)$$

$$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{OC}(\gamma,\mu_i) = \frac{\omega_{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{OC}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{OC}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{OC}(\gamma,-\mu_i)\right] \quad (3.3.6)$$

Or $$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{OC}(\gamma,\mu_i) = \frac{\omega_{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{OC}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{OC} \quad (3.3.7)$$

$$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{OC}(\gamma,-\mu_i) = \frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{OC}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{OC}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{OC}(\gamma,-\mu_i)\right] \quad (3.3.8)$$

Or $$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{OC}(\gamma,-\mu_i)=\frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{OC}\sum_{K=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{OC} \quad (3.3.9)$$

We have defined $D=\text{diag}\{1,1,-1,-1\}$ and for $i=1,2,3,.....,N$

$$\mathbf{\Psi}_{j,k}^{AT}=\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{AT}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{AT}(\gamma,-\mu_i)\right] \quad (3.3.10)$$

$$\mathbf{\Psi}_{j,k}^{OC}=\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{OC}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{OC}(\gamma,-\mu_i)\right] \quad (3.3.11)$$

However, the set of equations (3.3.2 –3.3.11) can be written in an equivalent but more elegant form by introducing following (4Nx1) vectors

$$\mathbf{H}_+^{AT/OC}(\gamma)=\left[\mathbf{H}^{AT/OC}(\gamma,\mu_1)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_2)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_3)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_4)^T,....,\mathbf{H}^{AT/OC}(\gamma,\mu_N)^T\right]^T$$

(3.3.12)

$$\mathbf{H}_-^{AT/OC}(\gamma)=\left[\mathbf{H}^{AT/OC}(\gamma,-\mu_1)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_2)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_3)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_4)^T,...,\mathbf{H}^{AT/OC}(\gamma,-\mu_N)^T\right]$$

(3.3.13)

Here each $\mathbf{H}^{AT/OC}(\gamma,\mu_i)^T$ is (4x1) vector. We now define

$$\mathbf{\Sigma}=\text{diag}(\omega_1\mathbf{\Gamma},\omega_2\mathbf{\Gamma},\omega_3\mathbf{\Gamma},......,\omega_N\mathbf{\Gamma}) \quad (3.3.14) \qquad \mathbf{X}=\text{diag}(\mu_1\mathbf{\Gamma},\mu_2\mathbf{\Gamma},\mu_3\mathbf{\Gamma},.......,\mu_N\mathbf{\Gamma}) \quad (3.3.15)$$

With $\mathbf{\Gamma}=\text{diagonal}(1,1,1,1)$

$$\mathbf{H}^{AT/OC}(\gamma,\pm\mu_i)=\left[\varphi(\gamma,\pm\mu_i)\quad \psi(\gamma,\pm\mu_i)\quad \theta(\gamma,\pm\mu_i)\quad \not\in(\gamma,\pm\mu_i)\right]^T \quad (3.3.16)$$

The matrices defined in (3.3.14) and (3.3.15) are of order (4Nx4N) but (3.3.16) is of order (Nx4). Using this formalism one can easily verify that the set (3.3.2 – 3.3.11) for each may be written compactly as follows.

$$\left(\mathbf{I}-\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_+^{AT}(\gamma)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}\mathbf{T}_J^S(AT,\gamma) \quad (3.3.17)$$

$$\left(\mathbf{I}+\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_-^{AT}(\gamma)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{AT}\mathbf{T}_J^S(AT,\gamma) \quad (3.3.18)$$

$$\left(\mathbf{I} - \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{+}^{OC}(\gamma) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_{J}^{OC}\mathbf{T}_{J}^{S}(OC,\gamma) \tag{3.3.19}$$

$$\left(\mathbf{I} + \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{-}^{OC}(\gamma) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_{J}^{OC}\mathbf{T}_{J}^{S}(OC,\gamma) \tag{3.3.20}$$

Here $\mathbf{I}$ is a (4Nx4N) identity matrix and (4Nx4) matrix $\mathbf{\Pi}(J,S)$ is given below.

$$\mathbf{\Pi}(J,S) = \left[\mathbf{P}_{J}^{S}(\mu_1),\mathbf{P}_{J}^{S}(\mu_2),\mathbf{P}_{J}^{S}(\mu_3),....,\mathbf{P}_{J}^{S}(\mu_N)\right]^{T} \tag{3.3.21}$$

$$\mathbf{T}_{J}^{S}(AT/OC,\gamma) = \mathbf{\Pi}^{T}(J,S)\mathbf{\Sigma}\mathbf{H}_{+}^{AT/OC}(\gamma) + (-1)^{J-S}D\mathbf{\Pi}^{T}(J,S)\mathbf{\Sigma}\mathbf{H}_{-}^{AT/OC}(\gamma) \tag{3.3.22}$$

Following Siewert's [163] approach we shall now derive equivalent set of relations by defining the following vectors for atmosphere and ocean

$$\mathbf{NA}^{AT} = \mathbf{H}_{+}^{AT}(\gamma) + \mathbf{H}_{-}^{AT}(\gamma)\ ;\ \mathbf{NB}^{AT} = \mathbf{H}_{+}^{AT}(\gamma) - \mathbf{H}_{-}^{AT}(\gamma) \tag{3.3.23}$$

$$\mathbf{NA}^{OC} = \mathbf{H}_{+}^{OC}(\gamma) + \mathbf{H}_{-}^{OC}(\gamma)\ ;\ \mathbf{NB}^{OC} = \mathbf{H}_{+}^{OC}(\gamma) - \mathbf{H}_{-}^{OC}(\gamma) \tag{3.3.24}$$

Taking sum and difference of (3.3.17) & (3.3.18) and (3.3.19) & (3.3.20) respectively for atmosphere and ocean it is easy to derive

$$\mathbf{A}^{AT}\mathbf{X}^{AT} = \frac{1}{\gamma}\mathbf{Y}^{AT}; \quad (3.3.25) \quad \mathbf{B}^{AT}\mathbf{Y}^{AT} = \frac{1}{\gamma}\mathbf{X}^{AT}. \tag{3.3.26}$$

$$\mathbf{A}^{OC}\mathbf{X}^{OC} = \frac{1}{\gamma}\mathbf{Y}^{OC}; \quad (3.3.27) \quad \mathbf{B}^{OC}\mathbf{Y}^{OC} = \frac{1}{\gamma}\mathbf{X}^{OC}. \tag{3.3.28}$$

