"POLARIZED RADIATIVE TRANSFER IN ATMOSPHERE OCEAN SYSTEM"

THE THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN SCIENCE (MATHEMATICS) UNDER THE UNIVERSITY OF NORTH BENGAL

SUBMITTED BY

Sri Apurva Kumar Sarkar

Under the supervision of

Dr. G.K. Biswas

Prof. S.B. Karanjai

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH BENGAL
DARJEELING, INDIA - 734013

ACKNOWLEDGMENT

Life often turns on small pivots. Such is the case with my initiation in Radiative transfer research. A brief visit to Dr. G.K.Biswas many years back about further studies in this field culminated in this work on polarized Radiative transfer theory. Without Dr.G.K.Biswas and Dr.S.Karanjai's patient guidance, this thesis would never have been possible. I especially appreciate the independence Dr.G.K.Biswas granted me in my research. I thank, "Department of Mathematics", "University of North Bengal", the University library for their relentless help and support. Finally, I thank my parents and wife, Debotri Sarkar, for their continual encouragement and for hours of proof reading. I am grateful to my wife and my child, Siddharth, for their love, and I dedicate this work to them.

INDEX

Chapter 2:

Formulation of the Problem.

chapter 3:

Solution of polarized homogeneous equation of transfer for coupled atmosphere ocean system.

chapter 4:

Particular & General Solution.

SUMMARY

In Chapter1, we discuss the Radiative transfer problem in detail. A description of the geometrical aspect of atmosphere and ocean and its parameters like extinction and absorption coefficients, Phase functions(OTHG, HG, TTHG, FF,), reflection and transmission functions for wind disturbed surface (Monte Carlo) have been taken up for study. Equations of transfer for non-polarized and polarized radiation field along with the boundary conditions for flat ocean surface and the bottom of the ocean
with the help of reflection and transmission matrix have been worked upon. A systematic approach towards determination of characteristic coefficients and reduction and modeling of the Radiative transfer equation in such system are dealt in detail.

In Chapter 2, we take up the task of solving the equation of transfer employing a new version of Chandrasekhar's discrete ordinate methods (DOM). Here we write the homogeneous version of the reduced Radiative transfer equations for atmosphere and ocean medium. We shall use the set of separated but coupled non-linear integro differential equations and for each component of stokes vector for discretization. We shall use Chandrasekhar discretization scheme to break the continuous radiation field into **2N** quadrature directions with corresponding weights keeping optical depth dependence exact, for $\mathbf{i,k} = \pm 1, \pm 2, \pm 3, \ldots, \pm N$.This enable us to form a set of n equations for each $\mathbf{i}$ (negative as well as positive). Each **homogeneous version of equations** is replaced by N equivalent equations, separated for positive and negative quadrature directions, with corresponding Gaussian weight functions expressed and interchanging the Gaussian summation with Fourier summation.
Computation for Eigen functions and Eigen values are worked upon in great detail along with the wind dependent albedo which finds a prominent place in further discussions.

In Chapter 3, we try to develop particular solution for the Radiative transfer equation. Here we proceed with the approach using a modified discrete ordinate method developed by C.E Siewert and accommodate for the inhomogeneous source term. The elementary solutions developed in the previous chapter is used to construct the green functions which in turn are required to find the particular solution i.e. to express it in terms of infinite-medium Green's function. The four boundary conditions are used to find the unknown constants. Finally the homogeneous solution added together with the particular solution found in this section gives the intensity of the Radiative transfer equation. We have established exact analytic forms of the polarized intensity from both the medium.

Chapter 4, deals with the computation of the positive and negative Eigen vectors along with the separation constants for various parameters as discussed under different sections for both the atmosphere and ocean. This section is also devoted towards meticulously found results of the four outgoing stokes parameters from ocean media for different directions.
Chapter 5 depicts the numerical results for certain specified dataset for atmosphere and ocean:

CHAPTER: I

INTRODUCTION

1.1. Importance of Radiative transfer theory in Biosphere:

About 30% of the Earth's surface is covered by land and till date most part of this is vegetated. Rest is ocean. Thus, land surface and ocean processes are important components of the terrestrial climate system which essentially maintains nurtures and controls evolution in the life support systems for all living organisms in the earth. The only natural machine which supplies the driving energy to maintain the huge macroscopic or microscopic harmony between each and every element, living or non-living, throughout their existence, is sun. The bulk of the solar energy provided to the troposphere transits through to the lower boundary (atmosphere, oceans and continents) first and is made available to the atmosphere through the fluxes of sensible and latent heat and thermal radiation. Accurate descriptions of the interaction of radiation with surface vegetation, atmosphere and ocean processes require quantitative as well as qualitative information on fluxes of energy (radiation) and mass (water vapor and CO), which are strong functions of photosynthetic and evapotransportation rates. The energy received from the sum in the form of electromagnetic energy primarily drives the whole engine that sustains the biosphere through numerous kinds of interactions.

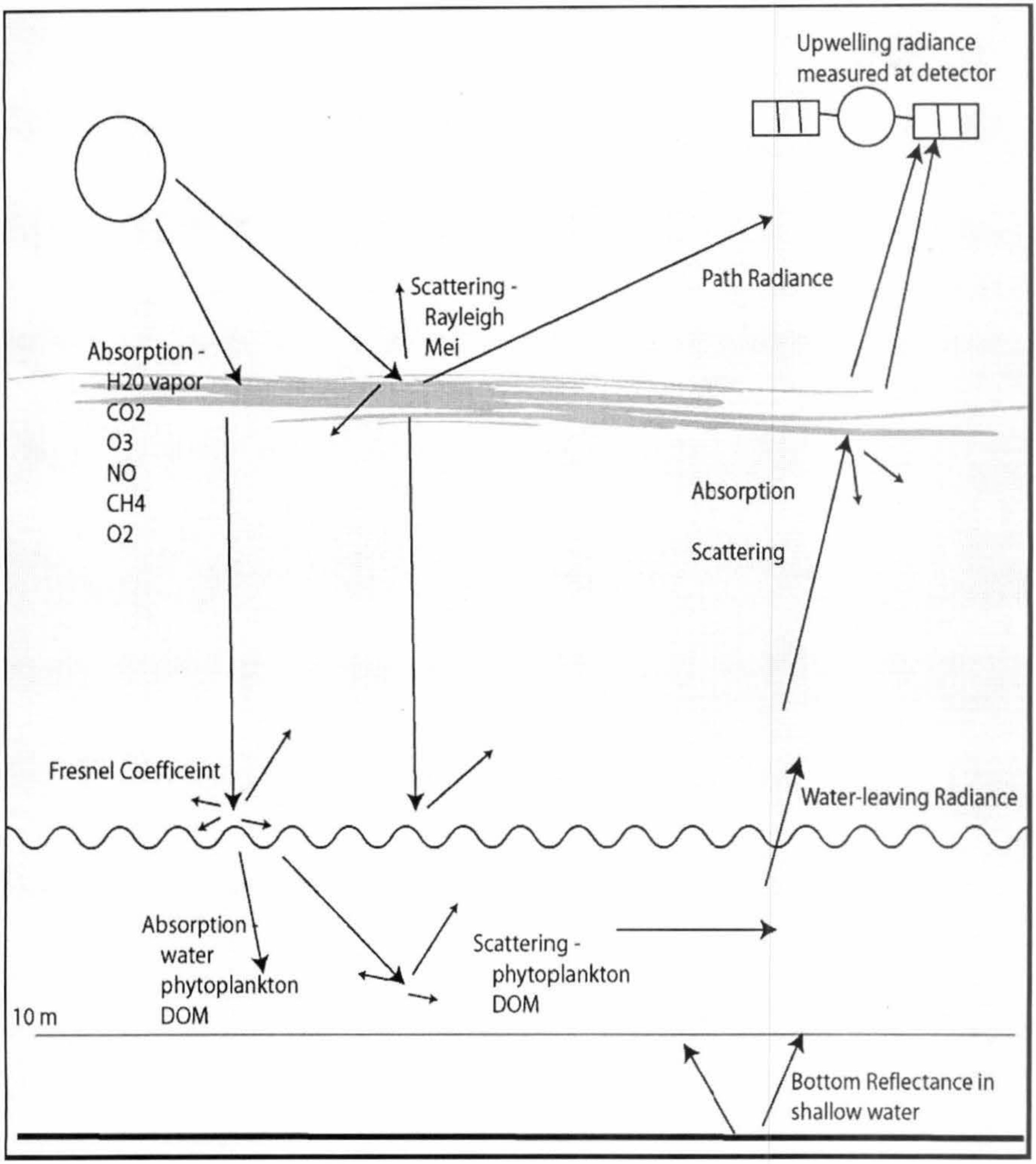
Solar radiation, as mentioned, is the primary energy source for the atmospheric general circulation and the hydrological cycle. As electromagnetic radiation (EMR) reaches the earth's surface, molecules and particles of the land, water, ocean and atmosphere environments interact with solar energy in the 400-2500nm spectral region through absorption, reflection, transmission and scattering. Some materials will reflect certain wavelengths of light, while other materials will absorb certain wavelengths. Many materials have unique patterns of reflectance and absorption across the electromagnetic spectrum. Before radiances are recorded at any sensor, the electromagnetic radiation has already passed through the atmosphere twice or more (sun to atmosphere then ocean and again after reflection to the sensor through atmosphere). During this process (Figure 1), EMR has been modified by the processes of scattering by air molecules and aerosols, oceanwater, oceanparticles and by absorption.

The processes of scattering require special citation. The incident solar radiation suffers scattering with appreciable changes in its initial direction of propagation several times before it reaches ocean surface. These scattered photons again interact with ocean water as well as suspended particles present within it. The science of atmospheric scattering is different from ocean scattering. However in general the scattering of light can be expressed mathematically in terms of phase functions. We have devoted one subsection to discuss the mathematical properties of phase functions in the next section. The main contributors to gaseous absorption are atmospheric water vapor, carbon dioxide, ozone, nitrous oxide, carbon monoxide, methane and oxygen (GAO, [1]). Major atmospheric bands, such as those of water vapor centered at approximately 940, 1140, 1380 and 1880 nm, the oxygen band at 760 nm, and the CO_2 band near 2006 nm can influence the relative brightness over an image. If the EMR interacts with the surface of the water, the reflection scattering and transmission of light across the air-water interface are governed by the Fresnel coefficient of reflection, which is modified by surface roughness, a function of wind speed (Hamilton et al, [2]). Photons can also penetrate the water column and can be absorbed by water itself, photosynthetic plant material, or dissolved organic substances. The fate of sunlight incident on the ocean or water surface may be represented by the diagram (Fig.2). Some fraction is reflected at the surface, the rest is refracted, scattered and finally absorbed in the water below. The diagram neglects diffuse illumination from the sky, waves, ripples, foam and the rest of the interesting detail just at the surface. We note that while the illumination of the surface may be a directed beam (as shown in Fig.2) during the middle of a bright day it will certainly be diffuse both on an overcast day and during sunrise and sunset. Just below the surface is a transition region in which the light experiences its first few scattering interactions with the water. Despite the sharp line shown dotted in the figure, the transition region actually has an indefinite lower boundary as the light distribution settles toward its asymptotic form. We shall also neglect this transition region in our calculational details but it is worth to mention that this region is likely to be narrower when the incident illumination is diffuse. Below the transition region lies the relatively orderly regime of downwelling light where many details of the surface illumination and sea state have been smoothed over and lost and the light is diffuse.

Multiple scattering effects by both photosynthetic and non photosynthetic particles can lead to increase probability of eventual absorption, or photons can be eventually ejected back through the surface (Hamilton et al, [2]). Upwelled light emerging from the surface carries information on all these processes. Because nearly all absorption takes place in the upper two to three attenuation lengths (a few to tens of meters), this information is confined to that region, and quantities derived from the water-leaving spectral radiance are generally referred to as surface values [72].

Transfer of solar radiation (Electromagnetic Radiation) and its spatial and temporal variations drive the general atmospheric and oceanic circulation and the hydrological cycle in the earth. The coupling between an atmospheric general circulation model (AGCM) and an oceanic general circulation model (OGCM) depends strongly on the radiative energy flow through the earth–atmosphere system. For the radiative energy budget near the surface the shortwave solar energy accounts for most of the heat flux transferred to the ocean. The solar radiation transferred into the upper-ocean layers affects the stability of the ocean mixed layer and the sea surface temperature. Consequently, the oceanic surface albedo (OSA) plays a key role in determining the energy flow exchange between atmosphere and ocean and so is an important issue for the coupling of atmosphere and ocean models.

One of the major sources of uncertainty in climate prediction lies in the radiative energy flow through the earth–atmosphere system and the radiative interactions between the atmosphere and the hydrosphere. The radiative energy budget is the most important component of the air–sea energy flux. In particular, the shortwave solar energy accounts for most of the heat flux transferred into the ocean. The absorption of solar radiation by the upper layers affects the stratification and stability of the ocean mixed layer, the sea surface temperature, and the general circulation of the ocean. Because solar radiation is the energy source for photosynthesis, it influences marine primary productivity directly, and impacts cascade throughout ocean ecosystems. The atmospheric radiation budget has long been recognized as fundamental to our understanding of the climate system ([3] Houghton et al. (1996)). Surface radiation measurements are essential for the validation of both radiative transfer models and flux retrieval algorithms using satellite data. Hence ground stations have been established to monitor the radiation budget. This has led to a series of comparisons between models and measurements (e.g., Kinne et al. [4]); Conant et al. [5]; Chou and Zhao [6]; Charlock and Alberta [7]; Kato et al. [8]). However, most measurements are over inhabited areas, and routine observations over the ocean are extremely limited. It is difficult to make accurate surface radiation measurements from a moving platform such as a ship or buoy.



Upwelling radiance measured at detector
Path Radiance
Scattering - Rayleigh Mei
Absorption - H20 vapor CO2 O3 NO CH4 O2
Absorption
Scattering
Fresnel Coefficeint
Water-leaving Radiance
Absorption - water phytoplankton DOM
Scattering - phytoplankton DOM
10 m
Bottom Reflectance in shallow water

(Fig.1)

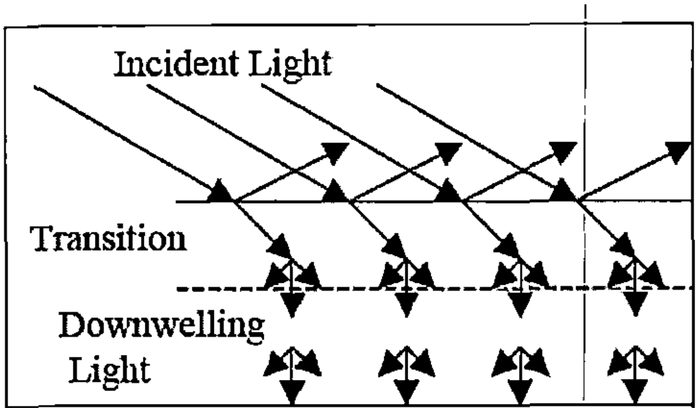


Fig (2)

1.2. What is radiative transfer?

The radiative transfer equation describes the propagation and scattering of electromagnetic radiation at any point inside an absorbing and scattering medium. Mathematically it consists of an integro-differential equation describing balance between the changes suffered by the energy (measured in terms of intensity) of radiations during its journey through any medium and the difference between the generation and loss of the energy. Since every medium (atmosphere, ocean, planetary-atmosphere, steller- atmosphere, etc.) have either some source of energy within or outside its periphery (like planets) this change in energy, if measurable or detectable, can be used to a large extent to retrieve the data containing informations about the nature and properties of the internal constituents of the prevailing medium through which the radiation have traversed. This is the key idea that drives the researcher to apply this equation in those fields where informations are to be collected remotely. The subject of radiative transfer was studied and formulated principally by astrophysicists for analysis of radiation coming from celestial bodies to retrieve information on the structure and constitution of the atmosphere of the astronomical objects. Soon it was realized that this equation is of much relevance to a variety of problems that arises in atmospheric sciences, nuclear reactor theory, optics, atmospheric-ocean processes and climatology or meteorology. Historically, it was developed in the early 1900's by astrophysicists and meteorolo-gists studying electromagnetic rotation in planetary, Stellar and terrestrial atmospheres (Schuster [41]). Later transfer theory became known by the name as transport theory in particle and Nano-paticle regime. Nowadays radiative transfer covers a variety of fields, including astrophysics, applied physics, optics, planetary sciences, atmospheric sciences, and meteorology, as well as various engineering disciplines.

Although the majority of the earth's surface is covered by water, in the early part of fifties of the last century, there had been remarkably little research done on the influence of various atmospheric and oceanic properties on the quantitative interaction between sunlight and the oceans. The path of a solar photon through the atmosphere and ocean can be very complex. As mentioned earlier, it may undergo absorption and multiple scattering by aerosols and atmospheric molecules, reflection and refraction at the ocean surface, and further absorption and multiple scattering by hydrosols and water molecules of the ocean. Additional complexity arises from scattering and absorption by the ocean floor, from waves on the ocean surface, and from the refraction and reflection (including total internal reflection at some angles) of the upwelling light at the ocean boundary. What is the distribution of the sunlight reflected from the ocean surface with waves? What is the angular distribution of the radiation within the ocean and what fraction of the radiation enters the upwelling stream and passes back into the atmosphere through the ocean surface? At what angles can an observer above the ocean surface see down into the ocean instead of seeing radiation reflected by the surface? Why can an observer in an airplane seated on the sunlit side easily see the shadows of isolated cumulus clouds upon the ocean, while an observer on the other side of the plane cannot see any trace of such shadows? A qualitative answer to some of these questions is given by Minnaert [264] in his delightful and provocative book on various types of phenomena about light and color. Both the theoretical and experimental knowledge about light in the oceans is reviewed later by Jerlov [265] and Jerlov and Nielsen [266] in their excellent books. The theory of radiative transfer in natural waters has been treated extensively (Preisendorfer [75, 348]; Prieur [349]). The aim of these studies is not only to describe natural and induced light fields under water but also to predict the composition of the water masses from their optical properties – the so-called inverse problems. Application of optical passive remote-sensing techniques concerns treatment of the Upwelled solar (IR) radiance and of irradiance reflectance, defined as the ratio of upwelling and downwelling irradiance. Various models have been developed and applied (Preisendorfer [75, 348]); Prieur [349]). Particularly attractive are irradiance models dealing with optical quantities that are relatively easy to measure as functions of depth and wave-length (see e.g., Tyler and Smith [350]; Spitzer and Wernand [351]). Two-flow (sunlight and fluorescence emissions) models have been previously applied and solved for homogeneous aquatic media (Preisendorfer [75, 348, 352]; Prieur [349]) including the fluorescence term (Kishino et al. [353]; Spitzer and Wernand [354]). The treatment described by Dirks, R. W. J., and Spitzer, D., [355] is restricted to the visible region (400 – 700 nm) and horizontally homogeneous media (i.e., horizontal irradiance variations being negligible compared to vertical variations). Calculations concern only subsurface phenomena and quantities, i.e., no atmospheric effects are included.

In general the transfer of radiation for all wavelengths that occurs in nature, fortunately obeys above mentioned wonderful mathematical equations which can be understood analyse and interpret qualitatively as well as quantitatively if appropriate boundary conditions can be formulated to obtain proper answers to the above mentioned questions. In remote sensing radiative transport theory basically serves as a mechanism for explaining the exchange of electro magnetic energy between the atmosphere and between the space and the earth atmosphere system. The radiation emitted by the atmosphere, ocean or in general target objects are identified and intercepted by the special purpose sensors stationed on board. The data collected through remote sensing are analyzed, interpreted and visualized suitably to make use of it. This technique is used as fundamental in the calculation of radiation energy budget of stars, planets, prediction of crop production, number counts and location of cattle and sheep, fish in ocean, atmospheric and ocean underwater fields, in the determination of reflecting properties of snows, ice, paint surfaces, photo emulsions, blood tissues viruses, bacterias, in the estimation of insulation properties of fiberglass materials, in the researches of thermal structure of blast furnaces and other high temperature machineries.

During the last six decades tremendous activities were undertaken by the scientific community to the development of solution of this RTE equation under different situations of practical interest. The mathematician relishes the technical challenge, whilst a representation of the radiation field is essential for the physicist wishing to understand the properties of stellar atmospheres. Schuster [41], Schwarzschild [88], Milne [51,51a], Eddington [50], Jeans [97], Hopf [52], Bronstein [78], Unsold [53] and others contributed substantially to the understanding of the theory of radiative transfer in connection with the problems of stellar atmospheres. One may find these initial investigations in the monographs stated above.

In the beginning most of the methods of solution were analytical involving beautiful mathematical techniques (Chandrasekhar [54], Sobolev [63], Busbridge [231] and Case & Zweifel [135]). In his landmark book, Chandrasekhar [54] presented the subject of radiative transfer in plane-parallel (one-dimensional) atmospheres as a branch of mathematical physics and developed numerous solution methods and techniques. The field of atmospheric radiation has evolved from the study of radiative transfer. It is now concerned with the study, understanding, and quantitative analysis of the interactions of solar and terrestrial radiation with molecules, aerosols, and cloud particles in planetary atmospheres as well as the surface on the basis of the theories of radiative transfer and radiometric observations made from the ground, the air, and space (Thomas and Stamnes [102] Liou [56, 57]). Fundamental understanding of radiative transfer processes is the key to understanding of the atmospheric greenhouse effects and global warming resulting from external radiative perturbations of the greenhouse gases and air pollution, and to the development of methodologies for inferring atmospheric and surface parameters by means of remote sensing.

Because of the diversity of topics that use radiative transfer theory, a wide range of solution methodologies of the radiative transfer equation (RTE) can be found in the literature. One can consult the monograph of Lenoble [58] and for radiative heat transfer Modest [114].The review article Stamens [59] is also very much informative.

1.3. Why radiative transfer theory for ocean is important?

There exists military and non-military importance of transport theory in ocean. We shall deal with non-military applications of this sub discipline. Photon absorption regulates photosynthesis which drives phytoplankton or primary production at the bottom of the food chain in ocean eco system. The correlation of optical radiance measurements from satellites with other ocean physics measurements (wave characteristics, temperature, salinity, etc.) is becoming more commonplace with recent enhanced efforts in ecological monitoring. Such applications include both remote and is situ sensing of ocean waters, typically done for a passive time-independent surface illumination, which is the emphasis of the discussion here. The solution of the time-dependent transport equation is needed, on the other hand, for active illumination imaging applications such as mine detection or communication by pulsed optical signal propagation.

The types of in-water problems being solved with the classic radiative transfer equation include (a) forward problems for determining the angle-dependent and angle-integrated light field, and (b) inverse problems for determining, for example, (c) the scattering and absorption properties in order to monitor the biological primary production of water or the ecology of coral reefs.

Good introductions to radiative transfer for oceanographic forward problem applications are available. The text by Kirk [323] is for people more interested in the biological aspects of radiative transfer. Mobley [222] gives details of how ocean optical transfer calculations can be done, as sell as a physicist's approach to biological oceanography. The text by Thomas and Stamnes [102] and the monograph by Walker [325] each contain an introduction to atmospheric and oceanic optics; with the former giving more details about atmospheric line-by-line radiative transfer calculations and the latter containing an extensive coverage of sea-surface and refracted light statistics. The monograph by Bukata et al. [321] pertains to water optical properties. For oceanographic inverse problem applications, an excellent review article is that by Gordon [322]. The key features (McCormick [49]) of optical transport in seawater are: Light in the 400 – 700 nm range is of primary interest, where most biological activity occurs and the optical transmission is the greatest; ocean waters are difficult to characterize because: a) they consist of water plus dissolved organic and inorganic matter not well-characterized as to type and location, b) the "microscopic cross sections," from which the effective absorption and scattering properties are determined, can only be idealized (e.g., with spherical or other regular shapes), c) the air-water interface condition is difficult to simulate because of refractive effects and surface waves; ocean waters usually are modeled with the plane parallel approximation for natural light illuminations because: a) the water column is assumed to be layered (i.e., there are minimal horizontal variations of the water constituents), b) the bottom is assumed flat and of uniform composition or very deep, c) the incident radiation from the atmosphere is assumed to be uniform over the sea surface.

The reasons ocean waters are difficult to characterize are:

A) The wavelength-dependent scattering of phytoplankton depends on the composition and particle size that can vary from approximately $0.7\,\mu m$ to $\sim 100\,\mu m$.

B) The suspended sediments are predominately scatterers, rather than absorbers, although strong absorption features have been observed in iron-rich sediment minerals.

C) Colored dissolved organic matter (CDOM) absorbs most strongly in ultraviolet wavelengths with various absorption peaks.

The similarities to other transport problems include: a) The linear Boltzmann equation for neutral particles is the governing equation, b) Active illumination problems (e.g., with a laser beam) usually are three-dimensional, c) The coefficients of individual, optically-active constituents of seawater are assumed additive. d) Phytoplankton and CDOM have the ability to fluoresce by re-emitting light within distinct wavebands that are somewhat longer than the absorbed light, which means a coupled wavelength analysis may be necessary. e) Raman (inelastic scattering events cause a wavelength increase (i.e., down-scattering in energy). f) Bioluminescence is analogous to an external source from neutron-gamma reactions in gamma transport or spontaneous fission in neutron transport analyses.

The differences from most other linear transport problems include: a) "Inverse Problems" (e.g., to characterize properties of medium) are very important, more so than for analyses of designed systems (e.g., nuclear engineering applications), b) a plane-parallel approximation usually is sufficient for waters illuminated by natural light so the radiance depends only on the water depth and the polar angle and azimuthal angle, c) the optical sensors most often used enable a simplified plane-parallel approximation so the azimuthal angular dependence need not be determined, d) The scattering is very strongly forward peaked (e.g., more like the scattering of atmospheric aerosols than neutrons), with peak forward-to-backward scattering ratios $O(10^5 - 10^6)$ (Petzold [324], Mobley [222]) the refractive index mis-match at the air-water interface causes angle-dependent internal reflection for some surface emerging directions, g) air-water interface waves are hard to characterize, h) problems with polarization effects have four radiance components for each wavelength.

It is worth emphasizing that ocean waters are dynamically changing systems that are hard to characterize, and hence the challenge of obtaining good input data for computations means the old adage "garbage in, garbage out" is a big concern. Furthermore, the environment for performing optical experiments in the field is often difficult, which leads to appreciable measurement uncertainties. For these reasons computational results are not needed with more precision than measurement uncertainties.

1.4. Why polarization:

The solar incident light interacts with all the components of the atmosphere-ocean system. Each phenomenon of scattering by molecules, aerosols, hydrosols and reflection over the sea surface introduces and modifies the polarization state of light. Therefore, the reflected and the transmitted solar radiation in any atmosphere-ocean system are polarized and contain embedded information about the intrinsic nature of aerosols and suspended matter in the ocean. All most all detail physical informations (i.e., size distribution, composition) about the particles present in the atmosphere-ocean system are available through the measurement and analysis of the spectral and angular polarization signature of the oceanic and atmospheric radiation.

For the description of fully polarized light propagating in a given direction Stokes vector convention is adopted in the literature. In the second chapter we have described in detail the Vector Radiative Transfer Equation (VRTE) for atmosphere –ocean system. There are four Stokes parameters which characterize the energy transported by the electromagnetic wave, its degree of polarization, the direction of polarization and the ellipticity. The first parameter can be any energetic quantity, as radiance, an irradiance, etc., and mathematically described as intensity intensity. The other three parameters are defined, as the first one from the two components of the electric vector on two arbitrary perpendicular axes in the wave plane. These quantities have the same energetic dimensions as intensity. Generally, a reference plane is chosen through the direction of propagation parallel to the reference plane and perpendicular to the same plane. The reference plane is taken as the vertical plane containing the direction of propagation.

It has been found that the principal reason for the greater effectiveness of remote sensing by means of polarization measurements is the significantly higher sensitivity of polarization features to particles size, shape and refractive index as a function of scattering angle and wavelength, than is the case for intensity measurements. The strength of polarization features has been widely demonstrated in the case of aerosol retrievals (Goloub et al., [9]; Chowdhary et al.,[10]); Li et al., [11]. However, retrievals of subsurface particulate matter properties using polarization and remotely sensed data have not been extensively studied yet. This is mainly because of practical difficulties in achieving reliable in-situ measurements. Most of the available observations were carried out decades ago Waterman, [13], Waterman [12]; Ivanoff and Waterman, [14]; Beardsley, [15]; Lundgren and Hojerslev, [16]; Voss and Fry, [17]. Another factor that contributes to reduce the number of studies about oceanic polarization is that most current methods of radiative transfer treat light as a scalar. As an example, the commonly used Hydrolight, CDISORT or Morel and Gentili's [370] Monte Carlo radiative transfer models do not account for the polarization of the oceanic radiation.

In general for the most practical situation consisting in coupling between atmospheric and ocean systems, the polarized radiative transfer equation has been developed and solved mainly to improve the modeling of bidirectional remote-sensing reflectance. The increasingly better performance of the new generations of ocean color sensors implies a more sophisticated description of the signal in which the full radiation field, including polarization is accounted for.

One such model (Chami, M. et al. [19]) of radiative transfer for the global ocean–atmosphere system has been developed (OSOA) which can predict the total and the polarized signals at the top of the atmosphere and at the ocean–atmosphere boundary. A previously developed radiative transfer code based on the successive-orders-of scattering was modified to account for various oceanic parameters. In the first part of the research article[19] a description of the new ocean– atmosphere radiative transfer model ,called the OSOA code, is given, where, in particular, attention is given to computation of the polarized component of the signal in the oceanic layer. The second part of the paper contains a nice description of the applications of the code to remote sensing of coastal and open ocean waters. The influence of marine particles on the polarization of water-leaving radiance is discussed. The radiance and the degree of polarization in the coupled ocean-atmosphere system helped in the study of the polarizing properties of the marine particles (namely phytoplankton and minerals) for different water conditions. Their analysis revealed that the use of the polarization of scattered energy in ocean color algorithms might significantly improve the retrieval of hydrosol properties, especially in coastal waters, [367].

Various techniques have been proposed to obtain numerical solutions of the equations of radiative transfer including multiple scattering. One of the most successful techniques, which have been used so far for modeling a system as complicated as coupled atmosphere-ocean system prevailing on earth, is the Monte-Carlo method. Later we have devoted a section on this method only.

Of the three fundamental properties of light intensity (or the rate of photon arrival), wavelength or spectrum (often interpreted as hue or color) and polarization, polarization is the least known to the general public. This is because humans are mostly insensitive to the polarization characteristics of light (although we use them in sunglasses, computer screens, digital displays, etc.). However, many animals, terrestrial and marine, are sensitive to the polarization of light and make use of this polarization sensitivity for a variety of tasks. In a simplified way, polarization can be described as the distribution of the planes of vibration (the orientation of polarization) of the electrical (or magnetic) fields of the electromagnetic waves within a light beam. Partially linearly polarized light can be conceived as a mixture of fully linearly polarized light, with the plane of vibration of its electric vector (e-vector) at a given orientation (angle), called the e-vector orientation, combined with fully depolarized light, having random e-vector orientations. The fraction of the fully polarized component is the partial polarization, often represented as a percentage (% polarization). Thus, linear polarization has three descriptors: intensity, partial polarization and orientation of polarization.

The distribution of polarized light in underwater condition is predominantly affected by the position of the sun (or the moon) in the sky, the optical properties of the water, the depth of viewing and reflections from surfaces, such as the sea floor or the surface of the water (Waterman, [12], Waterman [21]; Waterman and Westell, [22]; Tyler [23]; Timofeyeva, [24], Timofeyeva, V. A., [25]; Novales–Flamarique and Hawryshyn, [26]; Cronin. T., and Shashar, N. J., [27]). Measure-ments performed at depths of 5–6 m by Waterman [28] revealed that underwater there are two distinct polarization patterns, one inside Snell's window and one outside it. Generally, the polarization pattern inside Snell's window down to depths of a few meters is assumed to be determined by the same factors as those influencing the sky polarization. Therefore, sun position, amount of overcast, amount of atmospheric dust, the distance of the point observed from the zenith, multiple scattering, and depolarization due to anisotropy of air molecules will all influence the polarization pattern within Snell's window (Waterman [29], Eddington, A. S., [30]. [31] Horvath. G., and Varju D, [31]) modeled the underwater polarization pattern within Snell's window as it correlates to the celestial polarization pattern, taking into account refraction and repolarization of skylight at the air–water interface. However, due to the focusing and defocusing of sunlight by surface waves (Sehunck. H., [32]; Snyder and Dera, [33]; Stramska and Dickey, [34]; Maximov, [35]) and changes in polarization as the light propagates in water, certain distortions may well occur. Indeed, Cronin, T and Shashar, N. J. [27], measuring polarization at a depth of 15 m on a coral reef, found only small differences between the polarization patterns within Snell's window and outside it.

Underwater, factors such as turbidity, bottom reflection (Ivanoff and Waterman, [37]) and proximity to the shore line (Schwind, [38]) may diminish the percent polarization. In shallow waters, the percent polarization first decreases with depth (Ivanoff and Waterman [37]) and then reaches a depth-independent value (Timofeyeva, [40]). Assuming primarily Raleigh scattering, Waterman and Westell [22] proposed a model for the effect of the sun's position on the *e*-vector orientation outside Snell's window (see also illustration in Hawryshyn, [42]). However, with increasing depth, the pattern of *e*-vector orientation simplifies rapidly, tending to become horizontal everywhere (Waterman [12]; Tyler [23]; Timofeeva [24]). It also diverges from the predictions of Waterman and Westell's model (Waterman and Westell, [22]) suggesting an effect on polarization of other, non Raleigh, modes of scattering and of post-scattering processes.

1.5. Mathematical Methods in Treating Radiative Transfer Theory:

It is almost impossible to differentiate between mathematically purely analytical and purely numerical methods applied so far in the radiative transfer theory. Practically no method received attention until some numerically acceptable result is produced. In the beginning of investigation-era the analytical methods proposed were tested numerically to get first estimates of the outcome of the problem at hand. We shall now briefly describe the various important approaches made so far by eminent researchers to solve the RTE in different situations of practical and theoretical interest.

Principle of Invariance: The most fundamental characteristics of the radiation field in dispersive media such as stellar atmospheres, planetary atmospheres, planetary nebulae or 0cean are the diffuse radiation which arises from multiple scattering of radiation by the media. This has been studied through an approach called the principle of invariance, or invariant imbedding, due to Ambartzumian [62], Sobolev [63], Kourganoff [64], Wing [65], and Preisendorfer [66]. An important component of Chandrasekhar's work in radiative transfer was devoted to the development of the Invariant Principles. These principles were first introduced by Ambartzumian [62, 67].

Bellman and his collaborators have published several papers on this subject. The concept was developed by Sir George Stokes [47, 333] in his glass plate theory. In some remarkably simple papers, he derived the transmission and reflection factors when a ray of light passes through a system of glass plates. With such simple theoretical approach he was able to formulate the principle of invariance of reflectance when several glass plates are arranged parallel to each other, one on top of the other. He obtained difference equations for the reflection by a pile of identical glass plate and derived certain commutation relations for sets of glass plates. It is remarkable that he was able to obtain transmission and reflection factors which look similar to those obtained in more complicated media such as stellar atmospheres (Hottel, H. C., and Sarofim, [69]).

Ambartzumian's principle of invariance has been an essential tool for solving radiative transfer problems in semi-infinite homogeneous atmospheres and it was subsequently generalized by Chandrasekhar [54] to solve problems in finite homogeneous atmospheres. The idea of addition of layer of arbitrary optical thickness to a semi-infinite layer was proposed by Ambarzumian. The reflection characteristics remain invariant in such situations. This was taken up by Chandrasekhar and solved several problems which looked formidable until then. The emergent intensities are obtained through the H-functions in a semi-infinite medium and **X** and **Y** - function in finite medium. However, these are obtained by solving certain integral equations by iterative means. In plane parallel media, these **X** and **Y** functions have been tabulated for isotropic and anisotropic phase functions. The tables have a restricted use in general cases, such as inhomogeneous media or moving media. It would be ideal if one need not specify the thickness of the layers which are to be added. Layers with general properties can be added and their transmission and reflection properties can be calculated directly by utilizing what is known as Interaction Principle (see Redheffer, [71]; Preisendorfer, [66]; Grant and Hunt [68, 73]). The physical and geometrical properties of the layers would automatically determine the reflection and transmission properties. As there are no conditions imposed initially either on the size of the layer or its geometry or its physical properties, any in homogeneity can be introduced into the scheme at any point. One can easily obtain the internal radiation field in any given dispersive medium. This has been applied in non- stationary media such as the outer layers of supergiants stars, supernovae. The Interaction Principle which is a generalized form of the invariance principle is nothing but the manifestation of the conservation of the radiant flux. It balances the emergent radiation with that of the reflected and transmitted input radiation together with the internally generated radiation.

One of the most surprising long-term implications of the invariance principles has been their generalization and development into an entire mathematical field known as "invariant imbedding" by Bellman, R. E., Wing, G. M. [167]. This development was based on the recognition that the invariance principles had converted what was a boundary value problem (involving boundary conditions at two or more points) into an initial value problem (involving boundary conditions at a single point) through the introduction of the reflection and transmission functions and the integro-differential equations they satisfy. The invariant imbedding methods generalize this idea of transforming from a boundary value problem to an initial value problem to a much wider class of problems than just radiative transfer, including wave propagation and control theory, among others. This was an important practical advance, since initial value problems are generally much more numerically tractable than boundary value problems. In the introduction to their work, Bellman and Wing [167], after crediting earlier workers, clearly express the special influence of Chandrasekhar in the development of their methods.

Invariant Imbedding:

Invariant imbedding has been used extensively to solve astrophysical radiative transfer problems as well as in radiation dissymmetry calculations and radiative transfer in the atmosphere, ocean and atmosphere. This is the method of converting certain two-point boundary value problems to initial value problems. It is an outgrowth of the method of invariance, introduced by V.A. Ambarzumian and used successfully by Chandrasekhar's. in radiative transfer problems. Invariant imbedding is closely related to the sweep method of solving the differential equation. The radiative transfer equation may be classified as a linear transport equation that must be solved subject to boundary conditions at the top and the bottom of the medium. Mathematically, we refer to this problem as a linear two-point boundary value problem.. From a mathematical point of view this essentially amounts to the transformation of a "difficult" yet linear two-point boundary value problem into a set of "simpler" but partly nonlinear initial-value problems. From the general point of view this theory can be interpreted as the result of two different conceptual approaches to transport theory. One of these is considered as a physical approach whereby a particle counting procedure is developed to the basic physical concepts. These basic ideas of invariant imbedding can be traced back to an insight by the astrophysicist Ambarzumian [62]. The subsequent development of the theory and its application to radiative transfer problems was made by Chandrasekhar, Bellman, Preisendorfer, and others. The fullest exposition of invariant imbedding as applied to hydrologic optics is seen in Preisendorfer six-volume treatise Hydrologic optics (Preisendorfer [75]). They are applicable to the solution of the RTE including internal sources, depth-dependent inherent optical properties (IOP), arbitrary incident radiances, wind-blown air-water surfaces, and a finite or infinite-depth bottom. The only restriction is that of a plane parallel geometry. But there exist several simplicities. All quantities are computed with the same accuracy. In particular, there is no statistical noise in the numerical results. The methods are mathematically elegant and provide deep insights into the internal structure of radiative transfer theory. Many profound relationships are revealed among the building blocks of the theory. The methods are computationally efficient. The solution algorithms are fast and numerically stable. Moreover, computation time is a linear function of optical depth. The price one pays for the above benefits is mathematical complexity. Invariant imbedding methods require a considerable amount of mathematical development in going from the RTE to its solution, and the associated computer programming is much more tedious.

The first work with invariant imbedding in its title was by Bell man and Kalaba [76] where they combined the principles of invariance with the functional multistage processes of dynamic programming of Bellman [77]. A series of papers by Bellman and his co authors appeared showing the invariant imbedding formulation ands application to neutron transport problems with other particle transport processes. The summary paper is Bellman, Kalaba and Wing [39]. Rosescu [79] made a list of these extended works of Bellman as well as many other authors. An extended series of application papers were appeared (Dodson and Mingle [80], Mingle [81, 82], Timmons and Mingle [83], Kaiser and Mingle [84]), Shimizu and Mizuta [85]). Bellman et al. [86, 87] have studied the radiative transfer problem using the principles of invariant imbedding from the viewpoint of 'particle counting'. Addition of an infinitesimal layer and then counting the first order contributions to transmission and reflection leads to the derivation of integrodifferential equations for transmission and reflection coefficients in the limit as the layer thickness vanishes. The **X**-and **Y**-functions of Chandrasekhar are obtained in a similar way. They obtain a system of simultaneous non-linear ordinary differential equations by replacing the quadratures over angles by Gaussian sums. These are solved by the standard numerical methods such as fourth-order Adams–Moulton techniques or Runge-Kutta methods. Bellman, Kalaba, Prestrud [86] have tabulated the diffuse reflection function with fixed intervals of and for the albedo for single scattering. Kagiwada, et al. [89] has tabulated transmission and reflection factors for conservative isotropic scattering case. Kagiwada and Kalaba [90] estimated the local anisotropic function by using the invariance techniques. The first book containing considerable amount of numerical work was probably by Bellman, Kalaba and Prestrud [86] whereas the second book was by Mingle [92].

The second approach to invariant imbedding is a mathematical one that proceeded in parallel with the particle counting method. This approach transforms the basic transport equation into invariant imbedding form with the help of functional analysis. Wing [18] was first to proceed in this approach and details may be obtained from Bailey and Wing [48, 55, 94]. In addition books such as Bellman [95] and Denman [96] have sections based upon this mathematical procedure.

Discrete ordinate method:

The discrete ordinate method was first introduced by Wick, G. C. [168]. By 1944, fully analytic solutions of RTE had been presented by Chandrasekhar [91] for only a few problems e.g., the solution to the semi-infinite Milne problem by Wiener and Hopf [98], but these methods did not extend to all of the problems that were of interest to Chandrasekhar. This led him to adopt a scheme, introduced earlier by Wick, G. C. [168] that reduced an integrodifferential equation to an approximate, finite set of ordinary differential equations by the introduction of a quadrature scheme into the integral term. Because of Chandrasekhar's subsequent extensive development of this method, it is now often known as the Wick-Chandrasekhar discrete-ordinate method. These largely analytical calculations use the facts that (1) the transfer equation is linear in intensity-at least explicitly, and (2) the replacement of integral over angle / or frequency transforms it into a set of differential equations so that well known theorems of linear algebra and analysis can be applied to study the properties of these solutions. For these reasons Discrete-ordinate method (DOM) is regarded as one of the most useful and elegant approach. In most applications the angular dependency of the radiative transfer equation is discretized in the angular domain and the solution consists of a set of first-order differential equations. The solution of the radiative transfer equation can be derived explicitly and the intensity calculations do not depend on the total optical depth of cloud or aerosol layers. However the situation in particle transport theory is not so .The approaches for discretizing the direction variable in particle transport calculations are the discrete-ordinates method and function-expansion methods. Both approaches are limited if the transport solution is not smooth. Angular Discretization errors in the discrete-ordinates method arise from the inability of a given quadrature set to accurately perform the needed integrals over the direction ("angular") domain. Analytic two-stream and four-stream solutions can be derived in closed forms for cases of radiative transfer in atmospheric, astrophysical or oceanic contexts. Computational times are relatively small compared to other techniques. However, polarization effects are not included in the earlier works. Munch [100] applied this method in radiative transfer processes in stellar atmosphere considering broadening of spectral lines by electron scattering.

In a classic paper Chandrasekhar [93] explicitly formulated the equations of transfer for the two components of the polarized radiation field in a free-electron stellar atmosphere. The Thomson scattering of radiation by free electrons was recognized to be an important mechanism in the transfer of energy in a certain class of stars. This fact made necessary a more detailed description of the scattering laws in the formulation of the basic equations. In Chandrasekhar, S. [54] provided the new theory and presented an approximate solution for the outgoing angular distribution of the polarized light. In a following work, Chandrasekhar, S. [54] by passing to the infinite limit in a Wick-Chandrasekhar discrete-ordinate procedure, he was able to solve exactly for the laws of darkening in the Milne problem.

The discrete-ordinates method (DOM) has become one of the most popular methods for solving Boltzmann transport equations for radiation transfer and neutron transport. This is because the DOM can be accomplished to high-order accuracy, the derivation of DOM schemes is relatively simple, and the DOM is compatible with the finite-difference or finite-element schemes for specular or diffuse phenomena. Studies on almost every aspect of the DOM applicable to multidimensional radiative heat transfer have been reported. [168, 362-364] However, most of the previous DOM algorithms focused on the solution of steady-state RTE because the effect of time-dependent light propagation is negligible in traditional heat-transfer problems.

In most applicable astrophysical context the RTE in frequency domain is subject to the discretizationin either in angle and/or frequency variable. Intensive calculation in unpolarised regime with frequency dependence reveal the fact that general solution contains –in addition to terms which decreases sometimes exponentially with increasing optical depth particularly in finite media where coefficients are to be evaluated explicitly from the boundary conditions at two ends from a system of ill-conditioned algebraic equations. This disadvantages leads to the search for efficient algorithm for better numerical evaluation Mihalas, [173], Cannon, [174]. However despite several attempts several difficulties aroused particularly in the treatment of anisotropic media. By means of matrix formulation of discrete ordinate method (Wehrse [175], Wehrse & Kolkophen [176], Schmidt & Wehrse, [177]) the problem of increasing exponentials can be solved analytically and it was found that the resulting equations are numerically stable.

As indicated earlier computer implementations of this method were, however, plagued by numerical difficulties to such an extent that researchers made little use of it. Analytic two-stream (only two directions) solutions of the discrete ordinate equations have been worked out in detail and applied to a number of atmospheric and ocean scenarios (for a summary, see Thomas, G. E., Stamnes, K., [102]).The two-stream approximation dates back to 1905 when Schuster initiated the studies on the solution of RTE with an albedo in the context of simplest equation of radiative transfer A. Schuster [41] by considering the two directional streams of the radiation field there by converting the original integral equation into asset of two coupled differential equations with suitable breaking up of the boundary conditions. However Schwarzschild, K., [43, 74] considered the case of conserved scattering. An entertaining description of how the two-stream approximation may be used to explain numerous radiative transfer phenomena is found in Bohren, C. F., [20].

The Eddington approximation originated with A. S. Eddington [30]. The relationship between the two-stream and the Eddington approximation was discussed by Lyzenga, D. R., [133]. The accuracy of the Eddington and two-stream methods for anisotropic scattering was explored by W. J. Wiscombe and J. H. Joseph [178], who found that it was accurate for values of **g** less than **0.5**. This explains why the δ-Eddington and δ-two-stream methods are so valuable: The scaled asymmetry factor is always less than **0.5**. Attempts to combine two-stream solutions for several adjacent slabs with different optical properties date back more than thirty-nine years Shettle, E. P. and J. A. Weinman [326]. These solutions were intrinsically ill-conditioned, because the matrix (that had to be inverted to determine the constants of integration in the problem) contained a combination of very small and very large elements resulting from the negative and positive arguments of the exponential solutions. In the δ-Eddington method, subdivision of layers was employed to circumvent the ill-conditioning, but at the expense of increasing the computational burden substantially for thick layers (Wiscombe, W. U., [365]). The ill-containing problem was eliminated by a scaling transformation that removed the positive arguments of the exponential solutions (Stamnes, K. and P. Conklin [108]). This scaling transformation was eventually implemented into a general-purpose multistream (including two-stream) radiative transfer algorithm by Stamnes, K., S. C. Tsay, W. J. Wiscombe and K. Jayaweera [101]. The resulting code has been made generally available to interested users. A specific two-stream code that made use of this scaling transformation to remove the ill-conditioning has been developed (Toon, O. B, McKay, C. P., Ackerman, T. P., and Santhanam, K., [327]). Finally, a two-stream algorithm derived from the general-purpose multistream algorithm mentioned above has been extended for application to spherical geometry and to layers in which the internal source may vary rapidly (Kylling, A., K. Stamnes, and S. C. Tsay [328]). This two-stream code is generally available to interested users. Modern discussions of the two-stream method are due to Meador, W. E. and Weaver, W. R. [329]; Zdunkowski, W. G., Welch, R. M., and Korb, G., [330]; King, M. D., and Harshvardhan [331]; and Harshvardhan and King, M. D. [332].

A limited number of analytic 4-stream solutions have also developed Liou, K. N., [103]. With the advent of large and powerful computers in the 1970s, it became possible to develop multiple scattering RT models for multi-layer atmospheres. This was done for the 2N-stream plane-parallel discrete ordinate model by Knut Stamens and co-workers in a series of papers from 1980 onwards culminating in the release of the DISORT plane-parallel radiative transfer package in 1988 [101]. A new implement-tation of the discrete ordinate method for vertically inhomogeneous layered media which is free of these difficulties and to give a summary of its equations and its various advanced features DSORT (Stamens et al. [101]) was developed The resulting computer code represents the culmination of years of effort Stamnes et al. [104-109,101,110] to make it the finest available algorithm with the intension that the code be so well documented so versatile and error free so that other researchers can use it safely both in data analysis or as a component of large class of models. It solves the RTE for a scattering, absorbing, and emitting medium with an arbitrarily specified bidirectional reflectivity at the lower boundary. For multilayer applications one can see the article [241].

DISORT is the most widely used RT code available to the atmospheric community. It is a generic scattering formalism that does not require direct specification of atmospheric constituent inputs and their optical properties at the microphysical level. Instead, it is only necessary to specify three optical inputs for each layer - the total single scattering albedo, the vertical optical thickness and the total phase function moments.

The philosophy behind the DISORT work was to build a general-purpose and flexible radiative transfer package that could be used in a wide variety of atmospheric applications. The model applies to a plane-parallel medium and includes both thermal and solar beam sources. The model is called as a subroutine within an environment which the user tailors to his or her specific needs. The user will then create the DISORT inputs from the set of atmospheric constituents and parameters appropriate to the application. A pseudo-spherical version SDISORT has been developed [110], but unfortunately this has not been packaged as a general-purpose tool in the same manner as DISORT itself. The delta-M scaling is standard in DISORT. A second version of the code incorporates the single scatter correction procedure of Nakajima, T. and Tanaka, M., [112].

To extend the application of DISORT code for systems involving two or more media like atmosphere-ocean or atmosphere-cloud-ocean, Jin and Stamnes [113], developed CDISORT which account for the changes in the refractive indexes at the flat separating boundaries between the media. The study was extended for rough ocean surface.

On the basis of the linearization analysis, the numerical model LIDORT (Linearized Discrete Ordinate Radiative Transfer) has been developed and tested. The philosophy adopted for LIDORT is the same as that for DISORT - to make a general-purpose and flexible radiative transfer package that could be used in a wide variety of atmospheric applications, not just for simulating intensity, but also for generating weighting functions that are necessary in so many retrieval applications. Like DISORT, LIDORT is a subroutine called from a user-defined environment. LIDORT too is a scattering formalism; it does not need to know the number and nature of the atmospheric gases and particulates. Validation of LIDORT is straightforward: radiances may be compared directly with DISORT and SDISORT output, while weighting functions are validated using the finite-difference estimation and choosing optical depth small enough.

With the advent of the ultrafast laser and its broad applications in biomedical technologies, the study of time-dependent laser radiation transfer incorporating radiation propagation with the speed of light has become increasingly important. Recently the time-dependent DOM method has been explored to one-dimensional [118, 119] and multidimensional geometries. [120, 121] Yet the media were not characterized tissues. Fresnel's reflection was not taken into account, nor was the ultrafast laser pulse considered in these studies. Klose et al [122] used a DOM algorithm as a forward model for optical tomography. But the DOM there is time independent steady state and has only 24 discrete ordinates (equivalent to the S4 quadrature scheme) in the spatial angle direction. The DOM S4 method is easily subjected to the ray effect and false scattering as pointed out by Chai et al. [44] because of the limited number of discrete ordinates. Instead, high-order quadratures such as S8 (80 discrete ordinates) and S10 (120 discrete ordinates) are more commonly adopted in the radiation heat-transfer community. These high-order schemes have been applied to the study of ultrafast-laser-radiation transfer in anisotropically scattering, absorbing and emitting heterogeneous tissues in three dimensionally geometry. Guo, Z., and Kim, K., [123], Optical society of America.

Over the last fifty years polarized light travel through scattering media has been studied by the atmospheric optics and oceanography community in particular. An exact solution of the radiative transfer for a plane-parallel atmosphere with Raleigh scattering was derived by Chandrasekhar in [54]. More complicated geometries proved too complex to be solved analytically. He has also developed principle of invariance for both polarized and nonpolarised radiation field which serves as basic and fundamental in formulating physical interpretation of radiative transfer equation.

But if the discrete ordinate method has fallen out of favor for analytical radiative transfer, it remains to this day a very strong component of numerical work in stellar atmospheres and other astrophysical applications, because of its simplicity, accuracy, and adaptability to complex physical situations. In this way, the method continues to serve those seeking practical solutions to real physical problems, as Chandrasekhar himself was.

Cases Eigen function approach:

The discrete ordinate method was both Chandrasekhar's most transient and his most permanent contribution to the field. After DOM, the development of analytical radiative transfer rapidly moved toward full treatment of the angular dependence of the solutions, rather than discrete versions. This could already be seen in Chandrasekhar's own work, where he gradually (but not completely) shifted from the discrete ordinate method to the invariance principles to enable him to deduce analytical the structure of the solutions. Perhaps the most interesting development in this context was the singular eigenfunction method used in plasma physics by Van Kampen [134], and later applied to transfer theory by Case and others (see, e.g., Case & Zweifel [135]). In a sense, this is the true descendent of the discrete ordinate method, since it also starts by asking for solutions of exponential form, but now confronts the true nature of the continuous angular dependence in the scattering integral.

In this technique some singular eigenfunctions with specified eigenvalues are used to expand the unknown intensity. These normalized eigenfunctions are found to satisfy certain orthogonality and completeness conditions depending on the range of angular integrations. Analytical expression for singular and continuous eigenfunctions for both types of eigenvalues (for eigenvalues lying within the range of integration, another for lying outside the region) can be evaluated using orthogonality, normalization and completeness properties of the eigenfunctions. In radiative transfer this method of normal modes provides an elegant and systematic approach to the solution of one-dimensional, plane-parallel radiative transfer problems enabling the desired solution be written as a linear sum of the eigenfunctions of the homogeneous equation and a particular solution appropriate to the source function of interest. The solution to the problem is thus reduced to that of determining the unknown expansion coefficients appearing in the sum of elementary solutions These coefficients are determined by constraining the solution to meet the given boundary conditions and by then utilizing the orthogonality properties of these Case eigen functions. This procedure is completely analogous to the classical orthogonal expansion treatment of boundary value problems. This method is suitable when solution at any optical depth is sought. As a consequence of the method of singular Eigen solutions, a large class of problems in both radiative transfer and neutron-transport theory have become amenable to exact, closed form solution. The principal advantage of normal mode expansion technique is that solutions valid at any depth are obtained as well as results for various surface quantities. An additional merit is that determination of expansion coefficients requires minimum manipulations.

Case's normal-mode expansion technique Case, K. M., [136] is used to obtain solutions to the radiative transfer problem for an absorbing, emitting, isotropically scattering, nonisothermal gray medium bounded by specularly reflecting, diffusely emitting, gray parallel walls each held at uniform but different temperatures. Siewert, C. E and McCormick, N. J., [137] obtained a rigorous solution for an absorbing, emitting, anistropically scattering, semi-infinite medium with a linear source function and a free boundary. One of the earlier applications of the singular eigenfunction expansion technique to problems in finite geometry was made McCormick, N. J. and Mendelson, M. R. [138] who treated the slab albedo problem. Ferziger, J. H. and Simmons, G. M., [139] solved the radiative transfer problem for a non absorbing, non-emitting, perfectly scattering medium (or alternatively for a gray medium in radiative and local thermodynamic equilibrium) bounded by black heated parallel walls. Typical of transport problems with two boundaries, the results of Ferziger, J. H. and Simmons, G. M. [139] were not expressed in closed forms; however, they have shown that their analytical approximate solutions were highly accurate. Recently Heaslet, M. A. and Warming, R. F., [140] considered non-conservative radiative transfer in semi-infinite and finite media. The solution to the problem is thus reduced to that of determining the unknown expansion coefficients appearing in the sum of elementary solutions These coefficients are determined by constraining the solution to meet the given boundary conditions and by then utilizing the orthogonality properties of these Case eigen functions. This procedure is completely analogous to the classical orthogonal expansion treatment of boundary value problems.

In Case's eigen function approach ultimately one encounter a system of singular integral equations which are difficult to solve analytically and numerical implementations also require considerable cost. Siewert and Benoist [141] and Grandjean and Siewert [142] employed **Fn** method in which the unknown intensities appeared as integrand in the above mentioned equations is expanded in a series in angular variable with unknown coefficients. These coefficients are then determined from the resulting system of algebraic equations with coefficients which also involve some integration with singular eigenfunctions as integrand. These coefficients are found to obey certain recurrence relations from which they can be found easily. The values of the last coefficients are used to find the values of the expansion coefficients. However in doing so one has to choose discrete values for the continuous eigenvalues spectrum thus keeping the chance of affecting the numerical calculations in accuracy alive. Several authors Devaux and grandjean, [143], use several schemes to choose the eigenvalues depending upon the problems encountered. This method is considerably simpler than any other method. It has been shown that the desired numerical accuracy is achieved with few orders of terms in the expansion even in the cases of polarization studies in this method. (Siewert, C. E., [144] astrophysics & sp. science, Grandjean and Siewert, [145], Siewert, [144], Siewert and Benoist, [151], Siewert, (jqsrt), [150]. Despite its tremendous success in radiative transfer theory, neutron transport calculations and gas dynamics no attempt was seen so far as our knowledge is concerned to expand this technique in coupled atmosphere-ocean system.

Discrete Space Theory: As mentioned earlier Preisendorfer, R. W., [66], Grant, I. P. and Hunt, G. E., [73] generalizes the interactions principles into the invariance principles particularly in a finite medium. The basic idea of the interaction principle is to specify the radiation field in terms of the transmitted and reflected radiation at any given point in the medium.

Carlson [127] and Lathrop and Carlson [128] used a numerical version of the discrete ordinate technique in neutron reactor calculations. By integrating the radiative transfer equation over a finite volume in space coordinates and using the mean value theorem of integrals, we can develop difference equations that conserve flux. These difference equations are of quite general use in non-uniform media and curvilinear coordinate systems. This system of equations is solved by iterative methods. As these equations are the expression of the conservation of flux, invariance principles can be expected to be deduced from them. One needs to study the errors and stability factors of any system of equations. Carlson's S_n methods did not have a sell studied error and stability analysis. This can be overcome by rewriting equations in what is called 'invariant S_n' form. In this way, one can test the stability and estimate the errors due o truncation and round-off of the terms. The reflection and transmission operators can be expressed in the form of matrices. The matrix structure allows us to perform the desired analysis and to obtain an explicit solution which essentially expresses the results in terms of the Green's function of the transport operator which is related to the probability of quantum exit as defined and exploited by

Ueno [129]. The matrix structure is the discrete equivalent of the equation of Rybicki, G. B. and Usher, P. D., [130] and converges to it when we pass the limit of infinitesimally these segments. Discrete space theory Hunt, G. E., and Grant, I. P., [336] is actually a slight variation of doubling and matrix operator theory which will be described next.

The Doubling-Adding and the Matrix Operator Methods:

We shall now discuss a method that has been widely used to solve radiative transfer problems in planetary atmospheres. In this method doubling refers to how one finds the reflection and transmission matrices of two layers with identical optical properties from those of the individual layers, while Adding refers to the combination of two or more layers with different optical properties.

The doubling concept is rather old and seems to have originated with Stokes. It was rediscovered and put to practical use in atmospheric science by Van de Hulst and others.

The doubling concept seems to have originated in 1862 (Stokes, G., [333]). It was introduced into atmospheric physics one century later (Twomey, S., Jacobowitz, H., and Howell, J., [334]; Van de Hulst, H. C., and Grossman, K., [335]). The theoretical aspects as well as the numerical techniques have since been developed by a number of investigators; for references see, e.g., Wiscombe, W. J., [337].

The doubling method as commonly practiced today uses the known reflection and transmission properties of a single homogeneous layer to derive the resulting properties of two identical layers. To start the doubling procedure the initial layer is frequently taken to be thin enough that its reflection and transmission properties can be computed from single scattering. Repeated "Doublings" are then applied to reach the desired optical thickness. The division of an inhomogeneous slab into a series of adjacent sub-layers, each of which is homogeneous, (i.e., optical parameters do not vary with depth) but is principle different from all the others, is usually taken to be identical to that discussed previously for the discrete-ordinate method. The solution proceeds by first applying doubling to find the reflection and transmission matrices for each of the homogeneous layers, whereupon adding is subsequently used to find the solution for all the different layers combined.

The Spherical-Harmonics Method:

We have already discussed the doubling-adding method and how it is closely related to the discrete-ordinate method despite being seemingly quite different in concept. Another method that is closely related to the discrete-ordinate method is the spherical- harmonics method, which starts by expanding the intensity in Legendre polynomials in angular variables while the space

dependency is expressed through some space dependent coefficients to be determined by a set of differential equations. This type of expansion was first suggested by Eddington in 1916 and led to the widely used Eddington approximation by retaining only two terms in the expansion. The spherical harmonics method (SHM) is the spectral analogue of the discrete ordinates method. Much in the way one uses sines and cosines to represent functions with Fourier transforms, one uses spherical harmonics (Legendre polynomials in their simplest form) to represent the specific intensity and scattering phase function. Just as the orthogonality of sines and cosines makes the Fourier transform useful for solving differential equations, the orthogonality of the spherical harmonics makes it easier to solve RTE equation. One of the most important developments are by Karp et al. [276]. A very nice review article was written by Karp [276a].
It can be shown that the Eddington and two-stream methods are closely related. Since the generalization of the two-stream and Eddington methods leads to the discrete-ordinate method and the spherical-harmonics method, respectively, it is perhaps not surprising to learn that these two latter methods are also closely related. One reason for this similarity is that the spherical-harmonics method relies on an expansion of the intensity in Legendre polynomials, while the discrete-ordinate method relies on using quadrature, which in turn is based on approximating the intensity with an interpolating polynomial that makes essential use of the Legendre polynomials to achieve optimum accuracy. The difference between the two methods lies mainly in the implementation of boundary conditions. As we have seen, this is quite straightforward in the discrete-ordinate approach, but it appears somewhat more cumbersome in the spherical-harmonics method in which moments of the intensity are specified at each boundary instead of specifying the intensity in discrete directions as is done in the discrete-ordinate method. In spite of this difficulty the spherical-harmonics method has been developed into a reliable and efficient solution technique.

Long before Chandrasekhar proposed DOM, Jeans [97] used what we call today SPH method. The key idea (1*, 2*) is to replace the intensity or radiance function and phase function in orthogonal polynomials such that the RTE is replaced by a set of differential equations which can be converted easily to matrix differential equation. Solutions come out as matrix exponentials which are difficult to calculate. However knowledge of physical problems and use of linear algebra [276] allows one to obtain numerically accurate solutions. Another approach is to expand the unknown intensity in a series Karp, A. H., [277] of the following form (1*)

$$\mathbf{I}(z,\mu) = \sum_{n=0}^{n} \mathbf{f}_n(z)\mathbf{P}_n(\mu), \qquad (1^*)$$

$$\mathbf{P}(\mu,\mu') = \sum_{k=0}^{K} \beta_k \mathbf{P}_k(\mu)\mathbf{P}_k(\mu'), \quad (2^*)$$

$$I(z,\mu)=\sum_{k=0}^{K}\frac{2k+1}{2}P_k(\mu)\sum_{j=1}^{J}[A_j e^{-\lambda_j z}+(-1)^k B_j e^{-\lambda_j(z_0-z)}]g_k(\lambda_j),\ (1^{**})$$

Where A_j and B_j are constants determined by the boundary conditions and $g_k(\lambda)$ are the Chandrasekhar polynomials defined by 3-term recurrence

$$(k+1)g_{k+1}(\lambda)=\frac{2k+1-\omega\beta_k}{\lambda}g_k(\lambda)-kg_{k-1}(\lambda),\quad (2^{**})$$

The λ_j in equation (1**) Are the inverse of the positive roots of $g_{k+1}(\lambda)=0.$

with unknown coefficients determined by boundary conditions. The Chandrasekhar polynomials are defined by three terms recurrence having the following structure. Substitution of (1*) in homogeneous RTE and use of orthogonality relations satisfied by Legendre polynomials and (2**) yield set of algebraic system of equations which can be solved to obtain the unknown coefficients. Particular solutions are given for solar illuminations Benassi, M. et al. [278], diffuse and specular reflective boundaries McCormick, N. J. and Siewert, C. E., [279] and thermal radiations Barichello, L. B. et al. [280], as well as for polarization problems Siewert, C. E. and McCormick, N. J., [281] and spherical symmetry Siewert, C. E. and Thomas, J. R. Jr. [282].

Several other solution methods in SHM have been proposed since the linear algebra technique is published. Unfortunately most of these methods do not achieve the numerical accuracy as the initial researchers achieved. However each of them [283, 284, 285] has some interesting perspective worthy of attention. A combination of spherical harmonics and discrete ordinate method was developed by Evans [250] in three dimensional atmosphere problem. In this research spherical harmonic representation of angular variable to reduce the memory requirements. The transfer equation is integrated along discrete ordinates through spatial grid to model streaming of radiations. The solution method is of type successive order of scattering approach with adaptive grid approach to improve accuracy.

Several papers have appeared that examine the mathematical and computational properties of some aspects of the SHM. In case of reflecting boundary Settle [286] improved the Dave's [285] approach to evaluate certain integrals involving Legendre polynomial as integrand by introducing a 5-term recurrence relation for the generalization of the integral and thus enabling any form of reflectivity to be expressed as a polynomial in polar direction. Gander [287] viewed the same integral as integration of a polynomial over a finite interval with a positive weight and can be evaluated with Gauss type integration rule. However this approach is proved to be unstable can be implemented using ORTHOPOL package Gautschi, E., [288].

Chandrasekhar polynomials defined by equation (2**) is another area of interest. Dehesa et al. [289] derived differential equations using Chandrasekhar polynomials and found solutions, in some restricted sense, in the form of polynomials of hypergeometric type. Certain identities are derived using the observation that Chandrasekhar polynomials are the eigenvector of some matrix similar to some tridiagonal matrix Siewert, C. E. and McCormick, N. J., [290]. Application to the case of azimuthal dependence requires great care especially to the higher order terms in Fourier expansion Garcia, R. D. M. and Siewert, C. E., [291].

While the basic model of radiative transfer is useful in many domains, there is a great deal of interest in including other effects. Nuclear engineers want to keep track of neutrons that change energy, the multi-group problem Siewert, C. E., [292]. Atmospheric scientists want to include polarization Garcia, R. D. M. and Siewert, C. E., [293]. Many physical problems are not well represented by plane parallel layers. Everyone would like to find an effective solution in two Evans, K. F., [297] or three dimensions Evans, K. F., [298]. Extension of this method to heat transfer in spherical media Li, W. and Tong, W., [294] was reported. Siewert and Thomas [295] reported a particular solution. Tine et al. [296] used this method in inhomogeneous media. There exists one article [164] that establishes the equivalence between discrete-ordinate and spherical harmonics method. Extension of spherical harmonics method in two and three dimension was developed and experimented with numerical evaluation of Fortran code TWOTRAN by Lathrop and coworkers [116, 117].

1.6. Numerical approaches:

Iteration Methods:

There are several methods of an iterative nature, such as "successive orders of scattering", "lambda iteration" (or Neumann series) or operator perturbation method and "Gauss-Seidel iteration" that have been important both for the development and under standing of multiple-scattering theories. The advantage is that these approaches are physically based and this allows for easy intuitive interpretation of the results. The disadvantages are that they apply only under restrictive conditions such as optically thin media and non-conservative scattering. So we did not try to revisit these methods.

The Feautrier Method:

Another class of methods was first introduced by Feautrier, P., [338]. Feautrier's approach, which has gained prominence and popularity in astrophysics, is based on using symmetric and anti-symmetric averages of the radiation field as the dependent variable. The resulting equations ate discretized in both angle and optical depth and solved numerically using finite-difference techniques. The method was originally used mainly for problems with isotropic scattering, but it has been generalized to apply also to problems with anisotropic scattering. Since then it has been used extensively (e.g., Mihalas, D., [339]; see also articles in Kolkofen, W., [340]). Despite its success in astrophysical radiative problems, it did not received attention in the atmospheric ocean literature.

Integral Equation Approach:

One may convert the integro-differential radiative transfer equation into a Fredholm-type integral equation commonly referred to as the Schwarzschild-Milne integral equation. This approach is particularly appealing in line transfer problems occurring in astrophysics in which isotropic scattering and complete frequency redistribution prevail, since the resulting integral equation becomes angle and frequency independent. Although it can be readily generalized to anisotropic scattering, this approach has not received much in the non-astrophysical literature. The integral equation approach is described by Chandrasekhar [346] and by Cheyney, H., and A. Arking [341], and several investigators have applied this method to solve a variety of problems (e.g., Anderson, D. E., [342]; Hummer, D. F. and Rybicki, G. B., [343]; Strickland, D. J., [344]; Strickland, D. J. and T. M. Donahue, [345]).