Where

$$\mathbf{A}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_{J}^{AT}[I + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^{T}\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \tag{3.3.29}$$

$$\mathbf{B}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_{J}^{AT}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^{T}\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \tag{3.3.30}$$

$$\mathbf{A}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_{J}^{OC}[I + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^{T}\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \tag{3.3.31}$$

$$\mathbf{B}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_{J}^{OC}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^{T}\mathbf{\Sigma}\right)\mathbf{X}^{-1}. \tag{3.3.32}$$

We also have

$$\mathbf{X}^{AT} = \mathbf{X}\left[\mathbf{NA}^{AT}\right] \quad (3.3.33) \qquad \mathbf{Y}^{AT} = \mathbf{X}\left[\mathbf{NB}^{AT}\right]. \quad (3.3.34)$$

$$\mathbf{X}^{OC} = \mathbf{X}\left[\mathbf{NA}^{OC}\right]; \quad (3.3.35) \qquad \mathbf{Y}^{OC} = \mathbf{X}\left[\mathbf{NB}^{OC}\right]. \quad (3.3.36)$$

It is to be noted that equation (3.3.29)-(3.3.32) depend on optical depth of the medium. This makes the matrix A and B sensitive to optical depth .This is the difference between homogeneous and inhomogeneous atmosphere in this approach. In general matrix A and B are not symmetric. We now use equations (3.3.25) & (3.3.26) and (3.3.27) & (3.3.28) to get

$$(\mathbf{B}^{AT}\mathbf{A}^{AT})\mathbf{X}^{AT} = \Im^{AT}\mathbf{X}^{AT}; \quad (3.3.37) \qquad (\mathbf{B}^{OC}\mathbf{A}^{OC})\mathbf{X}^{OC} = \Im^{OC}\mathbf{X}^{OC} \quad (3.3.38)$$

$$(\mathbf{A}^{AT}\mathbf{B}^{AT})\mathbf{Y}^{AT} = \Im_{AT}\mathbf{Y}^{AT}; \quad (3.3.39) \qquad (\mathbf{A}^{OC}\mathbf{B}^{OC})\mathbf{Y}^{OC} = \Im_{OC}\mathbf{Y}^{OC} \quad (3.3.40)$$

Our task is now to evaluate the eigenvalues $\Im_{AT}, \Im_{OC}$ which will determine the separation constants γ_{AT}, γ_{OC} for atmosphere and ocean respectively. Separation constants occur in plus-minus pairs. It is to be noted that the eigenvalues may be complex. We can easily see that

$$\Im = \frac{1}{\gamma^2} \quad (3.3.41)$$

We shall first show some explicit numerical calculation for **A** and **B** (Both for atmosphere and ocean) matrix for the following set of values of **S** and **N**.
Let us set **S=0, M=2** and **N=2**, which implies **k=1, 2**.

$$\mathbf{A}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{AT}[I + (-1)^{J-0}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \quad (3.3.42)$$

$$\mathbf{B}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{AT}[I - (-1)^{J-0}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \quad (3.3.43)$$

$$\mathbf{A}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{OC}[I + (-1)^{J-0}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \quad (3.3.44)$$

$$\mathbf{B}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{OC}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \quad (3.3.45)$$

$$\mathbf{\Pi}(J,0) = \left[\mathbf{P}_J^0(\mu_1), \mathbf{P}_J^0(\mu_2), \mathbf{P}_J^0(\mu_3), \ldots, \mathbf{P}_J^0(\mu_N)\right]^T \quad (3.3.46)$$

For N=2, (Four stream approximation) we have the quadrature points given below

$$\mu_{\pm 1} = \pm 0.3399810; \; \mu_{\pm 2} = \pm 0.8611363; \quad (3.3.47)$$

Corresponding weights are given by $\omega_1 = \omega_{-1} = 0.6521452; \; \omega_2 = \omega_{-2} = 0.3478548.$ (3.3.48)

Hence we have from equation (3.3.10 – 3.3.11) & (3.3.12 – 3.3.13) the matrices given in the set (3.3.46)

$$\mathbf{\Pi}(0,0) = [\text{diag}\{1,0,0,1\}, \text{diag}\{1,0,0,1\}]^T; \quad (3.3.49)$$

$$\mathbf{\Pi}(1,0) = [\text{diag}\{\mu_1,0,0,\mu_1\}, \text{diag}\{\mu_2,0,0,\mu_2\}]^T; \quad (3.3.50)$$

$$\mathbf{\Pi}(2,0) = [\mathbf{P}_2^0(\mu_1), \mathbf{P}_2^0(\mu_2)]^T; \quad (3.3.51)$$

The last matrices explicitly look as below

$$\mathbf{\Pi}(2,0) = \left[\begin{pmatrix} -0.3266 & 0 & 0 & 0 \\ 0 & 0.5416 & 0 & 0 \\ 0 & 0 & 0.5416 & 0 \\ 0 & 0 & 0 & -0.3266 \end{pmatrix}, \begin{pmatrix} 0.6123 & 0 & 0 & 0 \\ 0 & 0.1583 & 0 & 0 \\ 0 & 0 & 0.1583 & 0 \\ 0 & 0 & 0 & 0.6123 \end{pmatrix}\right]^T \quad (3.3.52)$$

The numerical values corresponding to A-matrix and B-matrix have been calculated correct up to six decimal places from the set of equations (3.3.42 – 3.3.45). In using equation we have used values of the basic constants for **oblate spheroids particle** for atmosphere and suspended spherical particle for ocean. The values are given in the following table. **Table(2)** contains the values of the basic constants as measured by Wauben WMF and Hovenier JW [376] for scattering in the atmosphere by oblate spheroids with aspect ration 1.999987, size parameter 3 and index of refraction 1.53-0.006i. In **table (1)** we have tabulated the values of the basic constants for spherical particles. We shall use these data for ocean water. These values are taken from a problem collected and calculated by Vestrucci and Siewert [99] considering mie scattering of light, with wavelength 0.951 micro meter having gamma distribution with effective radius 0.2 micro meters, effective variance 0.07 and refractive index 1.34.

Table 1: Basic constants for spherical particle in ocean

l	α_l	β_l	γ_l	δ_l	ε_l	ζ_l
0	0.0	1.0	0.0	0.7120634246	0.0	0.0
1	0.0	1.4552931819	0.0	1.7601411931	0.0	0.0
2	3.3091220464	1.0540263128	−0.7552491518	1.0668243107	0.0420726875	2.5773207443
3	0.9633758276	0.3975899378	−0.3619934319	0.3965110389	0.0850671555	0.7574437604
4	0.2474124256	0.1165930161	−0.1155748816	0.0957641237	0.0154318420	0.1638177665
5	0.0452636955	0.0238747702	−0.0249815879	0.0176508810	0.0031534874	0.0278314781
6	0.0068892608	0.0039501033	−0.0041675362	0.0026154886	0.0004010299	0.0038897248
7	0.0008798202	0.0005388807	−0.0005739043	0.0003271332	0.0000460147	0.0004642654
8	0.0000987255	0.0000637172	−0.0000677900	0.0000358314	0.0000042875	0.0000490224
9	0.0000099029	0.0000066697	−0.0000070926	0.0000035142	0.0000003617	0.0000046647
10	0.0000009071	0.0000006329	−0.0000006712	0.0000003148	0.0000000272	0.0000004075
11	0.0000000769	0.0000000553	−0.0000000585	0.0000000261	0.0000000019	0.0000000331
12	0.0000000061	0.0000000045	−0.0000000047	0.0000000020	0.0000000001	0.0000000025
13	0.0000000005	0.0000000003	−0.0000000004	0.0000000001	0.0000000000	0.0000000002

Table 2: Basic Constants for oblate spheroid particle in atmosphere

l	α_l	β_l	γ_l	δ_l	ε_l	ζ_l
0	0.0	1.0	0.0	0.915207	0.0	0.0
1	0.0	2.104031	0.0	2.095727	0.0	0.0
2	3.726079	2.095158	−0.116688	2.008624	0.065456	3.615946
3	2.202868	1.414939	−0.209370	1.436545	0.221658	2.240516
4	1.190694	0.703593	−0.227137	0.706244	0.097752	1.139473
5	0.391203	0.235001	−0.144524	0.238475	0.052458	0.365605
6	0.105556	0.064039	−0.052640	0.056448	0.009239	0.082779
7	0.020484	0.012837	−0.012400	0.009703	0.001411	0.013649
8	0.003097	0.002010	−0.002093	0.001267	0.000133	0.001721
9	0.000366	0.000246	−0.000267	0.000130	0.000011	0.000172
10	0.000035	0.000024	−0.000027	0.000011	0.000001	0.000014
11	0.000003	0.000002	−0.000002	0.000001	0.000000	0.000001

In the following we have shown some sample results for the values of the eigenvectors corresponding to atmosphere and ocean with constant value 0.5 for albedo function in both the media. However in actual calculations these matrices are optical depth sensitive.

$$
\mathbf{A}^{AT} = \begin{pmatrix}
1.767907 & -0.019796 & 0 & 0 & -0.290508 & -0.003016 & 0 & 0 \\
-0.019796 & 1.893081 & 0 & 0 & 0.019348 & -0.159728 & 0 & 0 \\
0 & 0 & 2.941341 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.709012 & 0 & 0 & 0 & -0.306783 \\
-0.557247 & 0.037113 & 0 & 0 & 1.982013 & 0.005655 & 0 & 0 \\
-0.005786 & -0.306388 & 0 & 0 & 0.005655 & 2.828077 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2.874763 & 0 \\
0 & 0 & 0 & -0.588465 & 0 & 0 & 0 & 2.097714
\end{pmatrix} \tag{3.3.53}
$$

$$
\mathbf{B}^{AT} = \begin{pmatrix}
2.708091 & 0 & 0 & 0 & -0.307999 & 0 & 0 & 0 \\
0 & 2.941341 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.924065 & -0.011105 & 0 & 0 & -0.155007 & 0.010853 \\
0 & 0 & 0.011105 & 1.858084 & 0 & 0 & 0.001692 & -0.256764 \\
-0.590797 & 0 & 0 & 0 & 2.094635 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.874763 & 0 & 0 \\
0 & 0 & -0.297332 & -0.003246 & 0 & 0 & 2.829457 & 0.003172 \\
0 & 0 & -0.020819 & -0.492520 & 0 & 0 & -0.003172 & 2.040631
\end{pmatrix} \tag{3.3.54}
$$

$$
\mathbf{A}^{OC} = \begin{pmatrix} 1.874419 & -0.128128 & 0 & 0 & -0.394609 & -0.019523 & 0 & 0 \\ -0.128128 & 2.010384 & 0 & 0 & 0.125228 & 0.141854 & 0 & 0 \\ 0 & 0 & 2.941341 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.746214 & 0 & 0 & 0 & -0.257658 \\ -0.756932 & 0.240210 & 0 & 0 & 2.177180 & 0.036602 & 0 & 0 \\ -0.037450 & -0.272102 & 0 & 0 & 0.036602 & 2.833301 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.874763 & 0 \\ 0 & 0 & 0 & -0.494235 & 0 & 0 & 0 & 2.222141 \end{pmatrix}
$$

(3.3.55)

$$
\mathbf{B}^{OC} = \begin{pmatrix} 2.780009 & 0 & 0 & 0 & -0.213033 & 0 & 0 & 0 \\ 0 & 2.941341 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.216262 & -0.007134 & 0 & 0 & -0.110484 & 0.006976 \\ 0 & 0 & 0.007138 & 2.149267 & 0 & 0 & 0.001088 & -0.249361 \\ -0.408636 & 0 & 0 & 0 & 2.335172 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.874763 & 0 & 0 \\ 0 & 0 & -0.211928 & -0.002086 & 0 & 0 & 2.842470 & 0.002039 \\ 0 & 0 & -0.013381 & -0.478320 & 0 & 0 & -0.002039 & 2.318749 \end{pmatrix}
$$

(3.3.56)

3.4. Calculation of eigenvectors and eigen values from Eq. (3.3.41):

The eigenvectors and eigen values are now calculated for the present simple example. The eigenvectors (equations (3.3.33) and (3.3.36)) are column matrices (8x1) and eight in number one each for each of eight eigen values given diagonally in (3.3.37 – 3.3.40) for atmosphere and ocean respectively. The following calculated values for eigenvectors and eigen values are not optical depth sensitive but in actual case of our interest these eigenvectors and eigen values are optical depth sensitive.

$$\mathbf{X}^{AT}(\gamma) =$$
$$\begin{pmatrix} 0.552953 & 0.686688 & 0.074502 & 0 & -0.000777 & 0 & 0 & 0 \\ -0.005923 & -0.078648 & 0.946217 & 0 & 0.170586 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.002499 & 0 & 0.947002 & 0.042368 & -0.167458 \\ 0 & 0 & 0 & 0.467828 & 0 & -0.041798 & -0.636458 & -0.000438 \\ 0.833190 & -0.722172 & -0.070098 & 0 & 0.000896 & 0 & 0 & 0 \\ -0.001813 & -0.027227 & 0.306934 & 0 & -0.985342 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.000785 & 0 & 0.315756 & 0.015138 & 0.985879 \\ 0 & 0 & 0 & 0.883815 & 0 & 0.041684 & 0.769998 & 0.000617 \end{pmatrix}$$