1.7. Work done on atmosphere- ocean system:

Radiative transfer in aquatic media is a mature discipline in itself with its own nomenclature, terminology, and methodology. We have assumed that we are dealing with media for which the index of refraction is constant throughout the medium. The coupled atmosphere-ocean system provides an important exception to this situation because we have to consider the change in the index of refraction across the interface between the atmosphere (with $\mathbf{m}_r \approx 1$) and the ocean (with $\mathbf{m}_r \approx 1.33$). We should also point out that radiative transfer in aquatic media is similar in many respects to radiative transfer in gaseous media. In pure aquatic media, density fluctuations lead to Rayleigh-like scattering phenomena. Turbidity (which formally is a ratio of the scattering from particles to the scattering from the pure medium) in an aquatic medium is caused by dissolved organic and inorganic matter acting to scatter and absorb radiation in much the same way aerosol and cloud "particles" do in the atmosphere.

We shall now try to review the basic state of mathematical researches done in radiative transfer processes in atmosphere-ocean system under various boundary conditions. The main difference in solving the radiative transfer equation consistently in such a coupled system, from a solution that treats only the atmosphere is caused by the refractive index change across the air– water interface. This refractive index discontinuity at the air–water interface changes the formulation of the radiative transfer equation and complicates the interface radiance continuity conditions and therefore gives rise to a different solution of the radiative transfer equation. Several investigators have studied the transfer of radiation in coupled atmosphere ocean system.

For the plane-parallel geometry found in most hydrologic optics problems, there exist solution methods that are vastly more efficient than Monte Carlo simulation. We now begin the development of one of these analytical (meaning deterministic or non-statistical) methods for solving the RTE in one spatial dimension. Many such methods are available; they go by names such as discrete ordinates methods, spherical harmonics methods, iterative methods, matrix methods, and invariant imbedding methods. Van de Hulst [131] gave an excellent descriptive summary of the available solution methods, including the strengths and weaknesses of each. Some analytical methods are of great power and considerable generality. Others were developed for specific problems (such as Rayleigh scattering) and have found little or no application in hydrologic optics. Kattawar [132] has compiled 43 original papers on solution methods applicable to the plane-parallel geometry of interest here. His book is a good place to survey the richness of mathematical methods that has been brought to bear on solving the RTE.

Fraser and Walker [227] assumed a simple model of the ocean-atmosphere system (a standard gas on a smooth ocean) and reported the intensity and degree of polarization. For the same case of a smooth sea surface, Dave [228] and Kattawar et al. [229] conducted computations for more realistic atmospheric models. For a rough sea surface exhibiting the true complexity of the boundary conditions, Raschke [230], Plass et al. [216] and Quenzel and Kaestner [232] solved the problem, but neglected the polarization of diffuse radiation in the atmosphere is well as polarization of the reflected radiation. Ahmad and Fraser [233] and Takashima and Masuda [234] performed complete calculations accounting for the degree of polarization and also presented some limited comparisons. The difficulty of exact radiative transfer calculations for rough-ocean reflection is mainly numerical. Most radiative transfer calculations are made tractable by using Fourier series decomposition of the radiation field as a function of the azimuth. For the case of Lambertian ground or of a smooth sea surface, the boundary condition is compatible with this series expansion. On the other hand, this approach is not easy follow for the case of a rough ocean because of the complexity introduced by the wave slopes.

A detailed theoretical derivation of the radiative transfer equations and their solutions by discrete ordinate method for the case of flat ocean surface were given in Jin and Stamens [113]. The principal difficulty encountered in attempts to model radiative transfer throughout such a system with the discrete ordinate method originates from the bending or refraction of radiation across the interface between the media of different refractive properties. The Fresnel refraction and reflection will affect the form of the radiative-transfer equation and the particular solutions, and the continuity relations at the interface are totally different from the non refractive case. The atmosphere and the ocean are both assumed to be vertically stratified so that the optical properties depend only on the vertical coordinate. To account for the vertical inhomogeneity, the atmosphere and the ocean can be divided into any suitable number of horizontal layers, as required to resolve the vertical structure of the optical properties of each medium. In actual approach, the atmosphere and the ocean are divided into a suitable number of layers to adequately resolve the optical properties of each of the two media. Each layer is taken to be homogeneous, but the optical properties are allowed to vary from layer to layer. For a homogeneous medium, only one layer is required. At the interface between the ocean and the atmosphere (assumed to be flat), Fresnel's formula is used to compute the appropriate reflection and transmission coefficients, and Snell's law is applied to account for the refraction taking place there. The integral term in each of these azimuth-independent equations is then approximated by a Gaussian quadrature sum with terms (streams) in the atmosphere and terms in the ocean, so that there are streams in the refractive region of ocean that communicate directly with the atmosphere and streams in the total reflection region of the ocean. In this way the *integro*-differential equation is transformed into a system of coupled ordinary differential equations that is solved by the discrete ordinate method subject to appropriate boundary conditions. This method has the following unique features: (1) because the solution is analytic, the computational speed is completely independent of individual layer and total optical thickness, which may be taken to be arbitrarily large. The computational speed is directly proportional to the number of horizontal layers used to resolve the optical properties in the atmosphere and ocean. (2) Accurate Irradiances are obtained with just a few streams, which make the code very efficient. (3) Because the solution is analytic, radiances and irradiances can be returned at arbitrary optical depths unrelated to the computational levels. 4) The Discrete Ordinate method is essentially a matrix eigenvalues-eigenvector solution, from which the asymptotic solution is automatically obtained.

Cox and Munk [215] developed the statistical characteristics of reflection by wind driven ocean waves by modeling the sea surface as a collection of individual mirror like surface facets.Modelling the distribution of normals to each surface facets as wind speed dependent Gaussian function, roughness of the wind blown ocean surface have been successfully incorporated in_the radiative transfer problems in atmosphere-ocean system by many authors [215-220]. Extension to cases involving rough ocean surface using the method developed by Jin and Stamens [113] was made recently by Jin, Z., et al. [259]. Using computationally efficient DOM they successfully formulated analytical solutions of RTE in a coupled atmosphere ocean system with rough air-water interface and the results are compared with satellite observations. These studies reveal that surface roughness have significant effects on the upwelling radiations in the atmosphere and downwelling radiations in the ocean.

Radiative transfer calculations are used to quantify the effects of physical and biological processes on variations in the transmission of solar radiation through the upper ocean. Results indicate that net irradiance at 10 cm and 5 m can vary by 23 and 34 W m22, respectively, due to changes in the chlorophyll concentration, cloud amount, and solar zenith angle (when normalized to a climatological surface irradiance of 200 W m22). The thermal and dynamical evolution of the upper ocean is sensitive to the vertical distribution of the solar energy available for ocean radiant heating (Denman [185]; Simpson and Dickey, T. D., [186]; Charlock [187]; Kantha and Clayson [188]; Schneider et al. [189]; Brainerd and Gregg [190]; Ohlmann et al. [191]. A 10 W m22 change in the quantity of solar radiation absorbed within a 10-m layer can result in a temperature change of more than 0.68C month21. Simpson and Dickey, T. D., [186] reported a 0.58C change in mixed layer temperature over a 24-h period due to alteration of the solar attenuation coefficient. Such sensitivity to radiant heating processes demonstrates the need for upper ocean models that accurately represent the spatial and temporal variability in solar radiation transmission. Variations in solar transmission have been described primarily by Jerlov water type (Jerlov [192]), a subjective integer index used to indicate water turbidity, despite the continuous nature of solar attenuation (Kraus [193]; Paulson and Simpson [194]; Zaneveld and Spinrad [195]; Paulson and Simpson [196]; Woods et al. [197]; Simonot and Le Treut [198]). Models that rely upon continuous, measurable, physical and biological quantities on which solar transmission depends have been developed only recently (e.g., Morel [199]; Morel and Antoine [200]; Ohlmann et al. [201]). These models use the upper ocean chlorophyll concentration and, in one case, the cosine of the zenith angle of the in-water light field to describe solar attenuation. The models have been built upon existing bio-optical parameterizations because data sets with coincidentally measured optical, physical, and biological parameters are limited (Smith and Baker [202]; Morel [199]) To further improve ocean radiant heating rate parameterizations, a thorough understanding of relationships between solar transmission and the factors that regulate its variations must be developed. Chlorophyll concentration, cloud amount, solar zenith angle, and wind speed all influence solar transmission by altering the in-water solar flux divergence and the sea surface albedo. The quantity of attenuating materials, generally inferred from chlorophyll a concentration, has been shown to be the primary regulator of in-water solar transmission on mixed layer depth scales (Smith and Baker [202]; Siegel and Dickey [203]; Morel [199]; Lewis et al. [204]; Siegel et al. [205]; Ohlmann et al. [191]). However, the effect of chlorophyll biomass on solar transmission within the upper few meters (where a significant portion of solar energy exists outside the visible wavebands) is not well characterized. Clouds play a role in shaping the spectral composition of the incident irradiance (Nann and Riordan [206]; Ohlmann et al. [201]; Siegel et al. [207]) and influence the geometry of the incident light field (Liou [208]). In a later study by Siegel et al. [207] showed the radiant heating rate for the upper 10 cm of the ocean, normalized by the total incident irradiance, can decrease by 50% in the presence of clouds. Solar zenith angle can affect transmission through changes in the light field geometry. Dependence of the vertical decay of irradiance on sun angle has been illustrated for clear sky conditions using Monte Carlo simulations (Kirk [209]; Gordon [210]). Solar zenith angle and wind forcing of the sea surface have been shown to effect in-water radiative transfer through modification of the surface albedo (Payne [211]; Simpson and Paulson [214]; Katsaros et al. [212]; Preisendorfer and Mobley [213]. The relationships between solar transmission and chlorophyll concentration, cloud index, solar zenith angle, and wind speed must be quantified to determine the proper set of parameters for improved solar transmission parameterizations.

The HYDROLIGHT radiative transfer numerical model solves the radiance transfer equation for a plane parallel environment. A complete description of HYDROLIGHT is given in Mobley [222]. To solve the radiative transfer equation HYDROLIGHT discretizes the set of all directions into a finite set of quadrilateral regions, or quads, bounded by lines of constant directions both in azimuth and polar. Such a partitioning scheme is adequate for resolving changes in solar zenith angle and for the introduction of diffuse light due to clouds. When this directional discretization is applied to the radiance equation the fundamental quantity computed by HYDROLIGHT becomes the radiance averaged over each quad. Integration over all directions becomes a sum over all quads. Wavelength is similarly decomposed into finite wavelength bands. Invariant imbedding theory and Fourier analysis are used to reduce the set of equations for the quad- and band-averaged radiance to a set of Riccati differential equations governing transmittance and reflectance functions. Depth integration of Riccati equation by high order Runge-Kutta algorithm and incorporation of the boundary conditional the sea surface and bottom leads eventually to the radiances at desired levels A full description may be found in Mobley [221], Mobley and Preisendorfer [361]. Solution of these differential equations eventually gives the spectral radiance as a function of depth, direction, and wavelength. Inputs to HYDROLIGHT are absorption and scattering properties of the water column, which determine the beam attenuation coefficient and volume scattering function, the radiance distribution incident at the sea surface; and the wind speed, from which the sea surface roughness is computed. The standard version of HYDROLIGHT, which works from 350 to 700 nm, was modified to resolve the solar spectrum from 250 to 2500 nm. This requires the addition of absorption and scattering properties for the added ultraviolet and near-infrared wavebands. Total absorption and scattering are determined by summing the absorption and scattering coefficients for pure water and for chlorophyll biomass. A detailed comparison was described in by Mobley et al. [223].The chief advantage of this model is computational efficiency. Solution of the Riccati differential equations for radiance is an analytic process, and thus there are no Monte Carlo fluctuations in the computed radiances (except for a negligible amount introduced by the simulation of the sea surface). In particular, both upwelling and downwelling radiances are computed with the same accuracy. Moreover, computation time is a linear function of depth, so that accurate radiance distributions are easily obtained at great depths. Computation time depends only mildly on quantities such as the scattering-to-attenuation ratio, surface boundary conditions, and water stratification. The associated computer code is available and is documented by Mobley [224].

Energy transfer across the sea surface is crucial to the understanding of the general circulation of the ocean. Shortwave radiation from the sun contributes most the heat fluxes that penetrate the air-sea interface and are subsequently absorbed throughout the ocean mixed layer. Solar radiative transfer differs from other air-sea interaction processes such as wind stress, evaporation, precipitation, and sensible cooling that occur only at the sea surface. Ohlmann et al. [201] showed that the climatological value of solar flux penetrating the mixed layer can reach **40** $\mathbf{W\,m^{-2}}$ in the tropical regions and can produce a difference in the heating rate of a **20-m** mixed layer by about $0.33°C$ a month. The vertical distribution of solar flux also influences the stability and stratification of the mixed layer and the sea surface temperature. It is clear that a quantitative understanding of the solar flux profile is important to ocean model simulations.

The surface albedo, which includes the surface reflectivity and the upwelling radiation from the water surface, is critical to the energy budget in the atmospheric planetary boundary layer. However, the reflectivity of the wind-blown surface is difficult to evaluate. Mobley [222] showed that the surface reflectance may decrease by **50%** when the solar zenith angle (SZA) is $70°$ and the wind speed increases from **0** to $\mathbf{20\,m\,s^{-1}}$. The surface roughness also affects the upwelling radiation from the water surface and its determination requires accurate radiative transfer analysis. The impact of oceanic pigment of radiative transfer in the ocean is important, as discussed by Gordon et al. [238] and Morel [199]. A change of $\mathbf{0.10\,mg\,m^{-3}}$ in the phytoplankton concentration in the mixed layer can result in a corresponding change of the penetrative solar flux by about $\mathbf{10\ W\,m^{-2}}$ at a level of **20-m** depth (Siegel et al. [240]). Because solar radiation is the energy source for photosynthesis, it also directly affects the marine productivity.

A coupled atmospheric-ocean radiative transfer model based on analytic four stream approximation has been developed by Lee and Liou [366]. The objective of this study was to build an efficient coupled ocean-atmosphere radiative transfer model including consideration of the wind-blown sea surface. A typical coupled model either deals with radiative transfer in the atmosphere and ocean separately by considering one medium as the boundary condition for the other or is computationally expensive for solar flux calculation. The present coupled model is based on the delta-four-stream approximation, developed by Liou [225] and Liou et al. [226] that can provide an analytical solution for radiative flux calculation and at the same time maintain excellent accuracy. To include the effect of the wind-blown sea surface, a Monte Carlo method that simulates the traveling of photons is employed to calculate the surface reflectance and transmittance. Applying the results from the Monte Carlo simulation into the delta-four-stream approximation, radiative transfer in the atmosphere and the ocean can be treated simultaneously and consistently. The present model is computationally efficient and provides a physically consistent surface albedo and ocean heating rate profile in the mixed layer.

Various numerical models are now available for computing underwater irradiances and radiances to address a wide range of oceanographic problems. Using different levels of sophistication in simplifying various physical assumptions depending upon the physical situations various numerical schemes have been designed to solve RTE.

1.8. Monte Carlo Models:

A number of models, most of them based on the Monte Carlo technique, have been developed and are used for various studies of radiative transfer in the atmosphere-ocean system, [179-184]. For geometrics other than plane-parallel and/or media with irregular boundaries, Monte Carlo methods become attractive. In essence the Monte Carlo approach consists of simulating trajectories of individual photons using probabilistic methods and concepts such as those discussed in [6.13]. In order to get good statistics a large number or trajectories must be simulated. Such simulations can, in principle, yield very precise results. The accuracy is primarily limited by computer resources. Monte Carlo methods have been developed to a high degree of sophistication and used to solve a variety of radiative transfer problems in plane-parallel media as well as media with complicated geometrics. This approach has also been widely used to solve radiative transfer problems in the ocean including the coupled ocean-atmosphere problem in the presence of a non-planar (wavy) interface. Some of the advantages are: all relevant orders of multiple scattering are taken into account; accurate solutions can be obtained for optically deep layers that may be inhomogeneous; interior radiances can be calculated; waves on the ocean surface can be taken into account; highly asymmetric phase functions for the hydrosols and aerosols are easily incorporated into the theory.

The basic Monte Carlo method has been described by Plass and Kattawar [267, 268]. The method has been extended by them Plass, G. N and Kattawar, G. W., [269, 270, 271] to calculate the flux and radiance in an atmosphere-ocean system to include the Stokes vector so that the polarization and ellipticity of the radiation is obtained Kattawar, G. W. and Plass, G. N. and Guinn, J. A. Jr. [229] and to include the effect of waves on the ocean surface Plass, G. N., Kattawar, G. W. and Guinn, J. A. Jr. [216]. Raschke [273] has considered the effect of ocean waves. Gordon and Brown [274] have used Monte Carlo techniques to compute the radiation flux in calm ocean,

but their calculations are for either an isotropic radiance distribution or a solar beam without sky radiation incident on the ocean; they did not couple the radiation fields of the atmosphere and ocean. In this article a particular model of the atmosphere ocean. We shall describe below in detail five Monte Carlo models in atmospheric-ocean RTE.
First Monte Carlo model is due to Gordon [300-303, 368]. This model simulates radiative transfer in both the ocean and the atmosphere, as coupled across a wind-roughened interface. The code is designed to simulate irradiances as a function of depth for computation of the irradiance reflectance and diffuse attenuation functions such. The nadir-viewing radiance is also computed as a source. The optical properties of the ocean are continuously stratified in the vertical. They can be specified as discrete values as a function of depth (with linear interpolation between the given depths) or determined from formulas. Separate scattering phase functions are used for the particles and for the water itself. Variants of this code have been used for a number of studies of radiative transfer in the ocean.
The sea-surface roughness is modeled using the Cox and Munk [215] surface slope distribution for a given wind speed. The effect of the surface roughness is not simulated exactly because the possibility of shadowing of one facet by another is ignored. Multiple scattering, however, is included: e.g., if a downward-moving photon in the atmosphere encounters the sea surface and is still moving downward after reflection, it will undergo a second interaction with the sea surface. One important aspect of this model is the proper use of photon weights to account for the fact that not all facets are oriented in such a manner as to be able to interact with an incident photon, i.e., facets with normals making an angle less than $90°$ to the direction of the incident photon.
The atmospheric part of the model consists of fifty **1**-km layers with both molecular and aerosol scattering. The vertical distribution of the optical properties is taken from Elterman, L., [305]. The aerosol phase function at the given wavelength is determined from Mie, G., [306] theory with Deirmendjian, D., [307] Haze **C** size distribution. When a photon interacts with the atmosphere, the scattering angle is chosen from either the molecular or aerosol phase functions based on the ratio of their scattering coefficients for the layer in which the interaction takes place.
When inelastic processes are to be included, the above code is operated at the excitation wavelength, to determine the excitation radiance distribution. This is used an input to a second Monte Carlo code that computes the light field at the wavelength of interest Ge, Y., Gordon, H. and Voss, K., [303]. As with the elastically scattered radiation, the goal is to determine the irradiances of the inelastically scattered radiation. This is considerable simplification because the solution can be effected by working with the azimuthally averaged radiance, i.e., only the azimuthally averaged radiative transfer equation need be solved.
The second model also simulates a coupled ocean-atmosphere system. The Monte Carlo code relies heavily on several variance-reducing schemes to increase computational efficiency. We

give only a brief description of one of the most useful ones. The use of statistical weights allows us to treat each photon history as a packet of photons rather than as a single photon. Photons are never allowed to escape from the ocean-atmosphere system. The method of forced collisions is used, whereby we sample from biased distribution that ensures a collision along the path and the weight is then adjusted appropriately to unbias the result. This model can simulate inelastic scattering; the details are given in Kattawar, G. and Xu, X., [308]. The Monte Carlo method also been extended to include the full Stokes vector treatment of polarization [181, 309 – 311]; these papers show that substantial errors can occur if polarization is neglected.

The third Monte Carlo model is similar to those described by Plass and Kattawar [269 – 271] and by Gordon, H.R. and Brown, O. B., [274]. It is designed to simulate the radiance distribution at any level in the atmosphere and in the ocean. Between these two media, a wind-roughened interface is modeled with the isotropic Gaussian distribution of sea-surface slopes, as discussed under model MC1. The probability of occurrence of the various slopes is modified when considering nonvertically incident photons. This photon – facet interaction is modeled as in Plass et al., [216] it does not account for the possible occultation of a facet by an adjacent one. Transmitted and reflected photon packets resulting from interaction with the air-water surface are weighted according to Fresnel's law (including the possibility of total internal reflection). According to the problem under investigation, photon packets are introduced at the top of the atmosphere, or just above (or below) the ocean surface. For specific problems involving deep levels, packets can be reintroduced a intermediate depths inside the water body, according to a directional distribution that reproduces the downward radiance field as resulting from a previous Monte Carlo run. The bottom boundary is either an infinitely thick absorbing layer, in which photons are lost from the system, or a Lambertian reflecting bottom of a given albedo, from which weighted photon packets are reflected.

After each collision, the weight of each photon packet is multiplied by the local value of ω_0 that is pertinent to the altitude or the depth, to account for its partial absorption. A packet history is terminated when its weight falls below a predetermined value, typically 1×10^{-6}. For each collision a random number on the unit interval is compared with the local value of the ratio of the molecular scattering coefficient to the total scattering coefficient to determine if the scattering event will be of molecular type (air or water molecules) or is due to an aerosol or hydrosol particle. The appropriate phase function is then used to determine the scattering angle; the orientation of the scattering plane is chosen at random on the interval $(0,2\pi)$. The number of photons initiated depends on the single-scattering albedo value, so as to control the stochastic noise in the computed radiometric quantities (details can be found in Morel, A. and Gentili, B., [182, 312, 370]). The model is operated for its oceanic segment with the optical properties. For the atmospheric segment, fifty 1-km-thick layers are considered, with specified values for Rayleigh and aerosol scattering and for ozone absorption as in Elterman, L., [305]. The aerosol phase function (as computed by Mie scattering theory) for the maritime aerosol model defined by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics is used; (see the models of Tanre et al. [313] and Baker, K. and Frouin, R., [314]).

This forth model is intended primarily for simulation of the radiance distribution above and just below the surface, and for simulation of irradiances with the first five mean free paths of the surface. The model based on techniques described by Kirk, J., [183]. The model atmosphere is composed of 50 layers, each characterized by separate Rayleigh and particulate scattering coefficients and an albedo of single scattering, as given by Elterman, L., [305]. Weighted photon beams are projected into the atmosphere from the atmosphere-space boundary, and a collision is forced somewhere in the atmosphere along this original trajectory. The attenuated beam, which is the weight of the original trajectory. Beam losses that are due to absorption and scattering take place at the point of collision. There the absorbed portion is lost and the scattered portion exists the collision point in another single, weighted beam. A random number is compared with the ratio of the Rayleigh scattering cross section to the total scattering cross section to determine the type of volume scattering function governing the scattering event. In the case of an aerosol scattering, a two-term Henyey-Greenstein phase function is used to determine the scattering angle Gordon, H. and Castano, D., [315]. Otherwise the angle is determined by a Rayleigh phase function Blattern, W. et al. [316]. Once the trajectory of the scattered portion of the beam is calculated, the distance from the point of collision to the next encountered interface (air-water or air-space) is determined. A new collision is forced somewhere along this trajectory, and the process is repeated until the weigh of the scattered portion of the beam falls below a preset minimum fraction of the original beam weight. This minimum traceable weight is set to 1×10^{-6} of the original beam weight for the simulations presented below.

Some of the scattered trajectories encounter the atmosphere-space boundary and are forgotten; the others impinge on the sea-surface. For the latter, the angle of incidence depends on the nadir angle of the ray and the slope of the sea surface. The directions of the reflected and refracted rays are determined geometrically and the weights of the rays are calculated from the Fresnel formula. Although wave shadowing is neglected, multiple surface interactions may occur. A reflected ray that is still projected downward, or a transmitted ray that is still projected upward, must encounter the sea surface again immediately, without an intervening trajectory. Ray trajectories resulting from reflection are followed in the original manner. Transmitted portions of the beams are followed similarly until encountering the bottom or the sea surface, or until they are diminished to less than the minimum traceable weight. Those beams striking the bottom are lost;

those beams that are incident upon the sea surface from below are again subjected to the reflection and transmission calculations.

The Naval Research Laboratory optical model (referred to as the NORDA or NOARL optical model in earlier publications) uses standard Monte Carlo techniques Gordon, H. [300]; Kattawar, G. W. and Plass, G. N. [271]; Kirk, J., [183]. At each scattering event, a random number is used to determine if the scattering is due to molecular water, quartzlike particulates, algae, or organic detritus; the volume scattering functions of these components are treated separately, rather than using an average volume scattering function. The model includes the effects of Raman scattering. If a photon collision results in inelastic scattering (as determined by comparing a random number to the appropriate optical properties of the medium), the wavelength is shifted by an amount corresponding to the mean wave-number shift of $3357\ \mathrm{cm}^{-1}$, corresponding to Raman scatter by water molecules. The finite bandwidth of the Raman-shifted light is taken into account by averaging over 10-nm bandwidths (roughly corresponding to current oceanographic instruments); details of this averaging are described in Stavn, R. and Weidemann, A., [317, 318]. For the simulation of problem Mobley, C., [224], below, it was assumed the Raman scattering occurs in a very narrow waveband. The photons are tallied into zonal bands, as is convenient for computation of irradiances and the nadir-viewing radiance. There is no atmosphere per se implemented in the model. Atmospheric transmittances of solar irradiance needed for simulations are obtained from the nonlayred atmospheric model of Brine, D. and Iqbal, M., [319]. The model determines the skylight radiance pattern from the empirical model of Harrison, A. and Coombes, C., [320]. The present version of the code handles only homogeneous waters. A comparative study on different approaches with numerical presentation is found in Mobley, C. D. et al. [223].

There exist more or less detail information on radiative transfer calculation with field experiment data in aquatic media as documented in Kinne, S. et al. [4]; Ramanathan, W. C. et al. [5] and Houghton, J. T. et al. [3], Dera, J., Marine Physics, [347].

1.9. Polarization consideration:

The solar incident light interacts with all the components of the atmosphere-ocean system. Each phenomenon of scattering by molecules, aerosols, hydrosols and reflection over the sea surface introduces and modifies the polarization state of light. Therefore, the reflected solar radiation is polarized and contains embedded information about the intrinsic nature of aerosols and suspended matter in the ocean. Most of the detailed physical information (i.e., size distribution, composition) about the particles present in the atmosphere-ocean system is available through the measurement and analysis of the spectral and angular polarization signature of the oceanic and atmospheric radiation. The principal reason for the greater effectiveness of remote sensing by means of polarization measurements is the significantly higher sensitivity of polarization features to particles size, shape and refractive index as a function of scattering angle and wavelength, than is the case for intensity measurements.

The polarization processes from the surface and atmosphere can be simulated using various radiative transfer models. Eddington's model computes the intensity accurately with fast speed. Surface polarization can be taken into account by using a different surface emissivity at vertical and horizontal polarization. This is adequate for clear and nonprecipitating conditions. To allow for all polarization processes in a fast forward model, Liu and Weng [235] develop a polarimetric two stream approximation. It is an extension of a scalar two stream model (Liou [208]). The radiative transfer model developed in these studies is applicable to a plane-parallel medium with randomly oriented cloud particles.

Various accurate Stokes vector radiative transfer models have been developed in the past. For example, the vector discrete ordinate radiative transfer method (VDISORT) was originally developed by Weng [237] and later improved by Schultz et al. [241]. A matrix operator method was used to derive a rigorous solution of the vector radiative transfer equation (Evans and Stephen [242]; Waterman [243]; Liu and Ruprecht [244]. A computationally efficient scalar two-stream model is utilized in some applications (Coakley and Chylek [245]; Kerschgens et al. [246]; Schmetz [247]; Weng and Grody [248]). However, it is found that these models are either computationally expensive or exclude the interaction among Stokes components.

Unfortunately, the radiative transfer equation (RTE), or the vector radiative transfer equation (VRTE) if polarization is involved, has no analytical solution for most realistic source medium configurations and boundary conditions. Therefore, numerical solutions become the only means to solve the RTE. Due to the complexity of the problem and limitation of the computational resources, the RTE is traditionally solved in a plane-parallel medium in which the optical properties are only allowed to be inhomogeneous in one dimension but kept homogeneous in the other two dimensions. Due to the rapid development of modern computer technology, solving the RTE in a three-dimensional (3D) turbid medium has recently become a very active research topic since most natural systems are 3D [249–258]. As mentioned earlier, among the plethora of research topics in radiative transfer theory, numerical solutions to the RTE in an optically coupled atmosphere–ocean system (AOS) is especially important Mobley, C. D. et al. [222–223] in the studies of the terrestrial atmosphere and oceans. Most of these methods use one-dimensional (1D) models for the AOS. Understandably, the 1D model fails when the system is not intrinsically 1D. A few examples of such cases Gordon, H. R., [261] are in simulating the radiance under a ship, in littoral zones or in a region Reinersman, P. N. and Carder, K. L., [262] where the hydrosol distribution is horizontally inhomogeneous. To study the inhomogeneity effect in the AOS, one has to use a 3D model. Currently, the Monte Carlo method is the only effective method to solve the RTE in a 3D AOS.

A hybrid method is developed by Zhai, Kattawar and Yang [263] to solve the vector radiative transfer equation (VRTE) in a three dimensional atmosphere–ocean system (AOS). The system is divided into three parts: the atmosphere, the dielectric interface, and the ocean. The Monte Carlo method is employed to calculate the impulse response functions (Green functions) for the atmosphere and ocean. The impulse response function of the dielectric interface is calculated by the Fresnel formulas. The matrix operator method is then used to couple these impulse response functions to obtain the vector radiation field for the AOS. The primary advantage of this hybrid method is that it solves the VRTE efficiently in an AOS with different dielectric interfaces while keeping the same atmospheric and oceanic conditions. For the first time, we present the downward radiance field in an ocean with a sinusoidal ocean wave.

1.10. Method employed in this work:

Most of the credit, in our opinion, for the introduction and development of the discrete-ordinates method in the general area of particle and radiative transport theory should go to Chandrasekhar [54], who in his fundamental work on radiative transfer did much to define the method as an effective computational tool. However this method, as used by Chandrasekhar had one difficult computational aspect that kept the approach from being used effectively past a certain order. This practical limitation generates from the required separation constants that are defined in terms of the zeros of a certain polynomial. Here we do not intend to review the numerous works devoted, in general, to discrete-ordinates methods, but some particularly important computational improvements to Chandrasekhar's original formulation may be mentioned. For example, Barichello and Siewert [164] under certain restrictions on the quadrature scheme found that the discrete-ordinates method is equivalent to the spherical-harmonics method (often used in radiative transfer and neutron transport theory). For these special quadrature schemes, where the equivalence holds between the spherical-harmonics method and the discrete-ordinates method, the separation constants can be computed as the eigenvalues of a tridiagonal matrix (a much easier task than finding zeros of polynomials). A second improvement they mention have to do with a scaling of the discrete-ordinates solution so as to avoid all positive exponentials that cause unnecessary "overows" in numerical calculations and have lead many formulations to fail. Finally, it was found that the use of half-range" quadrature schemes, as used have made the discrete-ordinates method a much more powerful technique, since boundary conditions in most radiative and neutron transport applications are typically of the half-range type. To complement Chandrasekhar's version of the method, the discrete-ordinates method has been combined with finite-difference techniques (Lewis and Miller, [146]) that are useful when the spatial dependence of the problem can not be treated analytically.

Modern analytical version (Barichello and Siewert, [147]) of the discrete-ordinates method used following facts. (a) It does not depend on any special properties of the quadrature scheme and (b) for many applications, such as the case of isotropic scattering considered, it has the separation constants defined as the eigenvalues of a matrix with special properties (diagonal matrix plus a rank-one update) so that the basic eigenvalues computation is of a type generally considered even easier than the one for a tridiagonal matrix. This paper has been developed in the context of non-coherent scattering for applications related to stellar atmospheres, and so in order to demonstrate the development of the method for reactor-physics applications. However a much simpler version of the method has been demonstrated in Barichello and Siewert, [375].
A nice and simple setting for the development of analytical discrete-ordinates method Fireland, W. A., [115] is discussed for two simple, but basic, problems related to the nuclear field, and in order to complete the work they briefed the various problems that have been solved using this version of the method. These remarks are noted just to indicate how the analytical version of the method discussed in the mentioned papers has been used for problems considerable more challenging than the specific problems used to illustrate the method. First of all in regard to other applications in the nuclear field, we note that solutions basic to fully-coupled multigroup neutron transport theory are reported by Siewert [161], and another paper (Garcia and Siewert, [152]) concerns the transport of neutral hydrogen atoms in hydrogen plasma. Continuing, we can point out those references Siewert [162,163, 374]; Barichello and Siewert, [147, 148, 155]; Barichello et al., [166]) which are devoted to solve various radiative transfer problems (grey and non grey models, one and multidimensional applications, scalar and vector problems and with and without the inclusion of polarization effects) in atmospheric sciences, and all the following references (Siewert, C. E., [157, 158, 159, 160, 165]; Barichello and Siewert [149, 156], Barichello et al. [46, 147]; Siewert, C. E. and Valougeorgis, D., [153, 154]) reported solutions to classical problems in the area of rarefied gas dynamics. In solving this broad class of problems, these researchers have found it very convenient to make use of quadrature schemes defined specifically for the considered application. In this way they have made use of a fundamental aspect of discrete-ordinates method, and so they have been able to deal efficiently with problems defined by difficult characteristic functions and with boundary conditions that are not continuous. To conclude, it should be noted that in implementing this algorithms for the considered problems, it is found that the analytical discrete-ordinates method to be concise, easy to implement and especially accurate.

To the best of our knowledge we are the first to apply this method for polarized radiative transfer problem in the coupled atmosphere-ocean system. We have developed the full theory taking into consideration to the following facts. (A) **Inhomogeneity in the underlying medium** (atmosphere and ocean). (B) In the boundary conditions we have taken care for **change in the refractive index in the air water interface** with proper formulation (C) **Total internal reflections at the air-water interface** for the out coming rays from ocean(However we did not use it in our numerical consideration for simplicity) (D) The **wind velocity dependent albedo functions** for ocean. (E) **Different source functions** for atmosphere and ocean.

CHAPTER II:

FORMULATION OF THE PROBLEM

Description of the Problem:

In this chapter we shall elaborately describe the problem which we have decided to investigate. This include the geometrical setup chosen, brief description of the atmospheric parameters like extinction and absorption coefficients, Phase functions, Equations of transfer for non-polarized and polarized radiation field, boundary condition employed and reduction of the transfer equation and related equations in detail.

2.1. Geometrical set up for atmosphere-Ocean system:

We shall consider the general case of an absorbing and scattering atmosphere-ocean system in which the optical properties of atmosphere and ocean (extinction coefficient, single scattering coefficient albedo etc.) may vary with respect to optical depth of the concern medium. The $+\mathbf{z}$-axis is taken downwards such as $\mathbf{z}=\mathbf{0}$ and $\mathbf{z}_{\mathrm{w}}$ corresponds to the top of the atmosphere and ocean respectively when $\mathbf{z}_1$ represents the lower limit (surface) of the ocean. The modeling of radiative transfer through such system must consider for reflection and refraction at the interfaces between two such media having different reflecting and refracting properties. (Fig-3)

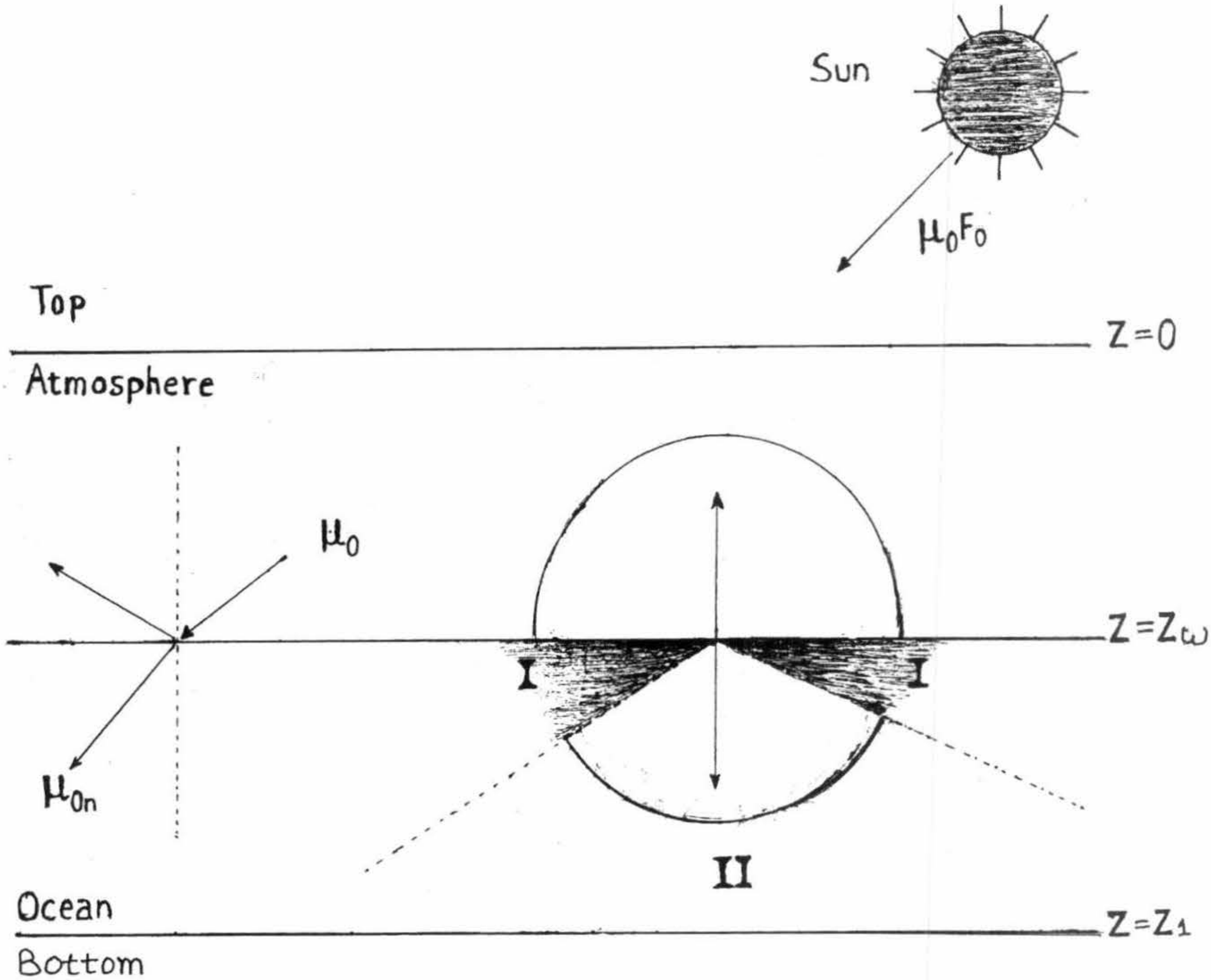


Fig (3)

We shall in most cases assume the boundary between these two media is a flat interface. The plane parallel model for both the media will be adopted so that the properties depend only on the vertical co-ordinates. Polarized radiation of intensity $\mathbf{I}(z,\mu,\varphi)$ at each point of the medium will be defined as a continuous function of z,μ,φ where μ is the absolute value of cosine of θ, the elevation angle relating to the $+z$-axis and φ is azimuthal angle measured relative to the $+x$-axis. These two angles specify the direction of the radiation field. The up and down radiation field without regard to direction will be denoted by $\pm\mu$. However we shall particularly denote $\mathbf{I}_{AT}(z,\mu,\varphi)$ for atmospheric medium and $\mathbf{I}_{OC}(z,\mu,\varphi)$ for ocean medium. The atmosphere is illuminated from above only by sunlight incident at an angle μ_0 to the radiation and the only solar flux crossing a plane perpendicular to the direction of incident is πF. The region I is the region of total internal reflection from which no radiation will go out from the ocean into the air. However region II represents the region of interactions between the two media.

2.2. Extinction and absorption Coefficient (in general for any type of media):

Radiation and matter have only two forms of interactions, extinction and emission. Lamberts, first proposed that the extinction (i.e., reduction) of radiation traversing an infinitesimal path is linearly proportional to the incident radiation and the amount of interacting matter along the path of radiation as follows

$$\frac{dI_\nu}{ds} = -K(\nu)I_\nu \tag{2.2.1}$$

Here $K(\nu)$ the extinction coefficient is a measurable property of the medium, and **s** is the absorption path long $K(\nu)$ is proportional to the local density of the medium and is positive-definite. The term extinction coefficient and the exact definition of $K(\nu)$ are somewhat ambiguous until their physical dimensions are specified. Path length, for example, can be measured in terms of column mass path, number path of molecules and geometric distance. Each path measure has a commensurate extinction coefficient: the mass extinction coefficient (i.e., optical cross-section per unit mass), the number extinction coefficient (i.e., optical cross-section per molecule) and the volume extinction coefficient (i.e., optical cross-section per unit concentration). These coefficients are interrelated.

Extinction includes all processes which reduce the radiant intensity. Hence these processes include absorption and scattering, both of which remove photons from the beam. Similarly the radiative emission is also proportional to the amount of matter along the path where S_ν is known as the source function. The source function plays an important role in radiative theory. We can show that if source function is known, then the full radiance field can be determined by an integration of the source function with the appropriate boundary conditions

$$\frac{dI_\nu}{ds} = K(\nu)S_\nu \tag{2.2.2}$$

Emission includes all processes which increase the radiant intensity. As will be described below, these processes include thermal emission and scattering which adds photons to the beam. Determination $K(\nu)$ of which contains all the information about the electromagnetic properties of the media, is the subject of active theoretical, laboratory and field research Extinction and emission are linear processes, and thus additive. Since they are the only two processes which alter the intensity of radiation, the equation of radiative transfer in its simplest differential form.

$$\frac{dI_\nu}{ds} = K(\nu)S_\nu - K(\nu)I_\nu \qquad (2.2.3)$$

2.3. Phase function:

Phase functions are responsible for the scattering of light radiation in the medium. Modeling of radiative transfer in either media requires an analytical form of phase functions. We shall first make a general discussion on phase functions. Later we shall try to give a brief review on types of phase function used in atmosphere and sea.

We shall consider that a beam of light is scattered by a single particle or a small volume-element of particles without causing any change in frequency of the light. Let ℓ and $\wp$ be the perpendicular to the scattering plane (the plane containing the incident and scattered beams) for both the incident and scattered beams in such a way that the direction of $\ell \times \wp$ coincides with the direction of propagation. The scattering process can be described by means of a 4×4 matrix, which we shall call the **scattering matrix (Muller matrix)** responsible for transformation of the Stokes parameters of the incident beam into those of the scattered beam apart from a constant factor (Van de Hulst, [124]). The scattering matrix may be described in the following form

$$F(\Theta) = \begin{bmatrix} a_1(\Theta) & b_1(\Theta) & 0 & 0 \\ b_1(\Theta) & a_2(\Theta) & 0 & 0 \\ 0 & 0 & a_3(\Theta) & b_2(\Theta) \\ 0 & 0 & -b_2(\Theta) & a_4(\Theta) \end{bmatrix}, \qquad (2.3.1)$$

Where $0 \le \Theta \le \pi$ is the **scattering angle**, i.e. the angle between the directions of the incident and scattered beams. This matrix contains 6 real functions and is valid in various situations, such as scattering by an (1) assembly of randomly oriented particles each of which has a plane of symmetry (e.g. homogeneous spheres or spheroids), an (2) assembly having particles and their mirror particles in equal numbers and with random orientation and (3) Rayleigh scattering with or without depolarization effects. Equation (2.3.1) gives

$$F(\Theta) = P\tilde{F}(\Theta)P, \qquad (2.3.2)$$

Where $P = \mathbf{diag}\ (1,1,-1,1)$, (2.3.3)

the well known reciprocity relation. As a consequence of symmetry with respect to the scattering plane one finds from equation (2.3.1)

$$\mathbf{F}(\Theta) = \mathbf{D}\mathbf{F}(\Theta)\mathbf{D}, \tag{2.3.4}$$

$$\text{Where} \quad \mathbf{D} = \mathbf{Diag}\ (1,1,-1,-1). \tag{2.3.5}$$

In addition to the symmetry relations (2.3.2) and (2.3.4) (Van de Hulst, [124]; Hovenier, [239]) we have for scattering angles $\mathbf{0}$ and π the special symmetry relations (Van de Hulst, [124]).

$\mathbf{a}_2(0) = \mathbf{a}_3(0)$, (2.3.6) $\quad \mathbf{b}_1(0) = \mathbf{b}_2(0) = 0$, (2.3.7) $\quad \mathbf{a}_2(\pi) = -\mathbf{a}_3(\pi)$, (2.3.8)

$\mathbf{b}_1(\pi) = \mathbf{b}_2(\pi) = 0$, (2.3.9)

The Stokes parameters of natural unpolarised light can be written as $\{\mathbf{I},0,0,0\}$. Hence, neglecting polarization in scattering problems amounts to keeping only $\mathbf{a}_1(\Theta) \neq 0$ in scattering matrix. Using the following representation

$$\mathbf{I}_C = \frac{1}{2}\begin{bmatrix} \mathbf{Q}+\mathbf{i}\mathbf{U} \\ \mathbf{I}+\mathbf{V} \\ \mathbf{I}-\mathbf{V} \\ \mathbf{Q}-\mathbf{i}\mathbf{U} \end{bmatrix}, \tag{2.3.10}$$

for both the incident and scattered beam, the scattering matrix in this representation via the following equation

$$\mathbf{G}_c = \mathbf{A}\mathbf{G}\mathbf{A}^{-1}. \tag{2.3.11}$$

can be written as

$$\mathbf{F}_c(\Theta) = \mathbf{A}\mathbf{F}(\Theta)\mathbf{A}^{-1}. \tag{2.3.12}$$

Matrix multiplications yields

$$\mathbf{F}_c(\Theta) = \frac{1}{2}\begin{bmatrix} \mathbf{a}_2(\Theta)+\mathbf{a}_3(\Theta) & \mathbf{b}_1(\Theta)+\mathbf{i}\mathbf{b}_2(\Theta) & \mathbf{b}_1(\Theta)-\mathbf{i}\mathbf{b}_2(\Theta) & \mathbf{a}_2(\Theta)-\mathbf{a}_3(\Theta) \\ \mathbf{b}_1(\Theta)+\mathbf{i}\mathbf{b}_2(\Theta) & \mathbf{a}_1(\Theta)+\mathbf{a}_4(\Theta) & \mathbf{a}_1(\Theta)-\mathbf{a}_4(\Theta) & \mathbf{b}_1(\Theta)-\mathbf{i}\mathbf{b}_2(\Theta) \\ \mathbf{b}_1(\Theta)-\mathbf{i}\mathbf{b}_2(\Theta) & \mathbf{a}_1(\Theta)-\mathbf{a}_4(\Theta) & \mathbf{a}_1(\Theta)+\mathbf{a}_4(\Theta) & \mathbf{b}_1(\Theta)+\mathbf{i}\mathbf{b}_2(\Theta) \\ \mathbf{a}_2(\Theta)-\mathbf{a}_3(\Theta) & \mathbf{b}_1(\Theta)-\mathbf{i}\mathbf{b}_2(\Theta) & \mathbf{b}_1(\Theta)+\mathbf{i}\mathbf{b}_2(\Theta) & \mathbf{a}_2(\Theta)+\mathbf{a}_3(\Theta) \end{bmatrix}. \tag{2.3.13}$$

This matrix contains four real functions (on both diagonals) and two complex functions which are conjugates. Obviously

$$\mathbf{F}_c(\Theta) = \widetilde{\mathbf{F}}_c(\Theta). \tag{2.3.14}$$

There exists a reciprocity relation as follows by substituting equation (2.3.2) in equation (2.3.12), taking the transpose on both sides and using the relations

$$\mathbf{P}\widetilde{\mathbf{A}} = \frac{1}{2}\mathbf{A}^{-1} \qquad (2.3.15) \qquad \text{and} \qquad \widetilde{\mathbf{A}}^{-1}\mathbf{P} = 2\mathbf{A} \qquad (2.3.16)$$

which result from the following two equations

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 0 & 1 & i & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -i & 0 \end{bmatrix}, \qquad (2.3.17) \qquad \mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -i & 0 & 0 & i \\ 0 & 1 & -1 & 0 \end{bmatrix}, \qquad (2.3.18)$$

As shown by equation (2.3.13) the matrix $\mathbf{F}_c(\Theta)$ is symmetric with respect to its center, i.e.

$$\mathbf{F}_c(\Theta) = \mathbf{M}\mathbf{F}_c(\Theta)\mathbf{M} \qquad (2.3.19)$$

$$\mathbf{M} = \widetilde{\mathbf{M}} = \mathbf{M}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \qquad (2.3.20)$$

This corresponds geometrically to symmetry with respect to the scattering plane as readily follows from equations (2.3.4) and (2.3.12) taking into account that

$$\mathbf{A}\mathbf{D}\mathbf{A}^{-1} = \mathbf{M}. \qquad (2.3.21)$$

Comparing equations (2.3.1) and (2.3.13) we see that, apparently, the price we must pay for simpler rotation properties is a greater complexity of the scattering matrix. We further assume that $\mathbf{F}(\Theta)$ is normalized in a such a way that

$$\frac{1}{4}\int_{(4\pi)} a_1(\Theta)d\omega = 1, \qquad (2.3.22)$$

where $d\omega$ is an element of solid angle.

In multiple scattering problems where polarization is neglected it is often advantageous to expand the phase function, $a_1(\Theta)$, in Legendre polynomials. This is true, in particular, for obtaining analytical expressions (Chandrasekhar, [54]; Sobolev, [377]; Van de Hulst, [131]) which can be used in various methods of solution. This we can write

$$a_1(\Theta) = \sum_{l=0}^{\infty}\omega_l P_l(\cos\Theta), \qquad (2.3.23)$$

where $\omega_0 = 1$, and $P_l(\cos\Theta)$ is the Legendre polynomial. We can assume that

$$\int_{-1}^{+1}[a_1(\Theta)]^2 d(\cos\Theta) < \infty. \qquad (2.3.24)$$

The convergence of the series in equation (2.3.23) is understood in the following sense:

$$\lim_{L\to\infty}\int_{-1}^{+1}\left|a_1(\Theta)-\sum_{l=0}^{L}\omega_l P_l(\cos\Theta)\right|^2 d(\cos\Theta)=0. \qquad (2.3.25)$$

The expansion coefficients ω_l may be found from the identity

$$\omega_l=\left(l+\frac{1}{2}\right)\int_{-1}^{+1}a_1(\Theta)P_l(\cos\Theta)\,d(\cos\Theta). \qquad (2.3.26)$$

Legendre polynomials are especially useful because they obey an addition theorem.
In applications of the series in equation (2.3.23) is usually truncated after the L-th term. As shown by Van der Mee [236], when polarization is neglected, the solution of the transport equation with phase function converges to the solution of the transport equation with untruncated phase function $a_1(\Theta)$

$$a_l^L(\Theta)=\sum_{l=0}^{L}\omega_l P_l(\cos\Theta) \qquad (2.3.27)$$

Unfortunately for nonnegative phase function $a_1(\Theta)$ the truncations in equation (2.3.27) may fail to be nonnegative, which means that $a_l^L(\Theta)$ may not correspond to a physical problem. As Feldman [272] showed, one may replace Eq. (2.3.27) by the non-negative approximants

$$\tilde{a}_l^L(\Theta)=\sum_{l=0}^{L}\omega_l\left(1-\frac{l}{L+1}\right)\left(1-\frac{l}{L+2}\right)P_l(\cos\Theta), \qquad (2.3.28)$$

and still the solution of the transport equation with phase function (2.3.28) converges to the solution of the equation with phase function $a_1(\Theta)$.
When polarization **is not neglected** a useful set of functions for making series expansions is provided by so-called generalized spherical functions. These functions are denoted by $P^l_{mm}(x)$. Here we limit m,n and l to be integers such that $m,n=-l,-l+1,\ldots,l$, or, in other words,

$$l\geq\max(|m|,|n|)=\frac{1}{2}(|m+n|+|m-n|). \qquad (2.3.29)$$

For other choices of l one defines $P^l_{mn}(x)=0$. The generalized spherical functions satisfy several important properties, one of which is the orthogonality relation.

$$\int_{-1}^{+1}P^l_{mn}(x)P^k_{mn}(x)dx$$

$$= (-1)^{m+n} \int_{-1}^{+1} P_{mn}^{l}(x) P_{mn}^{k}(x)^{*} dx = \frac{2}{2l+1} (-1)^{m+n} \delta_{lk'}, \qquad (2.3.30)$$

The asterisk denotes the complex conjugate, $k, l \geq \max(|m|,|n|)$, and $\delta_{lk} = 1$ if $l = k$ and vanishes if $l \neq k$. A precise description of the expansion of functions in generalized spherical functions is provided by the established fact. If the complex-valued function $h(x)$ on the closed interval $[-1,+1]$ is square integrable on this interval, i.e. if

$$\int_{-1}^{+1} |h(x)|^2 dx < \infty, \qquad (2.3.31)$$

then there exist unique coefficients $\eta_l [l \geq \max(|m|,|n|)]$ such that the series expansion

$$\sum_{l=\max(|m|,|n|)}^{\infty} \eta_l P_{mn}^{l}(x) = h(x) \qquad (2.3.32)$$

holds true in the following sense:

$$\lim_{L \to \infty} \int_{-1}^{+1} \left| h(x) - \sum_{l=\max(|m|,|n|)}^{\infty} \eta_l P_{mn}^{l}(x) \right|^2 dx = 0. \qquad (2.3.33)$$

We shall now turn our attention to expansions of the elements of the scattering matrices $F(\Theta)$ and $F_c(\Theta)$. Assume that the elements of $F(\Theta)$ satisfy the square integrability condition

$$\int_{-1}^{+1} |a_i(\Theta)|^2 d(\cos\Theta) < \infty \qquad (i = 1,2,3,4) \qquad (2.3.34)$$

and similarly for $b_1(\Theta)$ and $b_2(\Theta)$. Now the degree of polarization of any beam can never be larger than one. Applying this rule to the Stokes parameters of a beam of scattered light for incident light with Stokes parameters $\{1,0,1,0\}$ and $\{1,0,0,1\}$, respectively, we find

$$a_1(\Theta) \geq [|b_1(\Theta)|^2 + |b_2(\Theta)|^2 + |a_3(\Theta)|^2]^{\frac{1}{2}} \geq 0 \qquad (2.3.35)$$

and $$a_1(\Theta) \geq [|b_1(\Theta)|^2 + |b_2(\Theta)|^2 + |a_4(\Theta)|^2]^{\frac{1}{2}} \geq 0. \qquad (2.3.36)$$

On the other hand, if we take incident light with Stokes parameters $\{1,1,0,0\}$ and $\{1,-1,0,0\}$ we obtain

$$[a_1(\Theta)]^2 \pm 2a_1(\Theta)b_1(\Theta) \geq [a_2(\Theta)]^2 \pm 2a_2(\Theta)b_1(\Theta). \qquad (2.3.37)$$

By adding these two inequalities we find

$$a_1(\Theta) \geq |a_2(\Theta)| \geq 0. \qquad (2.3.38)$$

Therefore, it is sufficient to assume that, as for unpolarised light, condition (2.3.24) holds. In terms of the elements of the 2×2 matrix which transforms $\{\mathbf{E}_l, \mathbf{E}_r\}$ on scattering (Van de Hulst, [124]), necessary and sufficient conditions are

$$\int_{-1}^{+1} |\mathbf{S}_k(\Theta)|^4 \, d(\cos\Theta) < \infty, \tag{2.3.39}$$

where $\mathbf{k} = 1,2,3,4$. From equation (2.3.13), the preceding assumptions and the square integrability of sums and differences of square integrable functions it follows that the elements of $\mathbf{F}_c(\Theta)$, as function of $\cos\Theta$, are also square integrable on $[-1,+1]$. Thus, according to equation (2.3.32), we can expand each element of $\mathbf{F}(\Theta)$ and $\mathbf{F}_c(\Theta)$ in a series of generalized spherical functions $\mathbf{P}^l_{mn}(\cos\Theta)$ where, in principle, we can choose the integers $\mathbf{m}$ and $\mathbf{n}$ arbitrarily. However, a specific choice of $\mathbf{m}$ and $\mathbf{n}$ may be preferable in a numerical or analytical analysis of formulae containing $\mathbf{F}(\Theta)$ or $\mathbf{F}_c(\Theta)$.

To describe the state of polarization of a beam we first use Stokes parameters, but now the direction of $\mathbf{l}$ is along the meridian plane (plane through the beam and the $\mathbf{z}$-axis) and $\mathbf{r}$ is perpendicular to this plane. The direction of propagation is the direction of the vector product $\mathbf{r}\times\mathbf{l}$. The directions of the incident and scattered beams are represented in Fig (4) by points $\mathbf{P}_1$ and $\mathbf{P}_2$ respectively on the surface of a unit sphere, having $\mathbf{O}$ as its center. Suppose light traveling in a direction specified by υ' and φ' is scattered into a direction specified by υ and φ, the scattering angle being Θ. The positive $\mathbf{z}$-axis intersects the sphere in a point $\mathbf{N}$. On the surface of this sphere we have, in general, the spherical triangle $\mathbf{NP}_1\mathbf{P}_2$, with sides $\leq \pi$, namely υ, υ', and Θ. We assume the scattering in the volume-element to be governed by a scattering matrix of the form (2.3.1) with the normalization (2.3.22).

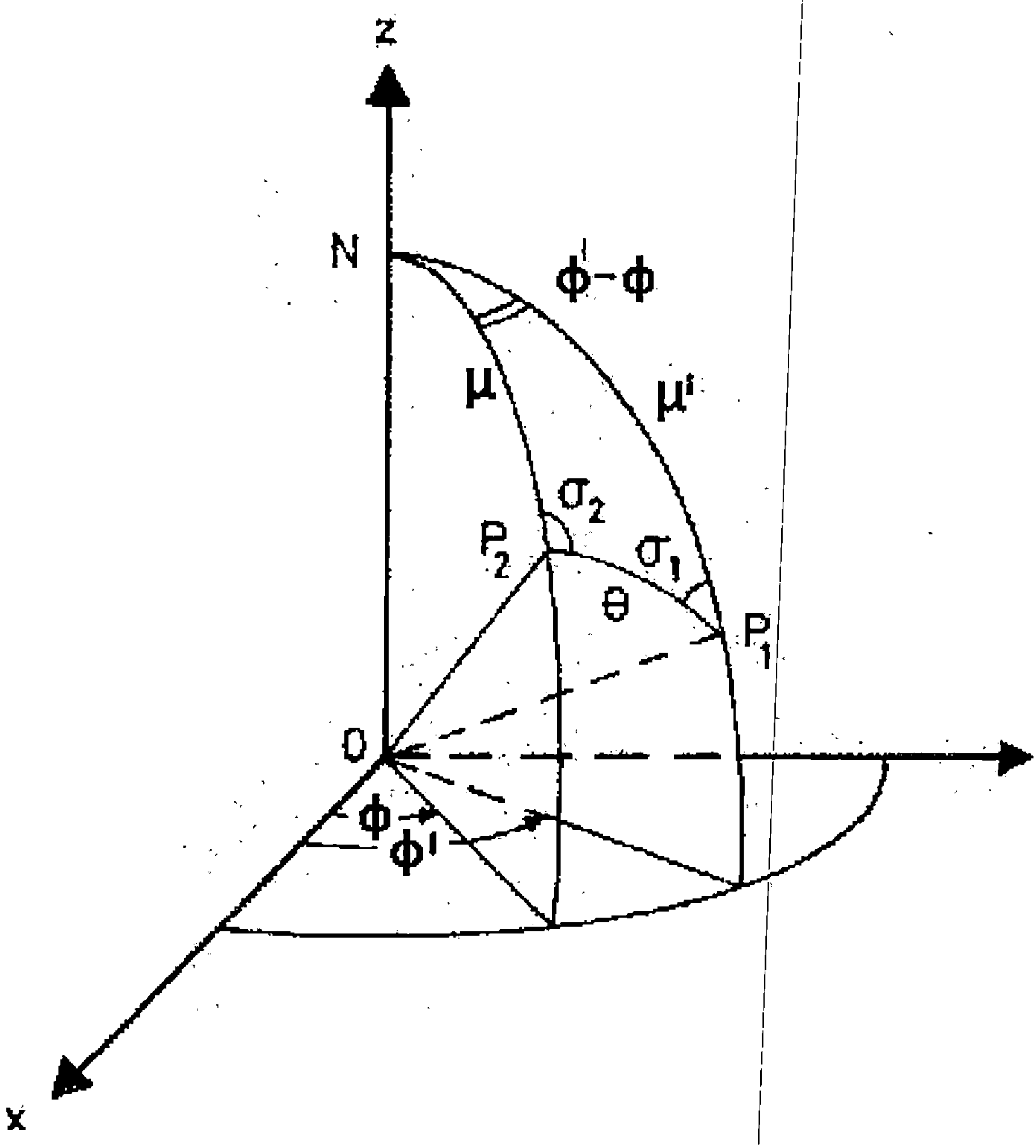


Fig (4)

First, we consider situations for which $0 < \varphi' - \varphi < \pi$. The scattering plane makes angles σ_1 (at P_1)

and σ_2 (at P_2) with the meridian plane, where $0<\sigma_1,\sigma_2<\pi$. Thus the angles of NP_1P_2 are σ_1,σ_2 and $\varphi'-\varphi$. The scattering process can now be described by means of a matrix which must be post multiplied by the Stokes vector of the incident beam to yield the Stokes vector of the scattered light (apart from a constant scalar depending on normalizations and physical units). We shall call this matrix the **phase matrix**. It may be written as

$$\mathbf{P}(\upsilon,\varphi;\upsilon',\varphi')=\mathbf{L}(\pi-\sigma_2)\mathbf{F}(\Theta)\mathbf{L}(-\sigma_1) \tag{2.3.40}$$

Rotation $\mathbf{L}(-\sigma_1)$ is responsible to turn the meridian plane at P_1 to the scattering plane while the second rotation $\mathbf{L}(\pi-\sigma_2)$ does the job of turning the scattering plane to the meridian plane at P_2. Using equation (1.3.1) and the following equation

$$\mathbf{L}(\chi)=\begin{bmatrix}1&0&0&0\\0&\cos2\chi&\sin2\chi&0\\0&-\sin2\chi&\cos2\chi&0\\0&0&0&1\end{bmatrix}, \tag{2.3.41}$$

we find

$$\mathbf{P}(\upsilon,\varphi;\upsilon',\varphi')=\begin{pmatrix}a_1(\Theta)&b_1(\Theta)C_1&-b_1(\Theta S_1&0\\ b_1(\Theta)C_2&C_2a_2(\Theta)C_1-S_2a_3(\Theta)S_1&-C_2a_2(\Theta)S_1-S_2a_3(\Theta)C_1&-b_2(\Theta)S_2\\ b_1(\Theta)S_2&S_2a_2(\Theta)C_1+C_2a_3(\Theta)S_1&-S_2a_2(\Theta)S_1+C_2a_3(\Theta)C_1&b_2(\Theta)C_2\\ 0&-b_2(\Theta)S_1&-b_2(\Theta)C_1&a_4(\Theta)\end{pmatrix}, \tag{2.4.42}$$

Where $\quad C_1=\cos2\sigma_1\ ,\ \ C_2=\cos2\sigma_2,\ \ S_1=\sin2\sigma_1\ ,\ \ S_2=\sin2\sigma_2 \qquad (2.3.43)$

Spherical trigonometry helps us to express the cosine rule for θ, υ and υ', successively

$$\cos\theta=\cos\upsilon\cos\upsilon'+\sin\upsilon\sin\upsilon'\cos(\varphi'-\varphi), \tag{2.3.44}$$

$$\cos\sigma_1=\frac{\cos\upsilon-\cos\upsilon'\cos\theta}{\sin\upsilon'\sin\theta}, \tag{2.3.45}$$

$$\cos\sigma_2=\frac{\cos\upsilon'-\cos\upsilon\cos\theta}{\sin\upsilon\sin\theta}. \tag{2.3.46}$$

We may further use

$$\cos2\sigma=2\cos^2\sigma-1, \quad (2.3.47) \qquad \sin2\sigma=2(1-\cos^2\sigma)^{\frac{1}{2}}\cos\sigma, \quad (2.3.48)$$

Where σ is σ_1 or σ_2. We should take σ_1 and σ_2 between $-\pi$ and 0 when executing the rotations of the co-ordinate axes when $0<\varphi-\varphi'<\pi$ or, $\pi<\varphi'-\varphi<2\pi$. One way of treating this problem is to leave the preceding formulae of this section as they are with the exception of the last one, which should be replaced by

$$\sin 2\sigma = -2(1-\cos^2\sigma_2)^{\frac{1}{2}}\cos\sigma\,, \tag{2.3.49}$$

Where $\sigma = \sigma_1$ or σ_2. It is natural to estimate limiting values when denominator of (2.3.40) and (2.3.46) becomes zero.

However in our context we shall use the following generalization of the phase function (2.3.40) with $\mathbf{F}(\boldsymbol{\Theta})$ given by following expression

$$\mathbf{P}(\cos\boldsymbol{\Omega}) = \begin{bmatrix} \sum_{r=0}^{N}\beta_r P_r(\cos\boldsymbol{\Omega}) & \sum_{r=2}^{N}\gamma_r P_r^2(\cos\boldsymbol{\Omega}) & 0 & 0 \\ \sum_{r=2}^{N}\gamma_r P_r^2(\cos\boldsymbol{\Omega}) & \sum_{r=0}^{N}\beta_r P_r(\cos\boldsymbol{\Omega}) & 0 & 0 \\ 0 & 0 & \sum_{r=0}^{N}\delta_r P_r(\cos\boldsymbol{\Omega}) & \sum_{r=2}^{N}\varepsilon_r P_r^2(\cos\boldsymbol{\Omega}) \\ 0 & 0 & \sum_{r=2}^{N}\varepsilon_r P_r^2(\cos\boldsymbol{\Omega}) & \sum_{r=0}^{N}\delta_r P_r(\cos\boldsymbol{\Omega}) \end{bmatrix} \tag{2.3.50}$$

In equation (2.3.50) we have developed phase matrix in Legendre polynomials $\mathbf{P}_r(\mathbf{Cos}\boldsymbol{\Omega})$ and associated Legendre polynomials $\mathbf{P}_r^2(\mathbf{Cos}\boldsymbol{\Omega})$. From the theory of spherical geometry it may be shown using the relationship between the scattering angle, the cosine of the zenith angles and the azimuth angles and addition theorem of spherical harmonics, that in general the phase function (kernel matrix) of the transfer equation $\mathbf{P}(\mu,\varphi,\mu',\varphi')$ may be expanded into the following form using cosine and sine series representation.

$$\mathbf{P}(\mu,\varphi,\mu',\varphi') = \sum_{S=0}^{L}\frac{1}{2}(2-\delta_{0,S})[\cos s(\varphi-\varphi')\mathbf{P}_C^S(\mu,\mu') + \sin s(\varphi-\varphi')\mathbf{P}_S^S(\mu,\mu')] \tag{2.3.51}$$

where linear combination of sin and cosine series in azimuth angle is associated with two functions of polar angles respectively thus enabling some decoupling of the transfer equation. These functions are given as follows

$$\mathbf{P}_C^S(\mu,\mu') = \begin{bmatrix} P^S{}_{11C} & P^S{}_{12C} & 0 & 0 \\ P^S{}_{21C} & P^S{}_{22C} & 0 & 0 \\ 0 & 0 & P^S{}_{33C} & P^S{}_{34C} \\ 0 & 0 & P^S{}_{43C} & P^S{}_{44C} \end{bmatrix} \tag{2.3.52}$$

$$
=\begin{bmatrix}
\sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu') & \sum_{J=S}^{M}\gamma_J P_J^S(\mu)R_J^S(\mu') & 0 & 0 \\
\sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu') & \sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu')+\xi_J T_J^S(\mu)T_J^S(\mu')) & 0 & 0 \\
0 & 0 & \sum_{J=S}^{M}(\alpha_J T_J^S(\mu)T_J^S(\mu')+\xi_J R_J^S(\mu)R_J^S(\mu')) & -\sum_{J=S}^{M}\varepsilon_J R_J^S(\mu)P_J^S(\mu') \\
0 & 0 & \sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)R_J^S(\mu') & \sum_{J=S}^{M}\delta_J P_J^S(\mu)P_J^S(\mu')
\end{bmatrix}
\tag{2.3.53}
$$

and

$$
P_S^S(\mu,\mu')=\begin{bmatrix}
0 & 0 & P^S{}_{13S} & P^S{}_{14S} \\
0 & 0 & P^S{}_{23S} & P^S{}_{24S} \\
P^S{}_{31S} & P^S{}_{32S} & 0 & 0 \\
P^S{}_{41S} & P^S{}_{42S} & 0 & 0
\end{bmatrix}
\tag{2.3.54}
$$

$$
=\begin{bmatrix}
0 & 0 & \sum_{J=S}^{M}\gamma_J P_J^S(\mu)T_J^S(\mu') & 0 \\
0 & 0 & \sum_{J=S}^{M}(\alpha_J R_J^S(\mu)T_J^S(\mu')+\xi_J T_J^S(\mu)R_J^S(\mu')) & -\sum_{J=S}^{M}\varepsilon_J T_J^S(\mu)P_J^S(\mu') \\
-\sum_{J=S}^{M}\gamma_J T_J^S(\mu)P_J^S(\mu') & -\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)R_J^S(\mu')+\xi_J R_J^S(\mu)T_J^S(\mu')) & 0 & 0 \\
0 & -\sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)T_J^S(\mu') & 0 & 0
\end{bmatrix}.
\tag{2.3.55}
$$

$R_J^S(\mu'), T_J^S(\mu')$, $R_J^S(\mu)$ and $T_J^S(\mu)$ are linear combinations of the generalized Legendre functions $P_{2,2}^S(\mu)$ and $P_{2,-2}^S(\mu)$. They are defined as

$$
P_{m,n}^l(\mu)=A_{m,n}^l(1-\mu)^{-\frac{(n-m)}{2}}(1+\mu)^{-\frac{(n+m)}{2}}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}].
\tag{2.3.56}
$$

$$A_{m,n}^{l} = \frac{(-1)^{l-m}}{2^{l}(l-m)!}\sqrt{\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}}. \tag{2.3.57}$$

These functions are normalized by $\frac{2}{(2l+1)}$. Discussion of coefficients appeared in the expansions will be made later. We can use following notations so that we can write phase matrix in more convenient form

$$A = \sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu'); \quad B = \sum_{J=S}^{M}\gamma_J P_J^S(\mu)R_J^S(\mu'); \quad C = \sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu')$$

$$D = \sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu') + \xi_J T_J^S(\mu)T_J^S(\mu')); \quad E = \sum_{J=S}^{M}(\alpha_J T_J^S(\mu)T_J^S(\mu') + \xi_J R_J^S(\mu)R_J^S(\mu'));$$

$$F = -\sum_{J=S}^{M}\varepsilon_J R_J^S(\mu)P_J^S(\mu') \quad G = \sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)R_J^S(\mu'); \quad H = \sum_{J=S}^{M}\delta_J P_J^S(\mu)P_J^S(\mu').$$

$$I = \sum_{J=S}^{M}\gamma_J P_J^S(\mu)T_J^S(\mu'); \quad J = \sum_{J=S}^{M}(\alpha_J R_J^S(\mu)T_J^S(\mu') + \xi_J T_J^S(\mu)R_J^S(\mu')); \quad K = -\sum_{J=S}^{M}\varepsilon_J T_J^S(\mu)P_J^S(\mu');$$

$$L = -\sum_{J=S}^{M}\gamma_J T_J^S(\mu)P_J^S(\mu'); \quad M = -\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)R_J^S(\mu') + \xi_J R_J^S(\mu)T_J^S(\mu')); \quad N = -\sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)T_J^S(\mu'). \tag{2.3.58}$$

$$A' = \sum_{J=S}^{M}\beta'_J P_J^S(\mu)P_J^S(\mu'); \quad B' = \sum_{J=S}^{M}\gamma'_J P_J^S(\mu)R_J^S(\mu'); \quad C' = \sum_{J=S}^{M}\gamma'_J R_J^S(\mu)P_J^S(\mu');$$

$$D' = \sum_{J=S}^{M}(\alpha'_J R_J^S(\mu)R_J^S(\mu') + \xi'_J T_J^S(\mu)T_J^S(\mu')); \quad E' = \sum_{J=S}^{M}(\alpha'_J T_J^S(\mu)T_J^S(\mu') + \xi'_J R_J^S(\mu)R_J^S(\mu'));$$

$$F' = -\sum_{J=S}^{M}\varepsilon'_J R_J^S(\mu)P_J^S(\mu'); G' = \sum_{J=S}^{M}\varepsilon'_J P_J^S(\mu)R_J^S(\mu'); H' = \sum_{J=S}^{M}\delta'_J P_J^S(\mu)P_J^S(\mu'); I' = \sum_{J=S}^{M}\gamma'_J P_J^S(\mu)T_J^S(\mu');$$

$$J' = \sum_{J=S}^{M}(\alpha'_J R_J^S(\mu)T_J^S(\mu') + \xi'_J T_J^S(\mu)R_J^S(\mu')); K' = -\sum_{J=S}^{M}\varepsilon'_J T_J^S(\mu)P_J^S(\mu'); L' = -\sum_{J=S}^{M}\gamma'_J T_J^S(\mu)P_J^S(\mu');$$

$$M' = -\sum_{J=S}^{M}(\alpha'_J T_J^S(\mu)R_J^S(\mu') + \xi'_J R_J^S(\mu)T_J^S(\mu')); \quad N' = -\sum_{J=S}^{M}\varepsilon'_J P_J^S(\mu)T_J^S(\mu'). \tag{2.3.59}$$

$$P^s_{ATC}(\mu,\mu') = \begin{bmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & E & F \\ 0 & 0 & G & H \end{bmatrix} \quad (2.3.60) \qquad P^s_{OCC}(\mu,\mu') = \begin{bmatrix} A' & B' & 0 & 0 \\ C' & D' & 0 & 0 \\ 0 & 0 & E' & F' \\ 0 & 0 & G' & H' \end{bmatrix} \quad (2.3.61)$$

$$P^s_{ATS}(\mu,\mu') = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & J & K \\ L & M & 0 & 0 \\ 0 & N & 0 & 0 \end{bmatrix} \quad (2.3.62) \qquad P^s_{OCS}(\mu,\mu') = \begin{bmatrix} 0 & 0 & I' & 0 \\ 0 & 0 & J' & K' \\ L' & M' & 0 & 0 \\ 0 & N' & 0 & 0 \end{bmatrix} \quad (2.3.63)$$

In the above representation for Fourier decompositions of the general phase function (2.3.51) we have separated the atmospheric and oceanic phase functions which are identified by the subscript **AT** and **OC.** We shall use this formulation in reducing the equation of transfer for atmosphere and ocean medium to render it azimuth free. However expressions (2.3.58-2.3.59) are used only to write some expressions in simplified form.

2.4. General discussion based on some experimentally derived sea-water phase functions:
Based on this general discussion on phase functions form the basis of the practical evaluation of phase functions for real atmospheric and ocean scattering cases of interest. In the light of the above discussion we shall now describe some of the particular phase functions for their physical realism, mathematical simplicity and properties and historic precedence. Together these phase functions mostly span the realm of functional forms seen in the literature of the oceanographic literature.

The **one-term Henyey-Greenstein (OTHG**) sea-water phase function is a one-parameter phase function that is frequently used because of its mathematical simplicity Haltrin, [260], [275]:

$$p_{HG}(\delta,g) = \frac{1}{2\pi}\frac{1-g^2}{(1-2g\delta+g^2)^{\frac{3}{2}}}. \qquad (2.4.1)$$

Where $\delta = \cos\Theta$, is a scattering angle in radians which is expressed through initial (θ,φ) and scattered (θ',φ') zenith and azimuth angles of light propagation, i.e.

$$\delta = \cos^{-1}(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\cos\theta\cos(\varphi-\varphi')), \qquad (2.4.2)$$

The parameter g, the **mean cosine of the scattering angle** is defined analytically as

$$g = \frac{1}{2}\int_0^{\pi} P(\Theta)\sin(\Theta)\cos(\Theta)d\Theta \qquad (2.4.3)$$

When integrated over the back-scatter directions equation (2.4.1) gives

$$B_p = \frac{1-g}{2g}\left(\frac{1+g}{\sqrt{1+g^2}}-1\right). \tag{2.4.4}$$

A value of $g = 0.9185$ gives $B_p = 0.0183$. The OTHG phase function levels out for Θ less than a few degrees, and it continues to decrease with increasing Θ at larger angles. Equation (2.4.1) is normalized according to

$$\int_{-1}^{1} p_{HG}(\delta, g)\, d\delta = 2. \tag{2.4.5}$$

Despite remarkable analytical properties of this phase function presented through the above equations, it can not really describe real scattering in seawater because the shape of HG phase functions differs significantly from real seawater real experimental scattering phase functions which shows prominent and distinct (and physically justified) first forward-directed peak and the backward-directed peak by ignoring the second one.

One can represent more realistically sea-water scattering phase function in the form of two anisotropic phase functions with peak forward and with the peak backward. There exist two such forms. Haltrin, [275, 356],

$$p_{HL}(\varepsilon, g, f, \delta) = f\, p_F(\varepsilon, \delta) + (1-f)\, p_B(g, \delta), \tag{2.4.6}$$

here $\delta = \cos\Theta$, Θ is a scattering angle, and f, ε and g are parameters. Both components of **PhF** (2.4.6) are normalized according to

$$\int_{-1}^{1} p_F(\varepsilon, \delta)\, d\delta = \int_{-1}^{1} p_B(\varepsilon, \delta)\, d\delta = \int_{-1}^{1} p_{HL}(\varepsilon, \delta)\, d\delta = 2. \tag{2.4.7}$$

The backward scattering portion of the phase function in equation (2.4.6) may be adequately represented by a Henyey-Greenstein function Haltrin, [369]

$$p_B(g, \delta) = \frac{1-g^2}{(1+g\delta+g^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} (2n+1)(-g)^n P_n(\delta), \quad 0 < g < 1. \tag{2.4.8}$$

here $P_n(\delta)$ are Legendre Polynomials. The mean or average cosine of **PhF** (2.4.8) is given by

$$\langle \cos\Theta \rangle = -g, \tag{2.4.9}$$

are backscattering probability by the formula

$$B_B(g) = \frac{1+g}{g}\left(1 - \frac{1-g}{\sqrt{1+g^2}}\right). \tag{2.4.10}$$

Another form proposed by Kattawar [357]

$$p_{TTHG}(\delta,\alpha,g,h) = \alpha p_{HG}(\delta,g) + (1-\alpha)p_{HG}(\delta,-h), \quad (0 \le \alpha, g, h \le 1). \qquad (2.4.11)$$

Here α is a weight of the forward-directed HG phase function, $(1-\alpha)$ is a weight of the backward-directed HG phase function, g having values near 1, is an asymmetry factor of the forward-directed HG phase function, thus indicating rapid increase at small scattering angle and h is an asymmetry factor of the backward-directed HG phase function, thus enabling the TTHG to increase with angle approaching 180 degree. The relation ship between g and h are given as Haltrin [275].

$$h = -0.30614 + 1.0006\,g - 0.01826\,g^2 + 0.03644\,g^3, \qquad (2.4.12)$$

$$\alpha = \frac{h(1+h)}{(g+h)(1+h-g)}. \qquad (2.4.13)$$

Choosing $g = 0.9809$ then gives $B_p = 0.0183$ for the Haltrin TTHG phase function.

The sign before the positive asymmetry factors determines the direction of elongation. A positive sign determines elongation forward and a negative sign determines elongation backward. So equation (2.4.11) defines the phase function that has two peaks: the forward peak with the weight α and asymmetry g and the backward peak with the weight $(1-\alpha)$ and asymmetry h.

The TTHG phase function of equation (2.4.11) has the following expansion into the Legendre polyno-mial series:

$$p_{TTHG}(\delta,\alpha,g,h) = \sum_{n=0}^{\infty}(2n+1)[\alpha g^n + (1-\alpha)\times(-h)^n]\,P_n(\delta). \qquad (2.4.14)$$

The integral parameters which include back scattering probability B, average cosine $\overline{\cos\Theta}$, and average square of cosine of $\overline{\cos^2\Theta}$ in the TTHG function are given by the following equations:

$$B \equiv \frac{1}{2}\int_{-1}^{0} p_{TTHG}(\mu,g)d\mu = \alpha\frac{1-g}{2g}\left[\frac{1-g}{(1+g^2)^{\frac{1}{2}}} - 1\right] + (1+\alpha)\frac{1+h}{2h}\left[1 - \frac{1-h}{(1+h^2)^{\frac{1}{2}}}\right], \qquad (2.4.15)$$

$$\overline{\cos\Theta} = \frac{1}{2}\int_{-1}^{1} p_{TTHG}(\mu)\,\mu d\mu = \alpha g + (1-\alpha)(-h) \equiv \alpha(g+h) - h, \qquad (2.4.16)$$

$$\overline{\cos^2\Theta} = \frac{1}{2}\int_{-1}^{1} p_{TTHG}(\mu)\,\mu^2 d\mu = \frac{1}{3} + \frac{2}{3}[\alpha(g^2-h^2) + h^2]. \qquad (2.4.17)$$

To reduce a number of independent parameters (α, g, h) in equation (2.4.11) for the TTHG function, one can use experimental data presented in the paper by Timofeyeva [358], in which a number of regression relationships between integral parameters of experimentally measured phase function are given. By using the data published by Timofeyeva [358], Haltrin [275] rederived the regression relationships between integral characteristics of the phase function in a form convenient for the elimination of the two extra parameters in equation (2.4.14). These new regressions are represented as follows

$$\overline{\cos\Theta} = 2\frac{1-2B}{2+B}, \qquad r^2 \cong 0.99999, \tag{2.4.18}$$

$$\overline{\cos^2\Theta} = \frac{6-7B}{3(2+B)}, \qquad r^2 \cong 0.99999, \tag{2.4.19}$$

The original measurements by Timofeyeva support equation (2.4.18) and (2.4.19) in the range of experimental data, i.e. $0.05 \le B \le 0.25$. This range encompasses the variability of B's corresponding to seawater; however, because of the specifics of the derivations, equations (78) (1.4.18) and (79) (1.4.19) are valid in the whole range of variability of backscattering probability $0 \le B \le 0.5$. The relationship of equations (2.4.18) and (2.4.19) should be considered as a better alternative to the original regressions by Timofeyeva. They are better because they give values of parameters that both lie in the range of experimental error for $0.05 \le B \le 0.25$ and satisfy two asymptotic conditions at delta-shaped scattering $(B=0)$ and isotropic scattering $(B=0.5)$. Equations (2.4.18) and (2.4.19) also give values almost identical to the values computed with Timofeyeva original regressions.

By solving the relationships of equations (2.4.18) and (2.4.19) with (2.4.15 – 2.4.16), we obtain the following coupling relationships between parameters (α, h) and the first asymmetry parameter g

$$\alpha = \frac{h(1+h)}{(g+h)(1+h-g)}, \tag{2.4.20}$$

$$h = -0.3061446 + 1.000568g - 0.01826332g^2 + 0.03643748g^3, \qquad 0.30664 < g \le 1. \tag{2.4.21}$$

Substitution of equations (2.4.20) and (2.4.21) into equation (2.4.11) or equation (2.4.14) given us a one-parameter **TTHG** phase function of light scattering in seawater with integral parameters $(B, \overline{\cos\Theta}, \overline{\cos^2\Theta})$ adjusted to the experimentally derived relationships given by equations (2.4.18) and (2.4.19).

The two-parametric analytic Fournier-Forand (FF) [359] scattering phase function was proposed by Fournier-Forand in 1994 to the ocean optics community. It was almost unnoticed by the optical oceanographic community possibly because the FF phase function has a more complex analytic form and the original paper did not include analysis of its properties. The major advantages of the

Fournier-Forand phase function are: 1) it depends only on two parameters; 2) it approximates almost all realistic marine phase functions with a very high degree of precision.
Fournier and Forand (FF) derived an approximate analytic form of the phase function of an ensemble of particles that have a hyperbolic (Junge-type) particle size distribution, with each particle scattering according to the anomalous diffraction approximation to the exact Mie theory. In its latest form this phase function is given by

$$\overline{\beta}_{FF}(\Theta)=\frac{1}{4\pi(1-\chi)^2\chi^{\nu}}\left[\nu(1-\chi)-(1-\chi^{\nu})+[\chi(1-\chi^{\nu})-\nu(1-\chi)]\sin^2\left(\frac{\Theta}{2}\right)\right]$$
$$+\frac{1-\chi_{180}^{\nu}}{16\pi(\chi_{180}-1)\chi_{180}^{\nu}}(3\cos^2\Theta-1), \tag{2.4.22}$$

where

$$\nu=\frac{3-\eta}{2}, \qquad \chi=\frac{4}{3(n-1)^2}\sin^2\left(\frac{\Theta}{2}\right). \tag{2.4.23}$$

Here n is the real index of refraction of the particles, η is the slope parameter of the hyperbolic distribution, and χ_{180} is χ evaluated at $\Theta = 180\,\mathrm{deg}$.

$$B_p = 1-\frac{1-\chi_{90}^{\nu+1}-0.5(1-\chi_{90}^{\nu})}{(1-\chi_{90})\chi_{90}^{\nu}}, \tag{2.4.24}$$

Where χ_{90} is χ evaluated at $\Theta = 90\,\mathrm{deg}$. Although equations (2.4.24) use only the real part of the index of refraction, the addition of moderate amounts of absorption does not significantly change the shape of the phase functions generated by equation (2.4.22). An analytic Fournier Forand scattering phase function was derived by Haltrin [371] as an alternative to the HG phase functions.

2.5. Reflection and Transmission:

The direction and energies of the reflected rays may be derived by Snell-Descartes' and Fresnel's respectively. Our broad aim in this study is to model radiative transfer in the ocean by following the fate of those solar photons which have passed down through the ocean surface, as they travel through the water column. In the following we first want to discuss the fate of a photon after it hits a flat ocean surface. For each photon, however, the first question which needs to be decided is whether it actually succeeds in passing through the air-water interface. Most photons do, but a proportion undergoes reflection. For a parallel beam of unpolarised light the surface-reflectance is determined by θ_a, the angle of the incident light in air to the upward normal to the water surface, and θ_t, the angle of the transmitted beam within the water to the downward normal to the surface, in accordance with Fresnel's equation

$$\mathbf{R_f} = \frac{1}{2}\left[\frac{\sin(\theta_a - \theta_w)}{\sin(\theta_a + \theta_w)}\right]^2 + \frac{1}{2}\left[\frac{\tan(\theta_a - \theta_w)}{\tan(\theta_a + \theta_w)}\right]^2. \tag{2.5.1}$$

So before deciding whether a photon passes through air into the water we must first determine what its angle would be in the event that it did pass through into the water. The angle, θ_w, in water is determined by the angle, θ_a, in air, and refractive index in accordance with Snell's Law

$$\sin\theta_w = \frac{\sin\theta_a}{\dfrac{n_w}{n_a}} \tag{2.5.2}$$

where $\frac{n_w}{n_a}$ is the ratio of the refractive index of seawater of that of air which is approximately given as ~ 1.33.If the surface of the water is smooth and flat, and then θ_a is simply the angle between the photon trajectory and the upward vertical. On a clear sunny day, when **80-85%** of the downward irradiance originates in the direct solar beam, θ_a for the great majority of the photons in the solar zenith angle. For such photons we calculate the surface reflectance, $\mathbf{R_f}$, with equations (2.5.1) and (2.5.2).However for flat water surface the Fresnel reflection coefficient is given by

$$\mathbf{R_F^0}(\theta_a) = \frac{1}{2}\left[\left(\frac{\cos\theta_a - \sqrt{n_w^2 - \sin^2\theta_a}}{\cos\theta_a + \sqrt{n_w^2 - \sin^2\theta_a}}\right)^2 + \left(\frac{n_w^2\cos\theta_a - \sqrt{n_w^2 - \sin^2\theta_a}}{n_w^2\cos\theta_a + \sqrt{n_w^2 - \sin^2\theta_a}}\right)^2\right], \tag{2.5.3}$$

here $n_w \approx 1.341$ is a refractive index of water. These above relations for reflection and transmission functions are useful for non-polarized treatment of radiative transfer processes.

For **polarized radiation transfer** the reflection and transmission Muller matrix (4x4 Matrix responsible for transformation of stokes vector when an interaction occurs) can be treated as follows. As we have described earlier the final stokes vector for the scattered radiation field can be written in terms of rotation matrix (2.3.41) which counts for clockwise rotation through the scattering angle. The rotation matrix has some interesting properties namely,

$$\mathbf{L}(-\chi) = \mathbf{L}(\pi - \chi); \tag{2.5.4}$$

$$\mathbf{L}(\chi_1)\mathbf{L}(\chi_2) = \mathbf{L}(\chi_1 + \chi_2); \tag{2.5.5}$$

$$\mathbf{L}^{-1}(\chi) = \mathbf{L}(-\chi). \tag{2.5.6}$$

In case of Raleigh scattering the Muller matrix takes the form

$$\mathbf{R}(\boldsymbol{\Theta}) = \begin{bmatrix} \frac{1+\alpha^2}{2} & \frac{\alpha^2-1}{2} & 0 & 0 \\ \frac{\alpha^2-1}{2} & \frac{1+\alpha^2}{2} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \quad (2.5.7)$$

with $\alpha = \cos\Theta$ and Θ is scattering angle. There are several interesting properties. (a) Complete linearization is possible at 90 degree scattering angle; (b) there exists no ellipticity in the multiple radiations if the source is unpolarised. In such cases the reflection and the transmission Mueller matrix for radiation going from **air to the other medium (like ocean)** may be given as

$$\mathbf{R}_{\mathbf{AM}} = \begin{bmatrix} \alpha+\eta & \alpha-\eta & 0 & 0 \\ \alpha-\eta & \alpha+\eta & 0 & 0 \\ 0 & 0 & \gamma_{\mathbf{Re}} & 0 \\ 0 & 0 & 0 & \gamma_{\mathbf{Re}} \end{bmatrix} \quad (2.5.8) \qquad \mathbf{T}_{\mathbf{AM}} = \begin{bmatrix} \alpha'+\eta' & \alpha'-\eta' & 0 & 0 \\ \alpha'-\eta' & \alpha'+\eta' & 0 & 0 \\ 0 & 0 & \gamma'_{\mathbf{Re}} & 0 \\ 0 & 0 & 0 & \gamma'_{\mathbf{Re}} \end{bmatrix} \quad (2.5.9)$$

where

$$\alpha = \frac{1}{2}\left[\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)}\right]^2 \quad (2.5.10) \qquad \eta = \frac{1}{2}\left[\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)}\right]^2 \quad (2.5.11)$$

$$\gamma_{\mathbf{Re}} = \frac{\tan(\theta_i-\theta_t)\sin(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)\sin(\theta_i+\theta_t)} \quad (2.5.12)$$

and

$$\alpha' = \frac{1}{2}\left[\frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}\right]^2 \quad (2.5.13) \qquad \eta' = \frac{1}{2}\left[\frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)}\right]^2 \quad (2.5.14)$$

$$\gamma'_{\mathbf{Re}} = \frac{4\sin^2\theta_t\cos^2\theta_i}{\sin^2(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} \quad (2.5.15)$$

θ_i And θ_t refer to the incident and transmitted angles and are related by Snell's law; namely, $\sin\theta_i = n\sin\theta_t$ where n is the refractive index of the medium relative to air. The element $R_{11} = \alpha+\eta$ is the effective reflectivity. It can be shown that the reflectivity is quite constant up to $\sim 60°$ at which point it rises sharply and goes to unity at an incidence angle of $90°$ for any refractive index. It should be noted that the R_{12} normalized element goes to -1.0 at the

Brewster angle (This is the angle at which incoming unpolarised radiation will become completely linearly polarized upon reflection).defined by $\theta_B^{AM} = \tan^{-1} n$ (for the tabulated values). In going from **medium into air**, as long as $\theta < \theta_{crit}$ where θ_{crit} is the critical angle ($\theta_{crit} = \sin^{-1} \bar{n}$ where $\bar{n}$ is the refractive index of air relative to the medium, i.e. $\bar{n} = \frac{1}{n}$), we can still use equations (2.5.8) and (2.5.9) to compute both R_{MA} and T_{MA}; namely

$$R_{MA} = \begin{bmatrix} \alpha+\eta & \alpha-\eta & 0 & 0 \\ \alpha-\eta & \alpha+\eta & 0 & 0 \\ 0 & 0 & \gamma_{Re} & -\gamma_{Im} \\ 0 & 0 & \gamma_{Im} & \gamma_{Re} \end{bmatrix}, \tag{2.5.16}$$

$$T_{MA} = \begin{bmatrix} \alpha'+\eta' & \alpha'-\eta' & 0 & 0 \\ \alpha'-\eta' & \alpha'+\eta' & 0 & 0 \\ 0 & 0 & \gamma'_{Re} & -\gamma'_{Im} \\ 0 & 0 & \gamma'_{Im} & \gamma'_{Re} \end{bmatrix}, \tag{2.5.17}$$

where for this region both $\gamma_{Im} = \gamma'_{Im} = 0$. However, for the region where $\theta > \theta_{crit}$ which is the region where total internal reflection takes place θ_t becomes complex, T_{MA} becomes the null matrix, and the following equations must be used to compute R_{MA}:

$$\alpha = \frac{1}{2}\left|\frac{\bar{n}^2 \cos\theta_i - i\sqrt{\sin^2\theta_i - \bar{n}^2}}{\bar{n}^2 \cos\theta_i + i\sqrt{\sin^2\theta_i - \bar{n}^2}}\right|^2 \quad (2.5.18) \qquad \eta = \frac{1}{2}\left|\frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - \bar{n}^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - \bar{n}^2}}\right|^2 \quad (2.5.19)$$

$$\gamma_{Im} = \mathrm{Im}\left\{\left[\frac{\bar{n}^2 \cos\theta_i + i\sqrt{\sin^2\theta_i - \bar{n}^2}}{\bar{n}^2 \cos\theta_i - i\sqrt{\sin^2\theta_i - \bar{n}^2}}\right]\cdot\left[\frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - \bar{n}^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - \bar{n}^2}}\right]\right\} \tag{2.5.20}$$

and

$$\gamma_{Re} = \mathrm{Re}\left\{\left[\frac{\bar{n}^2 \cos\theta_i + i\sqrt{\sin^2\theta_i - \bar{n}^2}}{\bar{n}^2 \cos\theta_i - i\sqrt{\sin^2\theta_i - \bar{n}^2}}\right]\cdot\left[\frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - \bar{n}^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - \bar{n}^2}}\right]\right\}, \tag{2.5.21}$$

Where now $\theta_i > \theta_{crit}$. The corresponding Brewster angle in going from medium into air is defined by $\theta_B^{MA} = \tan^{-1} \bar{n}$. We can obtain analytic expressions for both the location and the values of the matrix elements at the extremum. The results are

$$\theta_{max} = \cos^{-1}\left(\frac{1-\bar{n}^2}{1+\bar{n}^2}\right)^{\frac{1}{2}} \qquad (2.5.22) \qquad \frac{R_{33}^{min}}{R_{11}} = \frac{6\bar{n}^2 - \bar{n}^4 - 1}{(1+\bar{n}^2)^2} \qquad (2.5.23)$$

and

$$\frac{R_{34}^{min}}{R_{11}} = \frac{-4\bar{n}(1-\bar{n}^2)}{(1+\bar{n}^2)^2}. \qquad (2.5.24)$$

The location, however, of the extremum in terms of phase shifts is given by Born and Wolf [360]. The use of this angle is important in creating what is known as a Fresnel rhomb whereby linearly polarized light is converted into circularly polarized light. This is the region where we expect to see ellipticity reach a maximum.

One can write the above equation in the case of multiple scattering of electromagnetic radiation by a diffusive medium containing low density Raleigh scatters characterized by their cross sections so that mean free path is much larger than the wave length of radiation in the medium. The reflection and transmission matrix for such media have the following form.

$$R(\mu) = \begin{pmatrix} |r_R(\mu)|^2 & 0 & 0 & 0 \\ 0 & |r_L(\mu)|^2 & 0 & 0 \\ 0 & 0 & Re(r_R(\mu)r_L(\mu)^*) & -Im(r_R(\mu)r_L(\mu)^*) \\ 0 & 0 & Im(r_R(\mu)r_L(\mu)^*) & Re(r_R(\mu)r_L(\mu)^*) \end{pmatrix} \qquad (2.5.25)$$

$$T(\mu) = \frac{m\mu}{\sqrt{1-m^2v^2}} \begin{pmatrix} |t_R(\mu)|^2 & 0 & 0 & 0 \\ 0 & |t_L(\mu)|^2 & 0 & 0 \\ 0 & 0 & Re(t_R(\mu)t_L(\mu)^*) & -Im(t_R(\mu)t_L(\mu)^*) \\ 0 & 0 & Im(t_R(\mu)t_L(\mu)^*) & Re(t_R(\mu)t_L(\mu)^*) \end{pmatrix} \qquad (2.5.26)$$

$$r_R(\mu) = \frac{\mu - m\sqrt{1-m^2v^2}}{\mu + m\sqrt{1-m^2v^2}}, \qquad r_L(\mu) = \frac{\sqrt{1-m^2v^2} - m\mu}{\sqrt{1-m^2v^2} + m\mu},$$

$$t_R(\mu) = \frac{2\sqrt{1-m^2v^2}}{\mu + m\sqrt{1-m^2v^2}}, \qquad t_L(\mu) = \frac{2\sqrt{1-m^2v^2}}{\sqrt{1-m^2v^2} + m\mu}. \qquad (2.5.27)$$

In the above expressions $r_R(\mu)$, $r_L(\mu)$ and $t_R(\mu)$, $t_L(\mu)$ are the Fresnel reflection and transmission amplitude coefficients, respectively. The latter only depend on the inner incidence angle θ and on the index mismatch $\mathbf{m}$ .In the case of partial reflection, these coefficients are real, with absolute values less than unity. In the case of total reflection, the reflection coefficients are pure phases, i.e., complex numbers with unit modulus, while the transmission coefficients vanish by convention. The first two diagonal elements of the reflection matrix $\mathbf{R}(\mu)$ of equation (1.5.25) and equation (1.5.26) read

$$|\mathbf{r}_R(\mu)|^2 = \mathbf{R}_R(\mu) = 1 - \mathbf{T}_R(\mu), \qquad |\mathbf{r}_L(\mu)|^2 = \mathbf{R}_L(\mu) = 1 - \mathbf{T}_L(\mu) \tag{2.5.28}$$

in terms of the Fresnel reflection and transmission intensity coefficients.

2.6. Reflection and transmission in wind-disturbed surface. (Monte Carlo):

In the case of a wind-disturbed surface, θ_a, the angle between the photon trajectory and the normal to the surface at the point of impact, will most commonly not be equal to the solar zenith angle. To an extent depending on wind strength the different facets of water surface will vary from the horizontal by angular differences in the range $0°$ to $90°$. The relationship between wind speed and the angular variations in the surface was first systematically investigated over half a century ago by (Duntley) on Lake Winnipesaukee, New Hampshire in 1949 and by (Cox and Munk [215]) in the Pacific Ocean near Hawaii in 1951. Duntley used parallel pairs of vertical stainless steel wires, 25 mm apart, passing down through the water surface. The electrical impedance between them, which was continuously monitored, varied in accordance with their relative submerged lengths, which could therefore be calculated. The difference in submerged length was proportional to the slope of the wave passing the wires at a given moment, and so this slope could also be continuously monitored and from this the frequency distribution of slopes under given wind conditions could be obtained. Cox and Munk, by contrast, used aerial photography from a former World War II bomber flying at 2000 ft. They photographed the solar glitter pattern on the ocean surface over a range of wind conditions. By a detailed analysis of these photographs it was possible to arrive at the surface slope statistics as a function of wind speed.

The wave slope is $\tan\psi$ where ψ is the angle between the vertical and the normal to the sea surface at that point: ψ is of course also equal to the angle of the sea surface to the horizontal at the point in question. For every wave, corresponding to the slope on one side there is an opposite slope on the other side. We can choose to regard all slopes on the – say – downwind side as positive, in which case the corresponding slopes on the upwind side are negative. It is known from observation that the mean slope is negligible over an area the linear dimensions of which are mush greater than the longest ocean wavelength. The lake and ocean studies both led to the same fundamental conclusions which are –

1. At a given wind speed the distribution of wave slopes is Gaussian around the mean of zero

2. The mean square slope, σ^2, or $\overline{\tan^2\psi}$ increases linearly with wind speed.

Preisendorfer [75] showed that the Gaussian distribution of wave slope can be represented as a probability function for the occurrence of wave slope $\tan\psi$

$$P(\psi)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2}.\frac{\tan^2\psi}{\sigma^2}\right) \tag{2.6.1}$$

and that the fractional area of sea surface which has wave slopes lying in the range $(0,\tan\psi)$ at a given instant in time is

$$a_\psi{}'=1-\exp\left(-\frac{1}{2}.\frac{\tan^2\psi}{\sigma^2}\right), \tag{2.6.2}$$

Putting it another way, $a_\psi{}'$ is the probability that any given wave slope does not exceed the prescribed value, $\tan\psi$, and so $a_\psi{}'$ is the cumulative distribution function for wave slope

$$F(\tan\psi)=1-\exp\left(-\frac{1}{2}.\frac{\tan^2\psi}{\sigma^2}\right); \tag{2.6.3}$$

We shall now deduce some explicit expression for reflection and transmission matrix for rough ocean surface in a different context. The light incident at a flat water surface will be reflected or refracted directly. However, photons incident at the rough surface may scatter more than once among the surface wave facets before exit to air or water. We can write single-scattering reflectance-matrix at the air-water interface from (μ',φ') to (μ,φ)

$$R_0(\mu,\varphi,\mu',\varphi',n)=r(\cos\alpha_r,n)F(\mu',\varphi'\to\mu,\varphi,\mu_n^r,\sigma)s(\mu,\mu',\sigma), \tag{2.6.4}$$

where $r(\cos\alpha_r,n)$ is the (4x4) Fresnel reflection-matrix(coefficient) for relative refractive index n under incident angle α_r (may be computed from equations (2.6.1) & (2.6.3) with appropriate changes), $n=\frac{n_w}{n_a}$ for air incidence and $n=\frac{n_a}{n_w}$ for water incidence. The $F(\mu',\varphi'\to\mu,\varphi,\mu_n^r,\sigma)$ is the fraction of the sea surface (i.e., the effective area of the wave facets) with normal μ_n^r reflecting light from (μ',φ') to (μ,φ) and is given by

$$F(\mu',\varphi' \to \mu,\varphi,\mu_n^r,\sigma) = \frac{1}{4\mu(\mu_n^r)^4} f(\mu_n^r,\varphi_n^r), \tag{2.6.5}$$

With $f(\mu_n^r,\varphi_n^r)$ representing the probability distribution function for the surface slope distribution which is given accordingly

$$f(\mu_n^r,\varphi_n^r) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{1-\mu_n^2}{\sigma^2\mu_n^2}\right). \tag{2.6.6}$$

The required surface normal, μ_n^r, to fulfill the specular reflection from (μ',φ') to (μ,φ) is determined by μ',φ',μ and φ. Defining

$$\cos\alpha = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\varphi-\varphi'), \tag{2.6.7}$$

$\cos\alpha_r$ in equation (2.6.4) can be derived as

$$\cos\alpha_r = \sqrt{\frac{(1-\cos\alpha)}{2}}, \tag{2.6.8}$$

and μ_n^r can be derived as

$$\mu_n^r = \frac{\mu+\mu'}{\sqrt{2(1-\cos\alpha)}}. \tag{2.6.9}$$

In equation (2.6.2), the shadowing effect, representing the probability that the incident and the reflected lights are intercepted by other surface waves, is corrected by the function $s(\mu,\mu',\sigma)$.

Similarly, the single-scattering transmittance at the air-water interface from (μ',φ') to (μ,φ) can be written as

$$T_0(\mu,\varphi,\mu',\varphi',n) = t(\cos\alpha_t,n)F(\mu',\varphi' \to \mu,\varphi,\mu_n^t,\sigma)\, s(\mu,\mu',\sigma), \tag{2.6.10}$$

Where $t(\cos\alpha_t,n)$ is the (4x4) Fresnel transmission coefficient matrix for relative refractive index n for incident angle α_t (may be computed from equations (2.6.8), (2.6.9) with appropriate changes) $F(\mu',\varphi' \to \mu,\varphi,\mu_n^t,\sigma)$ is the fraction of the sea surface with the orientation to refracted light from (μ',φ') to (μ,φ) and is given by

$$F(\mu',\varphi' \to \mu,\varphi,\mu_n^t,\sigma) = \frac{n\sqrt{n^2+\cos^2\alpha_t - 1}}{4\mu(\mu_n^t)^4\cos\alpha_t} f(\mu_n^t), \tag{2.6.11}$$

The surface normal (μ_n^t) and the incident angle (α_t) required to fulfill the refraction are

$$\cos\alpha_t = \frac{|n\cos\alpha - 1|}{\sqrt{n^2 - 2n\cos\alpha + 1}}, \quad (2.6.12)$$

$$\mu_n^t = \mu'\cos\alpha_t + \sin\alpha_t\sqrt{1-\mu'^2}\sqrt{\frac{1-(1-\mu^2)\sin^2(\varphi-\varphi')}{\sin\alpha}}, \quad (2.6.13)$$

It is to be noted that in this work we did not consider this case in our boundary conditions. Hence in subsequent calculations we shall not use this formulation any further.

2.7. Radiative transfer equation:

Formulation and solution of the radiative transfer equation in the case of coupled atmosphere-ocean system do not have much difference from that of atmosphere. But certain points of differences should be mentioned. We shall only consider the solar radiation because it is strongly affected by the changes in refractive index and these changes affects the transfer processes considerably in the coupled system.

We shall first discuss the **nonpolarised case.**

In view of the abovementioned geometrical discussion, the non-coupled non-polarized radiative transfer equation (ref. 1a, 1b) for atmosphere and ocean system are given by following two equations

$$\mu\frac{dI_{AT}(z,\mu,\varphi)}{dz} = -I_{AT}(z,\mu,\varphi) + \frac{\omega_0^{AT}}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} P_{AT}(\Theta) I(z,\mu',\varphi')d\mu' d\varphi' + S_{AT}(z,\mu,\varphi). \quad (2.7.1)$$

$$\mu\frac{dI_{OC}(z,\mu,\varphi)}{dz} = -I_{OC}(z,\mu,\varphi) + \frac{\omega_0^{OC}}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} P_{OC}(\Phi) I(z,\mu',\varphi')d\mu' + S_{OC}(z,\mu,\varphi). \quad (2.7.2)$$

Where ω_0 is the single scattering albedo, and

$$P_{AT}(\Theta) = P_{AT}(z,\mu,\mu',\varphi,\varphi'), \quad (2.7.3)$$

is the phase function and Θ is the scattering angle in the atmosphere whose cosine is expressed through the relation by

$$\cos(\Theta) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'), \quad (2.7.4)$$

Similarly we can define ω as the ocean single scattering albedo and

$$P_{OC}(\Phi) = P_{OC}(z,\mu,\mu',\varphi,\varphi'), \quad (2.7.5)$$

as the ocean phase matrix with scattering angle given by Φ whose cosine is expressed through the relation

$$\cos(\Phi) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'), \tag{2.7.6}$$

For the sake of simplicity we did not distinguish the differences in the mode of scattering between the two media. However in reality the scattering characteristics in the two media are quite different.

At first, neglecting thermal emission, we can consider only a parallel beam of sunlight incident on a non-emitting atmosphere with phase function independent of depth term, i.e.,

$$S_{AT}(z,\mu,\varphi) = \frac{\omega_0^{AT} F}{4} P_{AT}(\mu,\mu_0,\varphi,-\varphi_0)\exp\left(-\frac{z}{\mu_0}\right), \tag{2.7.7}$$

$$S_{OC}(z,\mu,\varphi) = \frac{\omega_0^{OC} S}{4} P_{OC}(\mu,\mu_0,\varphi,\varphi_0)\exp\left(-\frac{z}{\mu_0}\right), \tag{2.7.8}$$

Where the cosine of the solar zenith angle is μ_0, φ_0 is the azimuthal angle for the incident solar beam and F is taken in such a way that $\pi\mu_0 F$ and $\pi\mu_0 S$ is the incident solar flux. Here and in the following the solar plane azimuth angle is considered to be zero, i.e., $\varphi = 0$ since there is no loss of generality when applying this assumption. The expansion of the intensity $I(z,\mu,\varphi)$ in a Fourier cosine series for both atmosphere and ocean

$$I(z,\mu,\varphi) = \sum_{n=0}^{M} I^n(z,\mu)\cos n\varphi, \tag{2.7.9}$$

leads to $M+1$ independent equations, one for each Fourier component $I^n(\tau,\mu)$. As functions $I^n(z,\mu)$ have a stepwise change at $\mu = 0$ it is more convenient for the numerical solution to consider downwelling $(\mu > 0)$ and upwelling $(\mu < 0)$ radiation separately. Thus, by introducing the notation

$$\left.\begin{aligned} I^{n+}(z,\mu) &= I^n(z,\mu) \quad \text{if} \quad \mu > 0, \\ I^{n+}(z,\mu) &= 0 \quad \text{if} \quad \mu < 0, \end{aligned}\right\} \tag{2.7.10}$$

and

$$\left.\begin{aligned} I^{n-}(z,\mu) &= I^n(z,\mu) \quad \text{if} \quad \mu < 0, \\ I^{n-}(z,\mu) &= 0 \quad \text{if} \quad \mu > 0, \end{aligned}\right\} \tag{2.7.11}$$

It is possible to write the Eqs. (2.7.1 – 2.7.2) for the n^{th} Fourier component in the form

$$\mu\frac{d}{dz} I_{AT}^{n\pm}(z,\mu) = -I_{AT}^{\pm}(z,\mu) + \frac{\omega_0^{AT}}{2}\int_{-1}^{0} P_{AT}^{n}(\mu,\mu')\ I_{AT}^{n-}(z,\mu')d\mu' + \frac{\omega_0^{AT}}{2}\int_{0}^{1} P_{AT}^{n}(\mu,\mu') I_{AT}^{n+}(z,\mu')d\mu' + S^{\pm}{}_{AT}. \tag{2.7.12}$$

$$\mu\frac{d}{dz}I_{OC}^{n\pm}(z,\mu)=-I_{OC}^{n\pm}(z,\mu)+\frac{\omega_0^{OC}}{2}\int_{-1}^{0}P_{OC}^{n}(\mu,\mu')I_{OC}^{n-}(\tau,\mu')d\mu'$$

$$+\frac{\omega_0^{OC}}{2}\int_{0}^{1}P_{OC}^{n}(\mu,\mu')I_{OC}^{n+}(z,\mu')d\mu'+S_{OC}^{n}(z,\mu'), \tag{2.7.13}$$

Where $S_{AT}^{\pm n}(z,\mu)=\frac{\omega_0^{AT}S}{4}\left[\left\{P_l^n(\pm\mu,\mu_0)\exp\left(-\frac{z}{\mu_0}\right)\right\}\right]$ and $P_{AT}^{n}(\mu,\mu')=\frac{1}{2\pi}\int_0^{2\pi}P_{AT}(\Theta)\cos n\,\varphi\,d\varphi$

will apply to every value of n. Similarly we can write for ocean

$S^{\pm n}{}_{OC}(z,\mu)=\frac{\omega_0^{OC}S}{4}\left[P_l^n(\pm\mu,\mu_0)\times\exp\left(-\frac{z_\omega}{\mu_0}\right)\right]$ and $P_{OC}^{n}(\mu,\mu')=\frac{1}{2\pi}\int_0^{2\pi}P_{OC}^{n}(\gamma)\cos n\,\varphi\,d\varphi.$

But these set of equations do not reflect true state of art of light scattering phenomena in the atmospheric-ocean system since neither light sensitive boundary conditions between sea and atmosphere is introduced nor any emission is allowed in either atmosphere or ocean. **To introduce light sensitive boundary condition in the air-water interface we have to deal with reflection and transmission properties of the interface region.** In view of the discussion made on reflection function in section (1.6) for flat and calm Ocean one can formulate more realistic approach as follows. We introduce internal source function (Jin, Z., and Stamens, [113]) as follows. The solar-beam source term in the atmosphere with depth dependent phase functions, can be expressed as

$$S_{AT}(z,\mu,\varphi)=\frac{\omega(z)}{4\pi}F_0P(z,\mu,\varphi,-\mu_0,\varphi_0)\exp\left(-\frac{z}{\mu_0}\right)$$

$$+\frac{\omega(z)}{4\pi}F_0R(-\mu_0,n)P(z,\mu,\varphi,\mu_0,\varphi_0)\exp\left(-\frac{(2z_a-z)}{\mu_0}\right), \tag{2.7.14}$$

Where μ_0 is the cosine of the solar zenith angle and is positive, φ_0 is the azimuthal angle for the incident solar beam, and F_0 is the solar-beam intensity at the top of atmosphere. Here n (to be changed) is the index of refraction of the ocean relative to the atmosphere, and z_ω is the total optical depth of the atmosphere. The first term in equation (2.7.14) represents the contribution from the downward, incident beam source, while the second term represents the contribution from the upward beam source reflected at the atmosphere – ocean interface because of the Fresnel reflection caused by the change in the refractive index between air and sea water. $R(-\mu_0,n)$ is the ocean-surface reflectance for the solar beam. In the ocean, the source term is

$$S_{OC}(z,\mu,\varphi)=\frac{\omega(z)}{4\pi}\frac{\mu_0}{\mu_{0n}(\mu_0,n)}F_0T(-\mu_0,n)P(z,\mu,\varphi,-\mu_{0n},\varphi_0)\times\exp\left(-\frac{z_a}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right),\quad(2.7.15)$$

Where $T(-\mu_0,n)$ is the transmittance through the interface, and μ_{0n} is the cosine of the solar zenith angle in the ocean, which is related to μ_0 by the Snell law

$$\mu_{0n}(\mu_0,n)=\sqrt{1-\frac{(1-\mu_0^2)}{n^2}}.\quad(2.7.16)$$

Expansion of the phase function $P(z,\cos\theta)$ and the intensity in a series of **2N** Legendre polynomials

$$I(z,\mu,\varphi)=\sum_{m=0}^{2N-1}I^m(z,\mu)\cos m(\varphi-\varphi_0),\quad(2.7.17)$$

$$P(z;\mu,\phi,\mu',\phi')\equiv P(z;\cos\theta)=\sum_{l=0}^{2N-1}(2l+1)g_l(z)P_l(\cos\theta),\quad(2.7.18)$$

This leads to the replacement of equations (2.7.12 – 2.7.13) with **2N** independent equations (one for each Fourier component)

$$\mu\frac{dI_{AT}^{\pm m}(z,\mu)}{dz}=-I_{AT}^{\pm m}(z,\mu)+\int_{-1}^{1}D_{AT}^m(z,\mu,\mu')I_{AT}^{\pm m}(z,\mu')d\mu'+S_{AT}^{\pm m}(z,\mu),$$

$$m=0,1,\ldots,2N-1,\quad(2.7.19)$$

$$D_{AT}^m(z,\mu,\mu')=\frac{\omega(z)}{2}\sum_{l=m}^{2N-1}(2l+1)g_{lat}(z)\frac{(l-m)!}{(l+m)!}P_{lat}^m(\mu)P_{lat}^m(\mu'),\quad(2.7.20)$$

Where θ is the scattering angle, $P_l^m(\mu)$ is the associated Legendre polynomial, $g_l(z)$ is the expansion coefficient, and S_{AT}^m is the **m**-th Fourier component of the beam source. In the atmosphère it is

$$S_{AT}^{\pm m}(z,\mu)=X_0{}^m(z,\mu)\exp\left(-\frac{z}{\mu_0}\right)+X_{01}{}^m(z,\mu)\exp\left(\frac{z}{\mu_0}\right),\quad(2.7.21)$$

Where

$$X_0{}^m(z,\mu)=\frac{\omega(z)}{4\pi}F_0(2-\delta_{m0})\sum_{l=0}^{2N-1}(-1)^{l+m}(2l+1)g_{lat}(z)\frac{(l-m)!}{(l+m)!}P_{lat}^m(\mu)P_{lat}^m(\mu'),\quad(2.7.22)$$

$$X_{01}{}^m(z,\mu)=\frac{\omega(z)}{4\pi}F_0R(-\mu_0,n)\exp\left(-\frac{2z_\omega}{\mu_0}\right)(2-\delta_{m0}),\quad(2.7.23)$$

$$\delta_{m0} = 1, \text{ if } m = 0$$
$$= 0, \text{ otherwise.}$$

For ocean the above equations become

$$\mu \frac{dI_{OC}^{\pm m}(z,\mu)}{dz} = -I_{OC}^{\pm m}(z,\mu) + \int_{-1}^{1} D_{OC}^{m}(z,\mu,\mu') I_{OC}^{\pm m}(z,\mu') d\mu' + S_{OC}^{\pm m}(z,\mu)$$
$$m = 0,1,\ldots,2N-1 \qquad (2.7.24)$$

$$D_{OC}^{m}(z,\mu,\mu') = \frac{\omega(z)}{2} \sum_{l=m}^{2N-1} (2l+1) g_{loc}(z) \frac{(l-m)!}{(l+m)!} P_{loc}^{m}(\mu) P_{loc}^{m}(\mu'). \qquad (2.7.25)$$

The source term in ocean can be expressed as

$$S_{OC}^{\pm m}(z,\mu) = X_{02}{}^{m}(z,\mu) \exp\left(-\frac{z}{\mu_{0n}}\right), \qquad (2.7.26)$$

where

$$X_{02}{}^{m}(z,\mu) = \frac{\omega(z)}{4\pi} \frac{\mu_0}{\mu_{0n}(\mu_0,n)} T(-\mu_0,n) F_0 \exp\left(-z_{\omega}\left(\frac{1}{\mu_0} - \frac{1}{\mu_{0n}}\right)\right)$$
$$\times (2-\delta_{m0}) \sum_{l=0}^{2N-1} (-1)^{l+m} (2l+1) g_{loc}(z) \frac{(l-m)!}{(l+m)!} P_{loc}^{m}(\mu) P_{loc}^{m}(\mu_{0n}). \qquad (2.7.27)$$

2.8. Boundary Conditions for non-polarized equation:

The boundary conditions to be applied at the top of the atmosphere; the continuity conditions at each interface between layers in the atmosphere and ocean; and finally the reflection and refraction occurring at the atmosphere ocean interface where we require Fresnel's equations to be satisfied.

At top we have with minus sign applies for the downward intensity and the positive sign for the upward intensity,

$$I_{AT}(0,-\mu) = I_{AT\infty}(-\mu), \qquad (2.8.1)$$

at the interface between atmosphere and the ocean we require

$$I_{AT}(z_{\omega},\mu) = I_{AT}(z_{\omega},-\mu) R_{AT}(-\mu,n) + \left\{\frac{I_{OC}(z_{\omega},\mu)}{n^2}\right\} T_{OC}(+\mu,n), \qquad (2.8.2)$$

z_{ω} being the air-water interface.

The first term is responsible for the reflection of the downward intensity reaching the interface through air at the air-water interface while the second term describes the transmitted upward intensity from inside the ocean at the interface. This situation is rather less complicated than when scattering is considered. The other surface condition is

$$\frac{I_{OC}(z_{\omega},-\mu)}{n^{2}}=\left\{\frac{I_{OC}(z_{\omega},\mu)}{n^{2}}\right\}R_{OC}(+\mu,n)+I_{AT}(z_{\omega},-\mu)T_{AT}(-\mu,n), \tag{2.8.3}$$

And finally at the bottom boundary we have

$$I_{OC}(z,\mu)=I_{OC}g(\mu). \tag{2.8.4}$$

We define $R(\pm\mu,n)$ and $T(\pm\mu,n)$ as the specular reflectance and transmittance, respectively, of the invariant intensity $\frac{I}{n^{2}_{abs}}$, where n_{abs} is the absolute index of refraction at the location where I is measured and n is the index of refraction of the ocean relative to the atmosphere.
The minus sign applies for the downward intensity and the positive sign for the upward intensity. Formulas for R and T can be derived from the basic Fresnel equations. The results are

$$R(-\mu,n)=\frac{1}{2}\left[\left(\frac{-\mu-n\mu}{\mu+n\mu}\right)^{2}+\left(\frac{\mu-n\mu}{\mu+n\mu}\right)^{2}\right], \tag{2.8.5}$$

$$R(+\mu,n)=R(-\mu,n), \tag{2.8.6}$$

$$T(-\mu,n)=2n\mu\,\mu\left[\left(\frac{1}{\mu+n\mu}\right)^{2}+\left(\frac{1}{\mu+n\mu}\right)^{2}\right], \tag{2.8.7}$$

$$T(+\mu,n)=T(-\mu,n), \tag{2.8.8}$$

$I_{AT\infty}(-\mu)$ is the intensity incident at the top of the atmosphere and $I_{OC}g(\mu)$ is determined by bidirectional reflectance of the underlying surface at the bottom of the ocean.
Although this formulation helps us to analyze the atmospheric-ocean radiative transfer processes more efficiently than the previous one but as a consequence of the presence of suspended matter in the sea water this type of model never produces adequate informations of practical interests. This requires consideration of polarized radiation field which is essential to make considerable improvements in the characteristic measurements of the optical properties of ocean and atmosphere as well. Next we shall consider the same.

2.9. Polarized Radiation Transfer Equation:

We shall now consider the **polarized radiative transfer** equation for coupled atmosphere-ocean system. The matrix equation of transfer for diffusion of polarized radiation for a medium with

scattering and absorption was developed by Kuscer and Riberic [36]. In this formulation a very general approach was undertaken for most general form of phase function. In our formulation we have employed different approach. **For atmosphere**, we have the following equation, keeping the notation preserved for intensity, phase functions (kernel matrix) and internal source, but now to be understood as matrix of appropriate order

$$\mu\frac{dI_{AT}(z,\mu,\varphi)}{dz} = -I_{AT}(z,\mu,\varphi) + \frac{\omega^{AT}(z)}{4\pi}\int_0^{2\pi} d\varphi'^{+}\int_{-1}^{+1} P_{AT}(\Theta)\, I_{AT}(z,\mu',\varphi')d\mu' + S_{AT}(z,\mu,\varphi). \quad (2.9.1)$$

Here the source matrix $S_{AT}(z,\mu,\phi)$ may be obtained using Snell–Descartes law to model the reflection matrix through air-seawater interface and sea-water to air interface.

We shall consider the following form of the source matrix having two distinct term .The first term represents the contribution from the downward beam source attenuated exponentially while the second term represents the contribution from the upward beam source reflected specularly at the atmosphere-ocean interface by the Fresnel reflection which is assumed to be accountable for the change of refractive index between the air and sea water interface and gets attenuated. The form of ocean-surface reflectance matrix $R(-\mu_0,n)$ will be dealt in later. Incident solar bean directions are specified by the solar zenith and azimuth angle (μ_0,φ_0) respectively. Solar beam intensity is considered given from actual measurement at the top of the atmosphere.

$$S_{AT}(z,\mu,\varphi) = \frac{\omega^{AT}(z)}{4\pi}F_0 P_{AT}(z,\mu,\phi,-\mu_0,\varphi_0)\exp\left(\frac{-z}{\mu_0}\right) +$$

$$\frac{\omega^{AT}(z)}{4\pi}F_0 R_{AT}(-\mu_0,n)P_{AT}(z,\mu,\varphi,\mu_0,\varphi_0)\exp\left(\frac{-(2z_\omega - z)}{\mu_0}\right)$$

$$= \frac{\omega^{AT}(z)}{4\pi}P_{AT}(z,\mu,\varphi,-\mu_0,\varphi_0)\begin{pmatrix}F_1\\F_2\\F_3\\F_4\end{pmatrix}\exp\left(-\frac{z}{\mu_0}\right) + \frac{\omega^{AT}(z)}{4\pi}\times$$

$$\frac{1}{2}\begin{pmatrix} r_L r_L^* + r_R r_R^* & r_L r_L^* - r_R r_R^* & 0 & 0 \\ r_L r_L^* - r_R r_R^* & r_L r_L^* + r_R r_R^* & 0 & 0 \\ 0 & 0 & r_L r_R^* + r_R r_L^* & r_L r_R^* - r_L r_R^* \\ 0 & 0 & r_L r_R^* - r_R r_L^* & r_L r_R^* + r_L r_R^* \end{pmatrix} P_{AT}(z,\mu,\varphi,\mu_0,\varphi_0)\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}\exp\left(-\frac{(2z_\omega - z)}{\mu_0}\right).$$

(2.9.2)

$$= \frac{\omega^{AT}(z)}{4\pi}[E_{AT}.F]. \tag{2.9.3}$$

We have introduced (4X4) E-matrix which will be reduced with the help of equations (2.3.51-2.3.54) in due course to get explicit expressions for source matrix elements.

$$E_{AT}(z,\mu,\varphi) = \left[P_{AT}(z,\mu,\varphi,-\mu_0,\varphi_0)\exp\left(-\frac{z}{\mu_0}\right) + P(z,\mu,\varphi;\mu_0,\varphi_0)R(-\mu_0,n)\exp\left(\frac{-2(z_\omega - z)}{\mu_0}\right)\right]$$

(2.9.4)

The intensity matrix $I_{AT}(z,\mu,\phi)$ has four stokes parameter each being a function of (z,μ,ϕ) and of order (4x1)

$$I_{AT}(z,\mu,\varphi) = \begin{bmatrix} L_{AT}(z,\mu,\varphi) \\ Q_{AT}(z,\mu,\varphi) \\ U_{AT}(z,\mu,\varphi) \\ V_{AT}(z,\mu,\varphi) \end{bmatrix}, \tag{2.9.5}$$

The phase matrix is now given by, $P_{AT}(z,\mu,\mu',\varphi,\varphi') = L(-\chi)P_{AT}(\cos\Theta)L(\chi')$, instead of equation (2.7.3). Scattering matrix for atmosphere is given by $P_{AT}(\cos\Theta)$. The rotation matrix $L(\chi)$ is required to turn the meridian plane at point **0** to the scattering plane and another rotation expressed by (2.3.41) to turn the scattering plane to the meridian at the point (P1). Fig (3)

$$L(\chi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.9.6}$$

Similarly for ocean we have

$$\mu\frac{dI_{OC}(z,\mu,\varphi)}{dz}=-I_{OC}(z,\mu,\varphi)+\frac{\omega^{OC}(z)}{4\pi}\int_{0}^{2\pi}d\varphi'^{+}\int_{-1}^{+1}P_{OC}(\Phi)\,I_{OC}(z,\mu',\varphi')d\mu'+S_{OC}(z,\mu,\varphi)\quad(2.9.7)$$

With same justification for the ocean scattering processes we can write for oceanic phase functions as

$$P_{OC}(z,\mu,\mu',\varphi,\varphi')=L(-\chi)P_{OC}(\cos\Phi)L(\chi')\qquad(2.9.8)$$

The source matrix in this case takes the form for downward direct solar radiation involving transmission matrix whose form will be considered later in detail.

$$S_{OC}(z,\mu,\varphi)=\frac{\omega^{OC}(z)}{4\pi}\frac{\mu_0}{\mu_{0n}(\mu_0,n)}F_0T_{OC}(-\mu_0,n)\times P_{OC}(z,\mu,\varphi,-\mu_{0n},\varphi_0)\exp\left(-\frac{z_\omega}{\mu_0}\right)\times\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)$$
(2.9.9)

μ_0 = Cosine of the solar zenith angle and is positive. φ_0=Azimuthal angle for incident solar beam. F_0 = Polarized Solar beam intensity at the top of the atmosphere. μ=Index of refraction of the ocean relative to the atmosphere. z_ω=Optical depth of the atmosphere. μ_{0n} = Cosine of the solar zenith angle in the ocean which is related to μ_0 by the Snell Law

$$\mu_{0n}(\mu_0,n)=\sqrt{1-\frac{(1-\mu_0^2)}{n^2}}.\qquad(2.9.10)$$

$$S_{OC}(z,\mu,\varphi)=\frac{\omega^{OC}(z)}{4\pi}\frac{\mu_0}{\mu_{0n}(\mu_0,n)}T(-\mu_0,n)\times P_{OC}(z,\mu,\varphi,-\mu_{0n},\varphi_0)F\exp\left(-\frac{z_\omega}{\mu_0}\right)\times\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)$$

$$=\frac{\omega^{OC}(z)}{4\pi}\exp\left(-\frac{z_\omega}{\mu_0}\right)T(-\mu_0,n)P_{OC}(z,\mu,\varphi,-\mu_{0n},\varphi_0)F.\left(\frac{\mu_0}{\mu_{0n}}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)$$

$$=\frac{\omega^{OC}(z)}{4\pi}[E_{OC}.F].\qquad(2.9.11)$$

As in the case of atmosphere the E-matrix (4X4) will also be reduced in due course to get explicit expressions for source matrix.

$$E_{OC}(z,\mu,\varphi)=\left[P_{OC}(z,\mu,\varphi,-\mu_{0n},\varphi_0).T(-\mu_0,n).\exp\left(\frac{-(z-z_\omega)}{\mu_{0n}}\right).\left(\frac{\mu_0}{\mu_n}\right).\exp\left(-\frac{z_\omega}{\mu_0}\right)\right]\quad(2.9.12)$$

2.10. Boundary Condition for polarized equation of transfer in Flat ocean surface:

This is rather simplified situation. The ocean surface is calm. The coupling between the two media is described by well known laws of reflection and refraction that apply at the interface as expressed by Snells law and Fresnel's law. The practical complications that arise are due to multiple scattering and total internal reflections. The downward radiation distributed over 2π **sr** in the atmosphere will be restricted to an angular cone less than 2π **sr** after being refracted across the interface into the ocean. Beams outside the refractive region in the ocean are in the total reflection region. The demarcation between the refractive and total reflective region in the ocean is given by the critical angle. Upward-traveling beams in total internal reflection region in ocean will be reflected back into the ocean upon reaching in interface. Thus, beams in total internal reflection region cannot reach the atmosphere directly; they must be scattered into other region in order to be returned to the atmosphere.

At the top of the atmosphere there is some prescribed incident radiation. The continuity conditions at each interface between layers in the atmosphere and ocean are omitted in this work as we have not considered layered atmosphere or ocean. Finally the reflection and refraction occurring at the atmosphere ocean interface where we require Fresnel's equation to be satisfied.

At the top we assume $I_{AT}(0,-\mu,\varphi) = I_{AT\infty}(-\mu,\varphi)$ In view of equation (2.9.5) this becomes,

$$\begin{pmatrix} L_{AT}(0,-\mu) \\ Q_{AT}(0,-\mu) \\ U_{AT}(0,-\mu) \\ V_{AT}(0,-\mu) \end{pmatrix} = \begin{pmatrix} L_{AT\infty}(0,-\mu) \\ Q_{AT\infty}(0,-\mu) \\ U_{AT\infty}(0,-\mu) \\ V_{AT\infty}(0,-\mu) \end{pmatrix}. \tag{2.10.1}$$

The exact expressions on the right hand side of (1.10.1) will be stated latter.

At the interface between atmosphere and the ocean we require the polarized version, i.e.

$$I_{AT}(z_\omega,\mu) = R_{AT}(-\mu,n) I_{AT}(z_\omega,-\mu) + T_{OC}(+\mu,n)\left\{\frac{I_{OC}(z_\omega,\mu)}{n^2}\right\}. \tag{2.10.2}$$

$$\begin{pmatrix} L_{AT}(z_\omega,\mu) \\ Q_{AT}(z_\omega,\mu) \\ U_{AT}(z_\omega,\mu) \\ V_{AT}(z_\omega,\mu) \end{pmatrix} = R_{AT}(-\mu,n)\begin{pmatrix} L_{AT\infty}(z_\omega,-\mu) \\ Q_{AT\infty}(z_\omega,-\mu) \\ U_{AT\infty}(z_\omega,-\mu) \\ V_{AT\infty}(z_\omega,-\mu) \end{pmatrix} + T_{OC}(+\mu,n)\begin{pmatrix} \dfrac{L_{OC}(z_\omega,-\mu)}{n^2} \\ \dfrac{Q_{OC}(z_\omega,-\mu)}{n^2} \\ \dfrac{U_{OC}(z_\omega,-\mu)}{n^2} \\ \dfrac{V_{OC}(z_\omega,-\mu)}{n^2} \end{pmatrix}. \tag{2.10.3}$$

Where the **air water interface** is z_ω and we also require the polarized version of the following equation

$$\frac{I_{OC}(z_\omega,-\mu)}{n^2} = R_{OC}(+\mu,n)\left\{\frac{I_{OC}(z_\omega,\mu)}{n^2}\right\} + T_{AT}(-\mu,n) I_{AT}(z_\omega,-\mu).$$

$$\begin{pmatrix} \frac{L_{OC}(z_\omega,\mu)}{n^2} \\ \frac{Q_{OC}(z_\omega,\mu)}{n^2} \\ \frac{U_{OC}(z_\omega,\mu)}{n^2} \\ \frac{V_{OC}(z_\omega,\mu)}{n^2} \end{pmatrix} = R_{OC}(+\mu,n)\left\{\begin{pmatrix} \frac{L_{OC}(z_\omega,\mu)}{n^2} \\ \frac{Q_{OC}(z_\omega,\mu)}{n^2} \\ \frac{U_{OC}(z_\omega,\mu)}{n^2} \\ \frac{V_{OC}(z_\omega,\mu)}{n^2} \end{pmatrix}\right\} + T_{AT}(-\mu,n)\begin{pmatrix} L_{AT}(z_\omega,-\mu) \\ Q_{AT}(z_\omega,-\mu) \\ U_{AT}(z_\omega,-\mu) \\ V_{AT}(z_\omega,-\mu) \end{pmatrix}. \quad (2.10.4)$$

Finally at the **bottom boundary** we have the polarized version of

$$I_{OC}(z,\mu) = I_{OC} \cdot g(\mu)$$

$$\begin{pmatrix} L_{OC}(z,\mu) \\ Q_{OC}(z,\mu) \\ U_{OC}(z,\mu) \\ V_{OC}(z,\mu) \end{pmatrix} = \begin{pmatrix} L_{OC} \cdot g(\mu) \\ Q_{OC} \cdot g(\mu) \\ U_{OC} \cdot g(\mu) \\ V_{OC} \cdot g(\mu) \end{pmatrix}. \quad (2.10.5)$$

We shall consider exact expression for the right hand side of (2.10.5) later.

We define $R(\pm\mu,n)$ and $T(\pm\mu,n)$ as the specular reflection and transmittance respectively of the invariant intensity. The minus sign applies for the downward intensity, and the positive sign for upward intensity. Formulas for R and T can be derived from the basic Fresnel's equations. The expression for R and T are

$$R = \frac{1}{2}\begin{bmatrix} r_L r_L^* + r_R r_R^* & r_L r_L^* - r_R r_R^* & 0 & 0 \\ r_L r_L^* - r_R r_R^* & r_L r_L^* + r_R r_R^* & 0 & 0 \\ 0 & 0 & r_L r_R^* + r_R r_L^* & r_L r_R^* - r_R r_L^* \\ 0 & 0 & r_L r_R^* - r_R r_L^* & r_L r_R^* + r_R r_L^* \end{bmatrix}. \quad (2.10.6)$$

and

$$T = \frac{1}{2}\begin{bmatrix} t_L t_L^* + t_R t_R^* & t_L t_L^* - t_R t_R^* & 0 & 0 \\ t_L t_L^* - t_R t_R^* & t_L t_L^* + t_R t_R^* & 0 & 0 \\ 0 & 0 & t_L t_R^* + t_R t_L^* & t_L t_R^* - t_R t_L^* \\ 0 & 0 & t_L t_R^* - t_R t_L^* & t_L r_R^* + t_L t_R^* \end{bmatrix}. \quad (2.10.7)$$

Here $\mathbf{r_L}$ and $\mathbf{r_r}$ depend on ω and on the sea water complex refractive index $\mathbf{m}$ and $\mu = \cos\theta$, according to

$$r_L = \frac{\sqrt{m^2 - \sin^2\mu} - m^2\cos\mu}{\sqrt{m^2 - \sin^2\mu} + m^2\cos\mu}, \qquad (2.10.8) \qquad r_R = \frac{\cos\mu - \sqrt{m^2 - \sin^2\mu}}{\cos\mu + \sqrt{m^2 - \sin^2\mu}}, \qquad (2.10.9)$$

$$t_L = \frac{2\sqrt{m^2 - \sin^2\mu}}{\sqrt{m^2 - \sin^2\mu} + m^2\cos\mu}, \qquad (2.10.10) \qquad t_R = \frac{2\sqrt{m^2 - \sin^2\mu}}{\cos\mu + \sqrt{m^2 - \sin^2\mu}} \qquad (2.10.11)$$

These equations are analogous to equations (2.5.27) and can be changed suitably for the cases **air to sea** and **sea to air** as mentioned in equations (2.5.8 – 2.5.21).

$$\begin{pmatrix} L_{AT}(-\mu) \\ Q_{AT}(-\mu) \\ U_{AT}(-\mu) \\ V_{AT}(-\mu) \end{pmatrix} \qquad (2.10.12)$$

is the intensity incident at the top of the atmosphere and $\mathbf{I_{OC}g(\mu)}$ is determined by bidirectional reflectance of the under lying surface at the bottom of the ocean.