(3.4.1)

$$\Im_j^{AT} = \begin{pmatrix} 2.854745 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6.436508 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.406996 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.090047 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.282509 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.513514 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6.523637 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8.282575 \end{pmatrix}$$

(3.4.2)

Corresponding separation constants for atmosphere and ocean (equation (3.4.4) and (3.4.6), are calculated in plus-minus pairs using (3.3.41).

$$\sqrt{\Im_j^{AT}} = \begin{pmatrix} 1.689599 & 2.537027 & 2.325295 & 1.757853 & 2.877935 & 2.348088 & 2.554141 & 2.877946 \end{pmatrix}$$

(3.4.3)

$$\gamma_j^{AT} = (\pm 0.591856 \quad \pm 0.394162 \quad \pm 0.430053 \quad \pm 0.568876 \quad \pm 0.347471 \quad \pm 0.425879 \quad \pm 0.391521 \quad \pm 0.3474$$

(3.4.4)

$$\mathbf{X}^{OC} =$$

$$\begin{pmatrix} 0.600968 & 0.257087 & -0.596022 & 0 & 0.012462 & 0 & 0 & 0 \\ -0.027447 & 0.881679 & 0.278474 & 0 & 0.177679 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.001788 & 0 & 0.942106 & -0.029407 & 0.177103 \\ 0 & 0 & 0 & -0.472312 & 0 & -0.029146 & 0.643225 & 0.000801 \\ 0.798757 & -0.271465 & 0.743344 & 0 & 0.015997 & 0 & 0 & 0 \\ -0.008454 & 0.287845 & 0.121035 & 0 & 0.983880 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.000570 & 0 & 0.332797 & -0.012205 & -0.984191 \\ 0 & 0 & 0 & -0.881430 & 0 & 0.028848 & -0.765015 & -0.001055 \end{pmatrix} \quad (3.4.5)$$

$$\Im_j^{OC} = \begin{pmatrix} 3.317108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.553703 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7.521719 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.957956 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.283257 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.407828 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7.342237 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8.283609 \end{pmatrix} \quad (3.4.6)$$

$$\sqrt{\Im_j^{OC}} = (1.821293 \quad 2.356630 \quad 2.742575 \quad 1.989461 \quad 2.878065 \quad 2.531369 \quad 2.709656 \quad 2.878126) \quad (3.4.7)$$

$$\gamma_j^{OC} = (0.549060 \quad 0.424335 \quad 0.364621 \quad 0.502649 \quad 0.347456 \quad 0.395043 \quad 0.369050 \quad 0.347448) \quad (3.4.8)$$

It is to be noted that the eigenvectors and eigenvalues are real for **N=2**. But in general we have found that both the eigenvector and eigenvalues may be real as well as complex.

3.5. The expression for eigenvectors:

We are now in a position to establish the eigen functions either from equations (3.3.33 – 3.3.34) or (3.3.35 – 3.3.36). Let us chose equation (3.3.33 – 3.3.34). Let $\Im_j^{AT/OC}$ and $\mathbf{X}^{AT/OC}(\Im_j^{AT/OC})$ denote the eigen-values and corresponding eigenvectors respectively for $j = 1,2,3,.....,4N$. Using equations (3.3.33 – 3.3.34), (3.3.35 – 3.3.36) with (3.3.17 – 3.3.18) we can easily deduce

$$\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC}) = \frac{1}{2}\mathbf{X}^{-1}\left(\mathbf{I} \pm \gamma_j^{AT/OC}\mathbf{A}^{AT/OC}\right)\mathbf{X}^{AT/OC}(\lambda_j^{AT/OC}) \qquad (3.5.1)$$

We also note that $\mathbf{H}_{+}^{AT/OC}(-\gamma_j^{AT/OC}) = \mathbf{H}_{-}^{AT/OC}(\gamma_j^{AT/OC})$ for $j = 1,2,3,.....,4N$. We shall now show some values of eigenvectors found numerically (corrected up to six decimal places) using the set of matrices (3.3.12 – 3.3.13). For **N=2**, and constant albedo function (set at value 0.5) each eigenvector is a (8x1) matrix. It is to be noted that For **N=2** we have eight eigenvector corresponding to eight separation constants (Eigen values). These atmospheric as well as the oceanic eigenvectors are also optical depth sensitive for inhomogeneous media.

$$\mathbf{H}_{+}^{AT}(\gamma_1^{AT}) = \begin{pmatrix} -3.901731 \\ 0.032845 \\ 0 \\ 0 \\ -6.708547 \\ 0.009994 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{+}^{AT}(\gamma_2^{AT}) = \begin{pmatrix} -14.503045 \\ 1.744800 \\ 0 \\ 0 \\ 17.849178 \\ 0.603121 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{+}^{AT}(\gamma_3^{AT}) = \begin{pmatrix} -1.162563 \\ -15.223190 \\ 0 \\ 0 \\ 1.216799 \\ -4.927890 \\ 0 \\ 0 \end{pmatrix};$$

$$\mathbf{H}_{+}^{AT}(\gamma_4^{AT}) = \begin{pmatrix} 0.025969 \\ -6.101985 \\ 0 \\ 0 \\ -0.036609 \\ 35.213518 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{+}^{AT}(\gamma_5^{AT}) = \begin{pmatrix} 0 \\ 0 \\ 0.037083 \\ -5.215253 \\ 0 \\ 0 \\ 0.011154 \\ -8.282254 \end{pmatrix}; \mathbf{H}_{+}^{AT}(\gamma_6^{AT}) = \begin{pmatrix} 0 \\ 0 \\ 23.978775 \\ -1.083295 \\ 0 \\ 0 \\ 7.647594 \\ 0.947807 \end{pmatrix};$$

$$\mathbf{H}_{+}^{AT}(\gamma_{7}^{AT})=\begin{pmatrix}0\\0\\1.257926\\-19.744291\\0\\0\\0.429838\\19.764744\end{pmatrix};\ \mathbf{H}_{+}^{AT}(\gamma_{8}^{AT})=\begin{pmatrix}0\\0\\6.246016\\0.017410\\0\\0\\-35.158474\\-0.019357\end{pmatrix}. \tag{3.5.2}$$

$$\mathbf{H}_{-}^{AT}(\gamma_{1}^{AT})=\begin{pmatrix}2.275308\\-0.015424\\0\\0\\4.313325\\-0.004782\\0\\0\end{pmatrix};\ \mathbf{H}_{-}^{AT}(\gamma_{2}^{AT})=\begin{pmatrix}12.483262\\-1.513470\\0\\0\\-15.773103\\-0.524850\\0\\0\end{pmatrix};\ \mathbf{H}_{-}^{AT}(\gamma_{3}^{AT})=\begin{pmatrix}0.943426\\12.440043\\0\\0\\-1.015283\\4.045526\\0\\0\end{pmatrix};$$

$$\mathbf{H}_{-}^{AT}(\gamma_{4}^{AT})=\begin{pmatrix}-0.023684\\5.600232\\0\\0\\0.034034\\-32.380893\\0\\0\end{pmatrix};\ \mathbf{H}_{-}^{AT}(\gamma_{5}^{AT})=\begin{pmatrix}0\\0\\-0.029731\\3.839210\\0\\0\\-0.008897\\5.741494\end{pmatrix};\ \mathbf{H}_{-}^{AT}(\gamma_{6}^{AT})=\begin{pmatrix}0\\0\\-0.029731\\3.839210\\0\\0\\-0.008897\\5.741494\end{pmatrix};$$

$$\mathbf{H}_{-}^{AT}(\gamma_7^{AT}) = \begin{pmatrix} 0 \\ 0 \\ -1.133307 \\ 17.872250 \\ 0 \\ 0 \\ -0.386319 \\ -17.551184 \end{pmatrix}; \mathbf{H}_{-}^{AT}(\gamma_8^{AT}) = \begin{pmatrix} 0 \\ 0 \\ -5.753465 \\ -0.016121 \\ 0 \\ 0 \\ 32.324306 \\ 0.017584 \end{pmatrix}. \tag{3.5.3}$$

$$\mathbf{H}_{+}^{OC}(\gamma_1^{OC}) = \begin{pmatrix} -4.305306 \\ 0.170320 \\ 0 \\ 0 \\ -6.389100 \\ 0.052181 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{+}^{OC}(\gamma_2^{OC}) = \begin{pmatrix} -4.831111 \\ -17.055099 \\ 0 \\ 0 \\ 5.601706 \\ -5.558499 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{+}^{OC}(\gamma_3^{OC}) = \begin{pmatrix} -12.395444 \\ 6.072300 \\ 0 \\ 0 \\ 17.707087 \\ 2.635219 \\ 0 \\ 0 \end{pmatrix};$$

$$\mathbf{H}_{+}^{OC}(\gamma_4^{OC}) = \begin{pmatrix} 0.423068 \\ -6.356229 \\ 0 \\ 0 \\ -0.629308 \\ 35.164305 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{+}^{OC}(\gamma_5^{OC}) = \begin{pmatrix} 0 \\ 0 \\ -0.026533 \\ 5.556992 \\ 0 \\ 0 \\ -0.008093 \\ 8.929680 \end{pmatrix}; \mathbf{H}_{+}^{OC}(\gamma_6^{OC}) = \begin{pmatrix} 0 \\ 0 \\ 23.854804 \\ -0.752152 \\ 0 \\ 0 \\ 8.060328 \\ 0.663645 \end{pmatrix};$$

$$\mathbf{H}_{+}^{OC}(\gamma_7^{OC}) = \begin{pmatrix} 0 \\ 0 \\ 0.873107 \\ -19.784462 \\ 0 \\ 0 \\ 0.346546 \\ 20.021149 \end{pmatrix}; \mathbf{H}_{+}^{OC}(\gamma_{8;}^{OC}) = \begin{pmatrix} 0 \\ 0 \\ -6.605761 \\ -0.031300 \\ 0 \\ 0 \\ 35.098295 \\ 0.034141 \end{pmatrix}. \qquad (3.5.4)$$

$$\mathbf{H}_{-}^{OC}(\gamma_1^{OC}) = \begin{pmatrix} 2.537654 \\ -0.089589 \\ 0 \\ 0 \\ 4.092864 \\ -0.027878 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{-}^{OC}(\gamma_2^{OC}) = \begin{pmatrix} 4.074931 \\ 14.461782 \\ 0 \\ 0 \\ -4.821307 \\ 4.731007 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{-}^{OC}(\gamma_3^{OC}) = \begin{pmatrix} 10.642340 \\ -5.253213 \\ 0 \\ 0 \\ -15.570149 \\ -2.287271 \\ 0 \\ 0 \end{pmatrix};$$

$$\mathbf{H}_{-}^{OC}(\gamma_4^{OC}) = \begin{pmatrix} -0.386414 \\ 5.833615 \\ 0 \\ 0 \\ 0.583321 \\ -32.335884 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{-}^{OC}(\gamma_5^{OC}) = \begin{pmatrix} 0 \\ 0 \\ 0.021273 \\ -4.167762 \\ 0 \\ 0 \\ 0.006455 \\ -6.395780 \end{pmatrix}; \mathbf{H}_{-}^{OC}(\gamma_6^{OC}) = \begin{pmatrix} 0 \\ 0 \\ -21.083748 \\ 0.666423 \\ 0 \\ 0 \\ -7.103615 \\ -0.580715 \end{pmatrix};$$

$$\mathbf{H}_{-}^{OC}(\gamma_7^{OC}) = \begin{pmatrix} 0 \\ 0 \\ -0.786610 \\ 17.892518 \\ 0 \\ 0 \\ -0.311460 \\ -17.821912 \end{pmatrix}; \mathbf{H}_{-}^{OC}(\gamma_8^{OC}) = \begin{pmatrix} 0 \\ 0 \\ 6.084840 \\ 0.028943 \\ 0 \\ 0 \\ -32.268979 \\ -0.031108 \end{pmatrix}; \qquad (3.5.5)$$

The above structure of the eigenvectors is due to our choice of the intensity vector representation (2.11.1) and (2.11.2).

We now have all that we require to write the solution of the homogeneous RTE (3.1.1 – 3.1.2) for both atmosphere and ocean. We let a (4Nx1) matrix as

$$\mathbf{H}_{AT/OC}(\pm) = \left[\mathbf{I}_{AT/OC}(z,\pm\mu_1)^T, \mathbf{I}_{AT/OC}(z,\pm\mu_2)^T, \mathbf{I}_{AT/OC}(z,\pm\mu_3)^T, \ldots\ldots, \mathbf{I}_{AT/OC}(z,\pm\mu_N)^T\right]^T \qquad (3.5.6)$$

The homogeneous solution can now be written as

$$\mathbf{H}_{AT}(+) = \sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_w - z}{\gamma_j^{AT}}\right) \qquad (3.5.7a)$$

$$\mathbf{H}_{OC}(+) = \sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_{+}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_{-}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right) \qquad (3.5.7b)$$

$$\mathbf{H}_{AT}(-) = \mathbf{\Delta}\sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right) \qquad (3.5.8a)$$

$$\mathbf{H}_{OC}(-) = \mathbf{\Delta}\sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_{-}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_{+}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right) \qquad (3.5.8b)$$

$\mathbf{\Delta} = \mathrm{diag}\{D, D, D, \ldots, D\}$ (2.5.9) is **(4Nx4N)** matrix.