2.11. Reduction of the Equation of transfer:

We shall now reduce the appropriate transfer equation using the Fourier series decompositions as mentioned earlier. We shall use following representation for intensity vector (2.11.1) and (2.11.2) and appropriate form of phase matrix (2.3.51) for concerned media in the radiative transfer equations (2.9.1) and (2.9.7) for atmosphere and ocean respectively.

$$I_{AT}(z,\mu,\varphi) = \begin{bmatrix} L_{AT}(z,\mu,\varphi) \\ Q_{AT}(z,\mu,\varphi) \\ U_{AT}(z,\mu,\varphi) \\ V_{AT}(z,\mu,\varphi) \end{bmatrix} = \sum_{S=0}^{L}(2-\delta_{0,S}) \begin{bmatrix} L^S_{AT}(z,\mu)\cos S(\varphi-\varphi_s) \\ Q^S_{AT}(z,\mu)\cos S(\varphi-\varphi_s) \\ U^S_{AT}(z,\mu)\sin S(\varphi-\varphi_s) \\ V^S_{AT}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix} \qquad (2.11.1)$$

$$\mathbf{I}_{OC}(z,\mu,\varphi)=\begin{bmatrix} L_{OC}(z,\mu,\varphi) \\ Q_{OC}(z,\mu,\varphi) \\ U_{OC}(z,\mu,\varphi) \\ V_{OC}(z,\mu,\varphi) \end{bmatrix}=\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} L_{OC}^{S}(z,\mu) & \cos S(\varphi-\varphi_s) \\ Q_{OC}^{S}(z,\mu) & \cos S(\varphi-\varphi_s) \\ U_{OC}^{S}(z,\mu) & \sin S(\varphi-\varphi_s) \\ V_{OC}^{S}(z,\mu) & \sin S(\varphi-\varphi_s) \end{bmatrix} \qquad (2.11.2)$$

We also use similar expressions for source matrix elements as given below. Substituting equations (2.11.1) & (2.11.2) in (2.9.1) & (2.9.7) respectively with appropriate phase function (2.3.51) for atmosphere and ocean

$$\mu\frac{d}{dz}\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} L_{AT}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ Q_{AT}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ U_{AT}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \\ V_{AT}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix}=-\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} L_{AT}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ Q_{AT}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ U_{AT}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \\ V_{AT}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix}$$

$$+\frac{\omega^{AT}(z)}{4\pi}\int_{0}^{2\pi}\int_{-1}^{+1}\sum_{S=0}^{L}(2-\delta_{0,S})[\cos S(\varphi-\varphi')P_{ATC}^{S}(\mu,\mu')+\sin S(\varphi-\varphi')P_{ATS}^{S}(\mu,\mu')].\times$$

$$\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} L_{AT}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ Q_{AT}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ U_{AT}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \\ V_{AT}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix}d\mu' d\varphi'+\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} S_{ATL}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ S_{ATQ}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ S_{ATU}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \\ S_{ATV}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix}. \qquad (2.11.3)$$

$$\mu\frac{d}{dz}\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} L_{OC}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ Q_{OC}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ U_{OC}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \\ V_{OC}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix}=-\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} L_{OC}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ Q_{OC}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ U_{OC}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \\ V_{OC}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix}$$

$$
+\frac{\omega^{OC}(z)}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}\sum_{S=0}^{L}(2-\delta_{0,S})[\cos S(\varphi-\varphi')P_{OCC}^{S}(\mu,\mu')+\sin S(\varphi-\varphi')P_{OCS}^{S}(\mu,\mu')].\times
$$

$$
\sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} L_{OC}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ Q_{OC}^{S}(z,\mu)\cos S(\varphi-\varphi_s) \\ U_{OC}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \\ V_{OC}^{S}(z,\mu)\sin S(\varphi-\varphi_s) \end{bmatrix} d\mu' d\varphi' + \sum_{S=0}^{L}(2-\delta_{0,S})\begin{bmatrix} S_{OCL}^{S}(z,\mu) & \cos S(\varphi-\varphi_s) \\ S_{OCQ}^{S}(z,\mu) & \cos S(\varphi-\varphi_s) \\ S_{OCU}^{S}(z,\mu) & \sin S(\varphi-\varphi_s) \\ S_{OCV}^{S}(z,\mu) & \sin S(\varphi-\varphi_s) \end{bmatrix}. \qquad (2.11.4)
$$

Using appropriately changed representation of phase function from equations (2.3.50, 2.3.51, 2.3.52) and following the procedures adopted in ([36], [45], [46], [60]) with (2.3.53), (2.3.55) appropriate for the two media and after a long but straightforward algebraic manipulation we arrive at the following set of reduced equation for atmosphere and ocean which can be separated for each stokes components.

$$
\mu\frac{dI_{AT}^{S}(z,\mu)}{dz} = -I_{AT}(z,\mu)+\frac{\omega^{AT}(z)}{4}\int_{-1}^{+1}P_{AT}(z,\mu,\mu')\,I_{AT}^{S}(z,\mu')d\mu' + S_{AT}^{S}(z,\mu) \qquad (2.11.5)
$$

where, $$S_{AT}^{S}(z,\mu) = \frac{\omega^{AT}(z)}{4}\exp\left(-\frac{z}{\mu_0}\right).[E_{AT}(z,\mu).F] \qquad (2.11.6)$$

We shall calculate (2.11.6) from the following expression.

$$
E_{AT}(z,\mu,\varphi) = \frac{\omega^{AT}(z)}{4\pi}\left[P_{AT}(z,\mu,\varphi,-\mu_0,\varphi_0)\exp\left(-\frac{z}{\mu_0}\right)+P(z,\mu,\varphi;\mu_0,\varphi_0)R(-\mu_0,n)\exp\left(\frac{-(2z_\omega-z)}{\mu_0}\right)\right]
$$

$S_{AT}^{S}(z,\mu)$ is a 4X1 matrix.

In the above equation the last tern is source function. The explicit form azimuth independent source matrix will be derived in the next section. Each equation for stoke components are coupled non-linear integro differential equation subject to the boundary conditions mentioned in the earlier sections. The reduced phase matrix can be expressed, dropping subscript "**s**" and restricting to optical depth independent phase function, as

$$
P_{AT}(z,\mu,\mu') =
$$

$$
\begin{bmatrix}
\sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu') & \sum_{J=S}^{M}\gamma_J P_J^S(\mu)R_J^S(\mu') & -\sum_{J=S}^{M}\gamma_J P_J^S(\mu)T_J^S(\mu') & 0 \\
\sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu') & \sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu')+\xi_J T_J^S(\mu)T_J^S(\mu')) & -\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)T_J^S(\mu')+\xi_J T_J^S(\mu)R_J^S(\mu')) & \sum_{J=S}^{M}\varepsilon_J T_J^S(\mu)P_J^S(\mu') \\
-\sum_{J=S}^{M}\gamma_J T_J^S(\mu)P_J^S(\mu') & -\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)R_J^S(\mu')+\xi_J R_J^S(\mu)T_J^S(\mu')) & \sum_{J=S}^{M}(\alpha_J T_J^S(\mu)T_J^S(\mu')+\xi_J R_J^S(\mu)R_J^S(\mu')) & -\sum_{J=S}^{M}\varepsilon_J R_J^S(\mu)P_J^S(\mu') \\
0 & -\sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)T_J^S(\mu') & \sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)R_J^S(\mu') & \sum_{J=S}^{M}\delta_J P_J^S(\mu)P_J^S(\mu')
\end{bmatrix}.
$$

(2.11.7)

$$=\sum_{J=S}^{M}\begin{bmatrix} P_J^S(\mu) & 0 & 0 & 0 \\ 0 & R_J^S(\mu) & -T_J^S(\mu) & 0 \\ 0 & -T_J^S(\mu) & R_J^S(\mu) & 0 \\ 0 & 0 & 0 & P_J^S(\mu)\end{bmatrix}\begin{bmatrix} \beta_J & \gamma_J & 0 & 0 \\ \gamma_J & \alpha_J & 0 & 0 \\ 0 & 0 & \xi_J & -\varepsilon_J \\ 0 & 0 & \varepsilon_J & \delta_J\end{bmatrix} X \begin{bmatrix} P_J^S(\mu') & 0 & 0 & 0 \\ 0 & R_J^S(\mu') & -T_J^S(\mu') & 0 \\ 0 & -T_J^S(\mu') & R_J^S(\mu') & 0 \\ 0 & 0 & 0 & P_J^S(\mu')\end{bmatrix}$$

$$= \quad \mathbf{P}_J^S(\mu)\prod_J^{AT}\mathbf{P}_J^S(\mu') \tag{2.11.8}$$

$$\prod_J^{AT} = \begin{bmatrix} \beta_J & \gamma_J & 0 & 0 \\ \gamma_J & \alpha_J & 0 & 0 \\ 0 & 0 & \xi_J & -\varepsilon_J \\ 0 & 0 & \varepsilon_J & \delta_J\end{bmatrix} \tag{2.11.9}$$

$$\mathbf{P}_J^S(\mu) = \begin{bmatrix} P_J^S(\mu) & 0 & 0 & 0 \\ 0 & R_J^S(\mu) & -T_J^S(\mu) & 0 \\ 0 & -T_J^S(\mu) & R_J^S(\mu) & 0 \\ 0 & 0 & 0 & P_J^S(\mu)\end{bmatrix} \tag{2.11.10}$$

It can be shown that

$$P_C^S(\mu,\mu') = \mathbf{P}_J^S(\mu)\prod_J^{AT}\mathbf{P}_J^S(\mu') + D\mathbf{P}_J^S(\mu)\prod_J^{AT}\mathbf{P}_J^S(\mu')D \tag{2.11.11}$$

We can also write using the definitions (2.3.58)

$$P_{AT}(z,\mu,\mu') = \begin{bmatrix} A & B & -I & 0 \\ C & D & -J & -K \\ L & M & E & F \\ 0 & N & G & H\end{bmatrix} \tag{2.11.12}$$

and

$$I_{ATS}(z,\mu,) = \begin{bmatrix} L_{ATS}(z,\mu,) \\ Q_{ATS}(z,\mu,) \\ U_{ATS}(z,\mu,) \\ V_{ATS}(z,\mu,)\end{bmatrix} \tag{2.11.13}$$

For the ocean media the corresponding set of equations are given compactly as

$$\mu\frac{dI_{OC}^S(z,\mu,)}{dz} = -I_{OC}^S(z,\mu,) + \frac{\omega^{OC}(z)}{4}\int_{-1}^{+1} P_{OCS}(z,\mu,\mu')\, I_{OC}^S(z,\mu')d\mu' + S_{OC}^S(z,\mu) \tag{2.11.14}$$

Where, $$S^{S}_{OC}(z,\mu)=\frac{\omega^{OC}(z)}{4}\exp\left(-\frac{z}{\mu_0}\right).[E_{OC}(z,\mu).F] \tag{2.11.15}$$

$S^{S}_{OC}(z,\mu)$ is a 4X1 matrix.

Explicit form of ocean source matrix will be derived in the next sub-section. The form of phase matrix with restriction mentioned in atmospheric case is given as

$$P_{OC}(z,\mu,\mu')=$$

$$=\begin{bmatrix} \sum\limits_{J=S}^{M}\beta'_J P^S_J(\mu)P^S_J(\mu') & \sum\limits_{J=S}^{M}\gamma'_J P^S_J(\mu)R^S_J(\mu') & -\sum\limits_{J=S}^{M}\gamma'_J P^S_J(\mu)T^S_J(\mu') & 0 \\ \sum\limits_{J=S}^{M}\gamma'_J R^S_J(\mu)P^S_J(\mu') & \sum\limits_{J=S}^{M}(\alpha'_J R^S_J(\mu)R^S_J(\mu')+\xi'_J T^S_J(\mu)T^S_J(\mu')) & -\sum\limits_{J=S}^{M}(\alpha'_J R^S_J(\mu)T^S_J(\mu')+\xi'_J T^S_J(\mu)R^S_J(\mu')) & \sum\limits_{J=S}^{M}\varepsilon'_J T^S_J(\mu)P^S_J(\mu') \\ -\sum\limits_{J=S}^{M}\gamma'_J T^S_J(\mu)P^S_J(\mu') & -\sum\limits_{J=S}^{M}(\alpha'_J T^S_J(\mu)R^S_J(\mu')+\xi'_J R^S_J(\mu)T^S_J(\mu')) & \sum\limits_{J=S}^{M}(\alpha'_J T^S_J(\mu)T^S_J(\mu')+\xi'_J R^S_J(\mu)R^S_J(\mu')) & -\sum\limits_{J=S}^{M}\varepsilon'_J R^S_J(\mu)P^S_J(\mu') \\ 0 & -\sum\limits_{J=S}^{M}\varepsilon'_J P^S_J(\mu)T^S_J(\mu') & \sum\limits_{J=S}^{M}\varepsilon'_J P^S_J(\mu)R^S_J(\mu') & \sum\limits_{J=S}^{M}\delta'_J P^S_J(\mu)P^S_J(\mu') \end{bmatrix}. \tag{2.11.16}$$

$$=\sum_{J=S}^{M}\begin{bmatrix} P^S_J(\mu) & 0 & 0 & 0 \\ 0 & R^S_J(\mu) & -T^S_J(\mu) & 0 \\ 0 & -T^S_J(\mu) & R^S_J(\mu) & 0 \\ 0 & 0 & 0 & P^S_J(\mu) \end{bmatrix}\begin{bmatrix} \beta'_J & \gamma'_J & 0 & 0 \\ \gamma'_J & \alpha'_J & 0 & 0 \\ 0 & 0 & \xi'_J & -\varepsilon'_J \\ 0 & 0 & \varepsilon'_J & \delta'_J \end{bmatrix} X \begin{bmatrix} P^S_J(\mu') & 0 & 0 & 0 \\ 0 & R^S_J(\mu') & -T^S_J(\mu') & 0 \\ 0 & -T^S_J(\mu') & R^S_J(\mu') & 0 \\ 0 & 0 & 0 & P^S_J(\mu') \end{bmatrix} \tag{2.11.17}$$

$$= \mathbf{P}^S_J(\mu)\Pi^{OC}_J\mathbf{P}^S_J(\mu') \tag{2.11.18}$$

Where, $$\Pi_J^{OC} = \begin{bmatrix} \beta'_J & \gamma'_J & 0 & 0 \\ \gamma'_J & \alpha'_J & 0 & 0 \\ 0 & 0 & \xi'_J & -\varepsilon'_J \\ 0 & 0 & \varepsilon'_J & \delta'_J \end{bmatrix} \tag{2.11.19}$$

We can show

$$P_S^S(\mu,\mu') = \mathbf{P}_J^S(\mu)\Pi_J^{OC}\mathbf{P}_J^S(\mu') - D\,\mathbf{P}_J^S(\mu)\Pi_J^{OC}\mathbf{P}_J^S(\mu') \tag{2.11.20}$$

We can also write using notations (2.3.59)

$$P_{OCS}(z,\mu,\mu') = \begin{bmatrix} A' & B' & -I' & 0 \\ C' & D' & -J' & -K' \\ L' & M' & E' & F' \\ 0 & N' & G' & H' \end{bmatrix} \quad (2.11.21) \text{ and } I_{OCS}(z,\mu,) = \begin{bmatrix} L_{OCS}(z,\mu,) \\ Q_{OCS}(z,\mu,) \\ U_{OCS}(z,\mu,) \\ V_{OCS}(z,\mu,) \end{bmatrix} \quad (2.11.22)$$

In (2.11.7) and (2.11.16) we have used Legendre expansion for each term in accordance with (2.3.53) and (2.3.55). However expressions (2.11.12) and (2.11.21) are used to write the equations involving phase functions in a simplified form.

2.12. Reduction of Source function:

We recall equations (2.9.4) & (2.9.12)

$$E_{AT}(z,\mu,\varphi) = \frac{\omega^{AT}(z)}{4\pi}\left[P_{AT}(z,\mu,\varphi,-\mu_0,\varphi_0)\exp\left[-\frac{z}{\mu_0}\right] + P(z,\mu,\varphi;\mu_0,\varphi_0)R(-\mu_0,n)\exp\left[\frac{-(2z_\omega - z)}{\mu_0}\right]\right]$$

and

$$E_{OC}(z,\mu,\varphi) = \frac{\omega^{OC}(z)}{4\pi}\left[P_{OC}(z,\mu,\varphi,-\mu_{0n},\varphi_0).T(-\mu_0,n).\exp\left[\frac{-(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).\exp\left[-\frac{z_\omega}{\mu_0}\right]\right]$$

Introducing the Fourier decomposed equations of corresponding phase functions for incident directions from (2.11.7 – 2.11.11) and (2.11.16 – 2.11.20) we can deduce the set of expressions for E-functions in both media. Let us first define some notations.

$$A_0=\sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu_0);\quad B_0=\sum_{J=S}^{M}\gamma_J P_J^S(\mu)R_J^S(\mu_0);\quad C_0=\sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu_0);$$

$$D_0=\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu_0)+\xi_J T_J^S(\mu)T_J^S(\mu_0));\quad E_0=\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)T_J^S(\mu_0)+\xi_J R_J^S(\mu)R_J^S(\mu_0));$$

$$F_0=-\sum_{J=S}^{M}\varepsilon_J R_J^S(\mu)P_J^S(\mu_0);\quad G_0=\sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)R_J^S(\mu_0);\quad H_0=\sum_{J=S}^{M}\delta_J P_J^S(\mu)P_J^S(\mu_0).$$

$$I_0=\sum_{J=S}^{M}\gamma_J P_J^S(\mu)T_J^S(\mu_0);\qquad J_0=\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)T_J^S(\mu_0)+\xi_J T_J^S(\mu)R_J^S(\mu_0));$$

$$K_0=-\sum_{J=S}^{M}\varepsilon_J T_J^S(\mu)P_J^S(\mu_0);\qquad L_0=-\sum_{J=S}^{M}\gamma_J T_J^S(\mu)P_J^S(\mu_0);$$

$$M_0=-\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)R_J^S(\mu_0)+\xi_J R_J^S(\mu)T_J^S(\mu_0));\quad N_0=-\sum_{J=S}^{M}\varepsilon_J P_J^S(\mu)T_J^S(\mu_0).\tag{2.12.1}$$

$$A_0'=\sum_{J=S}^{M}\beta_J' P_J^S(\mu)P_J^S(\mu_0);\quad B_0'=\sum_{J=S}^{M}\gamma_J' P_J^S(\mu)R_J^S(\mu_0);\quad C_0'=\sum_{J=S}^{M}\gamma_J' R_J^S(\mu)P_J^S(\mu_0);$$

$$D_0'=\sum_{J=S}^{M}(\alpha_J' R_J^S(\mu)R_J^S(\mu_0)+\xi_J' T_J^S(\mu)T_J^S(\mu_0));\quad E_0'=\sum_{J=S}^{M}(\alpha_J' T_J^S(\mu)T_J^S(\mu_0)+\xi_J' R_J^S(\mu)R_J^S(\mu_0));$$

$$F_0'=-\sum_{J=S}^{M}\varepsilon_J' R_J^S(\mu)P_J^S(\mu_0);\quad G_0'=\sum_{J=S}^{M}\varepsilon_J' P_J^S(\mu)R_J^S(\mu_0);\quad H_0'=\sum_{J=S}^{M}\delta_J' P_J^S(\mu)P_J^S(\mu_0);$$

$$I_0'=\sum_{J=S}^{M}\gamma_J' P_J^S(\mu)T_J^S(\mu_0);\quad J_0'=\sum_{J=S}^{M}(\alpha_J' R_J^S(\mu)T_J^S(\mu_0)+\xi_J' T_J^S(\mu)R_J^S(\mu_0));$$

$$K_0'=-\sum_{J=S}^{M}\varepsilon_J' T_J^S(\mu)P_J^S(\mu_0);\quad L_0'=-\sum_{J=S}^{M}\gamma_J' T_J^S(\mu)P_J^S(\mu_0);$$

$$M_0'=-\sum_{J=S}^{M}(\alpha_J' T_J^S(\mu)R_J^S(\mu_0)+\xi_J' R_J^S(\mu)T_J^S(\mu_0));\quad N_0'=-\sum_{J=S}^{M}\varepsilon_J' P_J^S(\mu)T_J^S(\mu_0)\tag{2.12.2}$$

$$P_{ATC}^s(\mu,\mu_0)=\begin{bmatrix}A_0 & B_0 & 0 & 0\\ C_0 & D_0 & 0 & 0\\ 0 & 0 & E_0 & F_0\\ 0 & 0 & G_0 & H_0\end{bmatrix}\quad(2.12.3)\qquad P_{OCC}^s(\mu,\mu_0)=\begin{bmatrix}A_0' & B_0' & 0 & 0\\ C_0' & D_0' & 0 & 0\\ 0 & 0 & E_0' & F_0'\\ 0 & 0 & G_0' & H_0'\end{bmatrix}\quad(2.12.4)$$

$$\mathbf{P}^{s}_{ATS}(\mu,\mu_0)=\begin{bmatrix}0 & 0 & I_0 & 0\\ 0 & 0 & J_0 & K_0\\ L_0 & M_0 & 0 & 0\\ 0 & N_0 & 0 & 0\end{bmatrix} \quad (2.12.5) \qquad \mathbf{P}^{s}_{OCS}(\mu,\mu_0)=\begin{bmatrix}0 & 0 & I'_0 & 0\\ 0 & 0 & J'_0 & K'_0\\ L'_0 & M'_0 & 0 & 0\\ 0 & N'_0 & 0 & 0\end{bmatrix} \quad (2.12.6)$$

$$\mathbf{P}_{ATS}(z,\mu,\mu_0)=\begin{bmatrix}A_0 & B_0 & -I_0 & 0\\ C_0 & D_0 & -J_0 & -K_0\\ L_0 & M_0 & E_0 & F_0\\ 0 & N_0 & G_0 & H_0\end{bmatrix} \quad (2.12.7) \qquad \mathbf{P}_{OCS}(z,\mu,\mu_0)=\begin{bmatrix}A'_0 & B'_0 & -I'_0 & 0\\ C'_0 & D'_0 & -J'_0 & -K'_0\\ L'_0 & M'_0 & E'_0 & F'_0\\ 0 & N' & G'_0 & H'_0\end{bmatrix} \quad (2.12.8)$$

As in the case of atmospheric and oceanic phase functions expressions (2.12.3 – 2.12.8) are used only for simplified look. From the representation of Phase function (2.11.7 – 2.11.11) and (2.11.16 – 2.11.20) we can reduce (2.9.4) & (2.9.12) in the following manner. First expand the R.H.S. of the source function (2.9.2) and (2.9.9) exactly as intensity vector using Fourier decomposition in azimuth angle. Secondly expand the L.H.S of (2.9.4) and (2.9.11) using the phase functions as given by equations (2.11.7 – 2.11.11) and (2.11.16 – 2.11.20) with proper changes in the arguments. The final outcomes of these exercises are given by the following set of equations (2.12.11) and (2.12.12) respectively for atmosphere and ocean.

From the right hand side of (2.9.4) we get using (2.10.6)

$$=\left[\begin{bmatrix}A_0 & B_0 & -I_0 & 0\\ C_0 & D_0 & -J_0 & -K_0\\ L_0 & M_0 & E_0 & F_0\\ 0 & N_0 & G_0 & H_0\end{bmatrix}\times\exp\left[-\frac{z}{\mu_0}\right]+\begin{bmatrix}A_0 & B_0 & -I_0 & 0\\ C_0 & D_0 & -J_0 & -K_0\\ L_0 & M_0 & E_0 & F_0\\ 0 & N_0 & G_0 & H_0\end{bmatrix}\times\right.$$

$$\left.\frac{1}{2}\begin{bmatrix}r_L r_L^* + r_R r_R^* & r_L r_L^* - r_R r_R^* & 0 & 0\\ r_L r_L^* - r_R r_R^* & r_L r_L^* + r_R r_R^* & 0 & 0\\ 0 & 0 & r_L r_R^* + r_R r_L^* & r_L r_R^* - r_L r_R^*\\ 0 & 0 & r_L r_R^* - r_R r_L^* & r_L r_R^* + r_L r_R^*\end{bmatrix}\times\exp\left[\frac{-(2z_\omega - z)}{\mu_0}\right]\right]$$

(2.12.9)

Similarly for the ocean we easily can get using (2.10.7) for the right hand side of (2.9.12) can be written as

$$
=\begin{bmatrix} A_0' & B_0' & -I_0' & 0 \\ C_0' & D_0' & -J_0' & -K_0' \\ L_0' & M_0' & E_0' & F_0' \\ 0 & N' & G_0' & H_0' \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} t_L t_L^* + t_R t_R^* & t_L t_L^* - t_R t_R^* & 0 & 0 \\ t_L t_L^* - t_R t_R^* & t_L t_L^* + t_R t_R^* & 0 & 0 \\ 0 & 0 & t_L t_R^* + t_R t_L^* & t_L t_R^* - t_L t_R^* \\ 0 & 0 & t_L t_R^* - t_R t_L^* & t_L r_R^* + t_L t_R^* \end{bmatrix}
$$

$$
\times \exp\left[\frac{-z_\omega}{\mu_0}\right] . \exp\left[\frac{-(z - z_\omega)}{\mu_{0n}}\right] . \left(\frac{\mu_0}{\mu_{0n}}\right). \tag{2.12.10}
$$

Evaluation the matrix multiplication in (2.12.9) and (2.12.10) gives us the expanded form of the matrix elements of the **E(AT)** and **E(OC)**. The matrix elements in each case can written as follows with "s"

Expanded matrix elements of $E_{AT}(z,\mu,\varphi)$:

$$
\begin{aligned}
E_{11}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\Bigg[& \cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\beta_J P_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right] \\
& + \Bigg\{\frac{1}{2}(r_L r_L^* + r_R r_R)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu_0) \\
& + \frac{1}{2}(r_L r_L^* - r_R r_R)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\gamma_J P_J^S(\mu)R_J^S(\mu_0)\Bigg\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
E_{12}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\Bigg[& \cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma_J P_J^S(\mu)R_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right] \\
& + \Bigg\{\frac{1}{2}(r_L r_L^* - r_R r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu_0) \\
& + \frac{1}{2}(r_L r_L^* + r_R r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\gamma_J P_J^S(\mu)R_J^S(\mu_0)\Bigg\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\Bigg].
\end{aligned}
$$

$$E_{13}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma_J P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$\left.+\left\{\frac{1}{2}(r_L r_R^* + r_R r_L^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}\gamma_J P_J^S(\mu)T_J^S(\mu_0)\right\}\exp\left[-\frac{2(z_\omega - z)}{\mu_0}\right]\right].$$

$$E_{14}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[(-\frac{1}{2})(r_L r_R^* - r_L r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}\gamma_J P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right].$$

$$E_{21}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma_J R_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$+\left\{\frac{1}{2}(r_L r_L^* + r_R r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu_0)\right.$$

$$\left.\left.+\frac{1}{2}(r_L r_L^* - r_R r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu_0) + \xi_J T_J^S(\mu)T_J^S(\mu_0))\right\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right].$$

$$E_{22}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J R_J^S(\mu)R_J^S(\mu_0) + \xi_J T_J^S(\mu)T_J^S(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$+\left\{\frac{1}{2}(r_L r_L^* - r_R r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu_0)\right.$$

$$\left.\left.+\frac{1}{2}(r_L r_L^* + r_R r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu_0) + \xi_J T_J^S(\mu)T_J^S(\mu_0))\right\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right].$$

$$E_{23}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J R_J^S(\mu)T_J^S(\mu_0) + \xi_J T_J^S(\mu)R_J^S(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$-\left\{\frac{1}{2}(r_L r_R^* + r_R r_L^*)\sin s(\varphi-\varphi_0)\times\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)T_J^S(\mu_0) + \xi_J T_J^S(\mu)R_J^S(\mu_0))\right.$$

$$\left.\left.-\frac{1}{2}(r_L r_R^* - r_R r_L^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}\epsilon_J\ T_J^S(\mu)P_J^S(\mu_0)\right\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right].$$

$$E_{24}^{AT}=(2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in_J\ T_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$-\left\{\frac{1}{2}(r_L r_R^*-r_L r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)T_J^S(\mu_0)+\xi_J T_J^S(\mu)R_J^S(\mu_0))\right.$$

$$\left.\left.-\frac{1}{2}(r_L r_R^*+r_L r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}\in_J\ T_J^S(\mu)P_J^S(\mu_0)\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right].$$

$$E_{31}^{AT}=(2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma_J T_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$+\left\{-\frac{1}{2}(r_L r_L^*+r_R r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}\gamma_J T_J^S(\mu)P_J^S(\mu_0)\right.$$

$$\left.\left.-\frac{1}{2}(r_L r_L^*-r_R r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)R_J^S(\mu_0)+\xi_J R_J^S(\mu)T_J^S(\mu_0))\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right].$$

$$E_{32}^{AT}=(2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J T_J^S(\mu)R_J^S(\mu_0)+\xi_J R_J^S(\mu)T_J^S(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$+\left\{-\frac{1}{2}(r_L r_R^*-r_R r_L^*)\sin s(\varphi-\varphi_0)\times\sum_{J=S}^{M}\gamma_J T_J^S(\mu)P_J^S(\mu_0)\right.$$

$$\left.\left.-\frac{1}{2}(r_L r_L^*+r_R r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)R_J^S(\mu_0)+\xi_1 R_J^S(\mu)T_J^S(\mu_0))\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right].$$

$$E_{33}^{AT}=(2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J T_J^S(\mu)T_J^S(\mu_0)+\xi_J R_J^S(\mu)R_J^S(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$+\left\{\frac{1}{2}(r_L r_R^*+r_R r_L^*)\cos s(\varphi-\varphi_0)\times\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)T_J^S(\mu_0)+\xi_J R_J^S(\mu)R_J^S(\mu_0))\right.$$

$$\left.\left.-\frac{1}{2}(r_L r_R^*-r_R r_L^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\in_J\ R_J^S(\mu)P_J^S(\mu_0)\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right].$$

$$E_{34}^{AT}=(2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[-\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in_J\ R_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]\right.$$

$$+\left\{\frac{1}{2}(r_L r_R^*-r_L r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(\alpha_J T_J^S(\mu)T_J^S(\mu_0)+\xi_J R_J^S(\mu)R_J^S(\mu_0))\right.$$

$$-\frac{1}{2}(r_L r_R^* + r_L r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\in_J\ R_J^S(\mu)P_J^S(\mu_0)\Big\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\Bigg].$$

$$E_{41}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\left[-\frac{1}{2}(r_L r_L^* - r_R r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}\in_J\ P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right].$$

$$E_{42}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\Bigg[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in_J\ P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]$$

$$-\frac{1}{2}(r_L r_L^* + r_R r_R^*)\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}\in_J\ P_J^S(\mu)T_J^S(\mu_0)\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\Bigg].$$

$$E_{43}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in_J\ P_J^S(\mu)R_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]$$

$$+\Big\{\frac{1}{2}(r_L r_R^* + r_R r_L^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\in_J\ P_J^S(\mu)R_J^S(\mu_0)$$

$$-\frac{1}{2}(r_L r_R^* - r_R r_L^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\delta_J P_J^S(\mu)P_J^S(\mu_0)\Big\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\Bigg]$$

$$E_{44}^{AT} = (2-\delta_{0,S})\frac{\omega^{AT}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\delta_J P_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]$$

$$+\Big[\frac{1}{2}(r_L r_R^* - r_L r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\in_J\ P_J^S(\mu)R_J^S(\mu_0)$$

$$-\frac{1}{2}(r_L r_R^* + r_L r_R^*)\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}\delta_J P_J^S(\mu)P_J^S(\mu_0)\Big\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\Bigg]. \qquad (2.12.11)$$

Similarly we get expanded matrix elements of $\mathbf{E_{OC}(z,\mu,\varphi)}$:

$$E_{11}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\beta'_J P_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}(t_L t_L^* + t_R t_R^*)$$

$$+\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P_J^S(\mu)R_J^S(\mu_{0n})$$

$$\times\frac{1}{2}(t_L t_L^* - t_R t_R^*)\Bigg]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{12}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\beta'_J P_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^* - t_R t_R^*)\}$$

$$+\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P_J^S(\mu)R_J^S(\mu_{0n})$$

$$\times\frac{1}{2}\{t_L t_L^* + t_R t_R^*)\}\Bigg]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{13}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P_J^S(\mu)T_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^* + t_R t_L^*)\}\right]\times$$

$$\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{14}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P_J^S(\mu)T_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^* - t_L t_R^*)\}\right]\times$$

$$\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{21}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J R_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^* + t_R t_R^*)\} + \cos s(\varphi-\varphi_0)$$

$$\times\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J R_J^S(\mu)R_J^S(\mu_{0n}) + \xi_J T_J^S(\mu)T_J^S(\mu_{0n}))\times\frac{1}{2}(t_L t_L^* - t_R t_R^*)\Bigg]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right)$$

.

$$E_{22}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J R_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^* - t_R t_R^*)\} + \cos s(\varphi-\varphi_0)$$

$$\times\sum_{J=S}^{M}(-1)^{J+S}\{\alpha'_J R_J^S(\mu)R_J^S(\mu_{0n}) + \xi'_J T_J^S(\mu)T_J^S(\mu_{0n})\}\times\frac{1}{2}\{t_L t_L^* + t_R t_R^*)\}\Bigg]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right.$$

.

$$E_{23}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J R_J^S(\mu)T_J^S(\mu_{0n})+\xi'_J T_J^S(\mu)R_J^S(\mu_{0n}))\times\frac{1}{2}\{t_L t_R^* + t_R t_L^*)\} - \sin s(\varphi-\varphi_0) \times\sum_{J=S}^{M}(-1)^{J+S}\varepsilon'_J T_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^* - t_R t_L^*)\}\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right]\cdot\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{24}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J R_J^S(\mu)T_J^S(\mu_{0n})+\xi'_J T_J^S(\mu)R_J^S(\mu_{0n}))\times \frac{1}{2}\{t_L t_R^* - t_L t_R^*)\} - \sin s(\varphi-\varphi_0) \times\sum_{J=S}^{M}(-1)^{J+S}\in'_J\ T_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^* + t_L t_R^*)\}\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right]\cdot\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{31}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J T_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^* + t_R t_R^*)\} - \sin s(\varphi-\varphi_0) \times\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J T_J^S(\mu)R_J^S(\mu_{0n})+\xi'_J R_J^S(\mu)T_J^S(\mu_{0n}))\times\frac{1}{2}\{t_L t_L^* - t_R t_R^*)\}\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right]\cdot\left(\frac{\mu_0}{\mu_{0n}}\right.$$

$$E_{32}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J T_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^* - t_R t_R^*)\} - \sin s(\varphi-\varphi_0) \times\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J T_J^S(\mu)R_J^S(\mu_{0n})+\xi'_J R_J^S(\mu)T_J^S(\mu_{0n}))\times\frac{1}{2}\{t_L t_L^* - t_R t_R^*)\}\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right]\cdot\left(\frac{\mu_0}{\mu_{0n}}\right.$$

$$E_{33}^{OC} = (2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J T_J^S(\mu)T_J^S(\mu_{0n})+\xi'_J R_J^S(\mu)R_J^S(\mu_{0n}))\times \frac{1}{2}\{t_L t_R^* - t_R t_L^*)\} - \cos s(\varphi-\varphi_0) \times\sum_{J=S}^{M}(-1)^{J+S}\in'_J\ R_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^* - t_R t_L^*\}\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right]\cdot\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{34}^{OC}=(2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}(\alpha'_J T_J^S(\mu)T_J^S(\mu_{0n})+\xi'_J R_J^S(\mu)R_J^S(\mu_{0n}))\times$$

$$\frac{1}{2}\{t_L t_R^{*}-t_L t_R^{*}\}-\cos s(\varphi-\varphi_0)$$

$$\times\sum_{J=S}^{M}(-1)^{J+S}\in'_J R_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^{*}+t_L t_R^{*}\}\Bigg]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{41}^{OC}=(2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in'_J P_J^S(\mu)T_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*}\}\right]\times$$

$$\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{42}^{OC}=(2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\left[-\sin s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in'_J P_J^S(\mu)T_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_L^{*}+t_R t_R^{*}\}\right]\times$$

$$\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{43}^{OC}=(2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in'_J P_J^S(\mu)R_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^{*}+t_R t_L^{*}\}+\cos s(\varphi-\varphi_0)$$

$$\times\sum_{J=S}^{M}(-1)^{J+S}\delta'_J P_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^{*}-t_R t_L^{*}\}\Bigg]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right).$$

$$E_{44}^{OC}=(2-\delta_{0,S})\frac{\omega^{OC}(z)}{4\pi}\Bigg[\cos s(\varphi-\varphi_0)\sum_{J=S}^{M}(-1)^{J+S}\in'_J P_J^S(\mu)R_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^{*}-t_L t_R^{*}\}+\cos s(\varphi-\varphi_0)$$

$$\times\sum_{J=S}^{M}(-1)^{J+S}\delta'_J P_J^S(\mu)P_J^S(\mu_{0n})\times\frac{1}{2}\{t_L t_R^{*}+t_L t_R^{*}\}\Bigg]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right). \quad (2.12.12)$$

We shall use the expanded source function in the following form as used earlier

$$S_{AT}(z,\mu,\varphi)=\begin{bmatrix}S_{ATL}(z,\mu,\varphi)\\ S_{ATQ}(z,\mu,\varphi)\\ S_{ATU}(z,\mu,\varphi)\\ S_{ATV}(z,\mu,\varphi)\end{bmatrix}=\sum_{S=0}^{\infty}(2-\delta_{0,S})\begin{bmatrix}S_{ATL}^S(z,\mu)\ \cos s(\varphi-\varphi_0)\\ S_{ATQ}^S(z,\mu)\ \cos s(\varphi-\varphi_0)\\ S_{ATU}^S(z,\mu)\ \sin s(\varphi-\varphi_0)\\ S_{ATV}^S(z,\mu)\ \sin s(\varphi-\varphi_0)\end{bmatrix}. \quad (2.12.13)$$

Using the following explicit form of the atmospheric E-matrix elements (2.12.11) for the atmospheric source function we next compute the following product

$$\mathbf{E}_{AT}(z,\mu,\varphi)\mathbf{F}=\begin{bmatrix}E_{11}^{AT}.F_1+E_{12}^{AT}.F_2+E_{13}^{AT}.F_3+E_{14}^{AT}.F_4\\ E_{21}^{AT}.F_1+E_{22}^{AT}.F_2+E_{23}^{AT}.F_3+E_{24}^{AT}.F_4\\ E_{31}^{AT}.F_1+E_{32}^{AT}.F_2+E_{33}^{AT}.F_3+E_{34}^{AT}.F_4\\ E_{41}^{AT}.F_1+E_{42}^{AT}.F_2+E_{43}^{AT}.F_3+E_{44}^{AT}.F_4\end{bmatrix}. \tag{2.12.14}$$

We now compare the coefficients of $\cos s(\phi-\phi_S)$ and $\sin s(\phi-\phi_S)$ from (2.12.13) and (2.12.14) with expressions (2.12.11) and we actually find following expressions for the elements of the source matrix in both media.

$$S_{ATL}^{S}(z,\mu)=\frac{\omega^{AT}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\beta_J P_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]+\left\{\frac{1}{2}(r_L r_L^{*}+r_R r_R^{*})\sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu_0)\right.\right.$$

$$\left.\left.+\frac{1}{2}(r_L r_L^{*}-r_R r_R^{*})\sum_{J=S}^{M}(\gamma_J P_J^S(\mu)R_J^S(\mu_0)\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right]F_1+\frac{\omega^{AT}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\gamma_J P_J^S(\mu)R_J^S(\mu_0)\exp\left[-\frac{}{\mu}\right.\right.$$

$$\left.\left.+\left\{\frac{1}{2}(r_L r_L^{*}-r_R r_R^{*})\sum_{J=S}^{M}\beta_J P_J^S(\mu)P_J^S(\mu_0)+\frac{1}{2}(r_L r_L^{*}+r_R r_R^{*})\sum_{J=S}^{M}\gamma_J P_J^S(\mu)R_J^S(\mu_0)\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right]F_2+0+($$

(2.12.15a)

$$S_{ATQ}^{S}(z,\mu)=\frac{\omega^{AT}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\gamma_J R_J^S(\mu)P_J^S(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]+\left\{\frac{1}{2}(r_L r_L^{*}+r_R r_R^{*})\sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu_0)\right.\right.$$

$$\left.\left.+\frac{1}{2}(r_L r_L^{*}-r_R r_R^{*})\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu_0)+\xi_J T_J^S(\mu)T_J^S(\mu_0))\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right]F_1+$$

$$\frac{\omega^{AT}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J R_J^S(\mu)R_J^S(\mu_0)+\xi_J T_J^S(\mu)T_J^S(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]+\left\{\frac{1}{2}(r_L r_L^{*}-r_R r_R^{*})\sum_{J=S}^{M}\gamma_J R_J^S(\mu)P_J^S(\mu_0)\right.\right.$$

$$\left.\left.+\frac{1}{2}(r_L r_L^{*}+r_R r_R^{*})\sum_{J=S}^{M}(\alpha_J R_J^S(\mu)R_J^S(\mu_0)+\xi_J T_J^S(\mu)T_J^S(\mu_0))\right\}\exp\left[-\frac{(2z_\omega-z)}{\mu_0}\right]\right]F_2+0+0.$$

(2.12.15b)

$$S^S_{ATU}(z,\mu)=\frac{\omega^{AT}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\gamma_J T^S_J(\mu)P^S_J(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]+\left\{-\frac{1}{2}(r_L r_L^* + r_R r_R^*)\sum_{J=S}^{M}\gamma_J T^S_J(\mu)P^S_J(\mu_0)\right.\right.$$

$$\left.\left.-\frac{1}{2}(r_L r_L^* - r_R r_R^*)\sum_{J=S}^{M}(\alpha_J T^S_J(\mu)R^S_J(\mu_0)+\xi_J R^S_J(\mu)T^S_J(\mu_0))\right\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right]F_1 +$$

$$\frac{\omega^{AT}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J T^S_J(\mu)R^S_J(\mu_0)+\xi_J R^S_J(\mu)T^S_J(\mu_0))\exp\left[-\frac{z}{\mu_0}\right]+\left\{-\frac{1}{2}(r_L r_R^* - r_R r_L^*)\sum_{J=S}^{M}\gamma_J T^S_J(\mu)P^S_J(\mu_0\right.\right.$$

$$\left.\left.-\frac{1}{2}(r_L r_L^* + r_R r_R^*)\sum_{J=S}^{M}(\alpha_J T^S_J(\mu)R^S_J(\mu_0)+\xi_1 R^S_J(\mu)T^S_J(\mu_0))\right\}\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right]F_2 + 0 + 0.$$

(2.12.15c)

$$S^S_{ATV}(z,\mu)=\frac{\omega^{AT}(z)}{4}\left[-\frac{1}{2}(r_L r_L^* - r_R r_R^*)\sum_{J=S}^{M}\epsilon_J\ P^S_J(\mu)T^S_J(\mu_0)\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right]F_1 +$$

$$\frac{\omega^{AT}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\epsilon_J\ P^S_J(\mu)T^S_J(\mu_0)\exp\left[-\frac{z}{\mu_0}\right]-\frac{1}{2}(r_L r_L^* + r_R r_R^*)\sum_{J=S}^{M}\epsilon_J\ P^S_J(\mu)T^S_J(\mu_0)\exp\left[-\frac{(2z_\omega - z)}{\mu_0}\right]\right]$$

$$+0+0.$$

(2.12.15d)

The azimuth free source for atmosphere can now be expressed as $S^S_{AT}(z,\mu)=\begin{bmatrix} S^S_{ATL}(z,\mu) \\ S^S_{ATL}(z,\mu) \\ S^S_{ATL}(z,\mu) \\ S^S_{ATL}(z,\mu) \end{bmatrix}$

(2.12.16)

Similarly for ocean cases we follow the same procedure using (2.12.17), (2.12.18) with (2.12.13) to get the expressions for ocean source function components.

$$S_{OC}(z,\mu,\varphi)=\begin{bmatrix}S_{OCL}(z,\mu,\varphi)\\ S_{OCQ}(z,\mu,\varphi)\\ S_{OCU}(z,\mu,\varphi)\\ S_{OCV}(z,\mu,\varphi)\end{bmatrix}=\sum_{S=0}^{\infty}(2-\delta_{0,s})\begin{bmatrix}S^S_{OCL}(z,\mu)\ \cos s(\varphi-\varphi_0)\\ S^S_{OCQ}(z,\mu)\ \cos s(\varphi-\varphi_0)\\ S^S_{OCU}(z,\mu)\ \sin s(\varphi-\varphi_0)\\ S^S_{OCV}(z,\mu)\ \sin s(\varphi-\varphi_0)\end{bmatrix}. \qquad (2.12.17)$$

$$E_{OC}(z,\mu,\varphi)F=\begin{bmatrix}E^{OC}_{11}.F_1+E^{OC}_{12}.F_2+E^{OC}_{13}.F_3+E^{OC}_{14}.F_4\\ E^{OC}_{21}.F_1+E^{OC}_{22}.F_2+E^{OC}_{23}.F_3+E^{OC}_{24}.F_4\\ E^{OC}_{31}.F_1+E^{OC}_{32}.F_2+E^{OC}_{33}.F_3+E^{OC}_{34}.F_4\\ E^{OC}_{41}.F_1+E^{OC}_{42}.F_2+E^{OC}_{43}.F_3+E^{OC}_{44}.F_4\end{bmatrix}. \qquad (2.12.18)$$

$$S^S_{OCL}(z,\mu)=\frac{\omega^{OC}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\beta'_J P^S_J(\mu)P^S_J(\mu_{0n})\times\frac{1}{2}(t_L t_L^*+t_R t_R^*)+\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P^S_J(\mu)R^S_J(\mu_{0n})\right.$$
$$\left.\times\frac{1}{2}(t_L t_L^*-t_R t_R^*)\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right)F_1+$$
$$\frac{\omega^{OC}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\beta'_J P^S_J(\mu)P^S_J(\mu_{0n})\times\frac{1}{2}\{t_L t_L^*-t_R t_R^*\}\}+\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J P^S_J(\mu)R^S_J(\mu_{0n})\right.$$
$$\left.\times\frac{1}{2}\{t_L t_L^*+t_R t_R^*\}\}\right]\exp\left[-\frac{z_\omega}{\mu_0}\right].\exp\left[-\frac{(z-z_\omega)}{\mu_{0n}}\right].\left(\frac{\mu_0}{\mu_{0n}}\right)F_2. \qquad (2.12.19a)$$

$$S^S_{OCQ}(z,\mu)=\left[\sum_{J=S}^{M}\gamma'_J R^S_J(\mu)P^S_J(\mu_{0n})\times\frac{1}{2}(t_L t_L^*+t_R t_R^*)+\sum_{J=S}^{M}(\alpha'_J R^S_J(\mu)R^S_J(\mu_{0n})+\xi'_J T^S_J(\mu)T^S_J(\mu_{0n}))\right.$$
$$\left.\times\frac{1}{2}(t_L t_L^*-t_R t_R^*)\right].\exp\left(-\frac{(z_\omega)}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right).\left(\frac{\mu_0}{\mu_{0n}}\right).F_1+$$
$$\frac{\omega^{OC}(z)}{4}\left[\sum_{J=S}^{M}(-1)^{J+S}\gamma'_J R^S_J(\mu)P^S_J(\mu_{0n})\frac{1}{2}\{t_L t_L^*-t_R t_R^*\}\}+\right.$$
$$\left.\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J R^S_J(\mu)R^S_J(\mu_{0n})+\xi_J T^S_J(\mu)T^S_J(\mu_{0n}))\frac{1}{2}(t_L t_L^*+t_R t_R^*)\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_2$$

.

(2.12.19b)

$$S_{OCU}^{S}(z,\mu)=\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\gamma_J' T_J^S(\mu)P_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L^{*}+t_R t_R^{*})\}-\right.$$
$$\left.\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J' T_J^S(\mu)R_J^S(\mu_{0n})+\xi_J' R_J^S(\mu)T_J^S(\mu_{0n}))\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*})\}\right]$$
$$\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_1+\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\gamma_J' T_J^S(\mu)P_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*})\}-\right.$$
$$\left.\sum_{J=S}^{M}(-1)^{J+S}(\alpha_J' T_J^S(\mu)R_J^S(\mu_{0n})+\xi_J' R_J^S(\mu)T_J^S(\mu_{0n}))\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*})\}\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_2$$

(2.12.19c)

$$S_{OCV}^{S}(z,\mu)=\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\in_J' P_J^S(\mu)T_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L^{*}-t_R t_R^{*}\}\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)$$
$$+\frac{\omega^{OC}(z)}{4}\left[-\sum_{J=S}^{M}(-1)^{J+S}\in_J' P_J^S(\mu)T_J^S(\mu_{0n})\frac{1}{2}\{t_L t_L^{*}+t_R t_R^{*}\}\right]\exp\left(-\frac{z_\omega}{\mu_0}\right)\exp\left(-\frac{(z-z_\omega)}{\mu_{0n}}\right)\left(\frac{\mu_0}{\mu_{0n}}\right)F_2.$$

(2.12.19d)

The azimuth free ocean source matrix now takes the form $S_{OC}^{S}(z,\mu)=\begin{bmatrix} S_{OCL}^{S}(z,\mu)\\ S_{OCQ}^{S}(z,\mu)\\ S_{OCU}^{S}(z,\mu)\\ S_{OCV}^{S}(z,\mu)\end{bmatrix}$ (2.12.20)

The presence of $(-1)^{J+S}$ in these expressions for source matrix elements indicates that the solar irradiances are pointing downwards.

We now have explicit expressions for source functions for use in transfer equations for each stokes component. Our next job is to specify the various coefficients used for representation of phase matrix elements. This is essential as these coefficients determine the actual scattering processes mathematically by the constituent materials present in the medium.

2.13. Details of the Reduction of Boundary Condition of flat ocean surface:

To reduce the boundary conditions described in earlier sections we have employed similar technique using equation (2.10.1)

Top boundary condition: $I_{AT}(0,-\mu,\varphi) = I_{AT\infty}(-\mu,\varphi)$

$$\begin{pmatrix} L^S_{AT}(0,-\mu) \\ Q^S_{AT}(0,-\mu) \\ U^S_{AT}(0,-\mu) \\ V^S_{AT}(0,-\mu) \end{pmatrix} = \begin{pmatrix} L^S_{AT}(\infty,0,-\mu) \\ Q^S_{AT}(\infty,0,-\mu) \\ U^S_{AT}(\infty,0,-\mu) \\ V^S_{AT}(\infty,0,-\mu) \end{pmatrix} \tag{2.13.1}$$

Inter face boundary condition at air-seasurface:

$$\begin{pmatrix} L^S_{AT}(z_\omega,\mu) \\ Q^S_{AT}(z_\omega,\mu) \\ U^S_{AT}(z_\omega,\mu) \\ V^S_{AT}(z_\omega,\mu) \end{pmatrix} = R_{AT}(-\mu,n) \begin{pmatrix} L^S_{AT}(z_\omega,\infty,-\mu) \\ Q^S_{AT}(z_\omega,\infty,-\mu) \\ U^S_{AT}(z_\omega,\infty,-\mu) \\ V^S_{AT}(z_\omega,\infty,-\mu) \end{pmatrix} + T_{OC}(+\mu,n) \begin{pmatrix} \dfrac{L^S_{OC}(z_\omega,\mu)}{n^2} \\ \dfrac{Q^S_{OC}(z_\omega,\mu)}{n^2} \\ \dfrac{U^S_{OC}(z_\omega,\mu)}{n^2} \\ \dfrac{V^S_{OC}(z_\omega,\mu)}{n^2} \end{pmatrix} \tag{2.13.2}$$

$$\begin{aligned} L^S_{AT}(z_\omega,\mu) = {} & \frac{1}{2}(r_L r_L^* + r_R r_R^*) L^S_{AT}(z_\omega,-\mu) + \frac{1}{2}(r_L r_L^* - r_R r_R^*) Q^S_{AT}(z_\omega,-\mu) + \\ & \frac{1}{2}(t_L t_L^* + t_R t_R^*) \frac{L^S_{OC}(z_\omega,\mu)}{n^2} + \frac{1}{2}(t_L t_L^* - t_R t_R^*) \frac{Q^S_{OC}(z_\omega,+\mu)}{n^2}. \end{aligned} \tag{2.13.3}$$

$$\begin{aligned} Q^S_{AT}(z_\omega,\mu) = {} & \frac{1}{2}(r_L r_L^* - r_R r_R^*) L^S_{AT}(z_\omega,-\mu) + \frac{1}{2}(r_L r_L^* + r_R r_R^*) Q^S_{AT}(z_\omega,-\mu) + \\ & \frac{1}{2}(t_L t_L^* - t_R t_R^*) \frac{L^S_{OC}(z_\omega,\mu)}{n^2} + \frac{1}{2}(t_L t_L^* + t_R t_R^*) \frac{Q^S_{OC}(z_\omega,+\mu)}{n^2}. \end{aligned} \tag{2.13.4}$$

$$U^S_{AT}(z_\omega,\mu) = \frac{1}{2}(r_L r_R^* + r_R r_L^*) U^S_{AT}(z_\omega,-\mu) + \frac{1}{2}(r_L r_R^* - r_L r_R^*) V^S_{AT}(z_\omega,-\mu) +$$

$$\frac{1}{2}(t_L t_R^* + t_R t_L^*)\frac{U_{OC}^S(z_\omega,\mu)}{n^2} + \frac{1}{2}(t_L t_R^* - t_R t_L^*)\frac{V_{OC}^S(z_\omega,\mu)}{n^2} \tag{2.13.5}$$

$$V_{AT}^S(z_\omega,\mu) = \frac{1}{2}(r_L r_R^* - r_R r_L^*)U_{AT}^S(z_\omega,-\mu) + \frac{1}{2}(r_L r_R^* + r_L r_R^*)V_{AT}^S(z_\omega,-\mu) +$$

$$\frac{1}{2}(t_L t_R^* - t_R t_L^*)\frac{U_{OC}^S(z_\omega,\mu)}{n^2} + \frac{1}{2}(t_L t_R^* + t_L t_R^*)\frac{V_{OC}^S(z_\omega,+\mu)}{n^2}. \tag{2.13.6}$$

Interface boundary condition at seasurface-air boundary:

$$\frac{I_{OC}(z_\omega,-\mu)}{n^2} = R_{OC}(+\mu,n)\left\{\frac{I_{OC}(z_\omega,\mu)}{n^2}\right\} + T_{AT}(-\mu,n)I_{AT}(Z_\omega,-\mu).$$

$$\begin{pmatrix} \dfrac{L_{OC}^S(z_\omega,-\mu)}{n^2} \\ \dfrac{Q_{OC}^S(z_\omega,\mu)}{n^2} \\ \dfrac{U_{OC}^S(z_\omega,\mu)}{n^2} \\ \dfrac{V_{OC}^S(z_\omega,\mu)}{n^2} \end{pmatrix} = R_{OC}(+\mu,n)\left\{\left(\begin{pmatrix} \dfrac{L_{OC}^S(z_\omega,\mu)}{n^2} \\ \dfrac{Q_{OC}^S(z_\omega,\mu)}{n^2} \\ \dfrac{U_{OC}^S(z_\omega,\mu)}{n^2} \\ \dfrac{V_{OC}^S(z_\omega,\mu)}{n^2} \end{pmatrix}\right)\right\} + T_{AT}(-\mu,n)\begin{pmatrix} L_{AT}^S(z_\omega,-\mu) \\ Q_{AT}^S(z_\omega,-\mu) \\ U_{AT}^S(z_\omega,-\mu) \\ V_{AT}^S(z_\omega,-\mu) \end{pmatrix} \tag{2.13.7}$$

$$\frac{L_{OC}^S(z_\omega,\mu)}{n^2} = \frac{1}{2}(r_L r_L^* + r_R r_R^*)\frac{L_{OC}^S(z_\omega,\mu)}{n^2} + \frac{1}{2}(r_L r_L^* - r_R r_R^*)\frac{Q_{OC}^S(z_\omega,\mu)}{n^2} +$$

$$\frac{1}{2}(t_L t_L^* + t_R t_R^*)L_{AT}^S(z_\omega,-\mu) + \frac{1}{2}(t_L t_L^* - t_R t_R^*)Q_{AT}^S(z_\omega,-\mu). \tag{2.13.8}$$

$$\frac{Q_{OC}^S(z_\omega,\mu)}{n^2} = \frac{1}{2}(r_L r_L^* - r_R r_R^*)\frac{L_{OC}^S(z_\omega,\mu)}{n^2} + \frac{1}{2}(r_L r_L^* + r_R r_R^*)\frac{Q_{OC}^S(z_\omega,\mu)}{n^2} +$$

$$\frac{1}{2}(t_L t_L^* - t_R t_R^*)L_{AT}^S(z_\omega,-\mu) + \frac{1}{2}(t_L t_L^* + t_R t_R^*)Q_{AT}^S(z_\omega,-\mu). \tag{2.13.9}$$

$$\frac{U_{OC}^S(z_\omega,\mu)}{n^2} = \frac{1}{2}(r_L r_R^* + r_R r_L^*)\frac{U_{OC}^S(z_\omega,\mu)}{n^2} + \frac{1}{2}(r_L r_R^* - r_L r_R^*)\frac{U_{OC}^S(z_\omega,\mu)}{n^2} +$$

$$\frac{1}{2}(t_L t_R^* + t_R t_L^*)U_{AT}^S(z_\omega,-\mu) + \frac{1}{2}(t_L t_R^* - t_L t_R^*)V_{AT}^S(z_\omega,-\mu). \tag{2.13.10}$$

$$\frac{V_{OC}^S(z_\omega,\mu)}{n^2} = \frac{1}{2}(r_L r_R^* - r_R r_L^*)\frac{U_{OC}^S(z_\omega,\mu)}{n^2} + \frac{1}{2}(r_L r_R^* + r_L r_R^*)\frac{U_{OC}^S(z_\omega,\mu)}{n^2} +$$

$$\frac{1}{2}(t_L t_R^* - t_R t_L^*) U_{AT}^S(z_\omega,-\mu) + \frac{1}{2}(t_L t_R^* + t_R t_L^*) V_{AT}^S(z_\omega,-\mu). \qquad (2.13.11)$$

Boundary condition at the bottom of the seasurface:

$$\begin{pmatrix} L_{OC}^S(z,\mu) \\ Q_{OC}^S(z,\mu) \\ U_{OC}^S(z,\mu) \\ V_{OC}^S(z,\mu) \end{pmatrix} = \begin{pmatrix} L_{OC}^S g(\mu) \\ Q_{OC}^S g(\mu) \\ U_{OC}^S g(\mu) \\ V_{OC}^S g(\mu) \end{pmatrix}. \qquad (2.13.12)$$

2.14. Final set of radiative transfer equation:

Since we are primarily interested in the azimuth independent picture of transfer of radiation in atmosphere and ocean system, we shall hence forth consider the case **S=0** i.e. we shall retain only the first term in the intensity expansion. We therefore now have final set of equations for atmosphere and ocean dropping superscript's'

$$\mu\frac{d}{dz}L_{AT}^S(z,\mu) = -L_{AT}^S(z,\mu) + \frac{\omega^{AT}(z)}{2}\left[\sum_{J=0}^{M}\beta_J P_J^S(\mu)\int_{-1}^{+1}P_J^S(\mu')L_{AT}^S(z,\mu')d\mu'\right.$$

$$\left.-\sum_{J=0}^{M}\gamma_J P_J^S(\mu)\int_{-1}^{+1}R_J^S(\mu')Q_{AT}^S(z,\mu')d\mu' + \sum_{J=0}^{M}\gamma_J P_J^S(\mu)\int_{-1}^{+1}T_J^S(\mu')U_{AT}^S(z,\mu')\right] + S_{ATL}^S(z,\mu) \qquad (2.14.1)$$

$$\mu\frac{d}{dz}Q_{AT}^S(z,\mu) = -Q_{AT}^S(z,\mu) + \frac{\omega^{AT}(z)}{2}\left[\sum_{J=S}^{M}\gamma_J R_J^S(\mu)\int_{-1}^{+1}P_J^S(\mu')L_{AT}^S(z,\mu')d\mu'\right.$$

$$+\sum_{J=S}^{M}\left(\alpha_J R_J^S(\mu)\int_{-1}^{+1}R_J^S(\mu')Q_{AT}^S(z,\mu')d\mu' + \xi_J T_J^S(\mu)\int_{-1}^{+1}T_J^S(\mu')Q_{AT}^S(z,\mu')d\mu'\right)$$

$$-\sum_{J=S}^{M}\left(\alpha_J R_J^S(\mu)\int_{-1}^{+1}T_J^S(\mu')U_{AT}^S(z,\mu')d\mu' + \xi_J T_J^S(\mu)\int_{-1}^{+1}R_J^S(\mu')U_{AT}^S(z,\mu')d\mu'\right).$$

$$\left.+\sum_{J=S}^{M}\varepsilon_J T_J^S(\mu)\int_{-1}^{+1}P_J^S(\mu')V_{AT}^S(z,\mu')d\mu'\right] + S_{ATQ}^S(z,\mu). \qquad (2.14.2)$$

$$\mu\frac{d}{dz}U^S_{AT}(z,\mu) = -U^S_{AT}(z,\mu) + \frac{\omega^{AT}(z)}{2}\left[-\sum_{J=S}^{M}\gamma_J T^S_J(\mu)\int_{-1}^{+1}P^S_J(\mu')L^S_{AT}(z,\mu')d\mu'\right.$$
$$-\sum_{J=S}^{M}\left(\alpha_J T^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')Q^S_{AT}(z,\mu')d\mu' + \xi_J R^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')Q^S_{AT}(z,\mu')d\mu'\right)$$
$$+\sum_{J=S}^{M}\left(\alpha_J T^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')U^S_{AT}(z,\mu')d\mu' + \xi_J R^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')U^S_{AT}(z,\mu')d\mu'\right)$$
$$\left.-\sum_{J=S}^{M}\varepsilon_J R^S_J(\mu)\int_{-1}^{+1}P^S_J(\mu')V^S_{AT}(z,\mu')d\mu'\right] + S^S_{ATU}(z,\mu). \tag{2.14.3}$$

$$\mu\frac{d}{dz}V^S_{AT}(z,\mu) = -V^S_{AT}(z,\mu) + \frac{\omega^{AT}(z)}{2}\left[-\sum_{J=S}^{M}\varepsilon_J P^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')Q^S_{AT}(z,\mu')d\mu'\right.$$
$$\left.+\sum_{J=S}^{M}\varepsilon_J P^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')U^S_{AT}(z,\mu')d\mu' + \sum_{J=S}^{M}\delta_J P^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')V^S_{AT}(z,\mu')d\mu'\right] + S^S_{ATV}(z,\mu). \tag{2.14.4}$$

$$\mu\frac{d}{dz}L^S_{OC}(z,\mu) = -L^S_{OC}(z,\mu) + \frac{\omega^{OC}(z)}{2}\left[\sum_{J=S}^{M}\beta'_J P^S_J(\mu)\int_{-1}^{+1}P^S_J(\mu')L^S_{OC}(z,\mu')d\mu'\right.$$
$$\left.+\sum_{J=S}^{M}\gamma'_J P^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')Q^S_{OC}(z,\mu')d\mu' - \sum_{J=S}^{M}\gamma'_J P^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')U^S_{OC}(z,\mu')d\mu'\right] + S^S_{OCL}(z,\mu). \tag{2.14.5}$$

$$\mu\frac{d}{dz}Q^S_{OC}(z,\mu) = -Q^S_{OC}(z,\mu) + \frac{\omega^{OC}(z)}{2}\left[\sum_{J=S}^{M}\gamma'_J R^S_J(\mu)\int_{-1}^{+1}P^S_J(\mu')L^S_{OC}(z,\mu')\right.$$
$$+\sum_{J=S}^{M}\left(\alpha'_J R^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')Q^S_{OC}(z,\mu')d\mu' + \xi'_J T^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')Q^S_{OC}(z,\mu')d\mu'\right)$$
$$-\sum_{J=S}^{M}\left(\alpha'_J R^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')U^S_{OC}(z,\mu')d\mu' + \xi'_J T^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')U^S_{OC}(z,\mu')d\mu'\right)$$
$$\left.+\sum_{J=S}^{M}\varepsilon'_J T^S_J(\mu)\int_{-1}^{+1}P^S_J(\mu')V^S_{OC}(z,\mu')d\mu'\right) + S^S_{OCQ}(z,\mu). \tag{2.14.6}$$

$$\mu\frac{d}{dz}U^S_{OC}(z,\mu)=-U^S_{OC}(z,\mu)+\frac{\omega^{OC}(z)}{2}\Bigg[-\sum_{J=S}^{M}\gamma'_J T^S_J(\mu)\int_{-1}^{+1}P^S_J(\mu')L^S_{OC}(z,\mu')d\mu'$$

$$-\sum_{J=S}^{M}\left(\alpha'_J T^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')Q^S_{OC}(z,\mu')d\mu'+\xi'_J R^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')Q^S_{OC}(z,\mu')d\mu'\right)$$

$$+\sum_{J=S}^{M}\left(\alpha'_J T^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')U^S_{OC}(z,\mu')d\mu'+\xi'_J R^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')U^S_{OC}(z,\mu')\right)$$

$$-\sum_{J=S}^{M}\varepsilon'_J R^S_J(\mu)\int_{-1}^{+1}P^S_J(\mu')V^S_{OC}(z,\mu')d\mu'\Bigg]+S^S_{OCU}(z,\mu). \qquad (2.14.7)$$

$$\mu\frac{d}{dz}V^S_{OC}(z,\mu)=-V^S_{OC}(z,\mu)+\frac{\omega^{OC}(z)}{2}\Bigg[-\sum_{J=S}^{M}\varepsilon'_J P^S_J(\mu)\int_{-1}^{+1}T^S_J(\mu')Q^S_{OC}(z,\mu')d\mu'$$

$$+\sum_{J=S}^{M}\varepsilon'_J P^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')U^S_{OC}(z,\mu')d\mu'+\sum_{J=S}^{M}\delta'_J P^S_J(\mu)\int_{-1}^{+1}R^S_J(\mu')V^S_{OC}(z,\mu')d\mu'\Bigg]+S^S_{OCV}(z,\mu).$$

$$(2.14.8)$$

The finally reduced compact atmosphere and ocean equations are given below. Here we have dropped "s" from intensity expression.

$$\mu\frac{d}{dz}I_{AT}(z,\mu)+I_{AT}(z,\mu)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}PAT^S_J(z,\mu)\int_{-1}^{+1}\mathbf{P}^S_J(\mu')I_{AT}(z,\mu')+S_{AT}(z,\mu) \qquad (2.14.9)$$

$$\mu\frac{d}{dz}I_{OC}(z,\mu)+I_{OC}(z,\mu)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}POC^S_J(z,\mu)\int_{-1}^{+1}\mathbf{P}^S_J(\mu')I_{OC}(z,\mu')+S_{OC}(z,\mu) \qquad (2.14.10)$$

$$PAT^S_J(z,\mu)=\mathbf{P}^S_J(\mu)\Pi^{AT}_J; \qquad (2.14.11)$$

$$POC^S_J(z,\mu)=\mathbf{P}^S_J(\mu)\Pi^{OC}_J; \qquad (2.14.12)$$

2.15. Determination of characteristic coefficients:

We are now in a position to define the coefficients used in the expansion of phase matrix elements. Actual numerical values of these coefficients are required for the measurement of the light scatted by the constituent particles present in the medium. Indeed the atmosphere and ocean have different characteristics particles from the viewpoint of light scattering properties and consequently we have used different coefficients for these two media. The determination of these coefficients requires the accurate estimation of the following integrals. Computations of phase functions which indicate the distribution of monochromatic scattered energy for single scattering should be carried out for fixed wavelength. Since phase functions (elements of phase matrix) are assumed to be a function of scattering angle, they can be expanded in Legendre polynomials (as used in (2.3.53), (2.3.55)) with characteristic coefficients.