Our new task is to find the arbitrary coefficients A and B. These coefficients are to be found from the boundary conditions. Equations (2.15.47 – 2.15.49) & (2.15.50 – 2.15.52) can be evaluated for each $\pm\mu_i$ which give **4N** equations for **4N** unknown coefficients A and B.

As for example for **N=2** we have

$$\mathbf{H}_{AT/OC}(\pm) = \left[\mathbf{I}_{AT/OC}(z,\pm\mu_1)^T, \mathbf{I}_{AT/OC}(z,\pm\mu_2)^T\right]^T \qquad (3.5.10)$$

These matrices are (8x1) in dimension.

From (3.5.7) & (3.5.8) and (3.5.6) we can write the form of the equations for positive eigenvectors

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}_+ = A_1\begin{bmatrix} a1 \\ a2 \\ a3 \\ a4 \\ a5 \\ a6 \\ a7 \\ a8 \end{bmatrix}_+ j1 + A_2\begin{bmatrix} b1 \\ b2 \\ b3 \\ b4 \\ b5 \\ b6 \\ b7 \\ b8 \end{bmatrix}_+ j2 + A_3\begin{bmatrix} c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ c7 \\ c8 \end{bmatrix}_+ j3 + A_4\begin{bmatrix} d1 \\ d2 \\ d3 \\ d4 \\ d5 \\ d6 \\ d7 \\ d8 \end{bmatrix}_+ j4 + A_5\begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \\ e5 \\ e6 \\ e7 \\ e8 \end{bmatrix}_+ j5 + A_6\begin{bmatrix} f1 \\ f2 \\ f3 \\ f4 \\ f5 \\ f6 \\ f7 \\ f8 \end{bmatrix}_+ j6 + A_7\begin{bmatrix} g1 \\ g2 \\ g3 \\ g4 \\ g5 \\ g6 \\ g7 \\ g8 \end{bmatrix}_+ j7 + A_8\begin{bmatrix} h1 \\ h2 \\ h3 \\ h4 \\ h5 \\ h6 \\ h7 \\ h8 \end{bmatrix}_+ j$$

$$+ B_1\begin{bmatrix} a1 \\ a2 \\ a3 \\ a4 \\ a5 \\ a6 \\ a7 \\ a8 \end{bmatrix}_- j9 + B_2\begin{bmatrix} b1 \\ b2 \\ b3 \\ b4 \\ b5 \\ b6 \\ b7 \\ b8 \end{bmatrix}_- j10 + B_3\begin{bmatrix} c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ c7 \\ c8 \end{bmatrix}_- j11 + B_4\begin{bmatrix} d1 \\ d2 \\ d3 \\ d4 \\ d5 \\ d6 \\ d7 \\ d8 \end{bmatrix}_- j12 + B_5\begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \\ e5 \\ e6 \\ e7 \\ e8 \end{bmatrix}_- j13 + B_6\begin{bmatrix} f1 \\ f2 \\ f3 \\ f4 \\ f5 \\ f6 \\ f7 \\ f8 \end{bmatrix}_- j14 + B_7\begin{bmatrix} g1 \\ g2 \\ g3 \\ g4 \\ g5 \\ g6 \\ g7 \\ g8 \end{bmatrix}_- j15 + B_8\begin{bmatrix} h1 \\ h2 \\ h3 \\ h4 \\ h5 \\ h6 \\ h7 \\ h8 \end{bmatrix}_- j16$$

(3.5.11)

In equation (3.5.11) RHS represent the values of the Stokes components. The first four values i.e. a, b, c, d are for μ_1 whereas e, f, g, h represent values for μ_2. In LHS the set {a1, a2, a3, a4, a5, a6, a7, a8} represents values of the eigenvector calculated from equation (3.5.1) for **j=1** and similarly for other values of j for appropriate cases. The suffix plus and minus represent the values corresponding for positive and negative eigenvectors. We also have for negative eigenvectors,

$$
\begin{bmatrix}a\\b\\c\\d\\e\\f\\g\\h\end{bmatrix}_{-} = \mathbf{\Delta}[A_1\begin{bmatrix}a1\\a2\\a3\\a4\\a5\\a6\\a7\\a8\end{bmatrix}_{-} j1 + A_2\begin{bmatrix}b1\\b2\\b3\\b4\\b5\\b6\\b7\\b8\end{bmatrix}_{-} j2 + A_3\begin{bmatrix}c1\\c2\\c3\\c4\\c5\\c6\\c7\\c8\end{bmatrix}_{-} j3 + A_4\begin{bmatrix}d1\\d2\\d3\\d4\\d5\\d6\\d7\\d8\end{bmatrix}_{-} j4 + A_5\begin{bmatrix}e1\\e2\\e3\\e4\\e5\\e6\\e7\\e8\end{bmatrix}_{-} j5 + A_6\begin{bmatrix}f1\\f2\\f3\\f4\\f5\\f6\\f7\\f8\end{bmatrix}_{-} j6 + A_7\begin{bmatrix}g1\\g2\\g3\\g4\\g5\\g6\\g7\\g8\end{bmatrix}_{-} j7 + A_8\begin{bmatrix}h1\\h2\\h3\\h4\\h5\\h6\\h7\\h8\end{bmatrix}_{-} j
$$

$$
+ B_1\begin{bmatrix}a1\\a2\\a3\\a4\\a5\\a6\\a7\\a8\end{bmatrix}_{+} j9 + B_2\begin{bmatrix}b1\\b2\\b3\\b4\\b5\\b6\\b7\\b8\end{bmatrix}_{+} j10 + B_3\begin{bmatrix}c1\\c2\\c3\\c4\\c5\\c6\\c7\\c8\end{bmatrix}_{+} j11 + B_4\begin{bmatrix}d1\\d2\\d3\\d4\\d5\\d6\\d7\\d8\end{bmatrix}_{+} j12 + B_5\begin{bmatrix}e1\\e2\\e3\\e4\\e5\\e6\\e7\\e8\end{bmatrix}_{+} j13 + B_6\begin{bmatrix}f1\\f2\\f3\\f4\\f5\\f6\\f7\\f8\end{bmatrix}_{+} j14 + B_7\begin{bmatrix}g1\\g2\\g3\\g4\\g5\\g6\\g7\\g8\end{bmatrix}_{+} j15 + B_8\begin{bmatrix}h1\\h2\\h3\\h4\\h5\\h6\\h7\\h8\end{bmatrix}_{+} j16 \tag{3.5.12}
$$