We have seen that the phase matrix can be expanded in the following form

$$P(\mu,\varphi,\mu',\varphi') = \sum_{S=0}^{L}(2-\delta_{0,S})[P_C^S(\mu,\mu')\cos S(\varphi-\varphi') + P_S^S(\mu,\mu')\sin S(\varphi-\varphi')] \tag{2.15.1}$$

Where $P_C^S(\mu,\mu')$ and $P_S^S(\mu,\mu')$ are given by expressions (2.3.53) and (2.3.55) respectively. The constant coefficients used in each of element of (2.3.53) and (2.3.55) must be calculated using the scattering matrix (2.3.50). The Legendre and associated Legendre polynomials ($P_r(\mu)$ and $P_r^S(\mu)$ respectively) obey the following orthogonality relations.

$$\int_{-1}^{+1} P_r(\mathrm{Cos}\,\boldsymbol{\Omega})P_{r'}(\cos\boldsymbol{\Omega})d(\cos(\boldsymbol{\Omega})) = \left(\frac{2}{2r+1}\right)\delta_{rr'} \tag{2.15.2}$$

$$\int_{-1}^{+1} P_r^2(\mathrm{Cos}\,\boldsymbol{\Omega})P_{r'}^2(\cos\boldsymbol{\Omega})d(\cos(\boldsymbol{\Omega})) = \left(\frac{2}{2r+1}\right)\delta_{rr'} \tag{2.15.3}$$

We also have for associated Legendre polynomials

$$P_r^S(\mu) = (1-\mu^2)^{\frac{S}{2}}\frac{d^S}{d\mu^S}P_r(\mu) \tag{2.15.4}$$

The explicit expressions for first five Legendre functions are given below

$$P_0(\mu) = 1, \tag{2.15.5}$$

$$P_1(\mu) = \mu \tag{2.15.6}$$

$$P_2(\mu) = \frac{1}{2}(3\mu^2-1) \tag{2.15.7}$$

$$P_3(\mu) = \frac{1}{2}(5\mu^3-3\mu) \tag{2.15.8}$$

$$P_4(\mu) = 35\mu^4 - 30\mu^2 + 3 \tag{2.15.9}$$

In (2.3.53) and (2.3.55) we have used functions $R_r^S(\mu)$ and $T_r^S(\mu)$. These functions can be expressed as linear combinations of the **generalized Legendre functions** [99, 169, and 372] which are given by (2.3.56) and (2.3.57). We rewrite these two equations here.

$$P_{r,r'}^S(\mu) = A_{m,n}^l(1-\mu)^{-\frac{(n-m)}{2}}(1+\mu)^{-\frac{(n+m)}{2}}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}] \tag{2.15.10}$$

$$A_{m,n}^{l} = \frac{(-1)^{l-m}}{2^{l}(l-m)!}\sqrt{\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}} \tag{2.15.11}$$

$$R_{r}^{S}(\mu) = \left[\frac{P_{S,2}^{r}(\mu) + P_{S,-2}^{r}(\mu)}{2}\right] \tag{2.15.12}$$

$$T_{r}^{S}(\mu) = \left[\frac{P_{S,2}^{r}(\mu) - P_{S,-2}^{r}(\mu)}{2}\right] \tag{2.15.13}$$

The **generalized spherical functions** are defined by for integers $\mathbf{r,r',S}$ with $\mathbf{S \geq 0}$ and $\mathbf{-S \leq r,r' \leq S}$

$$P_{rr'}^{S}(\mu) = \Gamma_{mn}^{l}(i)^{n-m}(1-\mu)^{-\frac{(n-m)}{2}}(1+\mu)^{-\frac{(n+m)}{2}}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}] \tag{2.15.14}$$

With $$\Gamma_{rr'}^{S} = \frac{(-1)^{l-m}}{2^{l}}\sqrt{\frac{(l+n)!}{(S-r)!(l+m)!(l-n)!}} \tag{2.15.15}$$

It is to be noted that up to the factor $(i)^{n-m}$ the generalized spherical functions $P_{rr'}^{S}(\mu)$ are real. However for other values of $\mathbf{r,r',S,S}$ one chooses $P_{rr'}^{S}(\mu) = 0$. These restrictions are also true for generalized Legendre functions given above. Complete calculation of the $(S-r)$-th derivative on the left hand side of (2.15.14) gives us an expression for $P_{rr'}^{S}(\mu)$ which remains unchanged when $\mathbf{r}$ and $\mathbf{r'}$ are reversed. On replacing the argument by its negative counterpart we get the parity relations defined by

$$P_{rr'}^{S}(-\mu) = (-1)^{S+r-r'}P_{r,r'}^{S}(\mu) \tag{2.15.16}$$

We can also find

$$P_{rr'}^{S}(\mu) = P_{-r,-r'}^{S}(\mu) = P_{r'r}^{S}(\mu) \tag{2.15.17}$$

The above mentioned two functions obey the two orthogonality relations

$$\int_{-1}^{1} P_{rr}^{S}(\mu)P_{rr}^{S'}(\mu)d\mu = \frac{2}{2S+1}\delta_{SS'} \tag{2.15.18}$$

$$\int_{-1}^{1} P_{r,-r}^{S}(\mu)P_{r,-r}^{S'}(\mu)d\mu = \frac{2}{2S+1}\delta_{SS'} \tag{2.15.19}$$

When we are interested in the azimuth free case we must set **S=0** in our calculations of phase matrix (2.11.8) and (2.11.18). We also need to fix the number of terms in the summation i.e. **M**.

Let us set

$$\mathbf{P}_J^0(\mu) = \mathbf{P}_J(\mu) \tag{2.15.20}$$

With the observation

$$\mathbf{P}_0(\mu) = \mathrm{diag}(1,0,0,1), \quad (2.15.21) \qquad \mathbf{P}_1(\mu) = \mathrm{diag}(\mu,0,0,\mu) \quad (2.15.22)$$

and $\mathbf{P}_2(\mu) = \mathrm{diag}(P_2(\mu), R_2(\mu), R_2(\mu), P_2(\mu))$ (2.15.23)

This is because $T_2(\mu)$ functions are zero.

We now set **s=0** and **r=2** in (2.15.10) and (2.15.11) and use (2.15.12) to get explicit expression for

$$R_2(\mu) = \frac{\sqrt{6}}{4}(1-\mu^2) \tag{2.15.24}$$

For $r \geq 2$ we find that

$$\mathbf{P}_{r+1}(\mu) = X_r^{-1}[(2r+1)\mu\mathbf{P}_r(\mu) - Y_r\mathbf{P}_{r-1}(\mu)], \tag{2.15.25}$$

$$\text{where } X_r = \mathrm{diag}\left\{r+1, [(r+3)(r-1)]^{\frac{1}{2}}, [(r+3)(r-1)]^{\frac{1}{2}}, r+1\right\} \tag{2.15.26}$$

$$\text{and } Y_r = \mathrm{diag}\left\{r, (r^2-4)^{\frac{1}{2}}, (r^2-4)^{\frac{1}{2}}, r\right\}. \tag{2.15.27}$$

$$\text{For } r=3,\ \mathbf{P}_4(\mu) = X_3^{-1}[7\mu\mathbf{P}_3(\mu) - Y_3\mathbf{P}_2(\mu)] \tag{2.15.28}$$

$$X_3 = \mathrm{diag}\left\{4, (6\times 2)^{\frac{1}{2}}, (6\times 2)^{\frac{1}{2}}, 4\right\} = \mathrm{diag}\{4, 2\sqrt{3}, 2\sqrt{3}, 4\}; \tag{2.15.29}$$

$$Y_3 = \mathrm{diag}\{3, \sqrt{5}, \sqrt{5}, 3\}. \tag{2.15.30}$$

$$\text{For } r=4,\ \mathbf{P}_5(\mu) = X_4^{-1}[9\mu\mathbf{P}_4(\mu) - Y_4\mathbf{P}_3(\mu)] \tag{2.15.31}$$

$$X_4 = \mathrm{diag}\left\{5, (7\times 3)^{\frac{1}{2}}, (7\times 3)^{\frac{1}{2}}, 5\right\} = \mathrm{diag}\{5, \sqrt{21}, \sqrt{21}, 5\}; \tag{2.15.32}$$

$$Y_4 = \mathrm{diag}\{4, \sqrt{12}, \sqrt{12}, 4\} = \mathrm{diag}\{4, 2\sqrt{3}, 2\sqrt{3}, 4\}. \tag{2 .15.33}$$

$$\text{For } r=5,\ \mathbf{P}_6(\mu) = X_5^{-1}[11\mu\mathbf{P}_5(\mu) - Y_5\mathbf{P}_4(\mu)] \tag{2.15.34}$$

$$X_5 = \mathrm{diag}\left\{6, (8\times 4)^{\frac{1}{2}}, (8\times 4)^{\frac{1}{2}}, 6\right\} = \mathrm{diag}\{6, 4\sqrt{2}, 4\sqrt{2}, 6\}; \tag{2.15.35}$$

$$Y_5 = \mathrm{diag}\,\{5, \sqrt{21}, \sqrt{21}, 5\}. \tag{2.15.36}$$

For $S = 1$ we find and **r=2**.

$$\mathbf{P}_1^1(\mu) = \mathrm{diag}\left\{(1-\mu^2)^{\frac{1}{2}}, 0, 0, (1-\mu^2)^{\frac{1}{2}}\right\} \tag{2.15.37}$$

and

$$\mathbf{P}_2^1(\mu) = \begin{vmatrix} 3\mu(1-\mu^2)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & R_2^1(\mu) & -T_2^1(\mu) & 0 \\ 0 & -T_2^1(\mu) & R_2^1(\mu) & 0 \\ 0 & 0 & 0 & 3\mu(1-\mu^2)^{\frac{1}{2}} \end{vmatrix} \tag{2.15.38}$$

with

$$R_2^1(\mu) = -\frac{1}{2}\mu\sqrt{6}(1-\mu^2)^{\frac{1}{2}} \quad (2.15.39) \quad \text{and} \quad T_2^1(\mu) = -\frac{1}{2}\sqrt{6}(1-\mu^2)^{\frac{1}{2}}, \tag{2.15.39}$$

and for $r \geq 2$ we find

$$\mathbf{P}_{r+1}^1(\mu) = [X_r^1]^{-1}[(2r+1)\mu\mathbf{P}_r^1(\mu) - X_r^1\mathbf{P}_{r-1}^1(\mu) + Z_r^1\,\mathbf{P}_r^1(\mu)] \tag{2.15.40}$$

where, in general,

$$X_r^S(\mu) = \mathrm{diag}\left\{r+1-S, \left(\frac{r-S+1}{r+1}\right)[(r+3)(r-1)]^{\frac{1}{2}}, \left(\frac{r-S+1}{r+1}\right)[(r+3)(r-1)]^{\frac{1}{2}}, r+1-S\right\}, \tag{2.15.41}$$

$$Y_r^S(\mu) = (1-\delta_{S,r})\mathrm{diag}\left\{r+S, \left(\frac{r+S}{r}\right)(r^2-4)^{\frac{1}{2}}, \left(\frac{r+S}{r}\right)(r^2-4)^{\frac{1}{2}}, r+S\right\}, \tag{2.15.42}$$

and

$$Z_r^S = \frac{2S(2r+1)}{r(r+1)}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \tag{2.15.43}$$

For $r = 2$,

$$\mathbf{P}_3^1(\mu) = [X_2^1]^{-1}[5\,\mu\,\mathbf{P}_2^1(\mu) - Y_2^1\,\mathbf{P}_1^1(\mu) + Z_2^1\,\mathbf{P}_2^1(\mu)], \tag{2.15.44}$$

For $r = 3$,

$$\mathbf{P}_4^1(\mu) = [X_3^1]^{-1}[7\,\mu\,\mathbf{P}_3^1(\mu) - Y_3^1\,\mathbf{P}_2^1(\mu) + Z_3^1\,\mathbf{P}_3^1(\mu)], \tag{2.15.45}$$

For $\mathbf{r}=4$,

$$\mathbf{P}_5^1(\mu)=[X_4^1]^{-1}[9\,\mu\,\mathbf{P}_4^1(\mu)-Y_4^1\,\mathbf{P}_3^1(\mu)+Z_4^1\,\mathbf{P}_4^1(\mu)], \qquad (2.15.46)$$

where $X_2^1=\mathrm{diag}\left\{2,\frac{2\sqrt{5}}{3},\frac{2\sqrt{5}}{3},2\right\}$, (2. 15.47) $\qquad X_3^1=\mathrm{diag}\left\{3,\frac{3\sqrt{3}}{2},\frac{3\sqrt{3}}{2},3\right\}$, (2.15.48)

$$X_4^1=\mathrm{diag}\left\{4,\frac{4\sqrt{21}}{5},\frac{4\sqrt{21}}{5},4\right\}. \qquad (2.15.49)$$

$Y_2^1=\mathrm{diag}\,\{3,0,0,3\}$, (2.15.50) $\qquad Y_3^1=\mathrm{diag}\left\{4,\frac{4\sqrt{5}}{3},\frac{4\sqrt{5}}{3},4\right\}$, (2.15.51)

$$Y_4^1=\mathrm{diag}\left\{5,\frac{5\sqrt{3}}{2},\frac{5\sqrt{3}}{2},5\right\}. \qquad (2.15.52)$$

$Z_2^1=\frac{5}{3}\begin{vmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{vmatrix}$, (2.15.53) $\qquad Z_3^1=\frac{7}{21}\begin{vmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{vmatrix}$, (2.15.54)

$$Z_4^1=\frac{9}{10}\begin{vmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{vmatrix}. \qquad (2.15.55)$$

Finally for the general case $S\geq 2$ we start with

$$\mathbf{P}_S^S=(1-\mu^2)^{\frac{S}{2}}\begin{vmatrix}(2S-1)!! & 0 & 0 & 0\\ 0 & K_S\left\{\frac{1+\mu^2}{1-\mu^2}\right\} & -K_S\left\{\frac{2\mu^2}{1-\mu^2}\right\} & 0\\ 0 & -K_S\left\{\frac{2\mu}{1-\mu^2}\right\} & K_S\left\{\frac{1+\mu^2}{1-\mu^2}\right\} & 0\\ 0 & 0 & 0 & (2S-1)!!\end{vmatrix} \qquad (2.15.56)$$

with

$$(2S-1)!!=1.3.5....(2S-1), \qquad (2.15.57)$$

$$K_S = \frac{(2S)!}{2^S}[(S-2)!(S+2)!]^{-\frac{1}{2}}, \tag{2.15.58}$$

and for $r \geq S$

$$\mathbf{P}^S_{r+1}(\mu) = [X^S_r]^{-1}[(2r+1)\mu\,\mathbf{P}^S_r(\mu) - Y^S_r\mathbf{P}^S_{r-1}(\mu) + Z^S_r\mathbf{P}^S_r(\mu)]. \tag{2.15.59}$$

We can now recall that we have used our decomposed phase matrix (In equations 2.11.8) as

$$\sum_{J=S}^{M}\begin{bmatrix} P^S_J(\mu) & 0 & 0 & 0 \\ 0 & R^m_J(\mu) & -T^m_J(\mu) & 0 \\ 0 & -T^m_J(\mu) & R^m_J(\mu) & 0 \\ 0 & 0 & 0 & P^m_J(\mu) \end{bmatrix}\begin{bmatrix} \beta_J & \gamma_J & 0 & 0 \\ \gamma_J & \alpha_J & 0 & 0 \\ 0 & 0 & \xi_J & -\varepsilon_J \\ 0 & 0 & \varepsilon_J & \delta_J \end{bmatrix} X$$

$$\begin{bmatrix} P^m_J(\mu') & 0 & 0 & 0 \\ 0 & R^m_J(\mu') & -T^m_J(\mu') & 0 \\ 0 & -T^m_J(\mu') & R^m_J(\mu') & 0 \\ 0 & 0 & 0 & P^m_J(\mu') \end{bmatrix} = \mathbf{P}^S_J(\mu)\prod{}^{AT}_J\mathbf{P}^S_J(\mu'). \tag{2.15.60}$$

We can easily derive similar formula for ocean. The general formula for the calculation of these coefficients appeared in (2.15.60) is given by

$$A^S_r = \frac{2}{2r+1}\int_{-1}^{+1} P_{ij}(\mu)P^S_r(\mu')d\mu' \tag{2.15.61}$$

Where $P_{ij}(\mu)$ are elements of the respective phase matrix given by equation (2.3.51). The coefficients used in (2.3.52) can be written explicitly in the following form with reference to (2.15.5 – 2.15.9).

$$\beta_r = \frac{2}{2l+1}\int_{-1}^{+1} P_{11}(\mu)P_r(\mu)d\mu, \tag{2.15.62}$$

$$\delta_r = \frac{2}{2l+1}\int_{-1}^{+1} P_{33}(\mu)P_r(\mu)d\mu, \tag{2.15.63}$$

$$\gamma_r = \frac{2}{2l+1}\int_{-1}^{+1} P_{12}(\mu)P^2_r(\mu)d\mu, \tag{2.15.64}$$

$$\varepsilon_r = \frac{2}{2l+1}\int_{-1}^{+1} P_{34}(\mu)P^2_r(\mu)d\mu, \tag{2.15.65}$$

$$\sum_{r=0}^{L}(\beta_r + \delta_r)P_r(\mu) = \sum_{r=2}^{L}(\alpha_r + \zeta_r)P^r_{2,2}(\mu) \tag{2.15.66}$$

$$\sum_{r=0}^{L}(\beta_r - \delta_r)P_r(\mu) = \sum_{r=2}^{L}(\alpha_r - \zeta_r)P^r_{2,-2}(\mu). \tag{2.15.67}$$

The associated Legendre polynomial $P^2_r(\mathrm{Cos}\boldsymbol{\Omega})$ can be obtained using Legendre polynomial expansion as given in (2.3.52) and using the form given in (2.3.56) by setting

CHAPTER III:

SOLUTION OF POLARIZED HOMOGENEOUS EQUATION OF TRANSFER FOR COUPLED ATMOSPHERE OCEAN SYSTEM

3.1. Discretization of equation of transfer:

In this section we shall solve the equation of transfer employing a new version of Chandrasekhar's discrete ordinate (DOM). First we rewrite (Dropping "s" in intensity) the homogeneous version of the reduced radiative transfer equations for atmosphere and ocean medium from equations (1.14.9) and (1.14.10).

$$\mu\frac{d}{dz}I_{AT}(z,\mu)+I_{AT}(z,\mu)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}PAT_J^S(z,\mu)\int_{-1}^{+1}\mathbf{P}_J^S(\mu')I_{AT}(z,\mu') \tag{3.1.1}$$

$$\mu\frac{d}{dz}I_{OC}(z,\mu)+I_{OC}(z,\mu)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}POC_J^S(z,\mu)\int_{-1}^{+1}\mathbf{P}_J^S(\mu')I_{OC}(z,\mu') \tag{3.1.2}$$

We shall use the set of separated but coupled non-linear integro differential equations (2.14.1 – 2.14.4) and (2.14.5 – 2.14.8) for each component of stokes vector for discretization. We shall use Chandrasekhar discretization (1960) scheme to break the continuous radiation field into **2N** quadrature directions with corresponding weights keeping optical depth dependence exact, for $i,k=\pm1,\pm2,\pm3,......,\pm N$. This enable us to form a set of n equations for each i (negative as well as positive). Each **homogeneous version of equations** of the set (2.14.1 – 2.14.4 and 2.14.5 – 2.14.8) is replaced by N equivalent equations, separated for positive and negative quadrature directions in the following form, with corresponding Gaussian weight functions expressed and interchanging the Gaussian summation with Fourier summation.

$$w_k=\frac{1}{P'^S(\mu_k)}\int_{-1}^{+1}\frac{P(\mu_k)}{\mu-\mu_k}d\mu. \tag{3.1.3}$$

$$\mu_i\frac{d}{dz}I_{AT}(z,\mu_i)+I_{AT}(z,\mu_i)=\frac{\omega_{AT}}{2}\sum_{J=S}^{M}PAT_J^S(z,\mu_i)\sum_{k=1}^{N}\omega_k(\mathbf{P}_J^S(\mu_k I_{AT}(z,\mu_K)+\mathbf{P}_k^S(-\mu_k)I_{AT}(z,-\mu_k)) \tag{3.1.4}$$

$$-\mu_i\frac{d}{dz}I_{AT}(z,-\mu_i)+I_{AT}(z,-\mu_i)=\frac{\omega_{AT}}{2}\sum_{J=S}^{M}PAT_J^S(z,-\mu_i)\sum_{k=1}^{N}\omega_k(\mathbf{P}_J^S(\mu_{k)}I_{AT}(z,\mu_K)+\mathbf{P}_k^S(-\mu_k)I_{AT}(z,-\mu_k)) \tag{3.1.5}$$

$$\mu_i \frac{d}{dz} I_{OC}(z,\mu_i) + I_{OC}(z,\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} POC_J^S(z,\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k) I_{OC}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{OC}(z,-\mu_k))$$

(3.1.6)

$$-\mu_i \frac{d}{dz} I_{OC}(z,-\mu_i) + I_{OC}(z,-\mu_i) = \frac{\omega_{AT}}{2} \sum_{J=S}^{M} POC_J^S(z,-\mu_i) \sum_{k=1}^{N} \omega_k (\mathbf{P}_J^S(\mu_k) I_{OC}(z,\mu_K) + \mathbf{P}_k^S(-\mu_k) I_{OC}(z,-\mu_k)$$

(3.1.7)

We shall now write down the exact expression explicitly for each stokes components for both the media from equations for "**S**"=**0**

3.2. Directionally Separated and Discretized equations for Atmospheric:

$$+\mu_i \frac{d}{dz} L_{AT}^S(z,+\mu_i) + L_{AT}^S(z,+\mu_i) = \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k)[L_{AT}^S(z,\mu_k) + (-1)^J L_{AT}^S(z,-\mu_k)]$$
$$+ \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k)[Q_{AT}^S(z,\mu_k) + (-1)^J Q_{AT}^S(z,-\mu_k)]$$
$$- \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)[U_{AT}^S(z,\mu_k) + (-1)^J U_{AT}^S(z,-\mu_k)]. \qquad (3.2.1)$$

$$-\mu_i \frac{d}{dz} L_{AT}^S(z,-\mu_i) + L_{AT}^S(z,-\mu_i) = \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k)[(-1)^J L_{AT}^S(z,\mu_k) + L_{AT}^S(z,-\mu_k)]$$
$$+ \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k)[(-1)^J Q_{AT}^S(z,\mu_k) + Q_{AT}^S(z,-\mu_k)]$$
$$- \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \beta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)[(-1)^J U_{AT}^S(z,\mu_k) + U_{AT}^S(z,-\mu_k)]. \qquad (3.2.2)$$

$$+\mu_i \frac{d}{dz} Q_{AT}^S(z,+\mu_i) + Q_{AT}^S(z,+\mu_i) = \frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \gamma_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k)[L_{AT}^S(z,\mu_k) + (-1)^J L_{AT}^S(z,-\mu_k)]$$

$$+ \frac{\omega^{AT}(z)}{2} [\sum_{J=0}^{M} \alpha_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) + \sum_{J=0}^{M} \xi_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)].[Q_{AT}^S(z,\mu_k) + (-1)^J Q_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)+\frac{\omega(z)}{2}\sum_{J=0}^{M}\xi_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)][U_{AT}^S(z,\mu_k)+(-1)^J U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\in_J\ T_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[V_{AT}^S(z,\mu_k)+(-1)^J V_{AT}^S(z,-\mu_k)]. \tag{3.2.3}$$

$$-\mu_i\frac{d}{dz}Q_{AT}^S(z,-\mu_i)+Q_{AT}^S(z,-\mu_i)=\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\gamma_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[(-1)^J L_{AT}^S(z,\mu_k)+L_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)].[(-1)^J Q_{AT}^S(z,\mu_k)+Q_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)][(-1)^J U_{AT}^S(z,\mu_k)+U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\in_J\ T_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[(-1)^J V_{AT}^S(z,\mu_k)+V_{AT}^S(z,-\mu_k)]. \tag{3.2.4}$$

$$+\mu_i\frac{d}{dz}U_{AT}^S(z,+\mu_i)+U_{AT}^S(z,+\mu_i)=-\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\gamma_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu_k)[L_{AT}^S(z,\mu_k)+(-1)^J L_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k R_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J R_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)][Q^{ATS}(z,\mu_k)+(-1)^J Q^{ATS}(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2}[\sum_{J=0}^{M}\alpha_J T_J^0(\mu_i)\sum_{k=1}^{N}w_k T_J^0(\mu_k)+\sum_{J=0}^{M}\xi_J R_J^0(\mu_i)\sum_{k=1}^{N}R_J^0(\mu_k)][U_{AT}^S(z,\mu_k)+(-1)^J U_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2}\sum_{J=0}^{M}\in_J\ R_J^0(\mu_i)\sum_{k=1}^{N}w_k P_J^0(\mu)[V_{AT}^S(z,\mu_k)+(-1)^J V_{AT}^S(z,-\mu_k)]. \tag{3.2.5}$$

$$-\mu_i \frac{d}{dz} U_{AT}^S(z,-\mu_i) + U_{AT}^S(z,-\mu_i) = -\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \gamma_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k) [(-1)^J L_{AT}^S(z,\mu_k) + L_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2} [\sum_{J=0}^{M} \alpha_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) + \sum_{J=0}^{M} \xi_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k)] [(-1)^J Q_{AT}^S(z,\mu_k) + Q_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} [\sum_{J=0}^{M} \alpha_J T_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k) + \sum_{J=0}^{M} \xi_J R_J^0(\mu_i) \sum_{k=1}^{N} R_J^0(\mu_k)] [(-1)^J U_{AT}^S(z,\mu_k) + U_{AT}^S(z,-\mu_k)]$$

$$-\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J R_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu) [(-1)^J V_{AT}^S(z,\mu_k) + V_{AT}^S(z,-\mu_k)]. \quad (3.2.6)$$

$$+\mu_i \frac{d}{dz} V_{AT}^S(z,+\mu_i) + V_{AT}^S(z,+\mu_i) = -\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k) [Q_{AT}^S(z,\mu_k) + (-1)^J Q_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) [U_{AT}^S(z,\mu_k) + (-1)^J U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \delta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k) [V_{AT}^S(z,\mu_k) + (-1)^J V_{AT}^S(z,-\mu_k)]. \quad (3.2.7)$$

$$-\mu_i \frac{d}{dz} V_{AT}^S(z,-\mu_i) + V_{AT}^S(z,-\mu_i) = -\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k) [(-1)^J Q_{AT}^S(z,\mu_k) + Q_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \in_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) [(-1)^J U_{AT}^S(z,\mu_k) + U_{AT}^S(z,-\mu_k)]$$

$$+\frac{\omega^{AT}(z)}{2} \sum_{J=0}^{M} \delta_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k) [(-1)^J V_{AT}^S(z,\mu_k) + V_{AT}^S(z,-\mu_k)]. \quad (3.2.8)$$

Directionally Separated and Discretized equations for Ocean:

$$+\mu_i \frac{d}{dz} L_{OC}^S(z,+\mu_i) = -L_{OC}^S(z,+\mu_i) + \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \beta'_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k P_J^0(\mu_k) [L_{OC}^S(z,\mu_k) + (-1)^J L_{OC}^S(z,-\mu_k)]$$

$$+\frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k R_J^0(\mu_k) [Q_{OC}^S(z,\mu_k) + (-1)^J Q_{OC}^S(z,-\mu_k)]$$

$$-\frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J P_J^0(\mu_i) \sum_{k=1}^{N} w_k T_J^0(\mu_k) [U_{OC}^S(z,\mu_k) + (-1)^J U_{OC}^S(z,-\mu_k)]. \quad (3.2.9)$$

$$-\mu_i \frac{d}{dz} L^S_{OC}(z,-\mu_i) = -L^S_{OC}(z,-\mu_i) + \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \beta'_J P^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[(-1)^J L^S_{OC}(z,\mu_k) + L^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J P^0_J(\mu_i) \sum_{k=1}^{N} w_k R^0_J(\mu_k)[(-1)^J Q^S_{OC}(z,\mu_k) + Q^S_{OC}(z,-\mu_k)]$$

$$- \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J P^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k)[(-1)^J U^S_{OC}(z,\mu_k) + U^S_{OC}(z,-\mu_k)]. \quad (3.2.10)$$

$$+\mu_i \frac{d}{dz} Q^S_{OC}(z,+\mu_i) = -Q^S_{OC}(z,+\mu_i) + \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[L^S_{OC}(z,\mu_k) + (-1)^J L^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k)][Q^S_{OC}(z,\mu_k) + (-1)^J Q^S_{OC}(z,-\mu_k)]$$

$$- \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k R^0_J(\mu_k)][U^S_{OC}(z,\mu_k) + (-1)^{JS} UOC(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \; T^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[V^S_{OC}(z,\mu_k) + (-1)^J V^S_{OC}(z,-\mu_k)]. \quad (3.2.11)$$

$$-\mu_i \frac{d}{dz} Q^S_{OC}(z,-\mu_i) = -Q^S_{OC}(z,-\mu_i) + \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[(-1)^J L^S_{OC}(z,\mu_k) + L^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k)][(-1)^J Q^{OCS}(z,\mu_k) + Q^{OCS}(z,-\mu_k)]$$

$$- \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k R^0_J(\mu_k)][(-1)^J U^S_{OC}(z,\mu_k) + U^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \; T^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[(-1)^J V^S_{OC}(z,\mu_k) + V^S_{OC}(z,-\mu_k)]. \quad (3.2.12)$$

$$+\mu_i \frac{d}{dz} U^S_{OC}(z,+\mu_i) = -U^S_{OC}(z,+\mu_i) - \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[L^S_{OC}(z,\mu_k) + (-1)^J L^S_{OC}(z,-\mu_k)]$$

$$- \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k R^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k)][Q^S_{OC}(z,\mu_k) + (-1)^J Q^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J R^0_J(\mu_i) \sum_{k=1}^{N} R^0_J(\mu_k)][U^S_{OC}(z,\mu_k) + (-1)^J U^S_{OC}(z,-\mu_k)]$$

$$- \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \ R^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu)[V^S_{OC}(z,\mu_k) + (-1)^J V^S_{OC}(z,-\mu_k)]. \qquad (3.2.13)$$

$$-\mu_i \frac{d}{dz} U^S_{OC}(z,-\mu_i) = -U^S_{OC}(z,-\mu_i) - \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \gamma'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[(-1)^J L^S_{OC}(z,\mu_k) + L^S_{OC}(z,-\mu_k)]$$

$$- \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k R^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J R^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k)][(-1)^J Q^S_{OC}(z,\mu_k) + Q^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} [\sum_{J=0}^{M} \alpha'_J T^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k) + \sum_{J=0}^{M} \xi'_J R^0_J(\mu_i) \sum_{k=1}^{N} R^0_J(\mu_k)][(-1)^J U^S_{OC}(z,\mu_k) + U^S_{OC}(z,-\mu_k)]$$

$$- \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \ R^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu)[(-1)^J V^S_{OC}(z,\mu_k) + V^S_{OC}(z,-\mu_k)]. \qquad (3.2.14)$$

$$+\mu_i \frac{d}{dz} V^S_{OC}(z,+\mu_i) = -V^S_{OC}(z,+\mu_i) - \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \ P^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k)[Q^S_{OC}(z,\mu_k) + (-1)^J Q^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \ P^0_J(\mu_i) \sum_{k=1}^{N} w_k R^{0S}_J(\mu_k)[UOC(z,\mu_k) + (-1)^J U^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \delta'_J P^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[V^S_{OC}(z,\mu_k) + (-1)^J V^S_{OC}(z,-\mu_k)]. \qquad (3.2.15)$$

$$-\mu_i \frac{d}{dz} V^S_{OC}(z,-\mu_i) = -V^S_{OC}(z,-\mu_i) - \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \ P^0_J(\mu_i) \sum_{k=1}^{N} w_k T^0_J(\mu_k)[(-1)^J Q^S_{OC}(z,\mu_k) + Q^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \in'_J \ P^0_J(\pm\mu_i) \sum_{k=1}^{N} w_k R^0_J(\mu_k)[(-1)^{JS} UOC(z,\mu_k) + U^S_{OC}(z,-\mu_k)]$$

$$+ \frac{\omega^{OC}(z)}{2} \sum_{J=0}^{M} \delta'_J P^0_J(\mu_i) \sum_{k=1}^{N} w_k P^0_J(\mu_k)[(-1)^J V^S_{OC}(z,\mu_k) + V^S_{OC}(z,-\mu_k)]. \qquad (3.2.16)$$

3.3. Eigen functions and Eigen values:

Hence forth we shall suppress the "**s**" dependence. We shall follow a slightly different approach using compact eigenfunction as follows

$$I_{AT/OC}(z,\pm\mu_i)=\mathbf{H}^{AT/OC}(\gamma,\pm\mu_i)\exp(-\frac{z}{\gamma}) \tag{3.3.1}$$

Using ansatz (3.3.1) in the set (3.2.1 – 3.2.16) we can arrive at the following set of equations for atmosphere and ocean respectively

$$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{AT}(\gamma,\mu_i)=\frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{AT}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{AT}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{AT}(\gamma,-\mu_i)\right] \tag{3.3.2}$$

Or $$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{AT}(\gamma,\mu_i)=\frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{AT}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k} \tag{3.3.3}$$

$$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{AT}(\gamma,-\mu_i)=\frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{AT}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{AT}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{AT}(\gamma,-\mu_i)\right] \tag{3.3.4}$$

Or $$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{AT}(\gamma,-\mu_i)=\frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{AT}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k} \tag{3.3.5}$$

$$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{OC}(\gamma,\mu_i)=\frac{\omega_{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{OC}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{OC}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{OC}(\gamma,-\mu_i)\right] \tag{3.3.6}$$

Or $$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{OC}(\gamma,\mu_i)=\frac{\omega_{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{OC}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{OC} \tag{3.3.7}$$

$$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{OC}(\gamma,-\mu_i)=\frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{OC}\sum_{k=1}^{N}\omega_k\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{OC}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{OC}(\gamma,-\mu_i)\right] \tag{3.3.8}$$

Or $$(1+\frac{\mu_i}{\gamma})\mathbf{D}\mathbf{H}^{OC}(\gamma,-\mu_i)=\frac{\omega_{AT}(z)}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{OC}\sum_{K=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{OC} \tag{3.3.9}$$

We have defined $\mathbf{D}=\mathbf{diag}\{1,1,-1,-1\}$ and for $\mathbf{i}=1,2,3,.....,N$

$$\mathbf{\Psi}_{j,k}^{AT}=\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{AT}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{AT}(\gamma,-\mu_i)\right] \tag{3.3.10}$$

$$\mathbf{\Psi}_{j,k}^{OC}=\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{OC}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{OC}(\gamma,-\mu_i)\right] \tag{3.3.11}$$

However, the set of equations (3.3.2 –3.3.11) can be written in an equivalent but more elegant form by introducing following (4Nx1) vectors

$$\mathbf{H}_{+}^{AT/OC}(\gamma)=\left[\mathbf{H}^{AT/OC}(\gamma,\mu_1)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_2)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_3)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_4)^T,....,\mathbf{H}^{AT/OC}(\gamma,\mu_N)^T\right]^T \tag{3.3.12}$$

$$\mathbf{H}_{-}^{AT/OC}(\gamma)=\left[\mathbf{H}^{AT/OC}(\gamma,-\mu_1)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_2)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_3)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_4)^T,...,\mathbf{H}^{AT/OC}(\gamma,-\mu_N)^T\right] \tag{3.3.13}$$

Here each $\mathbf{H}^{AT/OC}(\gamma,\mu_i)^T$ is (4x1) vector. We now define

$$\mathbf{\Sigma}=\mathbf{diag}(\omega_1\mathbf{\Gamma},\omega_2\mathbf{\Gamma},\omega_3\mathbf{\Gamma},......,\omega_N\mathbf{\Gamma}) \tag{3.3.14}$$ $$\mathbf{X}=\mathbf{diag}(\mu_1\mathbf{\Gamma},\mu_2\mathbf{\Gamma},\mu_3\mathbf{\Gamma},.......,\mu_N\mathbf{\Gamma}) \tag{3.3.15}$$

With $\mathbf{\Gamma}=\mathbf{diagonal(1,1,1,1)}$

$$\mathbf{H}^{AT/OC}(\gamma,\pm\mu_i)=\left[\varphi(\gamma,\pm\mu_i)\quad\psi(\gamma,\pm\mu_i)\quad\theta(\gamma,\pm\mu_i)\quad\not\in(\gamma,\pm\mu_i)\right]^T \tag{3.3.16}$$

The matrices defined in (3.3.14) and (3.3.15) are of order (4Nx4N) but (3.3.16) is of order (Nx4). Using this formalism one can easily verify that the set (3.3.2 – 3.3.11) for each may be written compactly as follows.

$$\left(\mathbf{I}-\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{+}^{AT}(\gamma)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}\mathbf{T}_J^S(AT,\gamma) \tag{3.3.17}$$

$$\left(\mathbf{I}+\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{-}^{AT}(\gamma)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{AT}\mathbf{T}_J^S(AT,\gamma) \tag{3.3.18}$$

$$\left(\mathbf{I} - \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{+}^{OC}(\gamma) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{T}_J^{S}(OC,\gamma) \tag{3.3.19}$$

$$\left(\mathbf{I} + \frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_{-}^{OC}(\gamma) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{T}_J^{S}(OC,\gamma) \tag{3.3.20}$$

Here $\mathbf{I}$ is a (4Nx4N) identity matrix and (4Nx4) matrix $\mathbf{\Pi}(J,S)$ is given below.

$$\mathbf{\Pi}(J,S) = \left[\mathbf{P}_J^S(\mu_1), \mathbf{P}_J^S(\mu_2), \mathbf{P}_J^S(\mu_3), \ldots, \mathbf{P}_J^S(\mu_N)\right]^T \tag{3.3.21}$$

$$\mathbf{T}_J^S(AT/OC,\gamma) = \mathbf{\Pi}^T(J,S)\mathbf{\Sigma}\mathbf{H}_{+}^{AT/OC}(\gamma) + (-1)^{J-S}D\mathbf{\Pi}^T(J,S)\mathbf{\Sigma}\mathbf{H}_{-}^{AT/OC}(\gamma) \tag{3.3.22}$$

Following Siewert's [163] approach we shall now derive equivalent set of relations by defining the following vectors for atmosphere and ocean

$$\mathbf{NA}^{AT} = \mathbf{H}_{+}^{AT}(\gamma) + \mathbf{H}_{-}^{AT}(\gamma) \; ; \; \mathbf{NB}^{AT} = \mathbf{H}_{+}^{AT}(\gamma) - \mathbf{H}_{-}^{AT}(\gamma) \tag{3.3.23}$$

$$\mathbf{NA}^{OC} = \mathbf{H}_{+}^{OC}(\gamma) + \mathbf{H}_{-}^{OC}(\gamma) \; ; \; \mathbf{NB}^{OC} = \mathbf{H}_{+}^{OC}(\gamma) - \mathbf{H}_{-}^{OC}(\gamma) \tag{3.3.24}$$

Taking sum and difference of (3.3.17) & (3.3.18) and (3.3.19) & (3.3.20) respectively for atmosphere and ocean it is easy to derive

$$\mathbf{A}^{AT}\mathbf{X}^{AT} = \frac{1}{\gamma}\mathbf{Y}^{AT}; \quad (3.3.25) \quad \mathbf{B}^{AT}\mathbf{Y}^{AT} = \frac{1}{\gamma}\mathbf{X}^{AT}. \tag{3.3.26}$$

$$\mathbf{A}^{OC}\mathbf{X}^{OC} = \frac{1}{\gamma}\mathbf{Y}^{OC}; \quad (3.3.27) \quad \mathbf{B}^{OC}\mathbf{Y}^{OC} = \frac{1}{\gamma}\mathbf{X}^{OC}. \tag{3.3.28}$$

Where

$$\mathbf{A}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}[I + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \tag{3.3.29}$$

$$\mathbf{B}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \tag{3.3.30}$$

$$\mathbf{A}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}[I + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \tag{3.3.31}$$

$$\mathbf{B}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}. \tag{3.3.32}$$

We also have

$$\mathbf{X}^{AT} = \mathbf{X}\ [\mathbf{NA}^{AT}] \qquad (3.3.33) \qquad\qquad \mathbf{Y}^{AT} = \mathbf{X}\ [\mathbf{NB}^{AT}]. \qquad (3.3.34)$$

$$\mathbf{X}^{OC} = \mathbf{X}\ [\mathbf{NA}^{OC}]; \qquad (3.3.35) \qquad\qquad \mathbf{Y}^{OC} = \mathbf{X}\ [\mathbf{NB}^{OC}]. \qquad (3.3.36)$$

It is to be noted that equation (3.3.29)-(3.3.32) depend on optical depth of the medium. This makes the matrix A and B sensitive to optical depth .This is the difference between homogeneous and inhomogeneous atmosphere in this approach. In general matrix A and B are not symmetric. We now use equations (3.3.25) & (3.3.26) and (3.3.27) & (3.3.28) to get

$$(\mathbf{B}^{AT}\mathbf{A}^{AT})\mathbf{X}^{AT} = \Im^{AT}\mathbf{X}^{AT}; \qquad (3.3.37) \qquad\qquad (\mathbf{B}^{OC}\mathbf{A}^{OC})\mathbf{X}^{OC} = \Im^{OC}\mathbf{X}^{OC} \qquad (3.3.38)$$

$$(\mathbf{A}^{AT}\mathbf{B}^{AT})\mathbf{Y}^{AT} = \Im_{AT}\mathbf{Y}^{AT}; \qquad (3.3.39) \qquad\qquad (\mathbf{A}^{OC}\mathbf{B}^{OC})\mathbf{Y}^{OC} = \Im_{OC}\mathbf{Y}^{OC} \qquad (3.3.40)$$

Our task is now to evaluate the eigenvalues $\Im_{AT},\Im_{OC}$ which will determine the separation constants γ_{AT},γ_{OC} for atmosphere and ocean respectively. Separation constants occur in plus-minus pairs. It is to be noted that the eigenvalues may be complex. We can easily see that

$$\Im = \frac{1}{\gamma^2} \qquad (3.3.41)$$

We shall first show some explicit numerical calculation for **A** and **B** (Both for atmosphere and ocean) matrix for the following set of values of **S** and **N**.
Let us set **S=0, M=2** and **N=2**, which implies **k=1, 2**.

$$\mathbf{A}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{AT}[I+(-1)^{J-0}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \qquad (3.3.42)$$

$$\mathbf{B}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{AT}[I-(-1)^{J-0}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \qquad (3.3.43)$$

$$\mathbf{A}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{OC}[I+(-1)^{J-0}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \qquad (3.3.44)$$

$$\mathbf{B}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}(z)}{2}\sum_{J=0}^{2}\mathbf{\Pi}(J,0)B_J^{OC}[I-(-1)^{J-S}D]\mathbf{\Pi}(J,0)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1} \qquad (3.3.45)$$

$$\mathbf{\Pi}(J,0) = \left[\mathbf{P}_J^0(\mu_1),\mathbf{P}_J^0(\mu_2),\mathbf{P}_J^0(\mu_3),....,\mathbf{P}_J^0(\mu_N)\right]^T \qquad (3.3.46)$$

For N=2, (Four stream approximation) we have the quadrature points given below

$$\mu_{\pm 1} = \pm 0.3399810; \quad \mu_{\pm 2} = \pm 0.8611363; \qquad (3.3.47)$$

Corresponding weights are given by $\omega_1 = \omega_{-1} = 0.6521452; \ \omega_2 = \omega_{-2} = 0.3478548.$ (3.3.48)

Hence we have from equation (3.3.10 – 3.3.11) & (3.3.12 – 3.3.13) the matrices given in the set (3.3.46)

$$\mathbf{\Pi}(0,0) = \left[\text{diag}\{1,0,0,1\}, \text{diag}\{1,0,0,1\}\right]^{T}; \tag{3.3.49}$$

$$\mathbf{\Pi}(1,0) = \left[\text{diag}\{\mu_1,0,0,\mu_1\}, \text{diag}\{\mu_2,0,0,\mu_2\}\right]^{T}; \tag{3.3.50}$$

$$\mathbf{\Pi}(2,0) = \left[\mathbf{P}_2^0(\mu_1), \mathbf{P}_2^0(\mu_2)\right]^{T}; \tag{3.3.51}$$

The last matrices explicitly look as below

$$\mathbf{\Pi}(2,0) = \left[\begin{pmatrix} -0.3266 & 0 & 0 & 0 \\ 0 & 0.5416 & 0 & 0 \\ 0 & 0 & 0.5416 & 0 \\ 0 & 0 & 0 & -0.3266 \end{pmatrix}, \begin{pmatrix} 0.6123 & 0 & 0 & 0 \\ 0 & 0.1583 & 0 & 0 \\ 0 & 0 & 0.1583 & 0 \\ 0 & 0 & 0 & 0.6123 \end{pmatrix}\right]^{T} \tag{3.3.52}$$

The numerical values corresponding to A-matrix and B-matrix have been calculated correct up to six decimal places from the set of equations (3.3.42 – 3.3.45). In using equation we have used values of the basic constants for **oblate spheroids particle** for atmosphere and suspended spherical particle for ocean. The values are given in the following table. **Table(2)** contains the values of the basic constants as measured by Wauben WMF and Hovenier JW [376] for scattering in the atmosphere by oblate spheroids with aspect ration 1.999987, size parameter 3 and index of refraction 1.53-0.006i. In **table (1)** we have tabulated the values of the basic constants for spherical particles. We shall use these data for ocean water. These values are taken from a problem collected and calculated by Vestrucci and Siewert [99] considering mie scattering of light, with wavelength 0.951 micro meter having gamma distribution with effective radius 0.2 micro meters, effective variance 0.07 and refractive index 1.34.

Table 1: Basic constants for spherical particle in ocean

l	α_l	β_l	γ_l	δ_l	ε_l	ζ_l
0	0.0	1.0	0.0	0.7120634246	0.0	0.0
1	0.0	1.4552931819	0.0	1.7601411931	0.0	0.0
2	3.3091220464	1.0540263128	− 0.7552491518	1.0668243107	0.0420726875	2.5773207443
3	0.9633758276	0.3975899378	− 0.3619934319	0.3965110389	0.0850671555	0.7574437604
4	0.2474124256	0.1165930161	− 0.1155748816	0.0957641237	0.0154318420	0.1638177665
5	0.0452636955	0.0238747702	− 0.0249815879	0.0176508810	0.0031534874	0.0278314781
6	0.0068892608	0.0039501033	− 0.0041675362	0.0026154886	0.0004010299	0.0038897248
7	0.0008798202	0.0005388807	− 0.0005739043	0.0003271332	0.0000460147	0.0004642654
8	0.0000987255	0.0000637172	− 0.0000677900	0.0000358314	0.0000042875	0.0000490224
9	0.0000099029	0.0000066697	− 0.0000070926	0.0000035142	0.0000003617	0.0000046647
10	0.0000009071	0.0000006329	− 0.0000006712	0.0000003148	0.0000000272	0.0000004075
11	0.0000000769	0.0000000553	− 0.0000000585	0.0000000261	0.0000000019	0.0000000331
12	0.0000000061	0.0000000045	− 0.0000000047	0.0000000020	0.0000000001	0.0000000025
13	0.0000000005	0.0000000003	− 0.0000000004	0.0000000001	0.0000000000	0.0000000002

Table 2: Basic Constants for oblate spheroid particle in atmosphere

l	α_l	β_l	γ_l	δ_l	ε_l	ζ_l
0	0.0	1.0	0.0	0.915207	0.0	0.0
1	0.0	2.104031	0.0	2.095727	0.0	0.0
2	3.726079	2.095158	−0.116688	2.008624	0.065456	3.615946
3	2.202868	1.414939	−0.209370	1.436545	0.221658	2.240516
4	1.190694	0.703593	−0.227137	0.706244	0.097752	1.139473
5	0.391203	0.235001	−0.144524	0.238475	0.052458	0.365605
6	0.105556	0.064039	−0.052640	0.056448	0.009239	0.082779
7	0.020484	0.012837	−0.012400	0.009703	0.001411	0.013649
8	0.003097	0.002010	−0.002093	0.001267	0.000133	0.001721
9	0.000366	0.000246	−0.000267	0.000130	0.000011	0.000172
10	0.000035	0.000024	−0.000027	0.000011	0.000001	0.000014
11	0.000003	0.000002	−0.000002	0.000001	0.000000	0.000001

In the following we have shown some sample results for the values of the eigenvectors corresponding to atmosphere and ocean with constant value 0.5 for albedo function in both the media. However in actual calculations these matrices are optical depth sensitive.

$$
\mathbf{A}^{AT} = \begin{pmatrix}
1.767907 & -0.019796 & 0 & 0 & -0.290508 & -0.003016 & 0 & 0 \\
-0.019796 & 1.893081 & 0 & 0 & 0.019348 & -0.159728 & 0 & 0 \\
0 & 0 & 2.941341 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.709012 & 0 & 0 & 0 & -0.306783 \\
-0.557247 & 0.037113 & 0 & 0 & 1.982013 & 0.005655 & 0 & 0 \\
-0.005786 & -0.306388 & 0 & 0 & 0.005655 & 2.828077 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2.874763 & 0 \\
0 & 0 & 0 & -0.588465 & 0 & 0 & 0 & 2.097714
\end{pmatrix} \tag{3.3.53}
$$

$$
\mathbf{B}^{AT} = \begin{pmatrix}
2.708091 & 0 & 0 & 0 & -0.307999 & 0 & 0 & 0 \\
0 & 2.941341 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.924065 & -0.011105 & 0 & 0 & -0.155007 & 0.010853 \\
0 & 0 & 0.011105 & 1.858084 & 0 & 0 & 0.001692 & -0.256764 \\
-0.590797 & 0 & 0 & 0 & 2.094635 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.874763 & 0 & 0 \\
0 & 0 & -0.297332 & -0.003246 & 0 & 0 & 2.829457 & 0.003172 \\
0 & 0 & -0.020819 & -0.492520 & 0 & 0 & -0.003172 & 2.040631
\end{pmatrix} \tag{3.3.54}
$$

$$
\mathbf{A}^{OC}=\begin{pmatrix}
1.874419 & -0.128128 & 0 & 0 & -0.394609 & -0.019523 & 0 & 0\\
-0.128128 & 2.010384 & 0 & 0 & 0.125228 & 0.141854 & 0 & 0\\
0 & 0 & 2.941341 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2.746214 & 0 & 0 & 0 & -0.257658\\
-0.756932 & 0.240210 & 0 & 0 & 2.177180 & 0.036602 & 0 & 0\\
-0.037450 & -0.272102 & 0 & 0 & 0.036602 & 2.833301 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2.874763 & 0\\
0 & 0 & 0 & -0.494235 & 0 & 0 & 0 & 2.222141
\end{pmatrix}
$$

(3.3.55)

$$
\mathbf{B}^{OC}=\begin{pmatrix}
2.780009 & 0 & 0 & 0 & -0.213033 & 0 & 0 & 0\\
0 & 2.941341 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2.216262 & -0.007134 & 0 & 0 & -0.110484 & 0.006976\\
0 & 0 & 0.007138 & 2.149267 & 0 & 0 & 0.001088 & -0.249361\\
-0.408636 & 0 & 0 & 0 & 2.335172 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 2.874763 & 0 & 0\\
0 & 0 & -0.211928 & -0.002086 & 0 & 0 & 2.842470 & 0.002039\\
0 & 0 & -0.013381 & -0.478320 & 0 & 0 & -0.002039 & 2.318749
\end{pmatrix}
$$

(3.3.56)

3.4. Calculation of eigenvectors and eigen values from Eq. (3.3.41):

The eigenvectors and eigen values are now calculated for the present simple example. The eigenvectors (equations (3.3.33) and (3.3.36)) are column matrices (8x1) and eight in number one each for each of eight eigen values given diagonally in (3.3.37 – 3.3.40) for atmosphere and ocean respectively. The following calculated values for eigenvectors and eigen values are not optical depth sensitive but in actual case of our interest these eigenvectors and eigen values are optical depth sensitive.

$$\mathbf{X}^{AT}(\gamma) =$$

$$\begin{pmatrix}
0.552953 & 0.686688 & 0.074502 & 0 & -0.000777 & 0 & 0 & 0 \\
-0.005923 & -0.078648 & 0.946217 & 0 & 0.170586 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.002499 & 0 & 0.947002 & 0.042368 & -0.167458 \\
0 & 0 & 0 & 0.467828 & 0 & -0.041798 & -0.636458 & -0.000438 \\
0.833190 & -0.722172 & -0.070098 & 0 & 0.000896 & 0 & 0 & 0 \\
-0.001813 & -0.027227 & 0.306934 & 0 & -0.985342 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.000785 & 0 & 0.315756 & 0.015138 & 0.985879 \\
0 & 0 & 0 & 0.883815 & 0 & 0.041684 & 0.769998 & 0.000617
\end{pmatrix}$$

(3.4.1)

$$\Im_j^{AT} = \begin{pmatrix}
2.854745 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6.436508 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5.406996 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.090047 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8.282509 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5.513514 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6.523637 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8.282575
\end{pmatrix}$$

(3.4.2)

Corresponding separation constants for atmosphere and ocean (equation (3.4.4) and (3.4.6), are calculated in plus-minus pairs using (3.3.41).

$$\sqrt{\Im_j^{AT}} = \begin{pmatrix} 1.689599 & 2.537027 & 2.325295 & 1.757853 & 2.877935 & 2.348088 & 2.554141 & 2.877946 \end{pmatrix}$$

(3.4.3)

$$\gamma_j^{AT} = (\pm 0.591856 \quad \pm 0.394162 \quad \pm 0.430053 \quad \pm 0.568876 \quad \pm 0.347471 \quad \pm 0.425879 \quad \pm 0.391521 \quad \pm 0.3474$$

(3.4.4)

$$\mathbf{X}^{OC} =$$

$$\begin{pmatrix} 0.600968 & 0.257087 & -0.596022 & 0 & 0.012462 & 0 & 0 & 0 \\ -0.027447 & 0.881679 & 0.278474 & 0 & 0.177679 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.001788 & 0 & 0.942106 & -0.029407 & 0.177103 \\ 0 & 0 & 0 & -0.472312 & 0 & -0.029146 & 0.643225 & 0.000801 \\ 0.798757 & -0.271465 & 0.743344 & 0 & 0.015997 & 0 & 0 & 0 \\ -0.008454 & 0.287845 & 0.121035 & 0 & 0.983880 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.000570 & 0 & 0.332797 & -0.012205 & -0.984191 \\ 0 & 0 & 0 & -0.881430 & 0 & 0.028848 & -0.765015 & -0.001055 \end{pmatrix}$$

(3.4.5)

$$\Im_j^{OC} = \begin{pmatrix} 3.317108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.553703 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7.521719 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.957956 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.283257 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.407828 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7.342237 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8.283609 \end{pmatrix}$$

(3.4.6)

$$\sqrt{\Im_j^{OC}} = \begin{pmatrix} 1.821293 & 2.356630 & 2.742575 & 1.989461 & 2.878065 & 2.531369 & 2.709656 & 2.878126 \end{pmatrix}$$

(3.4.7)

$$\gamma_j^{OC} = \begin{pmatrix} 0.549060 & 0.424335 & 0.364621 & 0.502649 & 0.347456 & 0.395043 & 0.369050 & 0.347448 \end{pmatrix}$$

(3.4.8)

It is to be noted that the eigenvectors and eigenvalues are real for **N=2**. But in general we have found that both the eigenvector and eigenvalues may be real as well as complex.

3.5. The expression for eigenvectors:

We are now in a position to establish the eigen functions either from equations (3.3.33 – 3.3.34) or (3.3.35 – 3.3.36). Let us chose equation (3.3.33 – 3.3.34). Let $\Im_j^{AT/OC}$ and $\mathbf{X}^{AT/OC}(\Im_j^{AT/OC})$ denote the eigen-values and corresponding eigenvectors respectively for $j = 1,2,3,.....,4N$. Using equations (3.3.33 – 3.3.34), (3.3.35 – 3.3.36) with (3.3.17 – 3.3.18) we can easily deduce

$$\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC}) = \frac{1}{2}\mathbf{X}^{-1}\left(\mathbf{I} \pm \gamma_j^{AT/OC}\mathbf{A}^{AT/OC}\right)\mathbf{X}^{AT/OC}(\lambda_j^{AT/OC}) \tag{3.5.1}$$

We also note that $\mathbf{H}_{+}^{AT/OC}(-\gamma_j^{AT/OC}) = \mathbf{H}_{-}^{AT/OC}(\gamma_j^{AT/OC})$ for $j = 1,2,3,.....,4N$. We shall now show some values of eigenvectors found numerically (corrected up to six decimal places) using the set of matrices (3.3.12 – 3.3.13). For **N=2**, and constant albedo function (set at value 0.5) each eigenvector is a (8x1) matrix. It is to be noted that For **N=2** we have eight eigenvector corresponding to eight separation constants (Eigen values). These atmospheric as well as the oceanic eigenvectors are also optical depth sensitive for inhomogeneous media.

$$\mathbf{H}_{+}^{AT}(\gamma_1^{AT}) = \begin{pmatrix} -3.901731 \\ 0.032845 \\ 0 \\ 0 \\ -6.708547 \\ 0.009994 \\ 0 \\ 0 \end{pmatrix}; \quad \mathbf{H}_{+}^{AT}(\gamma_2^{AT}) = \begin{pmatrix} -14.503045 \\ 1.744800 \\ 0 \\ 0 \\ 17.849178 \\ 0.603121 \\ 0 \\ 0 \end{pmatrix}; \quad \mathbf{H}_{+}^{AT}(\gamma_3^{AT}) = \begin{pmatrix} -1.162563 \\ -15.223190 \\ 0 \\ 0 \\ 1.216799 \\ -4.927890 \\ 0 \\ 0 \end{pmatrix};$$

$$\mathbf{H}_{+}^{AT}(\gamma_4^{AT}) = \begin{pmatrix} 0.025969 \\ -6.101985 \\ 0 \\ 0 \\ -0.036609 \\ 35.213518 \\ 0 \\ 0 \end{pmatrix}; \quad \mathbf{H}_{+}^{AT}(\gamma_5^{AT}) = \begin{pmatrix} 0 \\ 0 \\ 0.037083 \\ -5.215253 \\ 0 \\ 0 \\ 0.011154 \\ -8.282254 \end{pmatrix}; \quad \mathbf{H}_{+}^{AT}(\gamma_6^{AT}) = \begin{pmatrix} 0 \\ 0 \\ 23.978775 \\ -1.083295 \\ 0 \\ 0 \\ 7.647594 \\ 0.947807 \end{pmatrix};$$

$$\mathbf{H}_{+}^{AT}(\gamma_7^{AT}) = \begin{pmatrix} 0 \\ 0 \\ 1.257926 \\ -19.744291 \\ 0 \\ 0 \\ 0.429838 \\ 19.764744 \end{pmatrix}; \quad \mathbf{H}_{+}^{AT}(\gamma_8^{AT}) = \begin{pmatrix} 0 \\ 0 \\ 6.246016 \\ 0.017410 \\ 0 \\ 0 \\ -35.158474 \\ -0.019357 \end{pmatrix}. \tag{3.5.2}$$

$$\mathbf{H}_{-}^{AT}(\gamma_1^{AT}) = \begin{pmatrix} 2.275308 \\ -0.015424 \\ 0 \\ 0 \\ 4.313325 \\ -0.004782 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{-}^{AT}(\gamma_2^{AT}) = \begin{pmatrix} 12.483262 \\ -1.513470 \\ 0 \\ 0 \\ -15.773103 \\ -0.524850 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{-}^{AT}(\gamma_3^{AT}) = \begin{pmatrix} 0.943426 \\ 12.440043 \\ 0 \\ 0 \\ -1.015283 \\ 4.045526 \\ 0 \\ 0 \end{pmatrix};$$

$$\mathbf{H}_{-}^{AT}(\gamma_4^{AT}) = \begin{pmatrix} -0.023684 \\ 5.600232 \\ 0 \\ 0 \\ 0.034034 \\ -32.380893 \\ 0 \\ 0 \end{pmatrix}; \mathbf{H}_{-}^{AT}(\gamma_5^{AT}) = \begin{pmatrix} 0 \\ 0 \\ -0.029731 \\ 3.839210 \\ 0 \\ 0 \\ -0.008897 \\ 5.741494 \end{pmatrix}; \mathbf{H}_{-}^{AT}(\gamma_6^{AT}) = \begin{pmatrix} 0 \\ 0 \\ -0.029731 \\ 3.839210 \\ 0 \\ 0 \\ -0.008897 \\ 5.741494 \end{pmatrix};$$

$$\mathbf{H}_{-}^{AT}(\gamma_7^{AT})=\begin{pmatrix}0\\0\\-1.133307\\17.872250\\0\\0\\-0.386319\\-17.551184\end{pmatrix};\ \mathbf{H}_{-}^{AT}(\gamma_8^{AT})=\begin{pmatrix}0\\0\\-5.753465\\-0.016121\\0\\0\\32.324306\\0.017584\end{pmatrix}. \tag{3.5.3}$$

$$\mathbf{H}_{+}^{OC}(\gamma_1^{OC})=\begin{pmatrix}-4.305306\\0.170320\\0\\0\\-6.389100\\0.052181\\0\\0\end{pmatrix};\ \mathbf{H}_{+}^{OC}(\gamma_2^{OC})=\begin{pmatrix}-4.831111\\-17.055099\\0\\0\\5.601706\\-5.558499\\0\\0\end{pmatrix};\ \mathbf{H}_{+}^{OC}(\gamma_3^{OC})=\begin{pmatrix}-12.395444\\6.072300\\0\\0\\17.707087\\2.635219\\0\\0\end{pmatrix};$$

$$\mathbf{H}_{+}^{OC}(\gamma_4^{OC})=\begin{pmatrix}0.423068\\-6.356229\\0\\0\\-0.629308\\35.164305\\0\\0\end{pmatrix};\ \mathbf{H}_{+}^{OC}(\gamma_5^{OC})=\begin{pmatrix}0\\0\\-0.026533\\5.556992\\0\\0\\-0.008093\\8.929680\end{pmatrix};\ \mathbf{H}_{+}^{OC}(\gamma_6^{OC})=\begin{pmatrix}0\\0\\23.854804\\-0.752152\\0\\0\\8.060328\\0.663645\end{pmatrix};$$

$$\mathbf{H}_{+}^{OC}(\gamma_7^{OC})=\begin{pmatrix}0\\0\\0.873107\\-19.784462\\0\\0\\0.346546\\20.021149\end{pmatrix};\mathbf{H}_{+}^{OC}(\gamma_{8;}^{OC})=\begin{pmatrix}0\\0\\-6.605761\\-0.031300\\0\\0\\35.098295\\0.034141\end{pmatrix}. \tag{3.5.4}$$

$$\mathbf{H}_{-}^{OC}(\gamma_1^{OC})=\begin{pmatrix}2.537654\\-0.089589\\0\\0\\4.092864\\-0.027878\\0\\0\end{pmatrix};\mathbf{H}_{-}^{OC}(\gamma_2^{OC})=\begin{pmatrix}4.074931\\14.461782\\0\\0\\-4.821307\\4.731007\\0\\0\end{pmatrix};\mathbf{H}_{-}^{OC}(\gamma_3^{OC})=\begin{pmatrix}10.642340\\-5.253213\\0\\0\\-15.570149\\-2.287271\\0\\0\end{pmatrix};$$

$$\mathbf{H}_{-}^{OC}(\gamma_4^{OC})=\begin{pmatrix}-0.386414\\5.833615\\0\\0\\0.583321\\-32.335884\\0\\0\end{pmatrix};\mathbf{H}_{-}^{OC}(\gamma_5^{OC})=\begin{pmatrix}0\\0\\0.021273\\-4.167762\\0\\0\\0.006455\\-6.395780\end{pmatrix};\mathbf{H}_{-}^{OC}(\gamma_6^{OC})=\begin{pmatrix}0\\0\\-21.083748\\0.666423\\0\\0\\-7.103615\\-0.580715\end{pmatrix};$$

$$\mathbf{H}_{-}^{OC}(\gamma_7^{OC}) = \begin{pmatrix} 0 \\ 0 \\ -0.786610 \\ 17.892518 \\ 0 \\ 0 \\ -0.311460 \\ -17.821912 \end{pmatrix}; \mathbf{H}_{-}^{OC}(\gamma_8^{OC}) = \begin{pmatrix} 0 \\ 0 \\ 6.084840 \\ 0.028943 \\ 0 \\ 0 \\ -32.268979 \\ -0.031108 \end{pmatrix}; \quad (3.5.5)$$

The above structure of the eigenvectors is due to our choice of the intensity vector representation (2.11.1) and (2.11.2).

We now have all that we require to write the solution of the homogeneous RTE (3.1.1 – 3.1.2) for both atmosphere and ocean. We let a (4Nx1) matrix as

$$\mathbf{H}_{AT/OC}(\pm) = \left[\mathbf{I}_{AT/OC}(z,\pm\mu_1)^T, \mathbf{I}_{AT/OC}(z,\pm\mu_2)^T, \mathbf{I}_{AT/OC}(z,\pm\mu_3)^T, \ldots\ldots, \mathbf{I}_{AT/OC}(z,\pm\mu_N)^T\right]^T \quad (3.5.6)$$

The homogeneous solution can now be written as

$$\mathbf{H}_{AT}(+) = \sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_w - z}{\gamma_j^{AT}}\right) \quad (3.5.7a)$$

$$\mathbf{H}_{OC}(+) = \sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_{+}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_{-}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right) \quad (3.5.7b)$$

$$\mathbf{H}_{AT}(-) = \mathbf{\Delta}\sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right) \quad (3.5.8a)$$

$$\mathbf{H}_{OC}(-) = \mathbf{\Delta}\sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_{-}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_{+}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right) \quad (3.5.8b)$$

$\mathbf{\Delta} = \mathrm{diag}\{D, D, D, \ldots, D\}$ (2.5.9) is (**4Nx4N**) matrix.

Our new task is to find the arbitrary coefficients A and B. These coefficients are to be found from the boundary conditions. Equations (2.15.47 – 2.15.49) & (2.15.50 – 2.15.52) can be evaluated for each $\pm\mu_i$ which give **4N** equations for **4N** unknown coefficients A and B.

As for example for **N=2** we have

$$\mathbf{H}_{AT/OC}(\pm) = \left[\mathbf{I}_{AT/OC}(z,\pm\mu_1)^T, \mathbf{I}_{AT/OC}(z,\pm\mu_2)^T\right]^T \quad (3.5.10)$$

These matrices are (8x1) in dimension.

From (3.5.7) & (3.5.8) and (3.5.6) we can write the form of the equations for positive eigenvectors

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}_+ = A_1 \begin{bmatrix} a1 \\ a2 \\ a3 \\ a4 \\ a5 \\ a6 \\ a7 \\ a8 \end{bmatrix}_+ j1 + A_2 \begin{bmatrix} b1 \\ b2 \\ b3 \\ b4 \\ b5 \\ b6 \\ b7 \\ b8 \end{bmatrix}_+ j2 + A_3 \begin{bmatrix} c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ c7 \\ c8 \end{bmatrix}_+ j3 + A_4 \begin{bmatrix} d1 \\ d2 \\ d3 \\ d4 \\ d5 \\ d6 \\ d7 \\ d8 \end{bmatrix}_+ j4 + A_5 \begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \\ e5 \\ e6 \\ e7 \\ e8 \end{bmatrix}_+ j5 + A_6 \begin{bmatrix} f1 \\ f2 \\ f3 \\ f4 \\ f5 \\ f6 \\ f7 \\ f8 \end{bmatrix}_+ j6 + A_7 \begin{bmatrix} g1 \\ g2 \\ g3 \\ g4 \\ g5 \\ g6 \\ g7 \\ g8 \end{bmatrix}_+ j7 + A_8 \begin{bmatrix} h1 \\ h2 \\ h3 \\ h4 \\ h5 \\ h6 \\ h7 \\ h8 \end{bmatrix}_+ j$$

$$+ B_1 \begin{bmatrix} a1 \\ a2 \\ a3 \\ a4 \\ a5 \\ a6 \\ a7 \\ a8 \end{bmatrix}_- j9 + B_2 \begin{bmatrix} b1 \\ b2 \\ b3 \\ b4 \\ b5 \\ b6 \\ b7 \\ b8 \end{bmatrix}_- j10 + B_3 \begin{bmatrix} c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ c7 \\ c8 \end{bmatrix}_- j11 + B_4 \begin{bmatrix} d1 \\ d2 \\ d3 \\ d4 \\ d5 \\ d6 \\ d7 \\ d8 \end{bmatrix}_- j12 + B_5 \begin{bmatrix} e1 \\ e2 \\ e3 \\ e4 \\ e5 \\ e6 \\ e7 \\ e8 \end{bmatrix}_- j13 + B_6 \begin{bmatrix} f1 \\ f2 \\ f3 \\ f4 \\ f5 \\ f6 \\ f7 \\ f8 \end{bmatrix}_- j14 + B_7 \begin{bmatrix} g1 \\ g2 \\ g3 \\ g4 \\ g5 \\ g6 \\ g7 \\ g8 \end{bmatrix}_- j15 + B_8 \begin{bmatrix} h1 \\ h2 \\ h3 \\ h4 \\ h5 \\ h6 \\ h7 \\ h8 \end{bmatrix}_- j16$$

(3.5.11)

In equation (3.5.11) RHS represent the values of the Stokes components. The first four values i.e. a, b, c, d are for μ_1 whereas e, f, g, h represent values for μ_2. In LHS the set {a1, a2, a3, a4, a5, a6, a7, a8} represents values of the eigenvector calculated from equation (3.5.1) for **j=1** and similarly for other values of j for appropriate cases. The suffix plus and minus represent the values corresponding for positive and negative eigenvectors. We also have for negative eigenvectors,

$$
\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h \end{bmatrix}_{-}
= \mathbf{\Delta}[A_1 \begin{bmatrix} a1\\ a2\\ a3\\ a4\\ a5\\ a6\\ a7\\ a8 \end{bmatrix}_{-} j1 + A_2 \begin{bmatrix} b1\\ b2\\ b3\\ b4\\ b5\\ b6\\ b7\\ b8 \end{bmatrix}_{-} j2 + A_3 \begin{bmatrix} c1\\ c2\\ c3\\ c4\\ c5\\ c6\\ c7\\ c8 \end{bmatrix}_{-} j3 + A_4 \begin{bmatrix} d1\\ d2\\ d3\\ d4\\ d5\\ d6\\ d7\\ d8 \end{bmatrix}_{-} j4 + A_5 \begin{bmatrix} e1\\ e2\\ e3\\ e4\\ e5\\ e6\\ e7\\ e8 \end{bmatrix}_{-} j5 + A_6 \begin{bmatrix} f1\\ f2\\ f3\\ f4\\ f5\\ f6\\ f7\\ f8 \end{bmatrix}_{-} j6 + A_7 \begin{bmatrix} g1\\ g2\\ g3\\ g4\\ g5\\ g6\\ g7\\ g8 \end{bmatrix}_{-} j7 + A_8 \begin{bmatrix} h1\\ h2\\ h3\\ h4\\ h5\\ h6\\ h7\\ h8 \end{bmatrix}_{-} j8
$$

$$
+ B_1 \begin{bmatrix} a1\\ a2\\ a3\\ a4\\ a5\\ a6\\ a7\\ a8 \end{bmatrix}_{+} j9 + B_2 \begin{bmatrix} b1\\ b2\\ b3\\ b4\\ b5\\ b6\\ b7\\ b8 \end{bmatrix}_{+} j10 + B_3 \begin{bmatrix} c1\\ c2\\ c3\\ c4\\ c5\\ c6\\ c7\\ c8 \end{bmatrix}_{+} j11 + B_4 \begin{bmatrix} d1\\ d2\\ d3\\ d4\\ d5\\ d6\\ d7\\ d8 \end{bmatrix}_{+} j12 + B_5 \begin{bmatrix} e1\\ e2\\ e3\\ e4\\ e5\\ e6\\ e7\\ e8 \end{bmatrix}_{+} j13 + B_6 \begin{bmatrix} f1\\ f2\\ f3\\ f4\\ f5\\ f6\\ f7\\ f8 \end{bmatrix}_{+} j14 + B_7 \begin{bmatrix} g1\\ g2\\ g3\\ g4\\ g5\\ g6\\ g7\\ g8 \end{bmatrix}_{+} j15 + B_8 \begin{bmatrix} h1\\ h2\\ h3\\ h4\\ h5\\ h6\\ h7\\ h8 \end{bmatrix}_{+} j16
\tag{3.5.12}
$$

Required 16 unknown coefficients can now be calculated in general from these 16 equations (3.5.11) and (3.5.12) using boundary conditions on RHS of each equation.

Sine both the eigenvector and eigenvalues may be complex we want to write equations (3.5.7) and (3.5.8) in terms of real quantities. Let us represent **Nr** for number of real separation constants and **Nc** for the number of complex pairs of separation constants. With these we rewrite equations (3.5.7) and (3.5.8) as

$$
RE_{+}^{AT}(z) = \sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right), \tag{3.5.13a}
$$

$$
RE_{+}^{OC}(z) = \sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_{+}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_{-}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{AT/OC}}\right), \tag{3.5.13b}
$$

$$
CO_{+}^{AT}(z) = \sum_{j=1}^{Nc}\ \sum_{\alpha=1}^{2} A_j^{AT(\alpha)}F_{+}^{AT(\alpha)}(z,\gamma_j^{AT}) + B_j^{AT(\alpha)}F_{-}^{AT(\alpha)}(z_\omega - z,\gamma_j^{AT}), \tag{3.5.14a}
$$

$$
CO_{+}^{OC}(z) = \sum_{j=1}^{Nc}\ \sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_{+}^{OC(\alpha)}(z,\gamma_j^{OC}) + B_j^{OC(\alpha)}F_{-}^{OC(\alpha)}(z_1 - z,\gamma_j^{OC}), \tag{3.5.14b}
$$

$$RE_-^{AT}(z) = \boldsymbol{\Delta}\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right), \tag{3.5.15a}$$

$$RE_-^{OC}(z) = \boldsymbol{\Delta}\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right), \tag{3.5.15b}$$

$$CO_-^{AT}(z) = \sum_{j=1}^{Nc}\ \sum_{\alpha=1}^{2} A_j^{AT(\alpha)}F_-^{AT(\alpha)}(z,\gamma_j^{AT}) + B_j^{AT(\alpha)}F_+^{AT(\alpha)}(z_\omega - z,\gamma_j^{AT}), \tag{3.5.16a}$$

$$CO_-^{OC}(z) = \sum_{j=1}^{Nc}\ \sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_-^{OC(\alpha)}(z,\gamma_j^{OC}) + B_j^{OC(\alpha)}F_+^{OC(\alpha)}(z_1 - z,\gamma_j^{OC}), \tag{3.5.16b}$$

$$F_\pm^{AT/OC(1)}(z,\gamma_j^{AT/OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\ \right\} - \mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\}$$

$$= F^{AT/OC}1(\pm 1) - F^{AT/OC}1(\pm 2). \tag{3.5.17}$$

$$F_\pm^{AT/OC(2)}(z,\gamma_j^{AT/OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\}.$$

$$= F^{AT/OC}2(\pm 1) - F^{AT/OC}2(\pm 2). \tag{3.5.18}$$

We shall use following notations for different values of z in section 3.

$$F_\pm^{AT(1)}(z_\omega,\gamma_j^{AT}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\ \right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\right\}$$

$$= F^{AT}11(\pm 1) - F^{AT}11(\pm 2).$$

$$F_\pm^{AT(1)}(0,\gamma_j^{AT}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\ \right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\right\}$$

$$= \mathrm{Re}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\ \right\} - \mathrm{Im}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\right\} = F^{AT}12(\pm 1) - F^{AT}12(\pm 2).$$

$$F_\pm^{AT(2)}(z_\omega,\gamma_j^{AT}) = \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT}(\gamma_j^{AT})\right\}$$

$$= F^{AT}21(\pm 1) - F^{AT}21(\pm 2).$$

$$F_{\pm}^{AT(2)}(0,\gamma_j^{AT}) = \mathrm{Re}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} - \mathrm{Im}\left\{\mathbf{H}_{\pm}^{AT}(\gamma_j^{AT})\right\} = F^{AT}22(\pm 1) - F^{AT}22(\pm 2)$$

$$F_{\pm}^{OC(1)}(z_\omega,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\}$$

$$= F^{OC}11(\pm 1) - F^{OC}11(\pm 2).$$

$$F_{\pm}^{OC(1)}(z_1 - z_\omega,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\}$$

$$= F^{OC}12(\pm 1) - F^{OC}12(\pm 2).$$

$$F_{\pm}^{OC(2)}(z_\omega,\gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\}$$

$$= F^{OC}21(\pm 1) - F^{OC}21(\pm 2).$$

$$F_{\pm}^{OC(2)}(z_1 - z_\omega,\gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_1 - z_\omega}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\}$$

$$= F^{OC}22(\pm 1) - F^{OC}22(\pm 2).$$

$$F_{\pm}^{OC(1)}(z_1,\gamma_j^{OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\}$$

$$= \left[\ F^{OC}13(\pm 1) - F^{OC}13(\pm 2)\ \right]$$

$$F_{\pm}^{OC(2)}(z_1,\gamma_j^{OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Re}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_{\pm}^{OC}(\gamma_j^{OC}\right\}$$

$$= \left[\ F^{OC}23(\pm 1) - F^{OC}23(\pm 2)\ \right]$$

3.6. Albedo of ocean surface:

The oceanic surface albedo (OSA) plays very important role in the determination of the energy change processes between atmosphere and ocean. This is an important issue in the coupled atmosphere ocean system. Determination of OSA is an important task. In the last several decades several OSA schemes have been proposed and both observational and theoretical investigations are carried out. But analytical expressions and the dependent variable that comprises these schemes of investigations differ from one another to a great extent. There are models that depend only on solar zenith angle (SZA) while there are schemes that use additional parameters like wind speed or cloud optical depth or both. All most all the proposed investigations assumed valid for one albedo for the whole incident spectrum. However there exist schemes for albedo for differing wavelengths. Some schemes consider only clear or cloudy sky conditions while there are schemes which considered long time averages over both conditions.

One can assume that this wide variety of OSA schemes might be associated with differing radiative impact. This is generally a complicated process. Effective tools for such study are the one-dimensional radiative transfer models by which the influence of OSA on the upward flux at the top of the atmosphere (TOA) and the solar energy flow at the surface can be analyzed. This tool provides a controlled way of highlighting differences in the radiative forcing associated with each OSA scheme. It also provides a basis for understanding the ultimate response of the climate system in fully interactive global circulation model (GCM) climate integrations.

Canadian centre for climate modeling and analysis (CCCMA) have derived schemes for operational OSA parameterizations The CCCma second-generation atmospheric GCM (McFarlane et al. [170]) employed a relatively simple scheme for OSA, which depended on SZA and was independent of sky and surface wind conditions. The third-generation CCCma model, AGCM3 (McFarlane et al. [171]), employed the Hansen et al. [172] fit to Cox and Munk's approximate theory (Cox and Munk [215]. This fit was both a function of SZA and wind speed. We shall refer to Hansen et al. [172] as the H scheme. Validity of these schemes is discussed in detail by Barker and Li [219]. To correct this deficit Barker and Li simply adjusted the lead constant in the Hansen formulation, corresponding to a vertical shift or uniform increase, to enhance the albedo to more reasonable values.

In the Atmospheric general circulation model (AGCM4) the theory of Preisendorfer and Mobley (PM) [213] is used to help formulate the OSA. This theory is more accurate than Cox and Munks approximation. For example, it includes the reflection for the diffused rays as well as the orientation of the wind relative to the incoming solar flux. A comparative study was made in Li [378]. The parameterization used in AGCM4 is simply an approximate fit to the PM result. No attempt was made to account for the orientation of the wind relative to the direction of the incoming solar flux. This is only an issue at large SZA. The specific form of this fit is presented in [219]. We shall refer to the fit as the PM scheme. A comparison of the PM scheme to the H scheme used in AGCM3 is displayed in [378].Here we see the PM parameterization increases the albedo relative to the H scheme in a way that is arguably more physical than simply increasing all values by a constant.

The first two are simple in the sense that they depend solely on SZA. As such they represent time averages over other factors such as wind speed and direct versus diffuse conditions. The first scheme of Briegleb et al. [111] represents a fit to the observations of Payne [211]. The form of the OSA is expressed as

$$\omega^{OC}(\mu_0) = \frac{0.026}{1.1\mu_0^{1.7} + 0.065} + 0.15(\mu_0 - 0.1)(\mu_0 - 0.5)(\mu_0 - 1) \quad , \tag{3.6.1}$$

where $\omega(\mu_0)$ is the broadband OSA and μ_0 is the cosine of SZA (CSZA). Hereafter we shall refer to this formulation as the B scheme.

The second scheme of Taylor et al. [126] represents a fit to 5 yr of observations compiled by aircraft measurements. The form of the OSA in this scheme is expressed as

$$\omega^{OC}(\mu_0) = \frac{0.037}{1.1\mu_0^{1.4} + 0.15} . \tag{3.6.2}$$

This scheme is popularly known as T scheme. The third scheme of Jin et al. [372] derives its SZA and wind speed dependence from plane-parallel radiative transfer calculations that include the wind-blown roughened surface within the domain of the calculation. The surface is discretized by a set of inclined planes with random slopes that follow probability distributions given by Cox and Munk [215]. This scheme introduces an empirical dependence on aerosol/cloud optical depth to account for the effect of the direct and diffuse fluxes. Hereafter we shall refer to this formulation as the J scheme. Generally it has been found that the larger the aerosol/cloud optical depth the larger the fraction of the downward diffuse flux to the surface. There exist a number of factors that may contribute to the aerosol/cloud optical depth. Aerosol concentrations are typically largest in the lower troposphere while the location of cloud varies from surface to tropopause. For example, one value of optical depth could correspond to a variety of cloud and aerosol distributions. In principle the OSA will be sensitive to these different distributions since the fraction of direct relative to diffuse will be altered.

Analytical Formula for the PM Scheme

PM [213] is a scheme using the ray-tracing method based on Fresnel reflection on the ocean surface. The OSA of the PM scheme is parameterized as follows. A reference wind speed, U_0, is defined in terms of zenith angle as

$$U_0 = 180\mu_0^3(1-\mu_0^2) . \tag{3.6.3}$$

For wind speed $U_s \geq U_0$ the direct component of albedo is given as

$$\omega_{dir}^{OC}(\mu_0, U_s) = 0.021 + 0.0421(1-\mu_0)^2 + 0.128(1-\mu_0)^3 - 0.04(1-\mu_0)^6$$

$$+\left[\frac{4}{5.68+U_s-W_0}+\frac{0.074(1-\mu_0)}{1+3(U_s-U_0)}\right](1-\mu_0)^6, \tag{3.6.4}$$

but when $U_s < U_0$ the direct component is given as

$$\omega_{dir}^{OC}(\mu_0, U_s) = \left[1+\frac{5.4\mu_0^2(1-\mu_0^2)U_s(U_s-1.1U_0)^2}{W_0^3}\right]\times\left\{\ 0.021+0.0421(1-\mu_0)^2+0.128(1-\mu_0)^3\right.$$

$$\left.-0.04(1-\mu_0)^6+\left[\frac{4}{5.68+U_s-U_0}+\frac{0.074(1-\mu_0)}{1+3(U_s-U_0)}\right](1-\mu_0)^6\right\}. \tag{3.6.5}$$

In either case the diffuse component is given as

$$\omega_{dif}^{OC}(U_s) = 0.022\left\{1+0.55\exp\left[-\left(\frac{U_s}{7}\right)^2\right]+1.45\exp\left[-\left(\frac{U_s}{40}\right)^2\right]\right\}. \tag{3.6.6}$$

We have used above formula in our numerical consideration.

Daily Average for the 1D Model: It now desirable to discuss how to determine the incident directions of solar beam in different locations. We briefly discuss how to calculate the daily averaged results for different latitudes, since this has not been addressed in the one-dimensional radiative transfer study. This method provides a more realistic comparison to the results of GCM. The solar zenith angle θ_0 is generally given by

$$\cos\theta_0 = \sin\delta\sin\varphi+\cos\delta\cos\varphi\cos\omega, \tag{3.6.7}$$

where δ is declination angle, φ is the geographic latitude, and ω is hour angle with noon zero and morning positive. The declination angle is a function of day number (Iqbal 1983) as $\delta = f(\Gamma)$, where $\Gamma = \frac{2\pi(d_n-1)}{365}$, where d_n is the day number of the year, ranging from 1 on 1 January to 365 on 31 December. At the sunrise moment $\theta_0 = \frac{\pi}{2}$. From (3.6.7) $\omega_s = \cos^{-1}(-\tan\theta\tan\varphi)$, which is the hour angle for the sunrise, thus $-\omega_s$ is the hour angle for the sunset. The integral of ω over $\pm\omega_s$ generates the daily averaged results. Note, if $\cos\omega_s > 1$ there is no sunrise (polar night); if $\cos\omega_s \le -1$ there is no sunset and the solar zenith angle is given by (3.6.7) with $\pi \ge \omega - \pi$.

CHAPTER IV:

PARTICULAR & GENERAL SOLUTION

4.1. The Infinite Medium Green's Function:

In order to develop particular solution we shall proceed with the approach developed in (Barichello, L. B., Garcia, RDM, Siewert, C.E.) [70] to accommodate for the inhomogeneous source term $S^{J}_{AT/OC}(z,\mu)$ that appears in (2.12.16 & 2.12.20). The elementary solutions developed in the previous chapter can now be used to construct the green functions which in turn are required to find the particular solution i.e. to express it in terms of infinite-medium Green's function and therefore we take up the following two problems as mentioned below.

For atmosphere in regard to Green function $\mathbf{M}^{AT}(z,\pm\mu_i : x,\mu_k)$ & $\mathbf{M}^{AT}(z,\pm\mu_i : x,-\mu_k)$ for any source location x within atmosphere, we have

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT}(z,\mu_i : x,\mu_k) = \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT}(z,\mu_i : x,\mu_k) + I\delta(z-x)\delta_{i,k} \tag{4.1.1}$$

and

$$\left(-\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT}(z,-\mu_i : x,\mu_k) = \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT}(z,-\mu_i : x,\mu_k) \tag{4.1.2}$$

Similar equations for ocean are (Here also we used source location at x but to be understood as x is measured within ocean)

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{OC}(z,\mu_i : x,\mu_k) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{OC}(z,\mu_i : x,\mu_k) + I\delta(z-x)\delta_{i,k} \tag{4.1.3}$$

and
$$\left(-\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{OC}(z,-\mu_i : x,\mu_k) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{OC}(z,-\mu_i : x,\mu_k) \tag{4.1.4}$$

These equations constitute the first problem.

For the atmosphere the second problem is given by

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT}(z,\mu_i : x,-\mu_k) = \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT}(z,\mu_i : x,-\mu_k) \tag{4.1.5}$$

$$\left(-\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT}(z,-\mu_i : x,-\mu_k) = \frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT}(z,-\mu_i : x,-\mu_k) + I\delta(z-x)\delta_{i,k} \tag{4.1.6}$$

And for ocean

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{OC}(z,\mu_i : x,-\mu_k) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{OC}(z,\mu_i : x,-\mu_k) \tag{4.1.7}$$

$$\left(-\mu_i \frac{d}{dz}+I\right)\mathbf{M}^{OC}(z,-\mu_i : x,-\mu_k)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{K}_{OC}^{j}(z,-\mu_i : x,-\mu_k)+I\delta(z-x)\delta_{i,k} \quad (4.1.8)$$

For $i,k=1,2,\ldots,N$ with $S_{AT/OC}^{J}=\begin{pmatrix} S_{ATL/OCL}^{S}(z,\mu) \\ S_{ATQ/OCQ}^{S}(z,\mu) \\ S_{ATU?OCU}^{S}(z,\mu) \\ S_{ATV/OCV}^{S}(z,\mu) \end{pmatrix}$ (4.1.9)

Where I is as usual the (4X4) identity matrix and we have defined

$$\mathbf{K}_{AT/OC}^{J}(z,\pm\mu_i : x,\pm\mu_k)=\mathbf{P}_{J}^{S}(\pm\mu_i)B_{J}^{AT/OC}\sum_{\beta=1}^{N}w_{\beta}\mathbf{M}_{S,\beta}^{AT/OC}(z,x,\pm\mu_k) \quad (4.1.10)$$

and

$$\mathbf{M}_{S,\beta}^{AT/OC}(z : x,\pm\mu_k)=\mathbf{P}_{J}^{S}(\mu_\beta)\mathbf{M}^{AT/OC}(z,\mu_\beta : x,\pm\mu_k)+\mathbf{P}_{J}^{S}(-\mu_\beta)\mathbf{M}^{AT/OC}(z,-\mu_\beta : x,\pm\mu_k) \quad (4.1.11)$$

Source is located at $x\in(0,z_w)$ for atmosphere and at $x\in(z_w,z_1)$ in the ocean along with the source direction defined by $\mu_k\in\{\mu_i\}$. We have defined $\delta(z-x)$ as the Dirac delta "function" and $\delta_{i,k}$ as the Kronecker delta. Inclusion of the identity matrix in source term of equations (4.1.1) and (4.1.6) clearly indicates that each of the two Green's functions must come out as a (4X4) matrix.

We now proceed to develop the solution for $\mathbf{M}^{AT}(z,\xi_\beta : x,\pm\mu_k)$ by following Case and Zweifel [135].We can write one solution bounded as $z\to\infty$ and valid in the region $z>x$ for the homogeneous equation and another solution valid for $z<x$ and bounded as $z\to-\infty$). For matching up these two solutions with the notion of "jump" condition, valid when $i,k=1,2,\ldots,N$, we write, for atmosphere and ocean respectively

1. (+, +) equations

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,\mu_i : x,\mu_k)-\mathbf{M}^{AT}(z-\varepsilon,\mu_i : x,\mu_k)]=I\delta_{i,k} \quad (4.1.12)$$

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,\mu_i : x,\mu_k)-\mathbf{M}^{OC}(z-\varepsilon,\mu_i : x,\mu_k)]=I\delta_{i,k} \quad (4.1.13)$$

2. (–, +) equations

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,-\mu_i : x,\mu_k)-\mathbf{M}^{AT}(z-\varepsilon,-\mu_i : x,\mu_k)]=0 \quad (4.1.14)$$

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,-\mu_i : x,\mu_k)-\mathbf{M}^{OC}(z-\varepsilon,-\mu_i : x,\mu_k)]=0 \quad (4.1.15)$$

for $\mathbf{M}^{AT}(z,\pm\mu_i : x,\mu_k)$ & $\mathbf{M}^{OC}(z,\pm\mu_i : x,\mu_k)$

3. (+,–) equations

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,\mu_i : x,-\mu_k) - \mathbf{M}^{AT}(z-\varepsilon,\mu_i : x,-\mu_k)] = 0 \tag{4.1.16}$$

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,\mu_i : x,-\mu_k) - \mathbf{M}^{OC}(z-\varepsilon,\mu_i : x,-\mu_k)] = 0 \tag{4.1.17}$$

4. (–,–) equations

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,-\mu_i : x,-\mu_k) - \mathbf{M}^{AT}(z-\varepsilon,-\mu_i : x,-\mu_k)] = I\delta_{i,k} \tag{4.1.18}$$

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,-\mu_i : x,-\mu_k) - \mathbf{M}^{OC}(z-\varepsilon,-\mu_i : x,-\mu_k)] = I\delta_{i,k} \tag{4.1.19}$$

for $\mathbf{M}^{AT}(z,\pm\mu_i : x,-\mu_k)$ & $\mathbf{M}^{OC}(z,\pm\mu_i : x,-\mu_k)$.

Here in the above development we have used Z for any optical depth.

We shall now use elementary solution developed in section 2 in the above equations. Let us now denote our discrete ordinate solution for the two Green's functions for atmosphere and ocean as

$$\mathbf{M}_{\pm}^{AT}(z : x,\mu_k) = \left[[\mathbf{M}^{AT}(z,\pm\mu_1 : x,\mu_k)]^T,\ldots,[\mathbf{M}^{AT}(z,\pm\mu_N : x,\mu_k)]^T\right]^T \tag{4.1.20}$$

$$\mathbf{M}_{\pm}^{AT}(z : x,-\mu_k) = \left[[\mathbf{M}^{AT}(z,\pm\mu_1 : x,-\mu_k)]^T,\ldots,[\mathbf{M}^{AT}(z,\pm\mu_N : x,-\mu_k)]^T\right]^T \tag{4.1.21}$$

and

$$\mathbf{M}_{\pm}^{OC}(z : x,\mu_k) = \left[[\mathbf{M}^{OC}(z,\pm\mu_1 : x,\mu_k)]^T,\ldots,[\mathbf{M}^{OC}(z,\pm\mu_N : x,\mu_k)]^T\right]^T \tag{4.1.22}$$

$$\mathbf{M}_{\pm}^{OC}(z : x,-\mu_k) = \left[[\mathbf{M}^{OC}(z,\pm\mu_1 : x,-\mu_k)]^T,\ldots,[\mathbf{M}^{OC}(z,\pm\mu_N : x,-\mu_k)]^T\right]^T \tag{4.1.23}$$

We shall now consider following two cases separately.

Case A: For $z > x$:

For atmosphere the first set of solution:

$$\mathbf{M}_{+}^{AT}(z : x,\pm\mu_k) = \sum_{J=1}^{4N}\mathbf{H}_{+}^{AT}(\gamma_J^{AT})C^{AT}{}_j(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma_J^{AT}}\right), \quad z > x, \tag{4.1.24}$$

First we shall restrict us for **N=2**, Then we shall generalize. Using (4.1.24) we get from (4.1.12)

Case 1: Atmosphere for (+, +) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{AT}(z,+\mu_1;x,+\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,+\mu_2;x,+\mu_\alpha)]^T\end{pmatrix} = \mathbf{H}_{+}^{AT}(\gamma_1^{AT})C_1^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_1^{AT}}\right) + \mathbf{H}_{+}^{AT}(\gamma_2^{AT})C_2^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{AT}}\right)$$
$$+\mathbf{H}_{+}^{AT}(\gamma_3^{AT})C_3^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{AT}}\right) + \mathbf{H}_{+}^{AT}(\gamma_4^{AT})C_4^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{AT}}\right)$$

$$+\mathbf{H}_{+}^{AT}(\gamma_5^{AT})C_5^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_6^{AT})C_6^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{AT}}\right)$$

$$+\mathbf{H}_{+}^{AT}(\gamma_7^{AT})C_7^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_8^{AT})C_8^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{AT}}\right). \tag{4.1.25}$$

Case 2: Atmosphere for (+,–) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{AT}(z,+\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,+\mu_2;x,-\mu_\alpha)]^T\end{pmatrix}=\mathbf{H}_{+}^{AT}(\gamma_1^{AT})C_1^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_2^{AT})C_2^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{AT}}\right)$$

$$+\mathbf{H}_{+}^{T}(\gamma_3^{AT})C_3^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_4^{AT})C_4^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{AT}}\right)$$

$$+\mathbf{H}_{+}^{AT}(\gamma_5^{AT})C_5^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_5^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_6^{AT})C_6^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{AT}}\right)$$

$$+\mathbf{H}_{+}^{AT}(\gamma_7^{AT})C_7^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_8^{AT})C_8^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8^{AT}}\right). \tag{4.1.26}$$

The second set of solution:

$$\mathbf{M}_{-}^{AT}(z:x,\pm\mu_k)=\mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}_{-}^{AT}(\gamma_J^{AT})C^{AT}{}_j(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma_J^{AT}}\right),\quad z>x, \tag{4.1.27}$$

We shall again restrict us for **N=2** for the time being. Subscript (2x2) in the following expressions are used to denote consideration of 2 angle quadrature.

Case 3: Atmosphere for (–, +) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{AT}(z,-\mu_1;x,+\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,-\mu_2;x,+\mu_\alpha)]^T\end{pmatrix}=\mathbf{\Delta}_{2x2}\Bigg[\mathbf{H}_{-}^{AT}(\gamma_1^{AT})C_1^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{AT}}\right)+\mathbf{H}_{-}^{AT}(\gamma_2^{AT})C_2^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{AT}}\right)$$

$$+\mathbf{H}_{-}^{AT}(\gamma_3^{AT})C_3^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{AT}}\right)+\mathbf{H}_{-}^{AT}(\gamma_4^{AT})C_4^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{AT}}\right)$$

$$+\mathbf{H}_{-}^{AT}(\gamma_5^{AT})C_5^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_5^{AT}}\right)+\mathbf{H}_{-}^{AT}(\gamma_6^{AT})C_6^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{AT}}\right)$$

$$+\mathbf{H}_{-}^{AT}(\gamma_7)C_7^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7}\right)+\mathbf{H}_{-}^{AT}(\gamma_8)C_8^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8}\right)\Bigg]. \tag{4.1.28}$$

Case 4: Atmosphere for (–,–) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{AT}(z,-\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,-\mu_2;x,-\mu_\alpha)]^T\end{pmatrix}=\mathbf{\Delta}_{2x2}\Bigg[\mathbf{H}_{-}^{AT}(\gamma_1^{AT})C_1^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_1^{AT}}\right)+\mathbf{H}_{-}^{AT}(\gamma_2^{AT})C_2^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{AT}}\right)$$

$$+\mathbf{H}_{-}^{AT}(\gamma_3^{AT})C_3^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{AT}}\right)+\mathbf{H}_{-}^{AT}(\gamma_4^{AT})C_4^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{AT}}\right)$$

$$+\mathbf{H}_{-}^{AT}(\gamma_5^{AT})C_5^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{AT}}\right)+\mathbf{H}_{-}^{AT}(\gamma_6^{AT})C_6^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{AT}}\right)$$

$$+\mathbf{H}_{-}^{AT}(\gamma_7^{AT})C_7^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{AT}}\right)+\mathbf{H}_{-}^{AT}(\gamma_8^{AT})C_8^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{AT}}\right)\Bigg]. \tag{4.1.29}$$

For ocean the first set of solution:

$$\mathbf{M}_{+}^{OC}(z:x,\pm\mu_k)=\sum_{J=1}^{4N}\mathbf{H}_{+}^{OC}(\gamma_J^{OC})C^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma_j^{OC}}\right),\qquad z>x, \tag{4.1.30}$$

Case1: ocean for (+, +) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,+\mu_1;x,+\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,+\mu_2;x,+\mu_\alpha)]^T\end{pmatrix}=\mathbf{H}_{+}^{OC}(\gamma_1^{OC})C_1^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_1^{OC}}\right)+\mathbf{H}_{+}^{OC}(\gamma_2^{OC})C_2^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{OC}}\right)$$

$$+\mathbf{H}_{+}^{OC}(\gamma_3^{OC})C_3^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{OC}}\right)+\mathbf{H}_{+}^{OC}(\gamma_4^{OC})C_4^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{OC}}\right)$$

$$+\mathbf{H}_{+}^{OC}(\gamma_5^{OC})C_5^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{OC}}\right)+\mathbf{H}_{+}^{OC}(\gamma_6^{OC})C_6^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{OC}}\right)$$

$$+\mathbf{H}_{+}^{OC}(\gamma_7^{OC})C_7^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{OC}}\right)+\mathbf{H}_{+}^{OC}(\gamma_8^{OC})C_8^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{OC}}\right). \tag{4.1.31}$$

Case2: ocean for (+,–) pair in (μ_i, μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,+\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,+\mu_2;x,-\mu_\alpha)]^T\end{pmatrix} = \mathbf{H}_+^{OC}(\gamma_1^{OC})C_1^{OC}(-\mu_\alpha)\exp\left(-\frac{z-x}{\gamma_1^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_2^{OC})C_2^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{OC}}\right)$$
$$+\mathbf{H}_+^{OC}(\gamma_3^{OC})C_3^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_4^{OC})C_4^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{OC}}\right)$$
$$+\mathbf{H}_+^{OC}(\gamma_5^{OC})C_5^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_6^{OC})C_6^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{OC}}\right)$$
$$+\mathbf{H}_+^{OC}(\gamma_7^{OC})C_7^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_8^{OC})C_8^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{OC}}\right). \tag{4.1.32}$$

The second set of solution:

$$\mathbf{M}_-^{OC}(z:x,\pm\mu_k) = \mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}_-^{OC}(\gamma_J^{OC})C^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma_J^{OC}}\right), \quad z > x, \tag{4.1.33}$$

Case 3: ocean for (–, +) pair in (μ_i, μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,-\mu_1;x,+\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,-\mu_2;x,+\mu_\alpha)]^T\end{pmatrix} = \mathbf{\Delta}_{2x2}\left[\mathbf{H}_-^{OC}(\gamma_1^{OC})C_1^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_2^{OC})C_2^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{OC}}\right)\right.$$
$$+\mathbf{H}_-^{OC}(\gamma_3^{OC})C_3^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_4^{OC})C_4^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{OC}}\right)$$
$$+\mathbf{H}_-^{OC}(\gamma_5^{OC})C_5^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_5^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_6^{OC})C_6^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{OC}}\right)$$
$$\left.+\mathbf{H}_-^{OC}(\gamma_7^{OC})C_7^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_8^{OC})C_8^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8^{OC}}\right)\right]. \tag{4.1.34}$$

Case 4: ocean for (–,–) pair in (μ_i, μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,-\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,-\mu_2;x,-\mu_\alpha)]^T\end{pmatrix} = \mathbf{\Delta}_{2x2}\left[\mathbf{H}_-^{OC}(\gamma_1^{OC})C_1^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_2^{OC})C_2^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{OC}}\right)\right.$$

$$
+\mathbf{H}_{-}^{OC}(\gamma_3^{OC})C_3^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{OC}}\right)+\mathbf{H}_{-}^{OC}(\gamma_4^{OC})C_4^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{OC}}\right)
$$

$$
+\mathbf{H}_{-}^{OC}(\gamma_5^{OC})C_5^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_5^{OC}}\right)+\mathbf{H}_{-}^{OC}(\gamma_6^{OC})C_6^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{OC}}\right)
$$

$$
+\mathbf{H}_{-}^{OC}(\gamma_7^{OC})C_7^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7^{OC}}\right)+\mathbf{H}_{-}^{OC}(\gamma_8^{OC})C_8^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8^{OC}}\right)\Bigg]. \tag{4.1.35}
$$

Similarly for the second case we also get

Case B: For z<x:

For Atmosphere:

The first set of solution:

$$
\mathbf{M}_{+}^{AT}(z:x,\pm\mu_k)=-\sum_{J=1}^{4N}\mathbf{H}_{-}^{AT}(\gamma_J^{AT})D^{AT}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{AT}}\right),\quad z<x, \tag{4.1.36}
$$

Case 1: Atmosphere for (+, +) pair in (μ_i,μ_α)

$$
\begin{pmatrix}[\mathbf{M}^{AT}(z,+\mu_1;x,+\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,+\mu_2;x,+\mu_\alpha)]^T\end{pmatrix}=-\Bigg[\mathbf{H}_{+}^{AT}(\gamma_1^{AT})D_1^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_1^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_2^{AT})D_2^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{AT}}\right)
$$

$$
+\mathbf{H}_{+}^{AT}(\gamma_3^{AT})D_3^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_4^{AT})D_4^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{AT}}\right)
$$

$$
+\mathbf{H}_{+}^{AT}(\gamma_5^{AT})D_5^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_6^{AT})D_6^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{AT}}\right)
$$

$$
+\mathbf{H}_{+}^{AT}(\gamma_7^{AT})D_7^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{AT}}\right)+\mathbf{H}_{+}^{AT}(\gamma_8^{AT})D_8^{AT}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{AT}}\right)\Bigg]. \tag{4.1.37}
$$

Case 2: Atmosphere for (+,–) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{AT}(z,+\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,+\mu_2;x,-\mu_\alpha)]^T\end{pmatrix}=-\Bigg[\mathbf{H}_+^{AT}(\gamma_1^{AT})D_1^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_1^{AT}}\right)+\mathbf{H}_+^{AT}(\gamma_2^{AT})D_2^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{AT}}\right)$$

$$+\mathbf{H}_+^{AT}(\gamma_3^{AT})D_3^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{AT}}\right)+\mathbf{H}_+^{AT}(\gamma_4^{AT})D_4^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{AT}}\right)$$

$$+\mathbf{H}_+^{AT}(\gamma_5^{AT})D_5^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{AT}}\right)+\mathbf{H}_+^{AT}(\gamma_6^{AT})D_6^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{AT}}\right)$$

$$+\mathbf{H}_+^{AT}(\gamma_7^{AT})D_7^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{AT}}\right)+\mathbf{H}_+^{AT}(\gamma_8^{AT})D_8^{AT}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{AT}}\right)\Bigg].$$

(4.1.38)

The second set of solution:

$$\mathbf{M}_-^{AT}(z:x,\pm\mu_k)=-\mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}_+^{AT}(\gamma_J^{AT})D^{AT}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{AT}}\right),\quad z<x, \qquad (4.1.39)$$

Case 3: Atmosphere for (–, +) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{AT}(z,-\mu_1;x,+\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,-\mu_2;x,+\mu_\alpha)]^T\end{pmatrix}=-\mathbf{\Delta}_{2x2}\Bigg[\mathbf{H}_-^{AT}(\gamma_1^{AT})D_1^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{AT}}\right)+\mathbf{H}_-^{AT}(\gamma_2^{AT})D_2^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{AT}}\right)$$

$$+\mathbf{H}_-^{AT}(\gamma_3^{AT})D_3^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{AT}}\right)+\mathbf{H}_-^{AT}(\gamma_4^{AT})D_4^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{AT}}\right)$$

$$+\mathbf{H}_-^{AT}(\gamma_5^{AT})D_5^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_5^{AT}}\right)+\mathbf{H}_-^{AT}(\gamma_6^{AT})D_6^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{AT}}\right)$$

$$+\mathbf{H}_-^{AT}(\gamma_7^{AT})D_7^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7^{AT}}\right)+\mathbf{H}_-^{AT}(\gamma_8^{AT})D_8^{AT}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8^{AT}}\right)\Bigg].$$

(4.1.40)

Case 4: Atmosphere for (–,–) pair in (μ_i, μ_α)

$$\begin{pmatrix}[\mathbf{M}^{AT}(z,-\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{AT}(z,-\mu_2;x,-\mu_\alpha)]^T\end{pmatrix} = -\mathbf{\Delta}_{2x2}\Bigg[\mathbf{H}_-^{AT}(\gamma_1^{AT})D_1^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{AT}}\right) + \mathbf{H}_-^{AT}(\gamma_2^{AT})D_2^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{AT}}\right)$$

$$+\mathbf{H}_-^{AT}(\gamma_3^{AT})D_3^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{AT}}\right) + \mathbf{H}_-^{AT}(\gamma_4^{AT})D_4^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{AT}}\right)$$

$$+\mathbf{H}_-^{AT}(\gamma_5^{AT})D_5^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_5^{AT}}\right) + \mathbf{H}_-^{AT}(\gamma_6^{AT})D_6^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{AT}}\right)$$

$$+\mathbf{H}_-^{AT}(\gamma_7^{AT})D_7^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7^{AT}}\right) + \mathbf{H}_-^{AT}(\gamma_8^{AT})D_8^{AT}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8^{AT}}\right)\Bigg]. \tag{4.1.41}$$

For Ocean:
The first set of solution:

$$\mathbf{M}_+^{OC}(z:x,\pm\mu_k) = -\sum_{J=1}^{4N}\mathbf{H}_-^{OC}(\gamma_J^{OC})D^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{OC}}\right), \quad z < x, \tag{4.1.42}$$

Case 1: Ocean for (+, +) pair in (μ_i, μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,+\mu_1;x,\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,+\mu_2;x,\mu_\alpha)]^T\end{pmatrix} = -\Bigg[\mathbf{H}_+^{OC}(\gamma_1^{OC})D_1^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_1^{OC}}\right) + \mathbf{H}_+^{OC}(\gamma_2^{OC})D_2^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{OC}}\right)$$

$$+\mathbf{H}_+^{OC}(\gamma_3^{OC})D_3^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{OC}}\right) + \mathbf{H}_+^{OC}(\gamma_4^{OC})D_4^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{OC}}\right)$$

$$+\mathbf{H}_+^{OC}(\gamma_5^{OC})D_5^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{OC}}\right) + \mathbf{H}_+^{OC}(\gamma_6^{OC})D_6^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{OC}}\right)$$

$$+\mathbf{H}_+^{OC}(\gamma_7^{OC})D_7^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{OC}}\right) + \mathbf{H}_+^{OC}(\gamma_8^{OC})D_8^{OC}(+\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{OC}}\right)\Bigg]. \tag{4.1.43}$$

Case 2: Ocean for (+,–) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,+\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,+\mu_2;x,-\mu_\alpha)]^T\end{pmatrix} = -\left[\mathbf{H}_+^{OC}(\gamma_1^{OC})D_1^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_1^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_2^{OC})D_2^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_2^{OC}}\right)\right.$$

$$+\mathbf{H}_+^{OC}(\gamma_3^{OC})D_3^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_3^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_4^{OC})D_4^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_4^{OC}}\right)$$

$$+\mathbf{H}_+^{OC}(\gamma_5^{OC})D_5^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_5^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_6^{OC})D_6^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_6^{OC}}\right)$$

$$\left.+\mathbf{H}_+^{OC}(\gamma_7^{OC})D_7^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_7^{OC}}\right)+\mathbf{H}_+^{OC}(\gamma_8^{OC})D_8^{OC}(-\mu_\alpha)\exp\left(-\frac{(z-x)}{\gamma_8^{OC}}\right)\right]. \tag{4.1.44}$$

The second set of solution:

$$\mathbf{M}_-^{OC}(z:x,\pm\mu_k) = -\boldsymbol{\Delta}\sum_{J=1}^{4N}\mathbf{H}_+^{OC}(\gamma_J^{OC})D^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{OC}}\right), \quad z<x, \tag{4.1.45}$$

Case 3: Ocean for (–, +) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,-\mu_1;x,+\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,-\mu_2;x,+\mu_\alpha)]^T\end{pmatrix} = -\boldsymbol{\Delta}_{2x2}\left[\mathbf{H}_-^{OC}(\gamma_1^{OC})D_1^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_2^{OC})D_2^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{OC}}\right)\right.$$

$$+\mathbf{H}_-^{OC}(\gamma_3^{OC})D_3^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_4^{OC})D_4^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{OC}}\right)$$

$$+\mathbf{H}_-^{OC}(\gamma_5^{OC})D_5^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_5^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_6^{OC})D_6^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{OC}}\right)$$

$$\left.+\mathbf{H}_-^{OC}(\gamma_7^{OC})D_7^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_8^{OC})D_8^{OC}(+\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8^{OC}}\right)\right]. \tag{4.1.46}$$

Case 4: Ocean for (–,–) pair in (μ_i,μ_α)

$$\begin{pmatrix}[\mathbf{M}^{OC}(z,-\mu_1;x,-\mu_\alpha)]^T\\ [\mathbf{M}^{OC}(z,-\mu_2;x,-\mu_\alpha)]^T\end{pmatrix} = -\mathbf{\Delta}_{2x2}\left[\mathbf{H}_-^{OC}(\gamma_1^{OC})D_1^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_1^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_2^{OC})D_2^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_2^{OC}}\right)\right.$$

$$+\mathbf{H}_-^{OC}(\gamma_3^{OC})D_3^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_3^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_4^{OC})D_4^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_4^{OC}}\right)$$

$$+\mathbf{H}_-^{OC}(\gamma_5^{OC})D_5^{OC}(-\mu_\alpha)\exp\left(-\frac{x-z}{\gamma_5^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_6^{OC})D_6^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_6^{OC}}\right)$$

$$\left.+\mathbf{H}_-^{OC}(\gamma_7^{OC})D_7^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_7^{OC}}\right)+\mathbf{H}_-^{OC}(\gamma_8^{OC})D_8^{OC}(-\mu_\alpha)\exp\left(-\frac{(x-z)}{\gamma_8^{OC}}\right)\right]. \tag{4.1.47}$$

The arbitrary constants appear in the above equations are (1X4) vectors for each **j**. These constants are to be determined to find the corresponding green's functions.

$$C^{AT}{}_j(\pm\mu_k) = [C^{AT}{}_{1,j}(\pm\mu_k), C^{AT}{}_{2,j}(\pm\mu_k), C^{AT}{}_{3,j}(\pm\mu_k), C^{AT}{}_{4,j}(\pm\mu_k)],$$

$$C^{OC}{}_j(\pm\mu_k) = [C^{OC}{}_{1,j}(\pm\mu_k), C^{OC}{}_{2,j}(\pm\mu_k), C^{OC}{}_{3,j}(\pm\mu_k), C^{OC}{}_{4,j}(\pm\mu_k)]; \tag{4.1.48}$$

and

$$D^{AT}{}_j(\pm\mu_k) = [D^{AT}{}_{1,j}(\pm\mu_k), D^{AT}{}_{2,j}(\pm\mu_k), D^{AT}{}_{3,j}(\pm\mu_k), D^{AT}{}_{4,j}(\pm\mu_k)],$$

$$D^{OC}{}_j(\pm\mu_k) = [D^{OC}{}_{1,j}(\pm\mu_k), D^{OC}{}_{2,j}(\pm\mu_k), D^{OC}{}_{3,j}(\pm\mu_k), D^{OC}{}_{4,j}(\pm\mu_k)]. \tag{4.1.49}$$

4.2. Derivations of equations for the determination of arbitrary constants:

We shall now substitute equations (4.1.24) and (4.1.36) into limiting equations (4.1.12, 4.1.13) keeping appropriate consideration for positive μ_1 only for **z>x** and **z<x** to obtain the following equations after proper evaluation of limit as $\varepsilon \to \infty$, with all stokes components taken together for **n=2**

$$\begin{bmatrix}\mu_1 & 0 & 0 & 0\\ 0 & \mu_1 & 0 & 0\\ 0 & 0 & \mu_1 & 0\\ 0 & 0 & 0 & \mu_1\end{bmatrix} x[M^{AT}(z:x,\pm\mu_k) - M^{AT}(z:x,\pm\mu_k)] = \sum_{j=1}^{4}\mathbf{H}_+(\gamma_j^{AT})C_j(\mu_\alpha)+\mathbf{H}_-(\gamma_j^{AT})D_j(\mu_\alpha), \tag{4.2.1}$$

Here $\mathbf{H}_+$ are calculated for only μ_1 from (3.5.1). Hence this is a (4x1) matrix. Each component of **Cj** corresponds to each stokes components. This implies that each **Cj** is a (1x4) row vector. Hence the right hand side is a summation over (4x4) vectors. This is reason that in LHS the post factor of the matrix $\begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_1 \end{bmatrix}$ must be a (4x4) vector.

Similarly if we consider the same case for μ_2 only we shall get

$$\begin{bmatrix} \mu_2 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{bmatrix} \mathrm{x}[M^{AT}(z:x,\pm\mu_k) - M^{AT}(z:x,\pm\mu_k)] = \sum_{j=1}^{4} \mathbf{H}_+(\gamma_j^{AT})C_j(\mu_\alpha) + \mathbf{H}_-(\gamma_j^{AT})D_j(\mu_\alpha)$$

(4.2.2)

Combining these two cases we get for **N=2** a compact matrix equation

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \mathrm{x}\left[\begin{pmatrix} [\mathbf{M}^{AT}(z+\varepsilon,+\mu_1;x,\mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z+\varepsilon,+\mu_2;x,\mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{AT}(z-\varepsilon,+\mu_1;x,\mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z-\varepsilon,+\mu_2;x,\mu_\alpha)]^T \end{pmatrix}\right]$$

$$= \mathbf{X} \sum_{j=1}^{4x2=8} \mathbf{H}_+^{AT}(\gamma_j^{AT})C_j^{AT}(\mu_\alpha) + \mathbf{H}_-^{AT}(\gamma_j^{AT})D_j^{AT}(\mu_\alpha) = [I\delta_{1,\alpha}, I\delta_{2,\alpha}]^T$$

(4.2.3)

Where $\mathbf{X} = \mathrm{diag}(\mu_1\mathbf{\Gamma}, \mu_2\mathbf{\Gamma}, \mu_3\mathbf{\Gamma}, \ldots\ldots, \mu_N\mathbf{\Gamma})$ as given in (3.3.15). Extending the whole processes for **N=N** we now easily have

$$\mathbf{X}\sum_{J=1}^{4N}[\mathbf{H}_{+}^{AT}(\gamma_j^{AT})C_j^{AT}(\mu_\alpha) + \mathbf{H}_{-}^{AT}(\gamma_j^{AT})D_j^{AT}(\mu_\alpha)] = R_k \tag{4.2.4}$$

We now use equation (4.1.28) and (4.1.40) in the second atmospheric jump condition (4.1.14) for negative μ_i using the above mentioned procedure to get

$$-\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \times \left[\begin{pmatrix} [\mathbf{M}^{AT}(z+\varepsilon, -\mu_1; x, \mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z+\varepsilon, -\mu_2; x, \mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{AT}(z-\varepsilon, -\mu_1; x, \mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z-\varepsilon, -\mu_2; x, \mu_\alpha)]^T \end{pmatrix}\right]$$

$$= -\mathbf{X\Delta}\sum_{J=1}^{4\times2=8}[\mathbf{H}_{-}^{AT}(\gamma_j^{AT})C_j^{AT}(\mu_\alpha) + \mathbf{H}_{+}^{AT}(\gamma_j^{AT})D_j^{AT}(\mu_\alpha)] = 0. \tag{4.2.5}$$

We extend the case for the whole spectrum of **N** to get

$$-\mathbf{X\Delta}\sum_{J=1}^{4N}[\mathbf{H}_{-}^{AT}(\gamma_j^{AT})C_j^{AT}(\mu_\alpha) + \mathbf{H}_{+}^{AT}(\gamma_j^{AT})D_j^{AT}(\mu_\alpha)] = 0 \tag{4.2.6}$$

Similar application of equation (4.1.26) and (4.1.38) in atmospheric jump condition (4.1.16) for negative μ_α yields following eqs (4.3.25)

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \times \left[\begin{pmatrix} [\mathbf{M}^{AT}(z+\varepsilon,+\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z+\varepsilon,+\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{AT}(z-\varepsilon,+\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z-\varepsilon,+\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} \right]$$

$$= \mathbf{X} \sum_{J=1}^{4\times 2=8} [\mathbf{H}_+^{AT}(\gamma_j^{AT}) C_j^{AT}(-\mu_\alpha) + \mathbf{H}_-^{AT}(\gamma_j^{AT}) D_j^{AT}(-\mu_\alpha)] = 0. \qquad (4.2.7)$$

Similarly extending for **N=N** we have

$$\mathbf{X} \sum_{J=1}^{4N} [\mathbf{H}_+^{AT}(\gamma_j^{AT}) C_j^{AT}(-\mu_\alpha) + \mathbf{H}_-^{AT}(\gamma_j^{AT}) D_j^{AT}(-\mu_\alpha)] = 0 \qquad (4.2.8)$$

and for the second problem we get from (4.1.18) using (4.1.29) and (4.1.41)

$$-\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \times \left[\begin{pmatrix} [\mathbf{M}^{AT}(z+\varepsilon,-\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z+\varepsilon,-\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{AT}(z-\varepsilon,-\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{AT}(z-\varepsilon,-\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} \right]$$

$$= -\mathbf{X\Delta} \sum_{J=1}^{4\times 2=8} [\mathbf{H}_-^{AT}(\gamma_j^{AT}) C_j^{AT}(-\mu_\alpha) + \mathbf{H}_+^{AT}(\gamma_j^{AT}) D_j^{AT}(-\mu_\alpha)] = R_k. \qquad (4.2.9)$$

Similarly extending **N=N** we have

$$-\mathbf{X\Delta} \sum_{J=1}^{4N} [\mathbf{H}_-^{AT}(\gamma_j^{AT}) C_j^{AT}(-\mu_\alpha) + \mathbf{H}_+^{AT}(\gamma_j^{AT}) D_j^{AT}(-\mu_\alpha)] = R_k. \qquad (4.2.10)$$

For Ocean:

Similar expressions can be deduced for ocean using jump conditions appropriate for ocean cases with corresponding solutions. We write only the final expressions

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \times \left[\begin{pmatrix} [\mathbf{M}^{OC}(z+\varepsilon,+\mu_1;x,\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z+\varepsilon,+\mu_2;x,\mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{OC}(z-\varepsilon,+\mu_1;x,\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z-\varepsilon,+\mu_2;x,\mu_\alpha)]^T \end{pmatrix} \right]$$

$$= \mathbf{X} \sum_{j=1}^{4x2=8} \mathbf{H}_+^{OC}(\gamma_j^{OC}) C_j^{OC}(\mu_\alpha) + \mathbf{H}_-^{OC}(\gamma_j^{OC}) D_j^{OC}(\mu_\alpha) = [I\delta_{1,\alpha}, I\delta_{2,\alpha}]^T \qquad (4.2.11)$$

Extending the processes for **N=N** we have

$$\mathbf{X} \sum_{J=1}^{4N} [\mathbf{H}_+^{OC}(\gamma_j^{OC}) C_j^{OC}(\mu_\alpha) + \mathbf{H}_-^{OC}(\gamma_j^{OC}) D_j^{OC}(\mu_\alpha)] = R_k \qquad (4.2.12)$$

and

$$-\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \times \left[\begin{pmatrix} [\mathbf{M}^{OC}(z+\varepsilon,-\mu_1;x,\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z+\varepsilon,-\mu_2;x,\mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{OC}(z-\varepsilon,-\mu_1;x,\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z-\varepsilon,-\mu_2;x,\mu_\alpha)]^T \end{pmatrix} \right]$$

$$= -\mathbf{X}\boldsymbol{\Delta} \sum_{J=1}^{4\times2=8} [\mathbf{H}_{-}^{OC}(\gamma_j^{OC})C_j^{OC}(\mu_\alpha) + \mathbf{H}_{+}^{OC}(\gamma_j^{OC})D_j^{OC}(\mu_\alpha)] = 0. \qquad (4.2.13)$$

Similarly extending **N=N** we have, for the first problem

$$-\mathbf{X}\boldsymbol{\Delta} \sum_{J=1}^{4N} [\mathbf{H}_{-}^{OC}(\gamma_j^{OC})C_j^{OC}(\mu_\alpha) + \mathbf{H}_{+}^{OC}(\gamma_j^{OC})D_j^{OC}(\mu_\alpha)] = 0 \qquad (4.2.14)$$

For the second problem

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \times \left[\begin{pmatrix} [\mathbf{M}^{OC}(z+\varepsilon,+\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z+\varepsilon,+\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{OC}(z-\varepsilon,+\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z-\varepsilon,+\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} \right]$$

$$= \mathbf{X} \sum_{J=1}^{4\times2=8} [\mathbf{H}_{+}^{OC}(\gamma_j^{OC})C_j^{OC}(-\mu_\alpha) + \mathbf{H}_{-}^{OC}(\gamma_j^{OC})D_j^{OC}(-\mu_\alpha)] = 0. \qquad (4.2.15)$$

Similarly extending **N=N** we have, for the second problem

$$\mathbf{X} \sum_{J=1}^{4N} [\mathbf{H}_{+}^{OC}(\gamma_j^{OC})C_j^{OC}(-\mu_\alpha) + \mathbf{H}_{-}^{OC}(\gamma_j^{OC})D_j^{OC}(-\mu_\alpha)] = 0 \qquad (4.2.16)$$

$$
-\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 \end{bmatrix} \times \left[\begin{pmatrix} [\mathbf{M}^{OC}(z+\varepsilon,-\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z+\varepsilon,-\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} - \begin{pmatrix} [\mathbf{M}^{OC}(z-\varepsilon,-\mu_1;x,-\mu_\alpha)]^T \\ [\mathbf{M}^{OC}(z-\varepsilon,-\mu_2;x,-\mu_\alpha)]^T \end{pmatrix} \right]
$$

$$
= -\mathbf{X\Delta} \sum_{J=1}^{4\times 2=8} [\mathbf{H}_-^{OC}(\gamma_j^{OC})C_j^{OC}(-\mu_\alpha) + \mathbf{H}_+^{OC}(\gamma_j^{OC})D_j^{OC}(-\mu_\alpha)] = R_k . \tag{4.2.17}
$$

Similarly extending for **N=N** we have

$$
-\mathbf{X\Delta} \sum_{J=1}^{4N} [\mathbf{H}_-^{OC}(\gamma_j^{OC})C_j^{OC}(-\mu_\alpha) + \mathbf{H}_+^{OC}(\gamma_j^{OC})D_j^{OC}(-\mu_\alpha)] = R_k . \tag{4.2.18}
$$

Here we have introduced the (4NX4) matrix

$$
R_k = [I\delta_{1,k}, I\delta_{2,k}, \ldots, I\delta_{N,k}]^T \tag{4.2.19}
$$

We shall now solve the set of equations given by (4.2.3) and (4.2.5) [for N=2) or (4.2.4) and (4.2.5) to get the required coefficients for atmosphere. Similar consideration requires solution of equation pair (4.2.7) and (4.2.9) [For N=2] or equation (4.2.8) and (4.2.10). This requires certain orthogonality relations satisfied by the eigenvectors (3.5.1).

4.3. Derivation of orthogonality relations for eigenvectors:

Let us first define the **atmospheric adjoint problem** by replacing B_J^{AT} in (3.3.17& 3.3.18) with $[B_J^{AT}]^T$ and write the equation in the following form

$$
\left(\mathbf{I} - \frac{1}{\gamma}\mathbf{X}\right)\mathbf{AH}_+^{AT}(\gamma) = \frac{\omega^{AT}(z)}{2} \sum_{J=S}^{M} \mathbf{\Pi}(J,S)[B_J^{AT}]^T \mathbf{AT}_J^S(AT,\gamma) \tag{4.3.1}
$$

and

$$
\left(\mathbf{I} + \frac{1}{\gamma}\mathbf{X}\right)\mathbf{AH}_-^{AT}(\gamma) = \frac{\omega^{AT}(z)}{2} \sum_{J=S}^{M} (-1)^{l-J} \mathbf{\Pi}(l,J) D[B_J^{AT}]^T \mathbf{AT}_J^S(AT,\gamma) \tag{4.3.2}
$$

where

$$\mathbf{AT}_J^S(\mathrm{AT},\gamma)=\mathbf{\Pi}^T(\mathrm{l},\mathrm{J})\mathrm{W}\mathbf{AH}_+^{\mathrm{AT}}(\gamma)+(-1)^{\mathrm{l-J}}\mathrm{D}\mathbf{\Pi}^T(\mathrm{l},\mathrm{J})\mathrm{W}\mathbf{AH}_-^{\mathrm{AT}}(\gamma) \tag{4.3.3}$$

Similarly we can show for ocean

$$\left(\mathbf{I}-\frac{1}{\gamma}\mathbf{X}\right)\mathbf{AH}_+^{\mathrm{OC}}(\gamma)=\frac{\omega^{\mathrm{OC}}(z)}{2}\sum_{\mathrm{J=S}}^{\mathrm{M}}\mathbf{\Pi}(\mathrm{J},\mathrm{S})[\mathrm{B}_\mathrm{J}^{\mathrm{OC}}]^T\mathbf{AT}_J^S(\mathrm{AT},\gamma) \tag{4.3.4}$$

$$\left(\mathbf{I}+\frac{1}{\gamma}\mathbf{X}\right)\mathbf{AH}_-^{\mathrm{OC}}(\gamma)=\frac{\omega^{\mathrm{OC}}(z)}{2}\sum_{\mathrm{J=S}}^{\mathrm{M}}(-1)^{\mathrm{l-J}}\mathbf{\Pi}(\mathrm{J},\mathrm{S})\mathrm{D}[\mathrm{B}_\mathrm{J}^{\mathrm{OC}}]^T\mathbf{AT}_J^S(\mathrm{AT},\gamma) \tag{4.3.5}$$

$$\mathbf{AT}_J^S(\mathrm{AT},\gamma)=\mathbf{\Pi}^T(\mathrm{J},\mathrm{S})\mathrm{W}\mathbf{AH}_+^{\mathrm{OC}}(\gamma)+(-1)^{\mathrm{l-J}}\mathrm{D}\mathbf{\Pi}^T(\mathrm{l},\mathrm{J})\mathrm{W}\mathbf{AH}_-^{\mathrm{OC}}(\gamma) \tag{4.3.6}$$

We shall now show by actual calculation that the eigenvalues defined by the equations (3.3.37-3.3.40), where **A** and **B** matrices are involved, will not change if the above mentioned changes are introduced in the above equations. This means that the adjoint vectors $\mathbf{AH}_\pm^{\mathrm{AT}}(\gamma_j)$ are defined over the same spectrum as the vectors $\mathbf{H}_\pm^{\mathrm{AT}}(\gamma_j)$. This is also true for ocean also. We shall now deduce the orthogonality relations for the eigenvectors.

For Atmosphere:

Let us now evaluate equations (3.3.17) and (3.3.18) at $\gamma=\gamma_j$.

$$\left(\mathbf{I}-\frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_+^{\mathrm{AT}}(\gamma_j^{\mathrm{AT}})=\frac{\omega^{\mathrm{AT}}(z)}{2}\sum_{\mathrm{J=S}}^{\mathrm{M}}\mathbf{\Pi}(\mathrm{J},\mathrm{S})\mathrm{B}_\mathrm{J}^{\mathrm{AT}}\mathbf{T}_J^S(\mathrm{AT},\gamma_j^{\mathrm{AT}}) \tag{4.3.7}$$

$$\left(\mathbf{I}+\frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_-^{\mathrm{AT}}(\gamma_j^{\mathrm{AT}})=\frac{\omega^{\mathrm{AT}}(z)}{2}\sum_{\mathrm{J=S}}^{\mathrm{M}}\mathbf{\Pi}(\mathrm{J},\mathrm{S})(-1)^{\mathrm{J-S}}\mathrm{D}\mathrm{B}_\mathrm{J}^{\mathrm{AT}}\mathbf{T}_J^S(\mathrm{AT},\gamma_j^{\mathrm{AT}}) \tag{4.3.8}$$

After pre-multiplying (3.3.17) by $[\mathbf{AH}_+^{\mathrm{AT}}(\gamma_k)]^T\mathrm{W}$ $(\mathrm{k}\neq\mathrm{j})$ and equations (3.3.18) by $[\mathbf{AH}_-^{\mathrm{AT}}(\gamma_k)]^T\mathrm{W}$ we get

$$[\mathbf{AH}_+^{\mathrm{AT}}(\gamma_k)]^T\mathrm{W}\left(\mathbf{I}-\frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_+^{\mathrm{AT}}(\gamma_j^{\mathrm{AT}})=\frac{\omega^{\mathrm{AT}}(z)}{2}[\mathbf{AH}_+^{\mathrm{AT}}(\gamma_k)]^T\mathrm{W}\sum_{\mathrm{J=S}}^{\mathrm{M}}\mathbf{\Pi}(\mathrm{J},\mathrm{S})\mathrm{B}_\mathrm{J}^{\mathrm{AT}}\mathbf{T}_J^S(\mathrm{AT},\gamma_j^{\mathrm{AT}}) \tag{4.3.9}$$

$$[\mathbf{AH}_-^{\mathrm{AT}}(\gamma_k)]^T\mathrm{W}\left(\mathbf{I}+\frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_-^{\mathrm{AT}}(\gamma_j^{\mathrm{AT}})=\frac{\omega^{\mathrm{AT}}(z)}{2}[\mathbf{AH}_-^{\mathrm{AT}}(\gamma_k)]^T\mathrm{W}\sum_{\mathrm{J=S}}^{\mathrm{M}}\mathbf{\Pi}(\mathrm{J},\mathrm{S})(-1)^{\mathrm{J-S}}\mathrm{D}\mathrm{B}_\mathrm{J}^{\mathrm{AT}}\mathbf{T}_J^S(\mathrm{AT},\gamma_j^{\mathrm{AT}}). \tag{4.3.10}$$

We add these last two resulting equations to get the following equation

$$[\mathbf{AH}_+^{AT}(\gamma_k)]^T W\left(\mathbf{I} - \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_+^{AT}(\gamma_j^{AT}) + [\mathbf{AH}_-^{AT}(\gamma_k)]^T W\left(\mathbf{I} + \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_-^{AT}(\gamma_j^{AT}) =$$
$$\frac{\omega^{AT}(z)}{2}[\mathbf{AH}_+^{AT}(\gamma_k)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}\mathbf{T}_J^S(AT,\gamma_j^{AT}) +$$
$$\frac{\omega^{AT}(z)}{2}[\mathbf{AH}_-^{AT}(\gamma_k)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{AT}\mathbf{T}_J^S(AT,\gamma_j^{AT}). \tag{4.3.11}$$

Interchanging **j** and **k** in equation (4.3.11) we get

$$[\mathbf{AH}_+^{AT}(\gamma_j)]^T W\left(\mathbf{I} - \frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{H}_+^{AT}(\gamma_k^{AT}) + [\mathbf{AH}_-^{AT}(\gamma_j)]^T W\left(\mathbf{I} + \frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{H}_-^{AT}(\gamma_k^{AT}) =$$
$$\frac{\omega^{AT}(z)}{2}[\mathbf{AH}_+^{AT}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}\mathbf{T}_J^S(AT,\gamma_k^{AT}) +$$
$$\frac{\omega^{AT}(z)}{2}[\mathbf{AH}_-^{AT}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{AT}\mathbf{T}_J^S(AT,\gamma_k^{AT}). \tag{4.3.12}$$

We now pre multiply equation (4.3.11) with $[\mathbf{H}_+^{AT}(\gamma_j)]^T$ and equation (4.3.12) with $[\mathbf{H}_-^{AT}(\gamma_j)]^T$, then adding the two resulting equations and interchanging **j** and **k** we get,

$$[\mathbf{H}_+^{AT}(\gamma_j)]^T W\left(\mathbf{I} - \frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{AH}_+^{AT}(\gamma_k^{AT}) + [\mathbf{H}_-^{AT}(\gamma_j)]^T W\left(\mathbf{I} + \frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{AH}_-^{AT}(\gamma_k^{AT}) =$$
$$\frac{\omega^{AT}(z)}{2}[\mathbf{H}_-^{AT}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{AT}\mathbf{AT}_J^S(AT,\gamma_k^{AT}) +$$
$$\frac{\omega^{AT}(z)}{2}[\mathbf{H}_+^{AT}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}\mathbf{AT}_J^S(AT,\gamma_k^{AT}). \tag{4.3.13}$$

Next we set B_J^{AT} as $[B_J^{AT}]^T$ in (3.3.17) and take transpose resulting equation and subtract this equation from (4.3.11) to get,

$$\mathbf{AH}_+^{AT}(\gamma_k)^T W\mathbf{XH}_+^{AT}(\gamma_j) - \mathbf{AH}_-^{AT}(\gamma_k)^T W\mathbf{XH}_-^{AT}(\gamma_j) = 0, \quad \gamma_j \neq \gamma_k. \tag{4.3.14}$$

The same procedure when applied to the same set of equation but interchanging the order of the multiplier eigenvectors leads to the following equation

$$[\mathbf{AH}_{-}^{AT}(\gamma_k)]^T \mathbf{WXH}_{-}^{AT}(\gamma_j) - [\mathbf{AH}_{-}^{AT}(\gamma_k)]^T \mathbf{WXH}_{+}^{AT}(\gamma_j) = 0. \tag{4.3.15}$$

These last two equations are called orthogonality conditions satisfied by the eigenvectors. Now, we can again pre-multiply equation (4.2.1) by $[\mathbf{AH}_{+}^{AT}(\gamma_k)]^T \mathbf{W}$ and equation (4.2.6) by $[\mathbf{AH}_{-}^{AT}(\gamma_k)]^T \mathbf{W\Delta}$ and add the two resulting equations to find

$$\sum_{j=1}^{4N} \mathbf{AH}_{+}^{AT}(\gamma_k)\mathbf{WX}\left[\mathbf{H}_{+}^{AT}(\gamma_j)\mathbf{C}_j^{AT} + \mathbf{H}_{-}^{AT}(\gamma_j)\mathbf{D}_j^{AT}\right] + \mathbf{AH}_{-}^{AT}(\gamma_k)\mathbf{W\Delta}\left[-\mathbf{X\Delta}\left[\mathbf{H}_{-}^{AT}(\gamma_j)\mathbf{C}_j^{AT} + \mathbf{H}_{+}^{AT}(\gamma_j)\mathbf{D}_j^{AT}\right]\right]$$
$$= \mathbf{AH}_{+}^{AT}(\gamma_k)\mathbf{WR}_\alpha \tag{4.3.16}$$

For we have $\mathbf{j} = \mathbf{k}$ after noting the two orthogonality relations (4.3.14) and (4.3.15),

$$\mathbf{C}_j^{AT}(\mu_\alpha) = \frac{1}{\mathbf{NAT}(\gamma_j)}[\mathbf{AH}_{+}^{AT}(\gamma_j)]^T \mathbf{WR}_\alpha \tag{4.3.17}$$

Where

$$\mathbf{NAT}(\gamma_j) = [\mathbf{AH}_{+}^{AT}(\gamma_j)]^T \mathbf{WXH}_{+}^{AT}(\gamma_j) - [\mathbf{AH}_{-}^{AT}(\gamma_j)]^T \mathbf{WXH}_{-}^{AT}(\gamma_j). \tag{4.3.18}$$

We can now calculate the equations (4.3.17) and (4.3.18). In our numerical calculations we have tabulated the values of NAT since these values are required for (4.3.17).
Continuing, we shall pre-multiply equation (4.2.1) by $[\mathbf{AH}_{-}^{AT}(\gamma_k)]^T \mathbf{W}$ and equation (4.2.6) by $[\mathbf{AH}_{+}^{AT}(\gamma_k)]^T \mathbf{W\Delta}$, and add the two resulting equations and use equations (4.3.14) and (4.3.15) to obtain

$$\mathbf{D}_j^{AT}(\mu_\alpha) = -\frac{1}{\mathbf{NOC}(\gamma_j)}[\mathbf{AH}_{-}^{AT}(\gamma_j)]^T \mathbf{WR}_k \tag{4.3.19}$$

Similarly we find from equation (4.2.2)

$$\mathbf{C}_j^{AT}(-\mu_\alpha) = \frac{1}{\mathbf{NAT}(\gamma_j)}[\mathbf{AH}_{-}^{AT}(\gamma_j)]^T \mathbf{W\Delta R}_k \tag{4.3.20}$$

and

$$\mathbf{D}_j^{AT}(-\mu_\alpha) = -\frac{1}{\mathbf{NAT}(\gamma_j)}[\mathbf{AH}_{+}^{AT}(\gamma_j)]^T \mathbf{W\Delta R}_k \tag{4.3.21}$$

The required constants C and D in equations (4.1.24) and (4.1.36) are defined by equations (4.3.17) & (4.3.19) and (4.3.20 & 4.3.21).