Required 16 unknown coefficients can now be calculated in general from these 16 equations (3.5.11) and (3.5.12) using boundary conditions on RHS of each equation.
Sine both the eigenvector and eigenvalues may be complex we want to write equations (3.5.7) and (3.5.8) in terms of real quantities. Let us represent **Nr** for number of real separation constants and **Nc** for the number of complex pairs of separation constants. With these we rewrite equations (3.5.7) and (3.5.8) as

$$
\mathrm{RE}_{+}^{AT}(z) = \sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right), \tag{3.5.13a}
$$

$$
\mathrm{RE}_{+}^{OC}(z) = \sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_{+}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_{-}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{AT/OC}}\right), \tag{3.5.13b}
$$

$$
\mathrm{CO}_{+}^{AT}(z) = \sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{AT(\alpha)}F_{+}^{AT(\alpha)}(z,\gamma_j^{AT}) + B_j^{AT(\alpha)}F_{-}^{AT(\alpha)}(z_\omega - z,\gamma_j^{AT}), \tag{3.5.14a}
$$

$$
\mathrm{CO}_{+}^{OC}(z) = \sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_{+}^{OC(\alpha)}(z,\gamma_j^{OC}) + B_j^{OC(\alpha)}F_{-}^{OC(\alpha)}(z_1 - z,\gamma_j^{OC}), \tag{3.5.14b}
$$

$$RE_{-}^{AT}(z) = \Delta \sum_{j=1}^{Nr} A_j^{AT} \mathbf{H}_{-}^{AT}(\gamma_j^{AT}) \exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT} \mathbf{H}_{+}^{AT}(\gamma_j^{AT}) \exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right), \quad (3.5.15a)$$

$$RE_{-}^{OC}(z) = \Delta \sum_{j=1}^{Nr} A_j^{OC} \mathbf{H}_{-}^{OC}(\gamma_j^{OC}) \exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC} \mathbf{H}_{+}^{OC}(\gamma_j^{OC}) \exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right), \quad (3.5.15b)$$

$$CO_{-}^{AT}(z) = \sum_{j=1}^{Nc} \sum_{\alpha=1}^{2} A_j^{AT(\alpha)} F_{-}^{AT(\alpha)}(z, \gamma_j^{AT}) + B_j^{AT(\alpha)} F_{+}^{AT(\alpha)}(z_\omega - z, \gamma_j^{AT}), \quad (3.5.16a)$$

$$CO_{-}^{OC}(z) = \sum_{j=1}^{Nc} \sum_{\alpha=1}^{2} A_j^{OC(\alpha)} F_{-}^{OC(\alpha)}(z, \gamma_j^{OC}) + B_j^{OC(\alpha)} F_{+}^{OC(\alpha)}(z_1 - z, \gamma_j^{OC}), \quad (3.5.16b)$$

$$F_{\pm}^{AT/OC(1)}(z, \gamma_j^{AT/OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\} \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\} \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC})\right\}$$

$$= F^{AT/OC}1(\pm1) - F^{AT/OC}1(\pm2). \quad (3.5.17)$$

$$F_{\pm}^{AT/OC(2)}(z, \gamma_j^{AT/OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\} \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\} \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC})\right\}.$$

$$= F^{AT/OC}2(\pm1) - F^{AT/OC}2(\pm2). \quad (3.5.18)$$

We shall use following notations for different values of z in section 3.

$$F_{\pm}^{AT(1)}(z_\omega, \gamma_j^{AT}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\} \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\} \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\}$$

$$= F^{AT}11(\pm1) - F^{AT}11(\pm2).$$

$$F_{\pm}^{AT(1)}(0, \gamma_j^{AT}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\} \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\} \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\}$$

$$= \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} - \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} = F^{AT}12(\pm1) - F^{AT}12(\pm2).$$

$$F_{\pm}^{AT(2)}(z_\omega, \gamma_j^{AT}) = \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\} \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\} \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\}$$

$$= F^{AT}21(\pm 1) - F^{AT}21(\pm 2).$$

$$F_{\pm}^{AT(2)}(0,\gamma_j^{AT}) = \mathrm{Re}\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\} - \mathrm{Im}\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\} = F^{AT}22(\pm 1) - F^{AT}22(\pm 2)$$

$$F_{\pm}^{OC(1)}(z_\omega,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\}$$

$$= F^{OC}11(\pm 1) - F^{OC}11(\pm 2).$$

$$F_{\pm}^{OC(1)}(z_1 - z_\omega,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\} - \mathrm{Im}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\}$$

$$= F^{OC}12(\pm 1) - F^{OC}12(\pm 2).$$

$$F_{\pm}^{OC(2)}(z_\omega,\gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\} - \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\}$$

$$= F^{OC}21(\pm 1) - F^{OC}21(\pm 2).$$

$$F_{\pm}^{OC(2)}(z_1 - z_\omega,\gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\} - \mathrm{Re}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\}$$

$$= F^{OC}22(\pm 1) - F^{OC}22(\pm 2).$$

$$F_{\pm}^{OC(1)}(z_1,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\} - \mathrm{Im}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\}$$

$$= [\ F^{OC}13(\pm 1) - F^{OC}13(\pm 2)\ ]$$

$$F_{\pm}^{OC(2)}(z_1,\gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\} - \mathrm{Re}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\}$$

$$= [\ F^{OC}23(\pm 1) - F^{OC}23(\pm 2)\ ]$$

3.6. Albedo of ocean surface:

The oceanic surface albedo (OSA) plays very important role in the determination of the energy change processes between atmosphere and ocean. This is an important issue in the coupled atmosphere ocean system. Determination of OSA is an important task. In the last several decades several OSA schemes have been proposed and both observational and theoretical investigations are carried out. But analytical expressions and the dependent variable that comprises these schemes of investigations differ from one another to a great extent. There are models that depend only on solar zenith angle (SZA) while there are schemes that use additional parameters like wind speed or cloud optical depth or both. All most all the proposed investigations assumed valid for one albedo for the whole incident spectrum. However there exist schemes for albedo for differing wavelengths. Some schemes consider only clear or cloudy sky conditions while there are schemes which considered long time averages over both conditions.

One can assume that this wide variety of OSA schemes might be associated with differing radiative impact. This is generally a complicated process. Effective tools for such study are the one-dimensional radiative transfer models by which the influence of OSA on the upward flux at the top of the atmosphere (TOA) and the solar energy flow at the surface can be analyzed. This tool provides a controlled way of highlighting differences in the radiative forcing associated with each OSA scheme. It also provides a basis for understanding the ultimate response of the climate system in fully interactive global circulation model (GCM) climate integrations.