For ocean:

$$\left(\mathbf{I} - \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_{+}^{OC}(\gamma_j^{OC}) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{T}_J^S(OC,\gamma_j^{OC}) \qquad (4.3.22)$$

$$\left(\mathbf{I} + \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_{-}^{OC}(\gamma_j^{OC}) = \frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{T}_J^S(OC,\gamma_j^{OC}) \qquad (4.3.23)$$

After pre-multiplying (3.3.19) by $[\mathbf{AH}_{+}^{OC}(\gamma_k)]^T W$ $(k \neq j)$ and equation (3.3.20) by $[\mathbf{AH}_{-}^{OC}(\gamma_k)]^T W$ we get

$$[\mathbf{AH}_{+}^{OC}(\gamma_k)]^T W\left(\mathbf{I} - \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_{+}^{OC}(\gamma_j^{OC}) = \frac{\omega^{OC}(z)}{2}[\mathbf{AH}_{+}^{OC}(\gamma_k)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{T}_J^S(OC,\gamma_j^{OC}) \qquad (4.3.24)$$

$$[\mathbf{AH}_{-}^{OC}(\gamma_k)]^T W\left(\mathbf{I} + \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_{-}^{OC}(\gamma_j^{OC}) = \frac{\omega^{OC}(z)}{2}[\mathbf{AH}_{-}^{OC}(\gamma_k)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{T}_J^S(OC,\gamma_j^{OC}) \qquad (4.3.25)$$

We add these last two resulting equations to get the following equation

$$[\mathbf{AH}_{+}^{OC}(\gamma_k)]^T W\left(\mathbf{I} - \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_{+}^{OC}(\gamma_j^{OC}) + [\mathbf{AH}_{-}^{OC}(\gamma_k)]^T W\left(\mathbf{I} + \frac{1}{\gamma_j}\mathbf{X}\right)\mathbf{H}_{-}^{OC}(\gamma_j^{OC}) =$$

$$\frac{\omega^{OC}(z)}{2}[\mathbf{AH}_{+}^{OC}(\gamma_k)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{T}_J^S(OC,\gamma_j^{OC}) +$$
$$\frac{\omega^{OC}(z)}{2}[\mathbf{AH}_{-}^{OC}(\gamma_k)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{T}_J^S(OC,\gamma_j^{OC}). \qquad (4.3.26)$$

Interchanging **j** and **k** in (4.3.26) we get

$$[\mathbf{AH}_{+}^{OC}(\gamma_j)]^T W\left(\mathbf{I} - \frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{H}_{+}^{OC}(\gamma_k^{OC}) + [\mathbf{AH}_{-}^{OC}(\gamma_j)]^T W\left(\mathbf{I} + \frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{H}_{-}^{OC}(\gamma_k^{OC}) =$$
$$\frac{\omega^{OC}(z)}{2}[\mathbf{AH}_{+}^{OC}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{T}_J^S(OC,\gamma_k^{OC}) +$$

$$\frac{\omega^{OC}(z)}{2}[\mathbf{AH}_-^{OC}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{T}_J^S(OC,\gamma_k^{OC}). \tag{4.3.27}$$

We now pre multiply (3.3.19) with $[\mathbf{H}_+^{OC}(\gamma_j)]^T$ and (3.3.20) with $[\mathbf{H}_-^{OC}(\gamma_j)]^T$, and then adding the two resulting equations and interchange **j** and **k**, we get,

$$[\mathbf{H}_+^{OC}(\gamma_j)]^T W\left(\mathbf{I}-\frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{AH}_+^{OC}(\gamma_k^{OC})+[\mathbf{H}_-^{OC}(\gamma_j)]^T W\left(\mathbf{I}+\frac{1}{\gamma_k}\mathbf{X}\right)\mathbf{AH}_-^{OC}(\gamma_k^{OC})=$$
$$\frac{\omega^{OC}(z)}{2}[\mathbf{H}_-^{OC}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{AT}_J^S(OC,\gamma_k^{OC})+$$
$$\frac{\omega^{OC}(z)}{2}[\mathbf{H}_+^{OC}(\gamma_j)]^T W\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{AT}_J^S(OC,\gamma_k^{OC}). \tag{4.3.28}$$

Next we set B_J^{OC} as $[B_J^{OC}]^T$ in (4.3.28) and subtract the new transformed equation from (4.3.28) to get,

$$\mathbf{AH}_+^{OC}(\gamma_k)^T W\mathbf{XH}_+^{OC}(\gamma_j)-\mathbf{AH}_-^{OC}(\gamma_k)^T W\mathbf{XH}_-^{OC}(\gamma_j)=0,\quad \gamma_j\neq\gamma_k. \tag{4.3.29}$$

The same procedure when applied to the same set of equation but interchanging the order of the multiplier eigenvectors leads to the following equation

$$[\mathbf{AH}_-^{OC}(v_k)]^T W\mathbf{XH}_-^{OC}(v_j)-[\mathbf{AH}_-^{OC}(v_k)]^T W\mathbf{XH}_+^{OC}(v_j)=0. \tag{4.3.30}$$

These two equations are called orthogonality conditions satisfied by the eigenvectors.
Now, we can again pre-multiply equation (4.2.1) by $[\mathbf{AH}_+^{OC}(\gamma_k)]^T W$ and equation (4.2.6) by

$[\mathbf{AH}_-^{OC}(\gamma_k)]^T W\mathbf{\Delta}$

and add the two resulting equations to find

$$\sum_{j=1}^{4N}\mathbf{AH}_+^{OC}(\gamma_k)W\mathbf{X}\left[\mathbf{H}_+^{OC}(\gamma_j)C_j^{OC}+\mathbf{H}_-^{OC}(\gamma_j)D_j^{OC}\right]+\mathbf{AH}_-^{OC}(\gamma_k)W\mathbf{\Delta}\left[-\mathbf{X\Delta}\left[\mathbf{H}_-^{OC}(\gamma_j)C_j^{OC}+\mathbf{H}_+^{OC}(\gamma_j)D_j^{OC}\right]\right]$$
$$=\mathbf{AH}_+^{OC}(\gamma_k)WR_\alpha \tag{4.3.31}$$

For we have $\mathbf{j}=\mathbf{k}$ after noting the two orthogonality relations in equations (4.3.24-4.3.25).

$$C_j^{OC}(\mu_\alpha)=\frac{1}{NOC(\gamma_j)}[\mathbf{AH}_+^{OC}(\gamma_j)]^T WR_\alpha \tag{4.3.32}$$

Where

$$NOC(\gamma_j)=[\mathbf{AH}_+^{OC}(\gamma_j)]^T W\mathbf{XH}_+^{OC}(\gamma_j)-[\mathbf{AH}_-^{OC}(\gamma_j)]^T W\mathbf{XH}_-^{OC}(\gamma_j). \tag{4.3.33}$$

We shall now use to calculate the equations (4.3.32) and (4.3.33).
Continuing, we pre-multiply equation (4.2.1) by $[\mathbf{AH}_-^{OC}(\gamma_k)]^T W$ and equation (4.2.6) by

$[\mathbf{AH}_{+}^{OC}(\gamma_k)]^T \mathbf{W\Delta}$, add the two resulting equations and note equations (4.3.29) and (4.3.30) to obtain

$$D_j^{OC}(\mu_\alpha) = -\frac{1}{NOC(\gamma_j)}[\mathbf{AH}_{-}^{OC}(\gamma_j)]^T \mathbf{W} \mathbf{R}_k. \tag{4.3.34}$$

Similarly we find from equation (3.2.2)

$$C_j^{OC}(-\mu_\alpha) = \frac{1}{NOC(\gamma_j)}[\mathbf{AH}_{-}^{OC}(\gamma_j)]^T \mathbf{W\Delta} \mathbf{R}_k. \tag{4.3.35}$$

and

$$D_j^{OC}(-\mu_\alpha) = -\frac{1}{NOC(\gamma_j)}[\mathbf{AH}_{+}^{OC}(\gamma_j)]^T \mathbf{W\Delta} \mathbf{R}_k. \tag{4.3.36}$$

Having found the expansion coefficients for green functions we are now in a position to determine the particular solution for the inhomogeneous source term in the RTE. This is the objective of the next section. It is to be noted from equation (4.3.17) and (4.3.32) that determination of adjoint vectors is essential. Here we have adopted the standard text book procedure. If W and A are matrices such that W'*A = D*W', the columns of W are the left eigenvectors of A. We have used standard LAPACK algorithm used in Matlab 7 to calculate the left eigenvector as well as the right eigenvectors and corresponding eigenvalues. These left eigenvectors are used to find the adjoint vectors using equation (3.5.1).

4.4. Derivation of particular solution: To determine particular solution we recall our transfer equations in the following form.

For atmosphere

$$\mu_i \frac{d}{dz} I_{AT}(z,\mu_i) + I_{AT}(z,\mu_i) = \frac{\omega^{AT}(z)}{2} \sum_{J=S}^{M} \mathbf{P}_J^S(\mu_i) B_J^{AT} \sum_{\beta=1}^{N} \omega_\beta I_{J,\beta;AT}(z) + S_{AT}(z,\mu_i), \tag{4.4.1}$$

and

$$-\mu_i \frac{d}{dz} I_{AT}(z,-\mu_i) + I_{AT}(z,-\mu_i) = \frac{\omega^{AT}(z)}{2} \sum_{J=S}^{M} \mathbf{P}_J^S(-\mu_i) B_J^{AT} \sum_{\beta=1}^{N} \omega_\beta I_{J,\beta;AT}(z) + S_{AT}(z,-\mu_i), \tag{4.4.2}$$

for $i = 1,2,\ldots,N$. In writing equations (3.4.1 & 3.4.2), we have used

$$I_{J,\beta;AT}(z) = \mathbf{P}_J^S(\mu_\beta) I(z,\mu_\beta) + \mathbf{P}_J^S(-\mu_\beta) I(z,-\mu_\beta) \tag{4.4.3}$$

For ocean:

$$\mu_i \frac{d}{dz} I_{OC}(z,\mu_i) + I_{OC}(z,\mu_i) = \frac{\omega^{OC}(z)}{2} \sum_{J=S}^{M} \mathbf{P}_J^S(\mu_i) B_J^{OC} \sum_{\beta=1}^{N} \omega_\beta I_{J,\beta;OC}(z) + S_{OC}(z,\mu_i), \tag{4.4.4}$$

and

$$-\mu_i \frac{d}{dz} I_{OC}(z,-\mu_i) + I_{OC}(z,-\mu_i) = \frac{\omega^{OC}(z)}{2} \sum_{J=S}^{M} \mathbf{P}_J^S(-\mu_i) B_J^{OC} \sum_{\beta=1}^{N} \omega_\beta I_{J,\beta;OC}(z) + S_{OC}(z,-\mu_i), \quad (4.4.5)$$

for $i = 1,2,...,N$. Here we have defined

$$I_{J,\beta;OC}(z) = \mathbf{P}_J^S(\mu_\beta) I(z,\mu_\beta) + \mathbf{P}_J^S(-\mu_\beta) I(z,-\mu_\beta). \quad (4.4.6)$$

Particular solution for atmosphere:

The general solution to the homogeneous version of equations (4.4.1 - 4.4.2) is given by (3.5.13 – 3.5.16). With the help of green functions developed earlier for z>x and z<x, we can immediately write one particular solution in the following manner

$$I_{AT}^p(z,+\mu_i) = \sum_{\alpha=1}^{N} \int_0^{z_\omega} [\mathbf{M}^{AT}(z,+\mu_i : x,\mu_\alpha) S_{AT}(x,\mu_\alpha) + \mathbf{M}^{AT}(z,+\mu_i : x,-\mu_\alpha) S_{AT}(x,-\mu_\alpha)]\, dx \quad (4.4.7)$$

or

$$I_{AT}^p(+;z) = \sum_{\alpha=1}^{N} \int_0^{z_\omega} [\mathbf{M}_+^{AT}(z : x,\mu_\alpha) S_{AT}(x,\mu_\alpha) + \mathbf{M}_+^{AT}(z : x,-\mu_\alpha) S_{AT}(x,-\mu_\alpha)]\, dx \quad (4.4.8)$$

and

$$I_{AT}^p(z,-\mu_i) = \sum_{\alpha=1}^{N} \int_0^{z_\omega} [\mathbf{M}^{AT}(z,-\mu_i : x,\mu_\alpha) S_{AT}(x,\mu_\alpha) + \mathbf{M}^{AT}(z,-\mu_i : x,-\mu_\alpha) S_{AT}(x,-\mu_\alpha)]\, dx \quad (4.4.9)$$

or

$$I_{AT}^p(-;z) = \sum_{\alpha=1}^{N} \int_0^{z_\omega} [\mathbf{M}_-^{AT}(z : x,\mu_\alpha) S_{AT}(x,\mu_\alpha) + \mathbf{M}_-^{AT}(z : x,-\mu_\alpha) S_{AT}(x,-\mu_\alpha)]\, dx \quad (4.4.10)$$

Substituting from equations (4.1.24) and (4.1.36), we can rewrite equation (4.4.8 &4.4.10) as

$$I_{AT}^p(+;z) = \sum_{j=1}^{4N} \left[\Re_j^{AT}(z) \mathbf{H}_+^{AT}(\gamma_j) + \aleph_j^{AT}(z) \mathbf{H}_-^{AT}(\gamma_j)\right] \quad (4.4.11)$$

and

$$I_{AT}^p(-;z) = \mathbf{\Delta} \sum_{j=1}^{4N} [\Re_j^{AT}(z) \mathbf{H}_-^{AT}(\gamma_j) + \aleph_j^{AT}(z) \mathbf{H}_+^{AT}(\gamma_j)] \quad (4.4.12)$$

Where by simple substitution we find

$$\Re_j^{AT}(z) = \int_0^{z} \sum_{\alpha=1}^{N} C_j^{AT}(\mu_\alpha) S_{AT}(x,\mu_\alpha) + C_j^{AT}(-\mu_\alpha) S_{AT}(x,-\mu_\alpha)] \exp\left(-\frac{z-x}{\gamma_j}\right) dx \quad (4.4.13)$$

and

$$\aleph_j^{AT}(z) = -\int_z^{z_\omega} \sum_{\alpha=1}^{N} [D_j^{AT}(\mu_\alpha) S_{AT}(x,\mu_\alpha) + D_j^{AT}(-\mu_\alpha) S_{AT}(x,-\mu_\alpha)] \exp\left(-\frac{x-z}{\gamma_j}\right) dx. \quad (4.4.14)$$

To complete our general result we now use equations (4.3.17) and (4.320) and (4.3.19) and (4.3.21) in equations (4.4.13 &4.4.14) to obtain,

$$\Re_j^{AT}(z)=\int_0^z a_j^{AT}(x)\exp\left(-\frac{z-x}{\gamma_j}\right)dx \text{ and } \aleph_j^{AT}(z)=\int_z^{z_\omega} b_j^{AT}(x)\exp\left(-\frac{x-z}{\gamma_j}\right)dx. \qquad (4.4.15)$$

where

$$a_j^{AT}(x)=\frac{1}{NAT(\gamma_j)}\left[[\mathbf{AH}_+^{AT}(\gamma_j)]^T W S_{AT(+)}(x)+[\mathbf{AH}_-^{AT}(\gamma_j)]^T W\mathbf{\Delta} S_{AT(-)}(x)\right] \qquad (4.4.16)$$

and

$$b_j^{AT}(x)=\frac{1}{NAT(-\gamma_j)}\left[[\mathbf{AH}_-^{AT}(\gamma_j)]^T W S_{AT(+)}(x)+[\mathbf{AH}_+^{AT}(\gamma_j)]^T W\mathbf{\Delta} S_{AT(-)}(x)\right] \qquad (4.4.17)$$

In which we have set

$$S_{AT(+)}(x)=\left[[S_{AT}(x,\mu_1)]^T,[S_{AT}(x,\mu_2)]^T,\ldots,[S_{AT}(x,\mu_N)]^T\right]^T \qquad (4.4.18)$$

and

$$S_{AT(-)}(x)=\left[[S_{AT}(x,-\mu_1)]^T,[S_{AT}(x,-\mu_2)]^T,\ldots,[S_{AT}(x,-\mu_N)]^T\right]^T. \qquad (4.4.19)$$

Following similar steps we get similar sets of equations for ocean

$$I_{OC}^p(z,+\mu_i)=\sum_{\alpha=1}^{N}\int_0^{z_0}[\mathbf{M}^{OC}(z,+\mu_i:x,\mu_\alpha)S_{OC}(x,\mu_\alpha)+\mathbf{M}^{OC}(z,+\mu_i:x,-\mu_\alpha)S_{OC}(x,-\mu_\alpha)]\,dx \qquad (4.4.20)$$

$$I_{OC}^p(z,-\mu_i)=\sum_{\alpha=1}^{N}\int_0^{z_0}[\mathbf{M}^{OC}(z,-\mu_i:x,\mu_\alpha)S_{OC}(x,\mu_\alpha)+\mathbf{M}^{OC}(z,-\mu_i:x,-\mu_\alpha)S_{OC}(x,-\mu_\alpha)]\,dx \qquad (4.4.21)$$

Or,

$$I_{OC}^p(+;z)=\sum_{\alpha=1}^{N}\int_0^{z_0}[\mathbf{M}_+^{OC}(z:x,\mu_\alpha)S_{OC}(x,\mu_\alpha)+\mathbf{M}_+^{OC}(z:x,-\mu_\alpha)S_{OC}(x,-\mu_\alpha)]\,dx \qquad (4.4.22)$$

$$I_{OC}^p(-;z)=\sum_{\alpha=1}^{N}\int_0^{z_0}\left[\mathbf{M}_-^{OC}(z:x,\mu_\alpha)S_{OC}(x,\mu_\alpha)+\mathbf{M}_-^{OC}(z:x,-\mu_\alpha)S_{OC}(x,-\mu_\alpha)\right]dx. \qquad (4.4.23)$$

Now using equations (4.1.30) and (4.142), we can rewrite equations (4.4.22 & 4.4.23) as

$$I_{OC}^p(+;z)=\sum_{j=1}^{4N}\left[\Re_j^{OC}(z)\mathbf{H}_+^{OC}(\gamma_j)+\aleph_j^{OC}(z)\mathbf{H}_-^{OC}(\gamma_j)\right] \qquad (4.4.24)$$

and

$$I_{OC}^p(-;z)=\mathbf{\Delta}\sum_{j=1}^{4N}\left[\Re_j^{OC}(z)\mathbf{H}_-^{OC}(\gamma_j)+\aleph_j^{OC}(z)\mathbf{H}_+^{OC}(\gamma_j)\right] \qquad (4.4.25)$$

where

$$\Re_j^{OC}(z)=\int_0^z\sum_{\alpha=1}^{N}C_j^{OC}(\mu_\alpha)S_{OC}(x,\mu_\alpha)+C_j^{OC}(-\mu_\alpha)S_{OC}(x,-\mu_\alpha)]\exp\left(-\frac{z-x}{\gamma_j}\right)dx \tag{4.4.26}$$

and

$$\aleph_j^{OC}(z)=-\int_z^{z_1}\sum_{\alpha=1}^{N}D_j^{OC}(\mu_\alpha)S_{OC}(x,\mu_\alpha)+D_j^{OC}(-\mu_\alpha)S_{OC}(x,-\mu_\alpha)]\exp\left(-\frac{x-z}{\gamma_j}\right)dx. \tag{4.4.27}$$

To complete our general result we now use equations (4.3.32 – 4.3.36) in equation (4.4.26) to obtain

$$\Re_j^{OC}(z)=\int_0^z a_j^{OC}(x)\exp\left(-\frac{z-x}{\gamma_j}\right)dx \quad \text{and} \quad \aleph_j^{OC}(z)=\int_z^{z_1} b_j^{OC}(x)\exp\left(-\frac{x-z}{\gamma_j}\right)dx. \tag{4.4.28}$$

where

$$a_j^{OC}(x)=\frac{1}{NOC(\gamma_j)}\left[[\mathbf{AH}_+^{OC}(\gamma_j)]^T WS_{OC(+)}(x)+[\mathbf{AH}_-^{OC}(\gamma_j)]^T W\mathbf{\Delta}S_{OC(-)}(x)\right] \tag{4.4.29}$$

and

$$b_j^{OC}(x)=\frac{1}{NOC(-\gamma_j)}\left[[\mathbf{AH}_-^{OC}(\gamma_j)]^T WS_{OC\,(+)}(x)+[\mathbf{AH}_+^{OC}(\gamma_j)]^T W\mathbf{\Delta}S_{OC\,(-)}(x)\right]; \tag{4.4.30}$$

and where

$$S_{OC\,(+)}(x)=\left[[S_{OC}(x,\mu_1)]^T,[S_{OC}(x,\mu_2)]^T,\ldots,[S_{OC}(x,\mu_N)]^T\right]^T; \tag{4.4.31}$$

and

$$S_{OC\,(-)}(x)=\left[[S_{OC}(x,-\mu_1)]^T,[S_{OC}(x,-\mu_2)]^T,\ldots,[S_{OC}(x,-\mu_N)]^T\right]^T. \tag{4.4.32}$$

Our model of source function [vide equations (2.12.15) and (2.12.19)] can be written in the following form

$$S_{AT(\pm)}(x)=S_{AT}(\pm)\exp\left(-\frac{x}{\mu_0}\right) \tag{4.4.33}$$

$$S_{OC\,(\pm)}(x)=S_{OC}(\pm)\exp\left(-\frac{x}{\mu_{0n}}\right) \tag{4.4.34}$$

We can now evaluate $S_{AT/OC\,(\pm)}(x)$ for each μ_i from (2.12.15) and (2.12.19). Using these two expressions in (4.4.16) & (4.4.17) and (4.4.29) & (4.4.30) we easily get

For atmosphere:

$$a_j^{AT}=\frac{1}{NAT(\gamma_j)}\left[[\mathbf{AH}_+^{AT}(\gamma_j)]^T WS_{AT}(+)+[\mathbf{AH}_-^{AT}(\gamma_j)]^T W\mathbf{\Delta}S_{AT}(-)\right], \tag{4.4.35}$$

and

$$b_j^{AT} = \frac{1}{NAT(-\gamma_j)}\Big[[\mathbf{AH}_-^{AT}(\gamma_j)]^T WS_{AT}(+) + [\mathbf{AH}_+^{AT}(\gamma_j)]^T W\mathbf{\Delta} S_{AT}(-)\Big]. \qquad (4.4.36)$$

$$\Re_j^{AT}(z) = \mu_0 \gamma_j^{AT} a_j^{AT} CAT(z:\gamma_j^{AT},\mu_0) \qquad (4.4.37)$$

$$\aleph_j^{AT}(z) = \mu_0 \gamma_j^{AT} b_j^{AT} \exp(-z/\mu_0) SAT(z_\omega - z:\gamma_j^{AT},\mu_0). \qquad (4.4.38)$$

$$SAT(z:x,y) = \frac{1-\exp\left(-\frac{z}{x}\right)\exp\left(-\frac{z}{y}\right)}{x+y} \quad (4.4.39) \ \& \ CAT(z:x,y) = \frac{\exp\left(-\frac{z}{x}\right)-\exp\left(-\frac{z}{y}\right)}{x-y}. \qquad (4.4.40)$$

For ocean:

$$a_j^{OC} = \frac{1}{NOC(\gamma_j)}\Big[[\mathbf{AH}_+^{OC}(\gamma_j)]^T WS_{OC}(+) + [\mathbf{AH}_-^{OC}(\gamma_j)]^T W\mathbf{\Delta} S_{OC}(-)\Big] \qquad (4.4.41)$$

and

$$b_j^{OC} = \frac{1}{NOC(\gamma_j)}\Big[[\mathbf{AH}_-^{OC}(\gamma_j)]^T WS_{OC}(+) + [\mathbf{AH}_+^{OC}(\gamma_j)]^T W\mathbf{\Delta} S_{OC}(-)\Big]. \qquad (4.4.42)$$

$$\Re_j^{OC}(z) = \mu_{0n} \gamma_j^{OC} a_j^{OC} COC(z:\gamma_j,\mu_{0n}) \qquad (4.4.43)$$

$$\aleph_j^{OC}(z) = \mu_{0n} \gamma_j^{OC} b_j^{OC} \exp(-z/\mu_{0n}) SOC(z_1 - z:\gamma_j,\mu_{0n}). \qquad (4.4.44)$$

$$SOC(z:x,y) = \frac{1-\exp\left(-\frac{z}{x}\right)\exp\left(-\frac{z}{y}\right)}{x+y}; \quad (4.4.45) \ \& \ COC(z:x,y) = \frac{\exp\left(-\frac{z}{x}\right)-\exp\left(-\frac{z}{y}\right)}{x-y}. \qquad (4.4.46)$$

We shall now consider the case of **complex separation constants**. For real quantities if we let quantities with **asterisks** as **complex conjugates**

$$AZ_\pm^{AT/OC}(z,\gamma_j^{AT/OC}) = \Re_j^{AT/OC}(z)\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC}) + \Re_j^{AT/OC\,*}(z)\mathbf{H}_\pm^{AT/OC\,*}(\gamma_j^{AT/OC}); \qquad (4.4.47)$$

$$BZ_\pm^{AT/OC}(z,\lambda_j^{AT/OC}) = \aleph_j^{AT/OC}(z)\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC}) + \aleph_j^{AT/OC\,*}(z)\mathbf{H}_\pm^{AT/OC\,*}(\gamma_j^{AT/OC}). \qquad (4.4.48)$$

The **complete particular solution** can now be written as

$$\mathbf{I}^{p}_{AT}(+;z)=\sum_{j=1}^{Nr}[\Re_j^{AT}(z)\mathbf{H}_{+}^{AT}(\gamma_j)+\aleph_j^{AT}(z)\mathbf{H}_{-}^{AT}(\gamma_j)]+\sum_{j=1}^{Nc} AZ_{+}^{AT}(z,\gamma_j^{AT})+BZ_{-}^{AT}(z,\gamma_j^{AT}) \qquad (4.4.49)$$

$$\mathbf{I}^{p}_{AT}(-;z)=\mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(z)\mathbf{H}_{-}^{AT}(\gamma_j)+\aleph_j^{AT}(z)\mathbf{H}_{+}^{AT}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc} AZ_{-}^{AT}(z,\gamma_j^{AT})+BZ_{+}^{AT}(z,\lambda_j^{AT}) \qquad (4.4.50)$$

$$\mathbf{I}^{p}_{OC}(+;z)=\sum_{j=1}^{Nr}[\Re_j^{OC}(z)\mathbf{H}_{+}^{OC}(\gamma_j)+\aleph_j^{OC}(z)\mathbf{H}_{-}^{OC}(\gamma_j)]+\sum_{j=1}^{Nc} AZ_{+}^{OC}(z,\gamma_j^{OC})+BZ_{-}^{OC}(z,\lambda_j^{OC}) \qquad (4.4.51)$$

$$\mathbf{I}^{p}_{OC}(-;z)=\mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{OC}(z)\mathbf{H}_{-}^{OC}(\gamma_j)+\aleph_j^{OC}(z)\mathbf{H}_{+}^{OC}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc} [AZ_{-}^{OC}(z,\gamma_j^{OC})+BZ_{+}^{OC}(z,\lambda_j^{OC})] \qquad (4.4.52)$$

We now find out appropriate expressions of the particular solutions suitable for application in the boundary conditions.

For top boundary condition the following particular form will be used by setting **z=0** in (4.4.50)

$$\mathbf{I}^{p}_{AT}(-;0)=\mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(0)\mathbf{H}_{-}^{AT}(\gamma_j)+\aleph_j^{AT}(0)\mathbf{H}_{+}^{AT}(\gamma_j)]+\sum_{j=1}^{Nc} AZ_{+}^{AT}(0,\gamma_j^{AT})+BZ_{-}^{AT}(0,\gamma_j^{AT}) \qquad (4.4.53)$$

The particular form of solutions for application in the second and third boundary conditions are given by

$$[\mathbf{I}^{p}_{AT}(+;z_\omega)]=\sum_{j=1}^{Nr}[\Re_j^{AT}(z_\omega)\mathbf{H}_{+}^{AT}(\gamma_j)+\aleph_j^{AT}(z_\omega)\mathbf{H}_{-}^{AT}(\gamma_j)]+\sum_{j=1}^{Nc} AZ_{+}^{AT}(z_\omega,\gamma_j^{AT})+BZ_{-}^{AT}(z_\omega,\gamma_j^{AT}) \qquad (4.4.54)$$

$$[\mathbf{I}^{p}_{AT}(-;z_\omega)]=\mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(z_\omega)\mathbf{H}_{-}^{AT}(\gamma_j)+\aleph_j^{AT}(z_\omega)\mathbf{H}_{+}^{AT}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc} AZ_{-}^{AT}(z_\omega,\gamma_j^{AT})+BZ_{+}^{AT}(z_\omega,\lambda_j^{AT}) \qquad (4.4.55)$$

$$[\mathbf{I}^{p}_{OC}(+;z_\omega)]=\sum_{j=1}^{Nr}[\Re_j^{OC}(z_\omega)\mathbf{H}_{+}^{OC}(\gamma_j)+\aleph_j^{OC}(z_\omega)\mathbf{H}_{-}^{OC}(\gamma_j)]+\sum_{j=1}^{Nc} AZ_{+}^{OC}(z_\omega,\gamma_j^{OC})+BZ_{-}^{OC}(z_\omega,\lambda_j^{OC}) \qquad (4.4.56)$$

$$[\mathbf{I}^{p}_{OC}(-;z_\omega)]=\mathbf{\Delta}[\sum_{j=1}^{Nr}\Re_j^{OC}(z_\omega)\mathbf{H}_{-}^{OC}(\gamma_j)+\aleph_j^{OC}(z_\omega)\mathbf{H}_{+}^{OC}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc} [AZ_{-}^{OC}(z_\omega,\gamma_j^{OC})+BZ_{+}^{OC}(z\omega,\lambda_j^{OC})] \qquad (4.4.57)$$

The particular solution that will be used in the bottom boundary condition

$$[I_{OC}^{p}(+;z_1)]=\sum_{j=1}^{Nr}\Re_j^{OC}(z_1)\mathbf{H}_{+}^{OC}(\gamma_j)+\aleph_j^{OC}(z_1)\mathbf{H}_{-}^{OC}(\gamma_j)+\sum_{j=1}^{Nc}AZ_{+}^{OC}(z_1,\gamma_j^{OC})+BZ_{-}^{OC}(z_1,\lambda_j^{OC}) \quad (4.4.58)$$

We also have

$$\Re_j^{AT}(0)=\mu_0\gamma_j^{AT}a_j^{AT}CAT(0:\gamma_j,\mu_0)=0\ \forall\ j. \quad (4.4.59)$$

$$\aleph_j^{AT}(0)=\mu_0\gamma_j^{AT}\ b_j^{AT}\ SAT(z_\omega:\gamma_j,\mu_0). \quad (4.4.60)$$

$$\Re_j^{AT}(z_\omega)=\mu_0\gamma_j^{AT}a_j^{AT}CAT(z_\omega:\gamma_j,\mu_0). \quad (4.4.61)$$

$$\aleph_j^{AT}(z_\omega)=\mu_0\gamma_j^{AT}\ b_j^{AT}\ SAT(0:\gamma_j,\mu_0)=0\ \forall\ j. \quad (4.4.62)$$

4.5. Complete solution:

We can now write down the complete solution in the following form.

$$I_{AT+}(z)=RE_{+}^{AT}(z)+CO_{+}^{AT}(z)+I_{AT}^{p}(+;z); \quad (4.5.1)$$

$$I_{AT-}(z)=RE_{-}^{AT}(z)+CO_{-}^{AT}(z)+I_{AT}^{p}(-;z). \quad (4.5.2)$$

$$I_{OC+}(z)=RE_{+}^{OC}(z)+CO_{+}^{OC}(z)+I_{OC}^{p}(+;z); \quad (4.5.3)$$

$$I_{OC-}(z)=RE_{-}^{OC}(z)+CO_{-}^{OC}(z)+I_{OC}^{p}(-;z). \quad (4.5.4)$$

Boundary conditions for flat ocean surface: Now we shall recall the boundary conditions with slight modifications for flat ocean surface.

$$I_{AT}(0,-\mu)=f(-\mu); \quad (4.5.5)$$

$$I_{AT}(z_\omega,\mu)=R(-\mu,n)I_{AT}(z_\omega,-\mu)+T(\mu,n)\frac{I_{OC}(z_\omega,\mu)}{n^2}; \quad (4.5.6)$$

$$\frac{I_{OC}(z_\omega,-\mu)}{n^2}=R(\mu,n)\frac{I_{OC}(z_\omega,\mu)}{n^2}+T(-\mu,n)I_{AT}(z_\omega,-\mu); \quad (4.5.7)$$

$$I_{OC}(z_1,\mu)=g(\mu). \quad (4.5.8)$$

First of these equation indicates the radiation measured at the top of the atmosphere. The second condition describes the upward intensity at the interface in the atmosphere consisting of the specularly reflected downward atmospheric radiation from the flat ocean surface added with the transmitted upward radiation that has come out from the ocean. The third one represents in coming radiation at the ocean surface after reflection of the out coming ocean radiation added with the transmitted atmospheric radiation at the interface of the downward The last term can be evaluated using the reflecting properties of the ocean bottom surface. We now have to consider the case for total internal reflection region.

4.6. Consideration of total internal reflection at the interface between the two media:

The changes in the direction of light rays passing through the air-water interface are caused due to change in the refractive index of water mostly governed by Snell's law in the case of flat ocean surface. Analysis of those light rays passing from the air to water through a flat ocean surface require consideration of simple change in directions to measure the direction below the ocean surface. However for light rays that are coming from within the ocean and reach the ocean surface, only those having polar angle measured with respect to the down ward normal drawn at the point of contact on the ocean surface, less than the critical angle can cross through the air-water interface whereas those with polar angle less greater than critical angle will be totally reflected back within the water body. This natural constraint forces the researcher to use discrete numerical schemes. The mismatch between the refractive index of the two participating media affects the total intensity near the interface but at few optical depths below the interface where all most all light rays have undergone one or more scattering events the directional dependence of the total radiation intensity is mostly governed by the optical properties of the water (McCormick [49]).

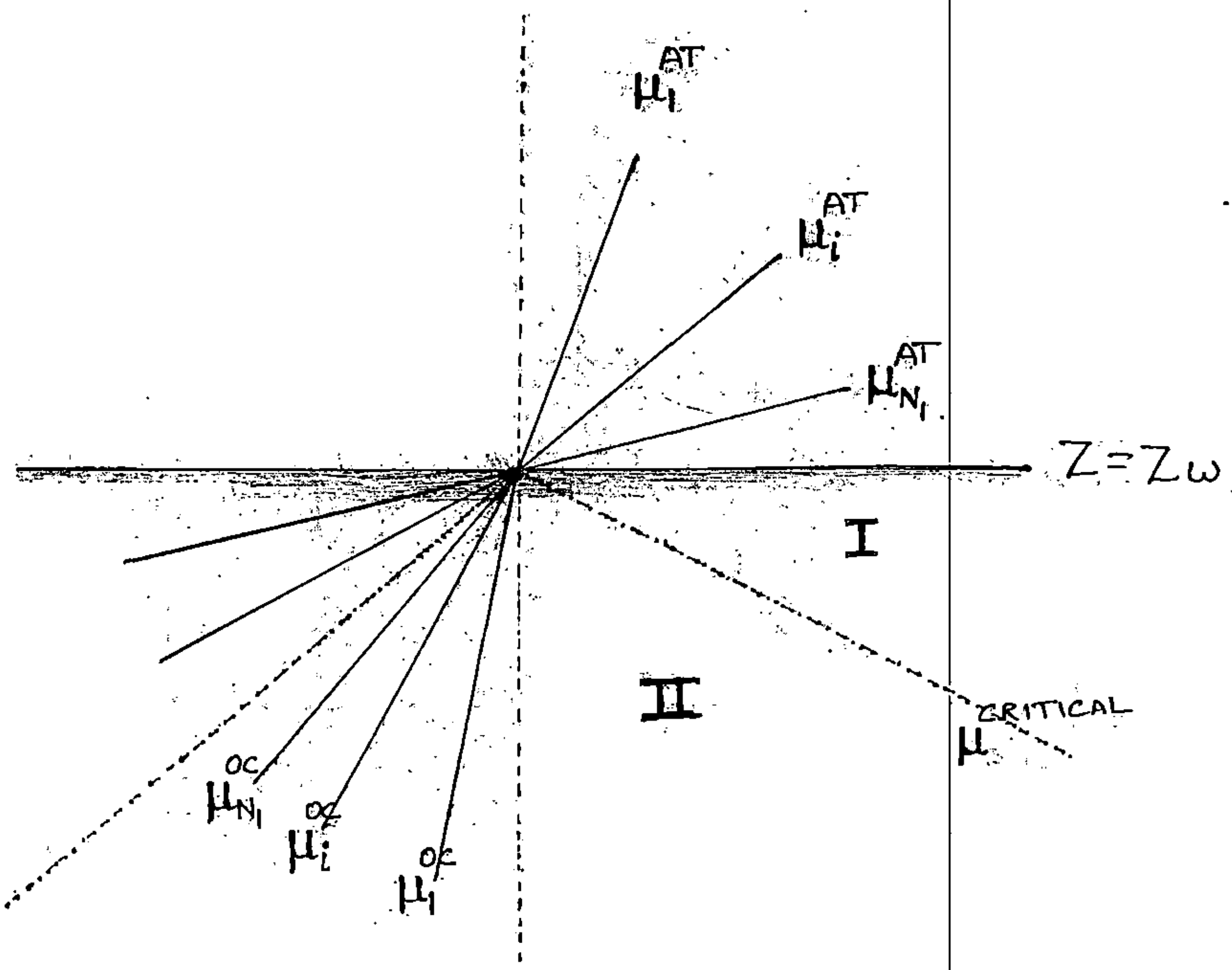


Fig (5)

The boundary condition appropriate for taking into consideration the total internal reflection at such interfaces will be given by the following equation

$$I^{OC}(z_{\omega},\mu) = I^{OC}(z_{\omega},-\mu) \quad \text{for} \quad \mu > \mu^{CRITICAL} \tag{4.6.1}$$

Now, instead of using a constant number of streams as is usual in cases of only one media, one can use Jin, Z., and Stamnes, K., [61] different numbers of streams for the atmosphere and the ocean ($2N_1$ and $2N_2$, respectively). In region II of the ocean [Fig 5.], which communicates directly with the atmosphere, one can use the same number of streams ($2N_1$) as is used in the atmosphere to properly account for directional narrowing caused by refraction of the angular domain in the ocean. In region I of the ocean, where total reflection of photons moving in the upward direction occurs at the ocean – atmosphere interface, one may use additional streams ($2N_2 - 2N_1$) to account for the scattering interaction between regions I and II in the ocean. There exist many options for choice of quadrature; this particular scheme will strongly affect the application of interface-continuity conditions and the accuracy of the solution (Jin, Z., and Stamnes, K., [61]). The quadrature used in this paper is not essentially the same as that adopted by Tanaka and Nakajima [184]. The Gauss quadrature rule is used to determine the quadrature points and weights, μ_i^{AT} and $w_i^{AT}(i = 1,2,...,N)$, in the atmosphere, as well as the quadrature points and weighs, μ_i^{OC} and $w_i^{OC} = (i = N_1 + 1,...,N_2)$, in the total-reflection region of the ocean. The quadrature points in the Fresnel cone of the ocean are obtained by simply refracting the downward streams in the atmosphere $(\mu_1^{AT},...,\mu_{N_1}^{AT})$, into the ocean.

Thus, in this region, μ_i^{OC} is related to μ_i^{AT} by the Snell law,

$$\mu_i^{OC} = S(\mu_i^{AT}) = \sqrt{\frac{1-[1-(\mu_i^{AT})^2]}{n^2}}, \quad i = 1,2,...,N; \tag{4.6.2}$$

and from this relation, the weights for this region can be derived as

$$w_i^{OC} = w_i^{AT}\left(\frac{dS(\mu^{AT})}{d\mu^{AT}}\right)_{\mu^{AT}=\mu_i^{AT}} = \frac{\mu_i^{AT}}{n^2 S(\mu_i^{AT})} w_i^{AT}, \quad i = 1,2,...,N. \tag{4.6.3}$$

The advantage of this choice of quadrature is that the points are clustered toward $\mu = 0$ both in the atmosphere and in the ocean, and in addition, toward the critical-angle direction in the ocean. This clustering gives superior results near these directions where the intensities vary rapidly. Also, this choice of quadrature will simplify the application of the interface-continuity condition and avoid the loss of accuracy incurred by the interpolation necessitated by adopting the same quadrature (i.e., the same number of streams) for the atmosphere and the ocean. In our numerical approach we did not consider scattering interaction between region I and II. However theoretically it is not difficult to consider the case in numerical calculations.

4.7. Boundary condition:

We shall now first discretize the boundary conditions (3.5.5-3.5.8) and then apply the complete solution to get linear equations for the determination of arbitrary unknown coefficients. To determine them we shall require 32 equations which we will get from the boundary conditions.

The top boundary condition:

$$\begin{bmatrix} \mathbf{I}_{AT}(0,-\mu_1) \\ \mathbf{I}_{AT}(0,-\mu_2) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(-\mu_1) \\ \mathbf{f}(-\mu_2) \end{bmatrix}. \tag{4.7.1}$$

Let us set the top boundary condition with following form for unknown function in RHS.

$$\mathbf{I}_{AT}(0,-\mu_1) = \mathbf{f}(-\mu_1) = \begin{bmatrix} f_1(-\mu_1) \\ f_2(-\mu_1) \\ f_3(-\mu_1) \\ f_4(-\mu_1) \end{bmatrix} = \begin{bmatrix} -\mu_1 \\ -\mu_1 \\ -\mu_1 \\ -\mu_1 \end{bmatrix}; (4.7.2)\ \mathbf{I}_{AT}(0,-\mu_2) = \mathbf{f}(-\mu_2) = \begin{bmatrix} f_1(-\mu_1) \\ f_2(-\mu_1) \\ f_3(-\mu_1) \\ f_4(-\mu_1) \end{bmatrix} = \begin{bmatrix} -\mu_2 \\ -\mu_2 \\ -\mu_2 \\ -\mu_2 \end{bmatrix}. \quad (4.7.3)$$

This is the case for N=2. Generalizing equation (4.7.2) and (4.7.3) can be written together as a compact equation for separate equations for each atmospheric stokes parameter and complete solutions from the set of equations (4.5.1-4.5.4) can then be used on the left hand side to get

$$\mathbf{\Delta}\left[\sum_{j=1}^{4x2} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT}) + B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right]$$
$$+\mathbf{\Delta}\left[\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{AT(\alpha)}\mathbf{F}_-^{AT(\alpha)}(z,\gamma_j^{AT}) + B_j^{AT(\alpha)}\mathbf{F}_+^{AT(\alpha)}(z_\omega,\gamma_j^{AT})\right] + \mathbf{I}_{AT}^{p}(-;0) = [\mu]. \tag{4.7.4}$$

$$\mathbf{\Delta}\left[\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT}) + B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\right] +$$
$$\mathbf{\Delta}\left[\sum_{j=1}^{Nc} A_j^{AT(1)}\left[\mathrm{Re}\{H_-^{AT}(\gamma_j^{AT})\} - \mathrm{Im}\{H_-^{AT}(\gamma_j^{AT})\}\right] + B_j^{AT(1)}\left[\mathrm{Re}\{H_+^{AT}(\gamma_j^{AT})\} - \mathrm{Im}\{H_+^{AT}(\gamma_j^{AT})\}\right]\right.$$
$$\left. + A_j^{AT(2)}\left[\mathrm{Re}\{H_-^{AT}(\gamma_j^{AT})\} - \mathrm{Im}\{H_-^{AT}(\gamma_j^{AT})\}\right] + B_j^{AT(2)}\left[\mathrm{Re}\{H_+^{AT}(\gamma_j^{AT})\} - \mathrm{Im}\{H_+^{AT}(\gamma_j^{AT})\}\right]\ \right] + \mathbf{I}_{AT}^{p}(-;0) = [\mu].$$

(4.7.5)

$$\sum_{j=1}^{Nr} \mathbf{\Sigma}_j^{AT} A_j^{AT} + \sum_{j=1}^{Nr} \mathbf{\Pi}_j^{AT} B_j^{AT} + \sum_{j=1}^{Nc} A_j^{AT(1)} \partial + \sum B_j^{AT(1)} \nabla + \sum A_j^{AT(2)} \partial + \sum B_j^{AT(2)} \nabla = [\mu] - I_{AT}^{p}(-;0) \tag{4.7.6}$$

Equation (4.7.6) is a set of equations for unknown constants.

$$\mathbf{\Sigma}_j^{AT} = \mathbf{\Delta H}_{-}^{AT}(\gamma_j^{AT}); \tag{4.7.7}$$

$$\mathbf{\Pi}_j^{AT} = \mathbf{\Delta H}_{+}^{AT}(\gamma_j^{AT}) \exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right); \tag{4.7.8}$$

$$\partial = \mathbf{\Delta}\left[\mathrm{Re}\{H_{-}^{AT}(\gamma_j^{AT})\} - \mathrm{Im}\{H_{-}^{AT}(\gamma_j^{AT})\}\right] \tag{4.7.9}$$

$$\nabla = \mathbf{\Delta}\left[\mathrm{Re}\{H_{+}^{AT}(\gamma_j^{AT})\} - \mathrm{Im}\{H_{+}^{AT}(\gamma_j^{AT})\}\right] \tag{4.7.10}$$

The second boundary condition:

The next boundary condition reads as for two quadrature angle

$$I_{AT}(z_\omega,\mu_1) = R(-\mu_1,n) I_{AT}(z_\omega,-\mu_1) + T(\mu_1,n)\frac{I_{OC}(z_\omega,\mu_1)}{n^2}; \tag{4.7.11}$$

$$I_{AT}(z_\omega,\mu_2) = R(-\mu_2,n) I_{AT}(z_\omega,-\mu_2) + T(\mu_2,n)\frac{I_{OC}(z_\omega,\mu_2)}{n^2}. \tag{4.7.12}$$

Combining these two equations we write

$$\begin{bmatrix} I_{AT}(z_\omega,\mu_1) \\ I_{AT}(z_\omega,\mu_2) \end{bmatrix} = \begin{bmatrix} R(-\mu_1,n) & 0 \\ 0 & R(-\mu_2,n) \end{bmatrix} \times \begin{bmatrix} I_{AT}(z_\omega,-\mu_1) \\ I_{AT}(z_\omega,-\mu_2) \end{bmatrix} + \begin{bmatrix} T(\mu_1,n) & 0 \\ 0 & T(\mu_2,n) \end{bmatrix} \times \begin{bmatrix} \frac{I_{OC}(z_\omega,\mu_1)}{n^2} \\ \frac{I_{OC}(z_\omega,\mu_1)}{n^2} \end{bmatrix} \tag{4.7.13}$$

Using now the appropriate complete solutions in the second boundary conditions for **N=N** in (4.7.13) to get

$$\left[\sum_{j=1}^{Nr} A_j^{AT} \mathbf{H}_{+}^{AT}(\gamma_j^{AT}) \exp(-\frac{z_\omega}{\gamma_j^{AT}}) + B_j^{AT} \mathbf{H}_{-}^{AT}(\gamma_j^{AT}) + \right.$$

$$\left. \sum_{j=1}^{N_c} \sum_{\alpha=1}^{2} A_j^{AT(\alpha)} F_{+}^{AT(\alpha)}(z_\omega,\gamma_j^{AT}) + B_j^{AT(\alpha)} F_{-}^{AT(\alpha)}(0,\gamma_j^{AT}) \right] + I_{AT}^{p}(+;z_\omega) =$$

$$R(-) X\ \mathbf{\Delta}\left[\sum_{j=1}^{4x2} A_j^{AT} \mathbf{H}_{-}^{AT}(\gamma_j^{AT}) \exp(-\frac{z_w}{\gamma_j^{AT}}) + B_j^{AT} \mathbf{H}_{+}^{AT}(\gamma_j^{AT}) + \right.$$

$$\sum_{j=1}^{N_c}\sum_{\alpha=1}^{2} A_j^{AT(\alpha)} F_-^{AT(\alpha)}(z_\omega,\gamma_j^{AT}) + B_j^{AT(\alpha)} F_+^{AT(\alpha)}(0,\gamma_j^{AT}) \Bigg] + R(-) X [I_{AT}^p(-;z_\omega)] +$$

$$T(+) X \left[\frac{1}{n^2}\sum_{j=1}^{4x2} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp(-\frac{z_w}{\gamma_j^{OC}}) + B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC}) + \right.$$

$$\left. \sum_{j=1}^{N_c}\sum_{\alpha=1}^{2} A_j^{OC(\alpha)} F_+^{OC(\alpha)}(z_\omega,\gamma_j^{OC}) + B_j^{OC(\alpha)} F_-^{OC(\alpha)}(z_1 - z_\omega,\gamma_j^{OC}) \right] + T(+) X [I_{OC}^p(+;z_\omega)]. \quad (4.7.14)$$

Rearranging terms

$$\left[\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\right] -$$

$$R(-)\times\mathbf{\Delta}\left[\sum_{j=1}^{4x2} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\right]$$

$$-T(+)\times\left[\frac{1}{n^2}\sum_{j=1}^{4x2} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC}) + \right.$$

$$\sum_{J=1}^{N_c}\Big[[F^{AT}11(+1) - F^{AT}11(+2)] - R(-)\mathbf{\Delta}[F^{AT}11(-1) - F^{AT}11(-2)]\Big] A_J^{AT(1)} +$$

$$\sum_{J=1}^{N_c}\Big[[F^{AT}21(+1) - F^{AT}21(+2)] + R(-)\mathbf{\Delta}[F^{AT}21(-1) - F^{AT}21(-2)]\Big] A_J^{AT(2)} +$$

$$\sum_{J=1}^{N_c}\Big[[F^{AT}12(-1) - F^{AT}12(-2)] + R(-)\mathbf{\Delta}[F^{AT}12(+1) - F^{AT}12(+2)]\Big] B_J^{AT(1)} +$$

$$\sum_{J=1}^{N_c}\Big[[F^{AT}22(-1) - F^{AT}22(-2)] + R(-)\mathbf{\Delta}[F^{AT}22(+1) - F^{AT}22(+2)]\Big] B_J^{AT(2)} -$$

$$\sum_{J=1}^{N_c} T(+)[F^{OC}11(+1) + F^{OC}11(+2)] A_J^{OC\,(1)} - \sum_{J=1}^{N_c} T(+)[F^{OC}21(+1) + F^{OC}21(+2)] A_J^{OC\,(2)} -$$

$$\sum_{J=1}^{N_c} T(+)[F^{OC}12(-1) + F^{OC}12(-2)] B_J^{OC\,(1)} - \sum_{J=1}^{N_c} T(+)[F^{OC}22(-1) + F^{OC}22(-2)] B_J^{OC\,(2)} =$$

$$R(-)\times[I_{AT}^p(-;z_\omega)] + T(+)\times[I_{OC}^p(+;z_\omega)] - I_{AT}^p(+;z_\omega). \quad (4.7.15)$$

Where we have defined reflection and transmission matrix for **N=2** quadrature angles. However this can be easily generalized.

$$\mathbf{R}(\pm)=\begin{bmatrix}\mathbf{R}(\pm\mu_1,\mathbf{n}) & \mathbf{0}\\ \mathbf{0} & \mathbf{R}(\pm\mu_2,\mathbf{n})\end{bmatrix} \quad (4.7.16) \qquad \mathbf{T}(\pm)=\begin{bmatrix}\mathbf{T}(\pm\mu_1,\mathbf{n}) & \mathbf{0}\\ \mathbf{0} & \mathbf{T}(\pm\mu_2,\mathbf{n})\end{bmatrix} \quad (4.7.17)$$

But now we have more concisely for **N** quadrature points

$$\sum_{j=1}^{4N}\mathbf{\Phi}_j^{AT}A_j^{AT}+\sum_{j=1}^{4N}\mathbf{\Psi}_j^{AT}B_j^{AT}-\sum_{j=1}^{4N}\mathbf{\Sigma}_j^{OC}A_j^{OC}-\sum_{j=1}^{4N}\mathbf{\Pi}_j^{OC}B_j^{OC}$$

$$+\sum_{j=1}^{Nc}[\mathbf{RF1}]\,\mathbf{A}_J^{AT(1)}+\sum_{j=1}^{Nc}[\mathbf{RF2}]\,\mathbf{A}_J^{AT(2)}+\sum_{j=1}^{Nc}[\mathbf{RF3}]\,\mathbf{B}_J^{AT(1)}+\sum_{j=1}^{Nc}[\mathbf{RF4}]\,\mathbf{B}_J^{AT(2)}$$

$$-\sum_{J=1}^{Nc}[\mathbf{TF1}]\,\mathbf{A}_J^{OC\,(1)}-\sum_{J=1}^{Nc}[\mathbf{TF2}]\,\mathbf{A}_J^{OC\,(2)}-\sum_{J=1}^{Nc}[\mathbf{TF3}]\,\mathbf{B}_J^{OC\,(1)}-\sum_{J=1}^{Nc}[\mathbf{TF4}]\,\mathbf{B}_J^{OC\,(2)}$$

$$=\mathbf{R}(-)\,[\mathbf{I}_{AT}^{p}(-;\mathbf{z}_\omega)]+\mathbf{T}(+)\,[\mathbf{I}_{OC}^{p}(+;\mathbf{z}_\omega)]-\mathbf{I}_{AT}^{p}(+;\mathbf{z}_\omega). \qquad (4.7.18)$$

This is a set of 4N linear algebraic equations for 8N unknown coefficients.
The following notations are used to write the last equation.

$$\mathbf{\Phi}_j^{AT}=[\mathbf{H}_+^{AT}(\gamma_j^{AT})-\mathbf{R}(-)\mathbf{\Delta H}_-^{AT}(\gamma_j^{AT})]\exp\left(-\frac{\mathbf{z}_\omega}{\gamma_j^{AT}}\right); \qquad (4.7.19)$$

$$\mathbf{\Psi}_j^{AT}=\mathbf{H}_-^{AT}(\gamma_j^{AT})-\mathbf{R}(-)\mathbf{\Delta H}_+^{AT}(\gamma_j^{AT}); \qquad (4.7.20)$$

$$\mathbf{\Sigma}_j^{OC}=\mathbf{T}(+)[\mathbf{H}_+^{OC}(\gamma_j^{OC})]\exp\left(-\frac{\mathbf{z}_\omega}{\gamma_j^{OC}}\right); \quad (4.7.21) \qquad \mathbf{\Pi}_j^{OC}=\mathbf{T}(+)\mathbf{H}_-^{OC}(\gamma_j^{OC}). \quad (4.7.22)$$

$$\mathbf{RF1}=[\mathbf{F}^{AT}11(+1)-\mathbf{F}^{AT}11(+2)]+\mathbf{R}(-)\mathbf{\Delta}[\mathbf{F}^{AT}11(-1)-\mathbf{F}^{AT}11(-2)]; \qquad (4.7.23)$$

$$\mathbf{RF2}=[\mathbf{F}^{AT}21(+1)-\mathbf{F}^{AT}21(+2)]+\mathbf{R}(-)\mathbf{\Delta}[\mathbf{F}^{AT}21(-1)-\mathbf{F}^{AT}21(-2)]; \qquad (4.7.24)$$

$$\mathbf{RF3}=[\mathbf{F}^{AT}12(-1)-\mathbf{F}^{AT}12(-2)]+\mathbf{R}(-)\mathbf{\Delta}[\mathbf{F}^{AT}12(+1)-\mathbf{F}^{AT}12(+2)]; \qquad (4.7.25)$$

$$\mathbf{RF4}=[\mathbf{F}^{AT}22(-1)-\mathbf{F}^{AT}22(-2)]+\mathbf{R}(-)\mathbf{\Delta}[\mathbf{F}^{AT}22(+1)-\mathbf{F}^{AT}22(+2)]; \qquad (4.7.26)$$

$$TF1 = T(+)[F^{OC}11(+1) + F^{OC}11(+2)]; \quad (4.7.27) \qquad TF2 = T(+)[F^{OC}21(+1) + F^{OC}21(+2)]; \quad (4.7.28)$$

$$TF3 = T(+)[F^{OC}12(-1) + F^{OC}12(-2)]; \quad (4.7.29) \qquad TF4 = T(+)[F^{OC}22(-1) + F^{OC}22(-2)]. \quad (4.7.30)$$

The third boundary condition:

The third boundary conditions for only two discretized angle are given as

$$\frac{I_{OC}(z_\omega,-\mu_1)}{n^2} = R(\mu_1,n)\frac{I_{OC}(z_\omega,\mu_1)}{n^2} + T(-\mu_1,n)I_{AT}(z_\omega,-\mu_1); \tag{4.7.31}$$

$$\frac{I_{OC}(z_\omega,-\mu_2)}{n^2} = R(\mu_1,n)\frac{I_{OC}(z_\omega,\mu_2)}{n^2} + T(-\mu_2,n)I_{AT}(z_\omega,-\mu_2). \tag{4.7.32}$$

This can be written as

$$\begin{bmatrix} \dfrac{I_{OC}(z_\omega,-\mu_1)}{n^2} \\ \dfrac{I_{OC}(z_\omega,-\mu_2)}{n^2} \end{bmatrix} = \begin{bmatrix} R(\mu_1,n) & 0 \\ 0 & R(\mu_2,n) \end{bmatrix} \times \frac{1}{n^2} \begin{bmatrix} I_{OC}(z_\omega,\mu_1) \\ I_{OC}(z_\omega,\mu_2) \end{bmatrix} + \begin{bmatrix} T(-\mu_1,n) & 0 \\ 0 & T(-\mu_2,n) \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} I_{AT}(z_\omega,-\mu_1) \\ I_{AT}(z_\omega,-\mu_2) \end{bmatrix} \end{bmatrix} \tag{4.7.33}$$

We now use the complete solution for the third boundary condition (4.7.31) with **N** quadrature angle to get

$$\frac{1}{n^2}\boldsymbol{\Delta}\left[\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC}) + \right.$$

$$\left.\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_-^{OC(\alpha)}(z_\omega,\gamma_j^{OC}) + B_j^{OC(\alpha)}F_+^{OC(\alpha)}(z_1 - z_\omega,\gamma_j^{OC})\right] + \frac{1}{n^2}I_{OC}^{p}(-;z_\omega) =$$

$$R(+)\left[\frac{1}{n^2}\sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC}) + \right.$$

$$\left.\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_+^{OC(\alpha)}(z_\omega,\gamma_j^{OC}) + B_j^{OC(\alpha)}F_-^{OC(\alpha)}(z_1 - z_\omega,\gamma_j^{OC})\right] + \frac{1}{n^2}R(+)\,[I_{OC}^{p}(+;z_\omega)]$$

$$+T(-).\boldsymbol{\Delta}\left[\sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT}) + \right.$$

$$\left.\sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{AT(\alpha)}F_-^{AT(\alpha)}(z_\omega,\gamma_j^{AT}) + B_j^{AT(\alpha)}F_+^{AT(\alpha)}(0,\gamma_j^{AT})\right] + T(-).[I_{AT}^{p}(-;z_\omega)]. \tag{4.7.34}$$

Again rearranging for like terms

$$\frac{1}{n^2}\boldsymbol{\Delta}\left[\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\right]-$$

$$R(+)\times\left[\frac{1}{n^2}\sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\right]-$$

$$T(-)\times\boldsymbol{\Delta}\left[\sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)+B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\right]+$$

$$\sum_{J=1}^{Nc}\left[\frac{1}{n^2}\boldsymbol{\Delta}[F^{OC}11(-1)-F^{OC}11(-2)]+\frac{1}{n^2}R(+)[F^{OC}11(+1)-F^{OC}11(+2)]\right]A_J^{OC\,(1)}+$$

$$\sum_{J=1}^{Nc}\left[\frac{1}{n^2}\boldsymbol{\Delta}[F^{OC}21(-1)-F^{OC}21(-2)]+\frac{1}{n^2}R(+)[F^{OC}21(+1)-F^{OC}21(+2)]\right]A_J^{OC\,(2)}+$$

$$\sum_{J=1}^{Nc}\left[\frac{1}{n^2}\boldsymbol{\Delta}[F^{OC}12(+1)-F^{OC}12(+2)]+\frac{1}{n^2}R(+)[F^{OC}12(-1)-F^{OC}12(-2)]\right]B_J^{OC\,(1)}+$$

$$\sum_{J=1}^{Nc}\left[\frac{1}{n^2}\boldsymbol{\Delta}[F^{OC}22(+1)-F^{OC}22(+2)]+\frac{1}{n^2}R(+)[F^{OC}22(-1)-F^{OC}22(-2)]\right]B_J^{OC\,(2)}-$$

$$\sum_{J=1}^{Nc}T(-)\boldsymbol{\Delta}[F^{AT}11(-1)-F^{AT}11(-2)]A_J^{AT(1)}+\sum_{J=1}^{N_c}T(-)\boldsymbol{\Delta}[F^{AT}21(-1)-F^{AT}21(-2)]A_J^{AT(2)}-$$

$$\sum_{J=1}^{Nc}T(-)\boldsymbol{\Delta}[F^{AT}12(+1)-F^{AT}12(+2)]B_J^{AT(1)}+\sum_{J=1}^{N_c}T(-)\boldsymbol{\Delta}[F^{AT}22(+1)-F^{AT}22(+2)]B_J^{AT(2)}=$$

$$\frac{1}{n^2}R(+)\times[I_{OC}^{p}(+;z_\omega)]+T(-)\times[I_{AT}^{p}(-;z_\omega)]-\frac{1}{n^2}I_{OC}^{p}(-;z_\omega). \qquad (4.7.35)$$

More conveniently we can rewrite equation (4.5.51)

$$\sum_{j=1}^{4N}\boldsymbol{\Omega}_j^{AT}A_j^{AT}+\sum_{j=1}^{4N}\boldsymbol{\Theta}_j^{AT}B_j^{AT}-\sum_{j=1}^{4N}\boldsymbol{\Phi}_j^{OC}A_j^{OC}-\sum_{j=1}^{4N}\boldsymbol{\Psi}_j^{OC}B_j^{OC}$$

$$+\sum_{j=1}^{Nc}[FR1]A_J^{OC(1)}+\sum_{j=1}^{Nc}[FR2]A_J^{OC(2)}+\sum_{j=1}^{Nc}[FR3]B_J^{OC(1)}+\sum_{j=1}^{Nc}[FR4]B_J^{OC(2)}$$

$$-\sum_{j=1}^{Nc}[FT1]A_J^{AT(1)}+\sum_{j=1}^{Nc}[FT2]A_J^{AT(2)}-\sum_{j=1}^{Nc}[FT3]B_J^{AT(1)}+\sum_{j=1}^{Nc}[FT4]B_J^{AT(1)}$$

$$=\frac{1}{n^2}R(+)\,[I_{OC}^{p}(+;z_\omega)]+T(-)\,[I_{AT}^{p}(-;z_\omega)]-\frac{1}{n^2}I_{AT}^{p}(-;z_\omega) \qquad (4.7.36)$$

Here we have used following notations

$$\mathbf{\Omega}_j^{AT} = T(-)\mathbf{\Delta H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right); \quad (4.7.37) \qquad \mathbf{\Theta}_j^{AT} = T(-)\mathbf{\Delta H}_+^{AT}(\gamma_j^{AT}); \quad (4.7.3$$

$$\mathbf{\Phi}_j^{OC} = \frac{1}{n^2}[\mathbf{\Delta H}_-^{OC}(\gamma_j^{OC}) + R(+)\mathbf{H}_+^{OC}(\gamma_j^{OC})]\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right); \quad (4.7.3$$

$$\mathbf{\Psi}_j^{OC} = \frac{1}{n^2}[\mathbf{\Delta H}_+^{OC}(\gamma_j^{OC}) + R(+)\mathbf{H}_-^{OC}(\gamma_j^{OC})]; \quad (4.7.4$$

$$FR1 = \frac{1}{n^2}\mathbf{\Delta}[F^{OC}11(-1) - F^{OC}11(-2)] + R(+)[F^{OC}11(+1) - F^{OC}11(+2)]; \quad (4.7.4$$

$$FR2 = \frac{1}{n^2}\mathbf{\Delta}[F^{OC}21(-1) - F^{OC}21(-2)] + R(+)[F^{OC}21(+1) - F^{OC}21(+2)] \quad (4.7.4$$

$$FR3 = \left[\frac{1}{n^2}\mathbf{\Delta}[F^{OC}12(+1) - F^{OC}12(+2)] + R(+)[F^{OC}12(-1) - F^{OC}12(-2)];\right] \quad (4.7.4$$

$$FR4 = \left[\frac{1}{n^2}\mathbf{\Delta}[F^{OC}22(+1) - F^{OC}22(+2)] + R(+)[F^{OC}22(-1) - F^{OC}22(-2)]\right]; \quad (4.7.4$$

$$FT1 = T(-)\mathbf{\Delta}[F^{AT}11(-1) - F^{AT}11(-2)]; \; (4.7.45) \quad FT2 = T(-)\mathbf{\Delta}[F^{AT}21(-1) - F^{AT}21(-2)]; \; (4.7.4$$

$$FT3 = T(-)\mathbf{\Delta}[F^{AT}12(+1) - F^{AT}12(+2)]; \; (4.7.47) \quad FT4 = T(-)\mathbf{\Delta}[F^{AT}22(+1) - F^{AT}22(+2)]; \; (4.7.48$$

The bottom boundary condition(fourth):

The bottom boundary condition takes the form

$$\begin{bmatrix} I(z_1,\mu_1) \\ I(z_1,\mu_2) \end{bmatrix} = \begin{bmatrix} g(\mu_1) \\ g(\mu_2) \end{bmatrix}. \quad (4.7.49)$$

We shall take $$g(\mu) = 0.3\sqrt{(1 - \frac{(1-\mu^2)}{1.34})} \quad (4.7.49a)$$

As usual we get

$$\sum \left[A_j^{OC} \mathbf{H}_+^{OC}(\gamma_j^{OC}) \exp\left(-\frac{z_1}{\gamma_j^{OC}}\right) + B_j^{OC} \mathbf{H}_-^{OC}(\gamma_j^{OC}) \right]$$

$$\sum_{j=1}^{Nc} \sum_{\alpha=1}^{2} A_j^{OC(\alpha)} F_+^{OC(\alpha)}(z_1, \gamma_j^{OC}) + B_j^{OC(\alpha)} F_-^{OC(\alpha)}(z_\omega - z_1, \gamma_j^{OC}) + [I_{OC}^p(+;z_1)] = [g(\mu)] \qquad (4.7.50)$$

$$\left[\sum_{j=1}^{Nr} A_j^{OC} \mathbf{H}_+^{OC}(\gamma_j^{OC}) \exp\left(-\frac{z_1}{\gamma_j^{OC}}\right) + B_j^{OC} \mathbf{H}_-^{OC}(\gamma_j^{OC}) + \right.$$

$$+ \sum_{j=1}^{Nc} A_j^{OC(1)} [\, F^{OC}13(+1) - F^{OC}13(+2) \,] + \sum_{j=1}^{Nc} A_j^{OC(2)} [\, F^{OC}23(+1) - F^{OC}23(+2) \,] +$$

$$\sum_{j=1}^{Nc} B_j^{OC(1)} [\, F^{OC}12(-1) - F^{OC}12(-2) \,] + \sum_{j=1}^{Nc} B_j^{OC(2)} [\, F^{OC}21(-1) - F^{OC}21(-2) \,] = g(\mu) - [I_{OC}^p(+;z_1)]$$

(4.7.51)

This is again a set of equations

$$\sum_{j=1}^{Nr} \mathbf{\Omega}_j^{OC} A_j^{OC} + \sum_{j=1}^{Nr} \mathbf{\Theta}_j^{OC} B_j^{OC} + + \sum_{j=1}^{Nc} A_j^{OC(1)} \forall + \sum_{j=1}^{Nc} A_j^{OC(2)} \exists + \sum_{j=1}^{Nc} B_j^{OC(1)} \ni + \sum_{j=1}^{Nc} B_j^{OC(2)} \vee \qquad (4.7.52)$$

$$\mathbf{\Omega}_j^{OC} = \mathbf{H}_+^{OC}(\gamma_j^{OC}) \exp\left(-\frac{z_1}{\gamma_j^{OC}}\right); \qquad (4.7.53) \qquad\qquad \mathbf{\Theta}_j^{OC} = \mathbf{H}_-^{OC}(\gamma_j^{OC}); \qquad (4.7.54)$$

$$\forall = [\, F^{OC}13(+1) - F^{OC}13(+2) \,] \qquad (4.7.55a) \qquad \exists = [\, F^{OC}23(+1) - F^{OC}23(+2) \,] \qquad (4.7.55b)$$

$$\ni = [\, F^{OC}12(-1) - F^{OC}12(-2) \,] \qquad (4.7.56a) \qquad \vee = [\, F^{OC}21(-1) - F^{OC}21(-2) \,] \qquad (4.7.56b)$$

We shall now consider the right hand side of the boundary conditions.
From equations (4.4.54-4.4.57) we arrive at the following set of equations which can be computed numerically as all the quantities involved are known. For no imaginary eigenvectors the

corresponding quantities will be dropped in the following expressions in actual numerical computations.

$$I_{AT}^{p}(-;0)=\mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(0)\mathbf{H}_{-}^{AT}(\gamma_j)+\aleph_j^{AT}(0)\mathbf{H}_{+}^{AT}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc}\ AZ_{-}^{AT}(0,\gamma_j^{AT})+BZ_{+}^{AT}(0,\gamma_j^{AT})$$

$$=\mathbf{\Delta}\sum_{j=1}^{Nr}[\aleph_j^{AT}(0)\mathbf{H}_{+}^{AT}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc}\ [\Re_j^{AT*}(0)\mathbf{H}_{-}^{AT*}(\gamma_j^{AT})+\aleph_j^{AT}(0)\mathbf{H}_{+}^{AT}(\gamma_j^{AT})+\aleph_j^{AT*}(0)\mathbf{H}_{+}^{AT*}(\gamma_j^{AT})]$$

$$=\mathbf{\Xi}_1 \tag{4.7.57}$$

Here we called

$$\Re_j^{AT}(0)=\mu_0\gamma_j^{AT}a_j^{AT}CAT(z:\gamma_j,\mu_0)=0\ \ \forall\, j. \tag{4.4.59}$$

$$\aleph_j^{AT}(0)=\mu_0\gamma_j^{OC}\ b_j^{OC}\ SAT(z_\omega:\gamma_j,\mu_0). \tag{4.4.60}$$

$$\Re_j^{AT}(z_\omega)=\mu_0\gamma_j^{AT}a_j^{AT}CAT(z_\omega:\gamma_j,\mu_0). \tag{4.4.61}$$

$$\aleph_j^{AT}(z_\omega)=\mu_0\gamma_j^{OC}\ b_j^{OC}\ SAT(0:\gamma_j,\mu_0)=0\ \ \forall\, j. \tag{4.4.62}$$

$$R(-)\,[I_{AT}^{p}(-;z_\omega)]-T(+)\,[I_{OC}^{p}(+;z_\omega)]-I_{AT}^{p}(+;z_\omega)$$

$$=R(-)[\mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(z_\omega)\mathbf{H}_{-}^{AT}(\gamma_j)+\aleph_j^{AT}(z_\omega)\mathbf{H}_{+}^{AT}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc}\ [AZ_{-}^{AT}(z_\omega,\gamma_j^{AT})+BZ_{+}^{AT}(z_\omega,\lambda_j^{AT/OC})]]$$

$$-T(+)[\sum_{j=1}^{Nr}[\Re_j^{OC}(z_\omega)\mathbf{H}_{+}^{OC}(\gamma_j)+\aleph_j^{OC}(z_\omega)\mathbf{H}_{-}^{OC}(\gamma_j)]+\sum_{j=1}^{Nc}\ [AZ_{+}^{OC}(z_\omega,\gamma_j^{OC})+\ BZ_{-}^{OC}(z_\omega,\gamma_j^{OC})]]$$

$$-\sum_{j=1}^{Nr}[\Re_j^{AT}(z_\omega)\mathbf{H}_{+}^{AT}(\gamma_j)+\aleph_j^{AT}(z_\omega)\mathbf{H}_{-}^{AT}(\gamma_j)]+\sum_{j=1}^{Nc}\ AZ_{+}^{AT}(z_\omega,\gamma_j^{AT})+BZ_{-}^{AT}(z_\omega,\gamma_j^{AT})$$

$$=\mathbf{\Xi}_2 \tag{4.7.58}$$

$$\frac{1}{n^2}R(+)\,[I_{OC}^{p}(+;z_\omega)]+T(-)\,[I_{AT}^{p}(-;z_\omega)]-\frac{1}{n^2}I_{AT}^{p}(-;z_\omega)$$

$$=\frac{1}{n^2}R(+)\ \ [\sum_{j=1}^{Nr}[\Re_j^{OC}(z)\mathbf{H}_{+}^{OC}(\gamma_j)+\aleph_j^{OC}(z)\mathbf{H}_{-}^{OC}(\gamma_j)]+\sum_{j=1}^{Nc}\ [AZ_{+}^{OC}(z,\gamma_j^{OC})+BZ_{-}^{OC}(z,\lambda_j^{OC})]]$$

$$+T(-)[\mathbf{\Delta}\sum_{j=1}^{Nr}[\Re_j^{AT}(z_\omega)\mathbf{H}_-^{AT}(\gamma_j)+\aleph_j^{AT}(z_\omega)\mathbf{H}_+^{AT}(\gamma_j)]+\mathbf{\Delta}\sum_{j=1}^{Nc}[AZ_-^{AT}(z,\gamma_j^{AT})+BZ_+^{AT}(z,\lambda_j^{AT/OC})]]$$

$$-\frac{1}{n^2}\sum_{j=1}^{Nr}[\Re_j^{AT}(z)\mathbf{H}_-^{AT}(\gamma_j)+\aleph_j^{AT}(z)\mathbf{H}_+^{AT}(\gamma_j)]+\sum_{j=1}^{Nc}AZ_-^{AT}(z,\gamma_j^{AT})+BZ_+^{AT}(z,\gamma_j^{AT})$$

$$=\mathbf{\Xi}_3 \tag{4.7.59}$$

$$[I_{OC}^p(+;z_1)]=[\sum_{j=1}^{Nr}[\Re_j^{OC}(z)\mathbf{H}_+^{OC}(\gamma_j)+\aleph_j^{OC}(z)\mathbf{H}_-^{OC}(\gamma_j)]+\sum_{j=1}^{Nc}[AZ_+^{OC}(z_1,\gamma_j^{OC})+BZ_-^{OC}(z,\lambda_j^{OC})]].$$

$$=\mathbf{\Xi}_4 \tag{4.7.60}$$

With this assumption we conclude that we can at least find all the unknown coefficients once we solve the above set of linear equations. Hence our essential requirement for the complete solutions of the problem is at hand. But these first versions of solutions are not general since they are only for discrete directions. We shall now develop general solutions for any desired angle with the help of discrete solutions in the next sub section.