Canadian centre for climate modeling and analysis (CCCMA) have derived schemes for operational OSA parameterizations The CCCma second-generation atmospheric GCM (McFarlane et al. [170]) employed a relatively simple scheme for OSA, which depended on SZA and was independent of sky and surface wind conditions. The third-generation CCCma model, AGCM3 (McFarlane et al. [171]), employed the Hansen et al. [172] fit to Cox and Munk's approximate theory (Cox and Munk [215]. This fit was both a function of SZA and wind speed. We shall refer to Hansen et al. [172] as the H scheme. Validity of these schemes is discussed in detail by Barker and Li [219]. To correct this deficit Barker and Li simply adjusted the lead constant in the Hansen formulation, corresponding to a vertical shift or uniform increase, to enhance the albedo to more reasonable values.

In the Atmospheric general circulation model (AGCM4) the theory of Preisendorfer and Mobley (PM) [213] is used to help formulate the OSA. This theory is more accurate than Cox and Munks approximation. For example, it includes the reflection for the diffused rays as well as the orientation of the wind relative to the incoming solar flux. A comparative study was made in Li [378]. The parameterization used in AGCM4 is simply an approximate fit to the PM result. No attempt was made to account for the orientation of the wind relative to the direction of the incoming solar flux. This is only an issue at large SZA. The specific form of this fit is presented in [219]. We shall refer to the fit as the PM scheme. A comparison of the PM scheme to the H scheme used in AGCM3 is displayed in [378].Here we see the PM parameterization increases the albedo relative to the H scheme in a way that is arguably more physical than simply increasing all values by a constant.

The first two are simple in the sense that they depend solely on SZA. As such they represent time averages over other factors such as wind speed and direct versus diffuse conditions. The first scheme of Briegleb et al. [111] represents a fit to the observations of Payne [211]. The form of the OSA is expressed as

$$\omega^{OC}(\mu_0) = \frac{0.026}{1.1\mu_0^{1.7} + 0.065} + 0.15(\mu_0 - 0.1)(\mu_0 - 0.5)(\mu_0 - 1) \quad , \tag{3.6.1}$$

where $\omega(\mu_0)$ is the broadband OSA and μ_0 is the cosine of SZA (CSZA). Hereafter we shall refer to this formulation as the B scheme.

The second scheme of Taylor et al. [126] represents a fit to 5 yr of observations compiled by aircraft measurements. The form of the OSA in this scheme is expressed as

$$\omega^{OC}(\mu_0) = \frac{0.037}{1.1\mu_0^{1.4} + 0.15} . \tag{3.6.2}$$

This scheme is popularly known as T scheme. The third scheme of Jin et al. [372] derives its SZA and wind speed dependence from plane-parallel radiative transfer calculations that include the wind-blown roughened surface within the domain of the calculation. The surface is discretized by a set of inclined planes with random slopes that follow probability distributions given by Cox and Munk [215]. This scheme introduces an empirical dependence on aerosol/cloud optical depth to account for the effect of the direct and diffuse fluxes. Hereafter we shall refer to this formulation as the J scheme. Generally it has been found that the larger the aerosol/cloud optical depth the larger the fraction of the downward diffuse flux to the surface. There exist a number of factors that may contribute to the aerosol/cloud optical depth. Aerosol concentrations are typically largest in the lower troposphere while the location of cloud varies from surface to tropopause. For example, one value of optical depth could correspond to a variety of cloud and aerosol distributions. In principle the OSA will be sensitive to these different distributions since the fraction of direct relative to diffuse will be altered.

Analytical Formula for the PM Scheme

PM [213] is a scheme using the ray-tracing method based on Fresnel reflection on the ocean surface. The OSA of the PM scheme is parameterized as follows. A reference wind speed, U_0, is defined in terms of zenith angle as

$$U_0 = 180\mu_0^3(1 - \mu_0^2) . \tag{3.6.3}$$

For wind speed $U_s \geq U_0$ the direct component of albedo is given as

$$\omega_{dir}^{OC}(\mu_0, U_s) = 0.021 + 0.0421(1-\mu_0)^2 + 0.128(1-\mu_0)^3 - 0.04(1-\mu_0)^6$$

$$+\left[\frac{4}{5.68+U_s-W_0}+\frac{0.074(1-\mu_0)}{1+3(U_s-U_0)}\right](1-\mu_0)^6, \quad (3.6.4)$$

but when $U_s < U_0$ the direct component is given as

$$\omega_{dir}^{OC}(\mu_0,U_s)=\left[1+\frac{5.4\mu_0^2(1-\mu_0^2)U_s(U_s-1.1U_0)^2}{W_0^3}\right]\times\left\{ 0.021+0.0421(1-\mu_0)^2+0.128(1-\mu_0)^3\right.$$

$$\left.-0.04(1-\mu_0)^6+\left[\frac{4}{5.68+U_s-U_0}+\frac{0.074(1-\mu_0)}{1+3(U_s-U_0)}\right](1-\mu_0)^6\right\}. \quad (3.6.5)$$

In either case the diffuse component is given as

$$\omega_{dif}^{OC}(U_s)=0.022\left\{1+0.55\exp\left[-\left(\frac{U_s}{7}\right)^2\right]+1.45\exp\left[-\left(\frac{U_s}{40}\right)^2\right]\right\}. \quad (3.6.6)$$

We have used above formula in our numerical consideration.

Daily Average for the 1D Model: It now desirable to discuss how to determine the incident directions of solar beam in different locations. We briefly discuss how to calculate the daily averaged results for different latitudes, since this has not been addressed in the one-dimensional radiative transfer study. This method provides a more realistic comparison to the results of GCM. The solar zenith angle θ_0 is generally given by

$$\cos\theta_0=\sin\delta\sin\varphi+\cos\delta\cos\varphi\cos\omega, \quad (3.6.7)$$

where δ is declination angle, φ is the geographic latitude, and ω is hour angle with noon zero and morning positive. The declination angle is a function of day number (Iqbal 1983) as $\delta=f(\Gamma)$, where $\Gamma=\frac{2\pi(d_n-1)}{365}$,where d_n is the day number of the year, ranging from 1 on 1 January to 365 on 31 December. At the sunrise moment $\theta_0=\frac{\pi}{2}$. From (3.6.7) $\omega_s=\cos^{-1}(-\tan\theta\tan\varphi)$, which is the hour angle for the sunrise, thus $-\omega_s$ is the hour angle for the sunset. The integral of ω over $\pm\omega_s$ generates the daily averaged results. Note, if $\cos\omega_s>1$ there is no sunrise (polar night); if $\cos\omega_s\le-1$ there is no sunset and the solar zenith angle is given by (3.6.7) with $\pi\ge\omega-\pi$.