4.8. Post processing procedure:

To get the solutions for any desired direction we shall adopt post processing procedure [54] by substituting the discrete solutions in the right hand side of the discretized equation of transfer in both the media. We recall our equation of transfer (2.9.1) and (2.9.7)

$$\mu\frac{d}{dz}I_{AT}(z,\mu)+I_{AT}(z,\mu)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_{J=S}^{M}(\mu)\mathbf{B}_J^{AT}\sum_{\alpha=1}^{N}\omega_\alpha I_{j,\alpha}^{AT}(z)+S_{AT}(z,\mu)$$

$$\mu\frac{d}{dz}I_{OC}(z,\mu)+I_{OC}(z,\mu)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_{J=S}^{M}(\mu)\mathbf{B}_J^{OC}\sum_{\alpha=1}^{N}\omega_\alpha I_{j,\alpha}^{OC}(z)+S_{OC}(z,\mu)$$

Where $I_{j,\alpha}^{AT}(z)=\mathbf{P}_j^m(\mu_\alpha)I_{AT}(z,\mu_\alpha)+\mathbf{P}_j^m(-\mu_\alpha)I_{AT}(z,-\mu_\alpha),$ (4.8.1)

$$I_{j,\alpha}^{OC}(z)=\mathbf{P}_j^m(\mu_\alpha)I_{OC}(z,\mu_\alpha)+\mathbf{P}_j^m(-\mu_\alpha)I_{OC}(z,-\mu_\alpha), \tag{4.8.2}$$

Let us represent the right hand side of the above equations of transfer as follows:

$$RSAT(z,\mu)=\frac{\omega^{AT}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_{J=S}^{M}(\mu)\sum_{\alpha=1}^{N}\omega_\alpha I_{j,\alpha}^{AT}(z)+S_{AT}(z,\mu), \tag{4.8.3}$$

$$RSOC(z,\mu)=\frac{\omega^{OC}(z)}{2}\sum_{J=S}^{M}\mathbf{P}_{J=S}^{M}(\mu)\sum_{\alpha=1}^{N}\omega_\alpha I_{j,\alpha}^{OC}(z)+S_{OC}(z,\mu), \tag{4.8.4}$$

$$\mu\frac{d}{dz}I_{AT}(z,\mu)+I_{AT}(z,\mu)=RSAT(z,\mu), \tag{4.8.5}$$

$$\mu\frac{d}{dz}I_{OC}(z,\mu)+I_{OC}(z,\mu)=RSOC(z,\mu), \tag{4.8.6}$$

Integrating these two equations we get solutions for any optical depth in the atmosphere and ocean from the following four equations depending on the direction.

$$I_{AT}(z(at),+\mu)=I^{AT}(z_\omega,\mu)\exp\left(-\frac{z_\omega-z(at)}{\mu}\right)+\int_{z(at)}^{z_\omega}RSAT(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}, \tag{4.8.7}$$

$$I_{AT}(z(at),-\mu)=I^{AT}(0,-\mu)\exp\left(-\frac{z(at)}{\mu}\right)+\int_{0}^{z(at)}RSAT(x,-\mu)\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}, \tag{4.8.8}$$

$$I_{OC}(z(oc),+\mu)=I^{OC}(z_1,\mu)\exp\left(-\frac{z_1-z(oc)}{\mu}\right)+\int_{z(oc)}^{z_1}RSOC(x,\mu)\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}, \tag{4.8.9}$$

$$I_{OC}(z(oc),-\mu)=I^{OC}(z_\omega,-\mu)\exp\left(-\frac{z(oc)}{\mu}\right)+\int_{z_\omega}^{z(oc)}RSOC(x,-\mu)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}. \tag{4.8.10}$$

We can now write the exit intensity distribution from the above consideration as follows:
Exit distributions from the top of the atmosphere: From equation (4.8.7) we set **z (at) =0** to get,

$$I(0,\mu)=I(z_\omega,\mu)\exp(-\frac{z_\omega}{\mu})+\int_0^{z_\omega}RSAT(x,\mu)\exp(-\frac{x}{\mu})\frac{dx}{\mu} \tag{4.8.11}$$

From boundary condition (4.5.6) we can substitute for $I(z_\omega,\mu)$ in to get the following expressions which involve ocean surface and bottom boundary quantities $I^{AT}(z_\omega,-\mu)$ and $I^{OC}(z_\omega,\mu)$.

$$I_{AT}(0,\mu)=R(-\mu,n)I_{AT}(z_\omega,-\mu)\exp(-\frac{z_\omega}{\mu})+\frac{T(\mu,n)}{n^2}I^{OC}(z_\omega,\mu)\exp(-\frac{z_\omega}{\mu})+$$

$$\int_0^{z_\omega}RSAT(x,\mu)\exp(-\frac{x}{\mu})\frac{dx}{\mu}$$

We now use equations (4.8.8) and (4.8.9) in the above expression with appropriate changes to get

$$\mathbf{I_{AT}(0,\mu) = R(-\mu,n)I^{AT}(0,-\mu)\exp(-\frac{z_\omega}{\mu})\exp(-\frac{z_\omega}{\mu}) + R(-\mu,n)\exp(-\frac{z_\omega}{\mu})}$$

$$\mathbf{\int_0^{z_\omega} RSAT(x,-\mu)\exp(-\frac{z_\omega - x}{\mu})\frac{dx}{\mu}}$$

$$\mathbf{+\frac{T(\mu,n)}{n^2}\exp(-\frac{z_\omega}{\mu})I^{OC}(z_1,\mu)\exp(-\frac{z_1 - z_\omega}{\mu}) + \frac{T(\mu,n)}{n^2}\exp(-\frac{z_\omega}{\mu})\int_{z_\omega}^{z_1} RSOC(x,\mu)\exp(-\frac{x - z_\omega}{\mu})\frac{dx}{\mu}}$$

$$\mathbf{+\int_0^{z_\omega} RSAT(x,\mu)\exp(-\frac{x}{\mu})\frac{dx}{\mu}} \qquad (4.8.12)$$

All the terms on the right hand side of are known provided the integrals involving are properly evaluated. In expressions (4.8.7)-(4.8.10) the term inside the integral sign on the right hand side, requires attention. It involve $\mathbf{I_{AT}(z,\pm\mu_\alpha)}$ and $\mathbf{I_{OC}(z,\pm\mu_\alpha)}$ which we have found in our previous analysis vide equations (4.5.1 – 4.5.4).

$$\mathbf{I_{AT+}(z) = RE_+^{AT}(z) + CO_+^{AT}(z) + I_{AT}^p(+;z);} \qquad (4.5.1)$$

$$\mathbf{I_{AT-}(z) = RE_-^{AT}(z) + CO_-^{AT}(z) + I_{AT}^p(-;z).} \qquad (4.5.2)$$

$$\mathbf{I_{OC+}(z) = RE_+^{OC}(z) + CO_+^{OC}(z) + I_{OC}^p(+;z);} \qquad (4.5.3)$$

$$\mathbf{I_{OC-}(z) = RE_-^{OC}(z) + CO_-^{OC}(z) + I_{OC}^p(-;z).} \qquad (4.5.4)$$

Substituting these on the right hand side of the integral part of (4.8.7) we get

$$\mathbf{\int_{z(at)}^{z_\omega} RSAT(x,\mu)\exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} =}$$

$$\mathbf{\sum_{J=S}^{M} P_J^S(\mu) B_J^{AT}\left[\int_{z(at)}^{z_\omega} \frac{\omega^{AT}(z)}{2}\sum_{\alpha=1}^{N} \omega_\alpha\left[P_j^S(\mu_\alpha)I_{AT}(x,\mu_\alpha) + P_j^S(-\mu_\alpha)I_{AT}(x,-\mu_\alpha)\right]\exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu}\right.}$$

$$\mathbf{\left. + \int S_{AT}(x,\mu)\exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu}\right]}$$

$$= \sum_{J=S}^{M} \mathbf{P}_J^S(\mu)\mathbf{B}_J^{AT}\left[\int_{z(at)}^{z_\omega} \frac{\omega^{AT}(z)}{2}\left[\mathbf{\Pi}(J,S)^T WI(x) + (-1)^{J-S} D\mathbf{\Pi}(J,S)^T WI(-x)\right]\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]$$

$$+ \int S_{AT}(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\Bigg]$$

$$= \sum_{J=S}^{M} \mathbf{P}_J^S(\mu)\mathbf{B}_J^{AT}\left[\int_{z(at)}^{z_\omega} \mathbf{\Pi}(J,S)^T \frac{\omega^{AT}(z)}{2} W\left\{RE_+^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} + \right.$$

$$\int_{z(at)}^{z_\omega} \mathbf{\Pi}(J,S)^T \frac{\omega^{AT}(z)}{2} W\left\{CO_+^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\int_{z(at)}^{z_\omega} \mathbf{\Pi}(J,S)^T W\left\{I_{AT}^P(+;x)\right\}\frac{\omega^{AT}(z)}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$+ \int_{z(at)}^{z_\omega} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{RE_-^{AT}(x)\right\}\frac{\omega^{AT}(z)}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\int_{z(at)}^{z_\omega} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{CO_-^{AT}(x)\right\}\frac{\omega^{AT}(z)}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\left.\int_{z(at)}^{z_\omega} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{I_{AT}^P(-;x)\right\}\frac{\omega^{AT}(z)}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]$$

$$+\left[\int_{z(at)}^{z_\omega} S_{AT}(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]. \tag{4.8.13}$$

These seven integral in equation (4.8.13) have to be evaluated.

4.9: Evaluation of integral in equation in (4.8.7):

Let us first consider the integrand of the **first integral** using equation (3.5.13a) for $RE_+^{AT}(x)$

$$\int_{z(at)}^{z_\omega} \left\{RE_+^{AT}(x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} =$$

$$\int_{z(at)}^{z_\omega}\left\{\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{x}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z_\omega - x}{\gamma_j^{AT}}\right)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$=\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\int_{z(at)}^{z_\omega}\exp(-\frac{x}{s})\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nr} B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\int_{z(at)}^{z_\omega}\exp(-\frac{x}{s})\exp\left(-\frac{z_\omega-x}{\gamma_j^{AT}}\right)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}.$$

$$=\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\frac{-s\gamma_j^{AT}}{s\mu+s\gamma_j^{AT}+\mu\gamma_j^{AT}}\left\{\exp\left(\frac{z(at)}{\mu}-(\frac{1}{\gamma_j^{AT}}+\frac{1}{\mu}+\frac{1}{s})zw\right)-\exp\left(-(\frac{1}{s}+\frac{1}{\gamma_j^{AT}})zat\right)\right\}$$

$$+\sum_{j=1}^{Nr} B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\left[\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}\mu+s\gamma_j^{AT}-s\mu)}\left\{\exp\left(-(\frac{1}{s}+\frac{1}{\mu})zw+\frac{zat}{\mu}\right)-\exp\left(-(\frac{1}{s}-\frac{1}{\gamma_j^{AT}})zat-\frac{zw}{\gamma_j^{AT}}\right)\right\}\right].$$

(4.9.1)

Second integral: Using equations (3.5.14a) for $CO_+^{AT}(x)$ we obtain

$$\int_{z(at)}^{z_\omega}\left\{CO_+^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}=$$

$$\sum_{j=1}^{Nc} A_j^{AT(1)}\int_{z(at)}^{z_\omega}F_+^{AT(1)}(x,\gamma_j^{AT})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)}\int_{z(at)}^{z_\omega}F_+^{AT(2)}(x,\gamma_j^{AT})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)}\int_{z(at)}^{z_\omega}F_-^{AT(1)}(z_0-x,\gamma_j^{AT})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$=\sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\int_{z(at)}^{z_\omega}\mathrm{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}-$$

$$\sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\int_{z(at)}^{z_\omega}\mathrm{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\int_{z(at)}^{z_\omega}\mathrm{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}-$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)} \operatorname{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \operatorname{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)} \operatorname{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \operatorname{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} -$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)} \operatorname{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \operatorname{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \operatorname{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \operatorname{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} -$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \operatorname{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \operatorname{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$\sum_{j=1}^{Nc} A_j^{AT(1)} \operatorname{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\operatorname{Im}(X1) +$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)} \operatorname{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\operatorname{Im}(X1) - \sum_{j=1}^{Nc} A_j^{AT(2)} \operatorname{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\operatorname{Re}(X1) +$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)} \operatorname{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\operatorname{Re}(X1) - \sum_{j=1}^{Nc} B_j^{AT(1)} \operatorname{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\operatorname{Im}(X1) +$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \operatorname{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\operatorname{Im}(X1) - \sum_{j=1}^{Nc} B_j^{AT(2)} \operatorname{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\operatorname{Re}(X1). \tag{4.9.2}$$

$$X1 = \frac{-\gamma_j^{AT}}{\mu + \gamma_j^{AT}}\left\{\exp\left(\frac{z(at)-z_\omega}{\mu}\right)\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right) - \exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}. \tag{4.9.3}$$

This integral will be relevant if there are complex eigen values. For N=2 we need not have to calculate this integral for there is no complex eigen values.

Third integral: We shall now use equations (4.4.49) for $I_{AT}^{P}(+;x)$ to evaluate

$$\int_{z(at)}^{z_\omega}\left\{I_{AT}^{P}(+;x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}=$$

$$\left[\sum_{j=1}^{Nr}\int_{z(at)}^{z_\omega}\Re_j^{AT}(x)\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+\sum_{j=1}^{Nr}\int_{z(at)}^{z_\omega}\aleph_j^{AT}(x)\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+\right.$$

$$\left.\sum_{j=1}^{Nc}\int_{z(at)}^{z_\omega}AZ_+^{AT}(x,\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+\sum_{j=1}^{Nc}\int_{z(at)}^{z_\omega}BZ_-^{AT}(x,\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]$$

$$=2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\int_{z(at)}^{z_\omega}\frac{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)-\exp\left(-\frac{x}{\mu_0}\right)}{\gamma_j^{AT}-\mu_0}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\int_{z(at)}^{z_\omega}\frac{1-\exp\left(-\frac{z_\omega-x}{\gamma_j^{AT}}\right)\exp\left(-\frac{z_\omega-x}{\mu_0}\right)}{\gamma_j^{AT}+\mu_0}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\exp(-\frac{x}{\mu_0})\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}a_j^{AT*}\mathbf{H}_+^{AT\,*}(\gamma_j^{AT*})\int_{z(at)}^{z_\omega}\frac{\exp\left(-\frac{x}{\gamma_j^{AT*}}\right)-\exp\left(-\frac{x}{\mu_0}\right)}{\gamma_j^{AT*}-\mu_0}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}b_j^{AT*}\mathbf{H}_-^{AT\,*}(\gamma_j^{AT*})\int_{z(at)}^{z_\omega}\frac{1-\exp\left(-\frac{z_\omega-x}{{}_j^{AT*}}\right)\exp\left(-\frac{z_\omega-x}{\mu_0}\right)}{\gamma_j^{AT*}{}_j+\mu_0}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$=2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})$$

$$\left[\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}-\mu_0)(s\gamma_j^{AT}+s\mu-\mu\gamma_j^{AT})}\left\{\exp\left(\left(\frac{z(at)}{\mu}\right)-\left(\frac{1}{\mu}+\frac{1}{s}+\frac{1}{\gamma_j^{AT}}\right)z_\omega\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT}}\right)z(at)\right)\right\}+\right.$$

$$\frac{s\mu_0}{(\gamma_j^{AT}-\mu_0)(s\mu+s\mu_0+\mu\mu_0)}\left\{\exp\left[-\left(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s}\right)z_\omega+\left(\frac{z(at)}{\mu}\right)\right]-\exp\left(\left[-\left(\frac{1}{\mu_0}+\frac{1}{s}\right)z(at)\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})$$

$$\left[\frac{-s\mu_0}{(\gamma_j^{AT}+\mu_0)(s\mu+\mu\mu_0+s\mu_0)}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_0}+\frac{1}{s}\right)z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_0}+\frac{1}{\mu_0}\right)z(at)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}+\mu_0)(s\mu+s\gamma_j^{AT}-\mu s)}\left\{\exp\left(-(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s})z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT}}\right)z-\left(\frac{1}{\gamma_j^{AT}}+\frac{1}{\mu_0}\right)z_\omega\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}a_j^{AT*}\mathbf{H}_+^{AT*}(\gamma_j^{AT*})$$

$$\left[\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}-\mu_0)(s\gamma_j^{AT*}+s\mu-\mu\gamma_j^{AT*})}\left\{\exp\left(\left(\frac{z(at)}{\mu}\right)-\left(\frac{1}{\mu}+\frac{1}{s}+\frac{1}{\gamma_j^{AT*}}\right)z_\omega\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT*}}\right)z(at)\right)\right\}+\right.$$

$$\left.\frac{s\mu_0}{(\gamma_j^{AT*}-\mu_0)(s\mu+s\mu_0+\mu\mu_0)}\left\{\exp\left[-\left(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s}\right)z_\omega+\left(\frac{z(at)}{\mu}\right)\right]-\exp\left(-\left(\frac{1}{\mu_0}+\frac{1}{s}\right)z(at)\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}b_j^{AT*}\mathbf{H}_-^{AT*}(\gamma_j^{AT*})$$

$$\left[\frac{-s\mu_0}{(\gamma_j^{AT*}+\mu_0)(s\mu+\mu\mu_0+s\mu_0)}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_0}+\frac{1}{s}\right)z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_0}+\frac{1}{\mu_0}\right)z(at)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}+\mu_0)(s\mu+s\gamma_j^{AT*}-\mu s)}\left\{\exp\left(-(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s})z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT*}}\right)z-\left(\frac{1}{\gamma_j^{AT*}}+\frac{1}{\mu_0}\right)z_\omega\right)\right\}\right].$$

(4.9.4)

The third and forth term of this expression are relevant when complex conjugate pairs of eigenvalues are present in the eigenvalues spectrum. For N=2 these terms are redundant for there exist no complex eigenvalues. We need only to evaluate the first two terms extending the summation over 8 terms.

Fourth integral: Using again equation (3.5.15a) we evaluate the integral

$$\int_{z(at)}^{z_\omega}\left\{RE_-^{AT}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}=$$

$$\mathbf{\Delta}\int_{z(at)}^{z_\omega}\left\{\sum_{j=1}^{Nr}A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x}{\gamma_j^{AT}}\right)+B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{z_\omega-x}{\gamma_j^{AT}}\right)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$=\mathbf{\Delta}\sum_{j=1}^{Nr}A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\int_{z(at)}^{z_\omega}\exp(-\frac{x}{s})\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\mathbf{\Delta}\sum_{j=1}^{Nr}B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\int_{z(at)}^{z_\omega}\exp(-\frac{x}{s})\exp\left(-\frac{z_\omega-x}{\gamma_j^{AT}}\right)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$=\mathbf{\Delta}\sum_{j=1}^{Nr}A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\frac{-s\gamma_j^{AT}}{s\mu+s\gamma_j^{AT}+\mu\gamma_j^{AT}}\left\{\exp\left(\frac{z(at)}{\mu}-(\frac{1}{\gamma_j^{AT}}+\frac{1}{\mu}+\frac{1}{s})zw\right)-\exp\left(-(\frac{1}{s}+\frac{1}{\gamma_j^{AT}})zat\right)\right\}$$

$$+\mathbf{\Delta}\sum_{j=1}^{Nr}B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\left[\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}\mu+s\gamma_j^{AT}-s\mu)}\left\{\exp\left(-(\frac{1}{s}+\frac{1}{\mu})zw+\frac{zat}{\mu}\right)-\exp\left(-(\frac{1}{s}-\frac{1}{\gamma_j^{AT}})zat-\frac{zw}{\gamma_j^{AT}}\right)\right\}\right]. \tag{4.9.5}$$

Fifth integral: Here we use (3.5.16a)

$$\int_{z(at)}^{z_\omega}\left\{CO_-^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}=$$

$$\sum_{j=1}^{Nc}A_j^{AT(1)}\int_{z(at)}^{z_\omega}F_-^{AT(1)}(x,\gamma_j^{AT})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nc}A_j^{AT(2)}\int_{z(at)}^{z_\omega}F_-^{AT(2)}(x,\gamma_j^{AT})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)} \int_{z(at)}^{z_\omega} F_+^{AT(1)}(z_0 - x, \gamma_j^{AT}) \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \int_{z(at)}^{z_\omega} F_+^{AT(2)}(z_0 - x, \gamma_j^{AT}) \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$= \sum_{j=1}^{Nc} A_j^{AT(1)} \mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} -$$

$$\sum_{j=1}^{Nc} A_j^{AT(1)} \mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)} \mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} -$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)} \mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)} \mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} -$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)} \mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Im}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu} -$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\} \int_{z(at)}^{z_\omega} \mathrm{Re}\left\{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)\right\} \exp\left(-\frac{x - z(at)}{\mu}\right)\frac{dx}{\mu}.$$

$$= \sum_{j=1}^{Nc} A_j^{AT(1)} \mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2) - \sum_{j=1}^{Nc} A_j^{AT(1)} \mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2) +$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)} \mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2) - \sum_{j=1}^{Nc} A_j^{AT(2)} \mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2) +$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)} \mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2) - \sum_{j=1}^{Nc} B_j^{AT(1)} \mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2) +$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2) - \sum_{j=1}^{Nc} B_j^{AT(2)} \mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2). \qquad (4.9.6)$$

$$X2=\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)-z_\omega}{\mu}-\frac{z_\omega}{\gamma_j^{AT}}\right)\right\}. \tag{4.9.7}$$

For N=2, the value of this integral need not have to be calculated since all the eigenvalues for this case is real and hence its value is incorporated already in 4th integral where the summation is carried out over Nr=8 terms.

Sixth integral: We use equation (3.5.16a) in deriving the following integral

$$\int_{z(at)}^{z_\omega}\left\{I_{AT}^{p}(-;x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\Bigg]=$$

$$\left[\mathbf{\Delta}\sum_{j=1}^{Nr}\int_{z(at)}^{z_\omega}\Re_j^{AT}(x)\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+\right.$$

$$\left.\mathbf{\Delta}\sum_{j=1}^{Nr}\int_{z(at)}^{z_\omega}\aleph_j^{AT}(x)\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]+$$

$$\mathbf{\Delta}\left[\sum_{j=1}^{Nc}\int_{z(at)}^{z_\omega}AZ_-^{AT}(x,\gamma_j^{AT})\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+\sum_{j=1}^{Nc}\int_{z(at)}^{z_\omega}BZ_+^{AT}(x,\gamma_j^{AT})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right].$$

$$=2\mathbf{\Delta}\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_-^{AT}(\gamma_j)\int_{z(at)}^{z_\omega}\frac{\exp\left(-\frac{x}{\gamma_j^{AT}}\right)-\exp\left(-\frac{x}{\mu_0}\right)}{\gamma_j^{AT}-\mu_0}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$2\mathbf{\Delta}\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\int_{z(at)}^{z_\omega}\frac{1-\exp\left(-\frac{z_\omega-x}{\gamma_j^{AT}}\right)\exp\left(-\frac{z_\omega-x}{\mu_0}\right)}{\gamma_j^{AT}+\mu_0}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\exp(-\frac{x}{\mu_0})\frac{dx}{\mu}+$$

$$\mathbf{\Delta}\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}a_j^{AT*}\mathbf{H}_-^{AT*}(\gamma_j^{AT*})\int_{z(at)}^{z_\omega}\frac{\exp\left(-\frac{x}{\gamma_j^{AT*}}\right)-\exp\left(-\frac{x}{\mu_0}\right)}{\gamma_j^{AT*}-\mu_0}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}+$$

$$\Delta\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}b_j^{AT*}\mathbf{H}_+^{AT\,*}(\gamma_j^{AT*})\int_{z(at)}^{z_\omega}\frac{1-\exp\left(-\frac{z_\omega-x}{\gamma_j^{AT*}}\right)\exp\left(-\frac{z_\omega-x}{\mu_0}\right)}{\gamma_j^{AT*}{}_j+\mu_0}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}.$$

$$=2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})$$

$$\Delta\left[\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}-\mu_0)(s\gamma_j^{AT}+s\mu-\mu\gamma_j^{AT})}\left\{\exp\left(\left(\frac{z(at)}{\mu}\right)-\left((\frac{1}{\mu})+(\frac{1}{s})+(\frac{1}{\gamma_j^{AT}})\right)z_\omega\right)-\exp\left(-\left((\frac{1}{s})+(\frac{1}{\gamma_j^{AT}})\right)z(at)\right)\right\}\right.$$

$$\left.\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}+\mu_0)(s\mu+s\gamma_j^{AT}-\mu s)}\left\{\exp\left(-(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s})z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT}}\right)z-\left(\frac{1}{\gamma_j^{AT}}+\frac{1}{\mu_0}\right)z_\omega\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})$$

$$\Delta\left[\frac{-s\mu_0}{(\gamma_j^{AT}+\mu_0)(s\mu+\mu\mu_0+s\mu_0)}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_o}+\frac{1}{s}\right)z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_0}+\frac{1}{\mu_0}\right)z(at)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}+\mu_0)(s\mu+s\gamma_j^{AT}-\mu s)}\left\{\exp\left(-(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s})z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT}}\right)z-\left(\frac{1}{\gamma_j^{AT}}+\frac{1}{\mu_0}\right)z_\omega\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}a_j^{AT*}\mathbf{H}_-^{AT*}(\gamma_j^{AT*})$$

$$\Delta\left[\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}-\mu_0)(s\gamma_j^{AT*}+s\mu-\mu\gamma_j^{AT*})}\left\{\exp\left(\left(\frac{z(at)}{\mu}\right)-\left((\frac{1}{\mu})+(\frac{1}{s})+(\frac{1}{\gamma_j^{AT*}})\right)z_\omega\right)-\exp\left(-\left((\frac{1}{s})+(\frac{1}{\gamma_j^{AT*}})\right)z(at)\right)\right\}\right.$$

$$\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}+\mu_0)(s\mu+s\gamma_j^{AT*}-\mu s)}\left\{\exp\left(-(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s})z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT*}}\right)z-\left(\frac{1}{\gamma_j^{AT*}}+\frac{1}{\mu_0}\right)z_\omega\right)\right\}\Bigg]-$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}b_j^{AT*}\mathbf{H}_+^{AT*}(\gamma_j^{AT*})$$

$$\mathbf{\Delta}\left[\frac{-s\mu_0}{(\gamma_j^{AT*}+\mu_0)(s\mu+\mu\mu_0+s\mu_0)}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_0}+\frac{1}{s}\right)z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_0}+\frac{1}{\mu_0}\right)z(at)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}+\mu_0)(s\mu+s\gamma_j^{AT*}-\mu s)}\left\{\exp\left(-(\frac{1}{\mu_0}+\frac{1}{\mu}+\frac{1}{s})z_\omega+\frac{z(at)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT*}}\right)z-\left(\frac{1}{\gamma_j^{AT*}}+\frac{1}{\mu_0}\right)z_\omega\right)\right\}\right].$$

(4.9.8)

In this integral the last two term are redundant because for N=2 there exist no complex eigen values. But the first two terms are calculated by extending the summation over 8 terms.

Seventh Integral: Using equation (4.4.33) we can evaluate the integral

$$\left[\int_{z(at)}^{z_\omega} S_{AT}(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]. = S_{AT}(+)\frac{-\mu_0}{\mu+\mu_0}\left[\exp(-(\frac{1}{\mu}+\frac{1}{\mu_0})z_\omega-\exp(-(\frac{1}{\mu_0})z(at)\right] \quad (4.9.9)$$

4.10: Evaluation of integral in equation in (4.8.8):

In this equation the integral part involves changes in the limits of integration as is evident from the following expressions.

$$\int_0^{z(at)} RSAT(x,-\mu)\exp(-\frac{x}{s})\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}$$

$$=\sum_{J=S}^{M}\mathbf{P}_J^S(-\mu)\mathbf{B}_J^{AT}\left[\int_0^{z(at)}\mathbf{\Pi}(J,S)^T W\left\{RE_+^{AT}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}+\right.$$

$$\int_0^{z(at)} \mathbf{\Pi}(J,S)^T W\left\{CO_+^{AT}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}+$$

$$\int_0^{z(at)} \mathbf{\Pi}(J,S)^T W\left\{I_{AT}^P(+;x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}$$

$$+\int_0^{z(at)} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{RE_-^{AT}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}+$$

$$\int_0^{z(at)} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{CO_-^{AT}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}+$$

$$\left.\int_0^{z(at)} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{I_{AT}^P(-;x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}\right]$$

$$+\left[\int_0^{z(at)} S_{AT}(x,\mu)\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}\right]. \tag{4.10.1}$$

Evaluation each individual integrals using appropriate expressions for required quantities we get the following results.

First Integral: We again use (3.5.13a)

$$=\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\frac{-s\gamma_j^{AT}}{\gamma_j^{AT}\mu+s\mu-s\gamma_j^{AT}}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT}}\right)z(at)\right)-\exp\left(-\frac{z(at)}{\mu}\right)\right\}$$

$$+\sum_{j=1}^{Nr} B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\frac{-s\gamma_j^{AT}}{\gamma_j^{AT}\mu-s\mu-s\gamma_j^{AT}}\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\left(\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT}}\right)z(at)\right)-\exp\left(-\frac{z(at)}{\mu}\right)\right). \tag{4.10.2}$$

Second Integral: We use (3.5.14a)

$$=\sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\,\mathrm{Re}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\,\mathrm{Im}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}+$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\,\mathrm{Im}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)}\,\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\,\mathrm{Re}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}+$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)}\,\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\,\mathrm{Re}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)}\,\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\,\mathrm{Im}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}+$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)}\,\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\,\mathrm{Im}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)}\,\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\,\mathrm{Re}\frac{\gamma_j^{AT}}{\gamma_j^{AT}-\mu}\left\{\exp\left(-\frac{z(at)}{\gamma_j^{AT}}\right)-\exp\left(\frac{z(at)}{\mu}\right)\right\}. \tag{4.10.3}$$

Third Integral: In this case we again use (3.5.16a)

$$=2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\left[\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}-\mu_0)(\mu\gamma_j^{AT}-s\gamma_j^{AT}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT}}\right)zat\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\mu_0}{(\gamma_j^{AT}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)z(at)\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\left[\frac{-s\mu_0}{(\gamma_j^{AT}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)z(at)\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}+\mu_0)(\gamma_j^{AT}\mu-s\mu-s\gamma_j^{AT})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT}}\right)z\right)-\exp\left(-\left(z_\omega(\frac{1}{\gamma_j^{AT}}+\frac{1}{\mu_0})\right)\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}a_j^{AT*}\mathbf{H}_+^{AT*}(\gamma_j^{AT*})\left[\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}-\mu_0)(\mu\gamma_j^{AT*}-s\gamma_j^{AT*}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT*}}\right)zat\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.$$

$$\frac{-s\mu_0}{(\gamma_j^{AT*}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)z(at)\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}\Bigg]+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}b_j^{AT*}\mathbf{H}_-^{AT*}(\gamma_j^{AT*})\left[\frac{-s\mu_0}{(\gamma_j^{AT*}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)z(at)\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}+\mu_0)(\gamma_j^{AT*}\mu-s\mu-s\gamma_j^{AT*})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT*}}\right)z\right)-\exp\left(-\left(z_\omega(\frac{1}{\gamma_j^{AT*}}+\frac{1}{\mu_0})\right)\right)\right\}\right]. \quad (4.10.4)$$

Fourth Integral:

$$=\mathbf{\Delta}\sum_{j=1}^{Nr}A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\frac{-s\gamma_j^{AT}}{\gamma_j^{AT}\mu+s\mu-s\gamma_j^{AT}}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT}}\right)zat\right)-\exp\left(-\frac{z(at)}{\mu}\right)\right\}$$

$$+\mathbf{\Delta}\sum_{j=1}^{Nr}B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\frac{-s\gamma_j^{AT}}{\gamma_j^{AT}\mu-s\mu-s\gamma_j^{AT}}\exp\left(-\frac{z_\omega}{\gamma_j^{AT}}\right)\left(\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT}}\right)zat\right)-\exp\left(-\frac{z(at)}{\mu}\right)\right).$$

(4.10.5)

Fifth Integral:

$$=\sum_{j=1}^{Nc}A_j^{AT(1)}\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}-$$

$$\sum_{j=1}^{Nc}A_j^{AT(1)}\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}+$$

$$\sum_{j=1}^{Nc}A_j^{AT(2)}\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}-$$

$$\sum_{j=1}^{Nc}A_j^{AT(2)}\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}+$$

$$\sum_{j=1}^{Nc}B_j^{AT(1)}\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Re}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}-$$

$$\sum_{j=1}^{Nc}B_j^{AT(1)}\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Im}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}+$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \operatorname{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\} \operatorname{Im}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}-$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)} \operatorname{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\} \operatorname{Re}\frac{-\gamma_j^{AT}}{\mu+\gamma_j^{AT}}\left\{1-\exp\left(\frac{-z(at)}{\mu}-\frac{z(at)}{\gamma_j^{AT}}\right)\right\}. \tag{4.10.6}$$

Sixth Integral:

$$=2\boldsymbol{\Delta}\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\left[\left[\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}-\mu_0)(\mu\gamma_j^{AT}-s\gamma_j^{AT}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT}}\right)zat\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.\right.$$

$$\left.\left.\frac{-s\mu_0}{(\gamma_j^{AT}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)zat\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}\right]\right]+$$

$$2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\left[\frac{-s\mu_0}{(\gamma_j^{AT}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)zat\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{AT}}{(\gamma_j^{AT}+\mu_0)(\gamma_j^{AT}\mu-s\mu-s\gamma_j^{AT})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT}}\right)z\right)-\exp\left(-\left(z_\omega(\frac{1}{\gamma_j^{AT}}+\frac{1}{\mu_0})\right)\right)\right\}\right]+$$

$$2\boldsymbol{\Delta}\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}a_j^{AT*}\mathbf{H}_-^{AT*}(\gamma_j^{AT*})\left[\left[\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}-\mu_0)(\mu\gamma_j^{AT*}-s\gamma_j^{AT*}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{AT*}}\right)z(at)\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.\right.$$

$$\left.\left.\frac{-s\mu_0}{(\gamma_j^{AT*}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)z(at)\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}\right]\right]+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}b_j^{AT*}\mathbf{H}_+^{AT*}(\gamma_j^{AT*})\left[\frac{-s\mu_0}{(\gamma_j^{AT*}-\mu_0)((\mu\mu_0-s\mu_0+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_0}\right)z(at)\right)-\exp\left(\frac{-z(at)}{\mu}\right)\right\}-\right.$$

$$\frac{-s\gamma_j^{AT*}}{(\gamma_j^{AT*}+\mu_0)(\gamma_j^{AT*}\mu-s\mu-s\gamma_j^{AT*})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{AT*}}\right)z\right)-\exp\left(-\left(z_\omega\left(\frac{1}{\gamma_j^{AT*}}+\frac{1}{\mu_0}\right)\right)\right)\right\}\Bigg]. \tag{4.10.7}$$

Seventh integral:

$$\left[\int_0^{z(at)} S_{AT}(x,\mu)\exp\left(-\frac{z(at)-x}{\mu}\right)\frac{dx}{\mu}\right]=S_{AT}(-)\frac{-\mu_0}{\mu+\mu_0}\left[\exp(-(\frac{1}{\mu_0})z(at)-\exp(-(\frac{1}{\mu})z(at)\right]. \tag{4.10.8}$$

OCEAN:

4.11. Evaluation of integral in equation in (4.8.9):

$$\int_{z(oc)}^{z_1} RSOC(x,\mu)\exp(-\frac{x}{s})\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}=$$

$$\sum_{J=S}^{M}\mathbf{P}_J^S(\mu)\mathbf{B}_J^{OC}\left[\int_{z(oc)}^{z_1}\exp(-\frac{x}{s})\frac{\omega^{OC}(z)}{2}\sum_{\alpha=1}^{N}\omega_\alpha\left[\mathbf{P}_j^S(\mu_\alpha)I_{OC}(x,\mu_\alpha)+\mathbf{P}_j^S(-\mu_\alpha)I_{OC}(x,-\mu_\alpha)\right]\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right.$$

$$\left.+\int_{z(oc)}^{z_1}S_{OC}(x,\mu)\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right]$$

$$=\sum_{J=S}^{M}\mathbf{P}_J^S(\mu)\mathbf{B}_J^{OC}\left[\int_{z(oc)}^{z_1}\frac{\omega^{OC}(z)}{2}\left[\mathbf{\Pi}(J,S)^T WI(x)+(-1)^{J-S}D\mathbf{\Pi}(J,S)^T WI(-x)\right]\exp(-\frac{x}{s})\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right.$$

$$\left.+\int_{z(oc)}^{z_1}S_{OC}(x,\mu)\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right]$$

$$= \sum_{J=S}^{M} \mathbf{P}_J^S(\mu)\mathbf{B}_J^{OC}\left[\int_{z(oc)}^{z_1} \mathbf{\Pi}(J,S)^T \frac{\omega^{OC}(z)}{2} W\left\{RE_+^{OC}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}+\right.$$

$$\int_{z(oc)}^{z_1} \mathbf{\Pi}(J,S)^T \frac{\omega^{OC}(z)}{2} W\left\{CO_+^{OC}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}+$$

$$\int_{z(oc)}^{z_1} \mathbf{\Pi}(J,S)^T W\left\{I_{OC}^P(+;x)\right\}\frac{\omega^{OC}(z)}{2}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}$$

$$+\int_{z(oc)}^{z_1} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{RE_-^{OC}(x)\right\}\exp(-\frac{x}{s})\frac{\omega^{OC}(z)}{2}\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}+$$

$$\int_{z(oc)}^{z_1} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{CO_-^{OC}(x)\right\}\exp(-\frac{x}{s})\frac{\omega^{OC}(z)}{2}\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}+$$

$$\left.\int_{z(oc)}^{z_1} (-1)^{J-S} D\mathbf{\Pi}(J,S)^T W\left\{I_{OC}^p(-;x)\right\}\exp(-\frac{x}{s})\frac{\omega^{OC}(z)}{2}\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right]$$

$$+\left[\int_{z(oc)}^{z_1} S_{OC}(x,\mu)\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right]. \tag{4.11.1}$$

As before using the expressions (3.5.13a–3.5.16b) for required functions for oceans we have evaluated the following set of integrals

First Integral:

$$\int_{z(oc)}^{z_1} \left\{RE_+^{OC}(x)\right\}\exp(-\frac{x}{s})\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu} =$$

$$\int_{z(oc)}^{z_1}\left(\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1-z}{\gamma_j^{AT/OC}}\right)\right)\exp(-\frac{x}{s})\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu} =$$

$$= \sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\frac{-s\gamma_j^{OC}}{s\mu+s\gamma_j^{OC}+\mu\gamma_j^{OC}}\left\{\exp\left(\frac{z(oc)}{\mu}-\left(\frac{1}{\gamma_j^{OC}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)\right\}$$

$$+\sum_{j=1}^{Nr} B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\left[\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}\mu+s\gamma_j^{OC}-s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z(oc)-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\right]. \tag{4.11.2}$$

Second Integral:

$$\begin{aligned}&=\sum_{j=1}^{Nc} A_j^{OC(1)}\,\mathrm{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\mathrm{Re}(X1)-\sum_{j=1}^{Nc} A_j^{OC(1)}\,\mathrm{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\mathrm{Im}(X1)+\\ &\sum_{j=1}^{Nc} A_j^{OC(2)}\,\mathrm{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\mathrm{Im}(X1)-\sum_{j=1}^{Nc} A_j^{OC(2)}\,\mathrm{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\mathrm{Re}(X1)+\\ &\sum_{j=1}^{Nc} B_j^{OC(1)}\,\mathrm{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\mathrm{Re}(X1)-\sum_{j=1}^{Nc} B_j^{OC(1)}\,\mathrm{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\mathrm{Im}(X1)+\\ &\sum_{j=1}^{Nc} B_j^{OC(2)}\,\mathrm{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\mathrm{Im}(X1)-\sum_{j=1}^{Nc} B_j^{OC(2)}\,\mathrm{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\mathrm{Re}(X1).\end{aligned} \tag{4.11.3}$$

$$X1=\frac{-\gamma_j^{OC}}{\mu+\gamma_j^{OC}}\left\{\exp\left(\frac{z(oc)-z_1}{\mu}\right)\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)-\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)\right\}. \tag{4.11.4}$$

Third Integral:

$$=2\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}a_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})$$

$$\left[\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}-\mu_{0n})(s\gamma_j^{OC}+s\mu+\mu\gamma_j^{OC})}\left\{\exp\left(\left(\frac{z(oc)}{\mu}\right)-\left(\frac{1}{\mu}+\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z_1\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)\right\}+\right.$$

$$\left.\frac{s\mu_{0n}}{(\gamma_j^{OC}-\mu_{0n})(s\mu+s\mu_{0n}+\mu\mu_{0n})}\left\{\exp\left[-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\left(\frac{z(oc)}{\mu}\right)\right]-\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{s}\right)z(oc)\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}b_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})$$

$$\left[\frac{-s\mu_{0n}}{(\gamma_j^{OC}+\mu_{0n})(s\mu+\mu\mu_{0n}+s\mu_{0n})}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_{0n}}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}\right)z(oc)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}+\mu_{0n})(s\mu+s\gamma_j^{OC}-\mu s)}\left\{\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z(oc)-\left(\frac{1}{\gamma_j^{OC}}+\frac{1}{\mu_{0n}}\right)z_1\right)\right\}\right]$$

$$2\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC^*}a_j^{OC^*}\mathbf{H}_+^{OC^*}(\gamma_j^{OC^*})$$

$$\left[\frac{-s\gamma_j^{OC^*}}{(\gamma_j^{OC^*}-\mu_{0n})(s\gamma_j^{OC^*}+s\mu+\mu\gamma_j^{OC^*})}\left\{\exp\left(\left(\frac{z(oc)}{\mu}\right)-\left(\frac{1}{\mu}+\frac{1}{s}+\frac{1}{\gamma_j^{OC^*}}\right)z_1\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC^*}}\right)z(oc)\right)\right\}+\right.$$

$$\left.\frac{s\mu_{0n}}{(\gamma_j^{OC^*}-\mu_{0n})(s\mu+s\mu_{0n}+\mu\mu_{0n})}\left\{\exp\left[-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\left(\frac{z(oc)}{\mu}\right)\right]-\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{s}\right)z(oc)\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC^*}b_j^{OC^*}\mathbf{H}_-^{OC^*}(\gamma_j^{OC^*})$$

$$\left[\frac{-s\mu_{0n}}{(\gamma_j^{OC^*}+\mu_{0n})(s\mu+\mu\mu_{0n}+s\mu_{0n})}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_{0n}}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}\right)z(oc)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{OC^*}}{(\gamma_j^{OC^*}+\mu_{0n})(s\mu+s\gamma_j^{OC^*}-\mu s)}\left\{\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC^*}}\right)z(oc)-\left(\frac{1}{\gamma_j^{OC^*}}+\frac{1}{\mu_{0n}}\right)z_1\right)\right\}\right]$$

(4.11.5)

Fourth Integral:

$$=\boldsymbol{\Delta}\sum_{j=1}^{Nr}A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\frac{-s\gamma_j^{OC}}{s\mu+s\gamma_j^{OC}+\mu\gamma_j^{OC}}\left\{\exp\left(\frac{z(oc)}{\mu}-\left(\frac{1}{\gamma_j^{OC}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)\right\}$$

$$+\boldsymbol{\Delta}\sum_{j=1}^{Nr}B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\left[\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}\mu+s\gamma_j^{OC}-s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z(oc)-\frac{z_1}{\gamma_j^{OC}}\right)\right\}\right.$$

(4.11.6)

Fifth Integral:

$$
\begin{aligned}
&=\sum_{j=1}^{Nc} A_j^{OC(1)}\operatorname{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\operatorname{Re}(X2)-\sum_{j=1}^{Nc} A_j^{OC(1)}\operatorname{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\operatorname{Im}(X2)+\\
&\sum_{j=1}^{Nc} A_j^{OC(2)}\operatorname{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\operatorname{Im}(X2)-\sum_{j=1}^{Nc} A_j^{OC(2)}\operatorname{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\operatorname{Re}(X2)+\\
&\sum_{j=1}^{Nc} B_j^{OC(1)}\operatorname{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\operatorname{Re}(X2)-\sum_{j=1}^{Nc} B_j^{OC(1)}\operatorname{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\operatorname{Im}(X2)+\\
&\sum_{j=1}^{Nc} B_j^{OC(2)}\operatorname{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\operatorname{Im}(X2)-\sum_{j=1}^{Nc} B_j^{OC(2)}\operatorname{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\operatorname{Re}(X2).
\end{aligned}
\tag{4.11.7}
$$

$$X2=\frac{-\gamma_j^{OC}}{\mu+\gamma_j^{OC}}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z(oc)-z_1}{\mu}-\frac{z_1}{\gamma_j^{OC}}\right)\right\}. \tag{4.11.8}$$

Sixth Integral:

$$=2\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}a_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})$$

$$\boldsymbol{\Delta}\left[\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}-\mu_{0n})(s\gamma_j^{OC}+s\mu-\mu\gamma_j^{OC})}\left\{\exp\left(\left(\frac{z(oc)}{\mu}\right)-\left(\frac{1}{\mu}+\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z_1\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)\right\}+\right.$$

$$\left.\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}+\mu_{0n})(s\mu+s\gamma_j^{OC}-\mu s)}\left\{\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z_1-\left(\frac{1}{\gamma_j^{OC}}+\frac{1}{\mu_{0n}}\right)z_1\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}b_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})$$

$$\mathbf{\Delta}\left[\frac{-s\mu_0}{(\gamma_j^{OC}+\mu_{0n})(s\mu+\mu\mu_{0n}+s\mu_{0n})}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_{0n}}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu_{0n}}\right)z(oc)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}+\mu_{0n})(s\mu+s\gamma_j^{OC}-\mu s)}\left\{\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z_1-\left(\frac{1}{\gamma_j^{OC}}+\frac{1}{\mu_{0n}}\right)z_1\right)\right\}\right]-$$

$$2\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC*}a_j^{OC*}\mathbf{H}_-^{OC*}(\gamma_j^{OC*})$$

$$\mathbf{\Delta}\left[\frac{-s\gamma_j^{OC*}}{(\gamma_j^{OC*}-\mu_{0n})(s\gamma_j^{OC*}+s\mu-\mu\gamma_j^{OC*})}\left\{\exp\left(\left(\frac{z(oc)}{\mu}\right)-\left(\frac{1}{\mu}+\frac{1}{s}+\frac{1}{\gamma_j^{OC*}}\right)z_1\right)-\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC*}}\right)z(oc)\right)\right\}+\right.$$

$$\left.\frac{-s\gamma_j^{OC*}}{(\gamma_j^{OC*}+\mu_{0n})(s\mu+s\gamma_j^{OC*}-\mu s)}\left\{\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC*}}\right)z_1-\left(\frac{1}{\gamma_j^{OC*}}+\frac{1}{\mu_{0n}}\right)z_1\right)\right\}\right]-$$

$$2\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC*}b_j^{OC*}\mathbf{H}_+^{OC*}(\gamma_j^{OC*})$$

$$\mathbf{\Delta}\left[\frac{-s\mu_{0n}}{(\gamma_j^{OC*}+\mu_{0n})(s\mu+\mu\mu_{0n}+s\mu_{0n})}\left\{\exp\left(-\left(\frac{1}{\mu}+\frac{1}{\mu_{0n}}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu_{0n}}\right)z(oc)\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{OC*}}{(\gamma_j^{OC*}+\mu_{0n})(s\mu+s\gamma_j^{OC*}-\mu s)}\left\{\exp\left(-\left(\frac{1}{\mu_{0n}}+\frac{1}{\mu}+\frac{1}{s}\right)z_1+\frac{z(oc)}{\mu}\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC*}}\right)z_1-\left(\frac{1}{\gamma_j^{OC*}}+\frac{1}{\mu_{0n}}\right)z_1\right)\right\}\right]$$

(4.11.9)

Seventh integral:

$$\left[\int_{z(oc)}^{z_1}S_{OC}(x,\mu)\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right]=\frac{-\mu_{0n}}{\mu+\mu_{0n}}S_{OC}(\pm)\left[\exp\left(\frac{z(oc)-z_1}{\mu}-\frac{z_1}{\mu_{0n}}\right)-\exp\left(-\frac{z(oc)}{\mu_{0n}}\right)\right].$$

(4.11.10)

4.12. Evaluation of integral in equation in (4.8.10):

$$\int_{z_\omega}^{z(oc)} RSOC(x,-\mu)\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x)}{\mu}\right)\frac{dx}{\mu}$$

$$= \sum_{J=S}^{M}\mathbf{P}_J^S(-\mu)\mathbf{B}_J^{OC}\left[\int_0^{z(oc)}\mathbf{\Pi}(J,S)^T W\left\{RE_+^{OC}(x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}+\right.$$

$$\int_0^{z(oc)}\mathbf{\Pi}(J,S)^T W\left\{CO_+^{OC}(x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}+$$

$$\int_0^{z(oc)}\mathbf{\Pi}(J,S)^T W\left\{I_{OC}^P(+;x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}$$

$$+\int_0^{z(oc)}(-1)^{J-S}D\mathbf{\Pi}(J,S)^T W\left\{RE_-^{OC}(x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}+$$

$$\int_0^{z(oc)}(-1)^{J-S}D\mathbf{\Pi}(J,S)^T W\left\{CO_-^{OC}(x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}+$$

$$\left.\int_0^{z(oc)}(-1)^{J-S}D\mathbf{\Pi}(J,S)^T W\left\{I_{OC}^P(-;x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}\right]$$

$$+\left[\int_0^{z(oc)}S_{OC}(x,\mu)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}\right]. \tag{4.12.1}$$

First Integral:

$$\int_{z_\omega}^{z(oc)}\left\{RE_+^{OC}(x)\right\}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}=$$

$$\int_{z_\omega}^{z(oc)}\left(\sum_{j=1}^{Nr}A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{x}{\gamma_j^{OC}}\right)+B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1-x}{\gamma_j^{AT/OC}}\right)\right)\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{z(oc)-x}{\mu}\right)\frac{dx}{\mu}=$$

$$=\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\frac{-s\gamma_j^{OC}}{\gamma_j^{OC}\mu+s\mu-s\gamma_j^{OC}}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(-\left(\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z_\omega+\frac{z(oc)+z_\omega}{\mu}\right)\right)\right\}$$

$$+\sum_{j=1}^{Nr} B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\frac{-s\gamma_j^{OC}}{\gamma_j^{OC}\mu-s\mu-s\gamma_j^{OC}}$$

$$\exp\left(-\frac{z_1}{\gamma_j^{OC}}\right)\left(\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(-\left(\frac{1}{s}-\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)z_\omega-\frac{z(oc)}{\mu}\right)\right).$$

(4.12.2)

Second Integral:

$$=\sum_{j=1}^{Nc} A_j^{OC(1)}\,\mathrm{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} A_j^{OC(1)}\,\mathrm{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}+$$

$$\sum_{j=1}^{Nc} A_j^{OC(2)}\,\mathrm{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} A_j^{OC(2)}\,\mathrm{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}+$$

$$\sum_{j=1}^{Nc} B_j^{OC(1)}\,\mathrm{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} B_j^{OC(1)}\,\mathrm{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}+$$

$$\sum_{j=1}^{Nc} B_j^{OC(2)}\,\mathrm{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}-$$

$$\sum_{j=1}^{Nc} B_j^{OC(2)}\,\mathrm{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(-\frac{z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{z_\omega(\gamma_j^{OC}-\mu)-\gamma_j^{OC}z(oc)}{\gamma_j^{OC}\mu}\right)\right\}.$$

(4.12.3)

Third Integral:

$$= 2\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}a_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\left[\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}-\mu_{0n})(\mu\gamma_j^{OC}-s\gamma_j^{OC}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\mu_{0,n}}{(\gamma_j^{OC}-\mu_0)(\mu\mu_{0,n}-s\mu_{0,n}+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0,n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}b_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\left[\frac{-s\mu_{0n}}{(\gamma_j^{OC}-\mu_{0n})(\mu\mu_{0n}-s\mu_{0n}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}+\mu_{0n})(\gamma_j^{OC}\mu-s\mu-s\gamma_j^{OC})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(-\left(z_1(\frac{1}{\gamma_j^{OC}}+\frac{1}{\mu_{0n}})\right)\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC^*}a_j^{OC^*}\mathbf{H}_+^{OC^*}(\gamma_j^{OC^*})\left[\frac{-s\gamma_j^{OC^*}}{(\gamma_j^{OC^*}-\mu_0)(\mu\gamma_j^{OC^*}-s\gamma_j^{OC^*}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC^*}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\mu_{0n}}{(\gamma_j^{OC^*}-\mu_{0n})(\mu\mu_{0n}-s\mu_{0n}+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC^*}b_j^{OC^*}\mathbf{H}_-^{OC^*}(\gamma_j^{OC^*})\left[\frac{-s\mu_{0n}}{(\gamma_j^{OC^*}-\mu_{0n})(\mu\mu_{0n}-s\mu_{0n}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}-\right.$$

$$\left.\frac{-s\gamma_j^{OC^*}}{(\gamma_j^{OC^*}+\mu_{0n})(\gamma_j^{OC^*}\mu-s\mu-s\gamma_j^{OC^*})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC^*}}\right)z(oc)\right)-\exp\left(-\left(z_1(\frac{1}{\gamma_j^{OC^*}}+\frac{1}{\mu_{0n}})\right)\right)\right\}\right].$$

(4.12.4)

Fourth Integral:

$$=\boldsymbol{\Delta}\sum_{j=1}^{Nr} A_j^{OC}\,\mathbf{H}_-^{OC}(\gamma_j^{OC})\frac{-s\gamma_j^{OC}}{\gamma_j^{OC}\mu+s\mu-s\gamma_j^{OC}}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(-\frac{z(oc)}{\mu}\right)\right\}$$

$$+\boldsymbol{\Delta}\sum_{j=1}^{Nr} B_j^{OC}\,\mathbf{H}_+^{OC}(\gamma_j^{OC})\frac{-s\gamma_j^{OC}}{\gamma_j^{OC}\mu-s\mu-s\gamma_j^{OC}}\exp\left(-\frac{z_\omega}{\gamma_j^{OC}}\right)\left(\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(-\frac{z(oc)}{\mu}\right)\right).$$

(4.12.5)

Fifth Integral:

$$=\sum_{j=1}^{Nc} A_j^{OC(1)}\,\mathrm{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}-$$

$$\sum_{j=1}^{Nc} A_j^{OC(1)}\,\mathrm{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}+$$

$$\sum_{j=1}^{Nc} A_j^{OC(2)}\,\mathrm{Re}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}-$$

$$\sum_{j=1}^{Nc} A_j^{OC(2)}\,\mathrm{Im}\{\mathbf{H}_-^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}+$$

$$\sum_{j=1}^{Nc} B_j^{OC(1)}\,\mathrm{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}-$$

$$\sum_{j=1}^{Nc} B_j^{OC(1)}\,\mathrm{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}+$$

$$\sum_{j=1}^{Nc} B_j^{OC(2)}\,\mathrm{Re}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Im}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}-$$

$$\sum_{j=1}^{Nc} B_j^{OC(2)}\,\mathrm{Im}\{\mathbf{H}_+^{OC}(\gamma_j^{OC})\}\,\mathrm{Re}\frac{\gamma_j^{OC}}{\gamma_j^{OC}-\mu}\left\{\exp\left(\frac{-z(oc)}{\gamma_j^{OC}}\right)-\exp\left(\frac{-z(oc)}{\mu}+z_\omega\left(\frac{1}{\mu}-\frac{1}{\gamma_j^{OC}}\right)\right)\right\}.$$

(4.12.6)

Sixth Integral:

$$
\begin{aligned}
&= 2\boldsymbol{\Delta}\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}a_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\Bigg[\Bigg[\frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}-\mu_{0n})(\mu\gamma_j^{OC}-s\gamma_j^{OC}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\} \\
&\qquad \frac{-s\mu_{0n}}{(\gamma_j^{OC}-\mu_{0n})((\mu\mu_{0n}-s\mu_{0n}+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}\Bigg]+ \\
&2\sum_{j=1}^{Nr}\mu_{0n}\gamma_j^{OC}b_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\Bigg[\frac{-s\mu_{0n}}{(\gamma_j^{OC}-\mu_{0,n})((\mu\mu_{0n}-s\mu_{0,n}+s\mu))}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}- \\
&\qquad \frac{-s\gamma_j^{OC}}{(\gamma_j^{OC}+\mu_{0n})(\gamma_j^{OC}\mu-s\mu-s\gamma_j^{OC})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC}}\right)z(oc)\right)-\exp\left(-\left(z_\omega\left(\frac{1}{\gamma_j^{OC}}+\frac{1}{\mu_{0n}}\right)\right)\right)\right\}\Bigg]\Bigg]+ \\
&2\boldsymbol{\Delta}\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC*}a_j^{OC*}\mathbf{H}_-^{OC*}(\gamma_j^{OC*})\Bigg[\Bigg[\frac{-s\gamma_j^{OC*}}{(\gamma_j^{OC*}-\mu_{0n})(\mu\gamma_j^{OC*}-s\gamma_j^{OC*}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\gamma_j^{OC*}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\} \\
&\qquad \frac{-s\mu_{0n}}{(\gamma_j^{OC*}-\mu_{0n})(\mu\mu_{0n}-s\mu_{0n}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}\Bigg]+ \\
&2\sum_{j=1}^{Nc}\mu_{0n}\gamma_j^{OC*}b_j^{OC*}\mathbf{H}_+^{OC*}(\gamma_j^{OC*})\Bigg[\frac{-s\mu_{0n}}{(\gamma_j^{OC*}-\mu_{0n})(\mu\mu_{0n}-s\mu_{0n}+s\mu)}\left\{\exp\left(-\left(\frac{1}{s}+\frac{1}{\mu_{0n}}\right)z(oc)\right)-\exp\left(\frac{-z(oc)}{\mu}\right)\right\}- \\
&\qquad \frac{-s\gamma_j^{OC*}}{(\gamma_j^{OC*}+\mu_{0n})(\gamma_j^{OC}\mu-s\mu-s\gamma_j^{OC*})}\left\{\exp\left(-\left(\frac{1}{s}-\frac{1}{\gamma_j^{OC*}}\right)z(oc)\right)-\exp\left(-\left(z_\omega\left(\frac{1}{\gamma_j^{OC*}}+\frac{1}{\mu_{0n}}\right)\right)\right)\right\}\Bigg]\Bigg].
\end{aligned}
\tag{4.12.7}
$$

Seventh integral:

$$
\left[\int_{z_\omega}^{z(oc)}S_{OC}(x,\mu)\exp\left(-\frac{x-z(oc)}{\mu}\right)\frac{dx}{\mu}\right]=\frac{-\mu_{0n}}{\mu+\mu_{0n}}S_{OC}(\pm)\left[\exp\left(\frac{-z(oc)}{\mu_{0n}}\right)-\exp\left(\left(\frac{z(oc)-z_\omega}{\mu}\right)-\frac{z_\omega}{\mu_{0n}}\right)\right].
\tag{4.12.8}
$$

With this we have completed the evaluation of the integrals. Since all the quantities are over known functions they can be evaluated. We can explore equation (4.8.12) if we want to find intensity at the top of the atmosphere.

Chapter V:

Numerical Consideration

Numerical results for certain specified dataset for atmosphere and ocean:
In this section we shall reproduce numerical results of our calculations presented in section 2 &3. Following subsections will be devoted to describe the model parameters chosen for atmosphere and Ocean, eigenvectors and eigenvalues etc and corresponding numerical results.

5.1: Model parameters: The model parameters are as follows.
(A) **Albedo** for single scattering both for atmosphere an ocean. (1). For **atmosphere** we have chosen an inhomogeneous albedo function of following form depending on optical depth and a parameters 'ω_0' and 's' whose value is chosen as 0,(homogeneous),10,100,1000 etc.

$$\omega^{AT}(z) = \omega_0 \exp(-z(at)/s) \tag{5.1.1}$$

(2)The inhomogeneity in the **oceanic albedo function** is expressed through the following equations where ocean direct and diffuse surface albedo $\omega_{dir}^{OC}(\mu_0, U_s)$ depend on underlying wind speed over the ocean surface and solar zenith angle.

$$\omega^{OC}(z) = \omega_{dir}^{OC}(\mu_0, U_s)\exp(-z(oc)/s) \tag{5.1.2}$$

Here z (at) and z (oc) are desired representative optical depth in atmosphere and ocean respectively.
(B) Model 1: quadrature value (N) =2; s=10; refractive index=1.34; ω_0=0.5; incident direction=0.50; wind speed=3.0;
Atmosphere: Atmosphere is composed of spheroid particles having phase function expansion coefficients for Legendre series given by Table.
Ocean: Ocean is assumed to be composed of spherical particles having phase function coefficients of Legendre series as given in table.

First we show that the eigen values are optical depth sensitive for both the medium. This is obvious for an inhomogeneous media. It is to be noted that no numerical values of any sort are available in the literature so that our results can be verified. We have used equation (3.5.1) for atmosphere and ocean.
Standard textbook procedures are used to find the inverse of the matrix involved in equation (3.5.1). We have extensively used Matlab 7 software where LAPACK (DZEEV) algorithm is used to calculate the eigenvectors and eigen values of a real asymmetric matrix. We have taken first

five Legendre polynomials to calculate the R-functions and T-functions given in equations (2.15.12) and (2.15.13) respectively. In calculating (2.14.11) and (2.14.12) we have used twelve values of the basic constants for spherical and oblate spheroidal particles in the chosen ocean and atmosphere model given in table 2 and table 1 in Chapter 3 respectively. In calculation the summation in equation (2.11.7) and (2.11.16) we therefore have used 12 values for J starting from 0 t011. In all calculations we have taken S=0.

We have already stated earlier that eigenvalues are sensitive to optical depth z in the case of inhomogeneous atmosphere. We have chosen optical depth (Atmosphere) value starting from 0.3, 0.5, 0.7 [illegible] and 0.9 for atmosphere. However for ocean the corresponding values are taken as 1.0, 1.6, [illegible] For each optical depth the values of inhomogeneous parameter,'s' are chosen as 10, 100, and 1000. For these calculations we have chosen fixed albedo value 0.5 for ω_0. For N=2, i.e, 4-stream approximation, there are eight eigenvectors corresponding to eight eigenvalues. We have seen that for this case the eigenvalues as well as eigenvectors are real. However this is not the case when value of N is chosen greater than 2.

In the tables given the first set of boxed data give the values of separation constants. The second set of arrayed data represents the positive eigenvectors whereas the third similar set is for negative eigenvectors. The last set of boxed data represents the calculated values of the quantity NAT and NOC from equation (4.3.18) and (4.3.33) respectively. These sets are used in subsequent calculations.

Atmosphere Tables

Table 1

Z(At)=0.3,   S=10

Separation Constants:

5.7922e-001	3.9228e-001	4.2631e-001	3.4747e-001	5.5796e-001	4.2236e-001	3.8975e-001	3.4747e-001

Eigen vector (positive):

-1.4518e+000	-1.8431e+000	-1.9503e-001	2.2713e-003	-1.3982e-018	-6.9143e-016	-1.2901e-016	4.5905e-016
1.3860e-002	2.1650e-001	-2.5003e+000	-4.9834e-001	1.7381e-016	-8.0439e-015	-1.6614e-016	-1.0050e-013
6.8774e-018	3.6882e-017	-3.8789e-016	5.0890e-015	9.8537e-003	3.1216e+000	-1.3436e-001	5.0004e-001
2.5961e-017	-1.1653e-017	1.6691e-016	-1.1357e-015	-1.5134e+000	-1.4038e-001	2.0669e+000	1.4022e-003
-2.3499e+000	2.0662e+000	1.9020e-001	-2.8720e-003	-4.5362e-018	6.9860e-016	1.4641e-016	-5.8060e-016
4.1889e-003	7.4492e-002	-8.0412e-001	2.8339e+000	-6.1713e-016	-2.8353e-015	1.9231e-015	5.7156e-013
-6.0923e-020	-2.7461e-020	2.0086e-019	-4.7263e-018	2.9880e-003	1.0088e+000	-4.6651e-002	-2.8323e+000
4.3843e-017	-2.0024e-017	2.8574e-016	-1.9620e-015	-2.5600e+000	1.2787e-001	-2.2196e+000	-1.7234e-003

Eigenvector (Negative):

-1.6487e-001	-1.7757e-001	-2.5023e-002	1.0145e-004	-1.2402e-020	-8.6280e-017	-1.2158e-017	2.0626e-017
3.6074e-003	1.5464e-002	-2.8169e-001	-5.4288e-003	-2.6176e-017	-9.5228e-016	3.9383e-017	-1.0936e-015
-1.7900e-018	-2.6343e-018	4.3701e-017	-5.5438e-017	-2.3921e-003	-3.3734e-001	9.1645e-003	-5.4462e-003
-2.8540e-018	-9.8323e-019	7.1648e-018	-1.6383e-016	1.3836e-001	1.6780e-002	-1.8601e-001	-5.9269e-005
-5.1564e-002	8.9456e-003	1.1629e-002	1.9369e-004	-6.2783e-019	3.9655e-017	3.1585e-019	3.8880e-017
1.0454e-003	4.4716e-003	-8.1495e-002	-1.5688e-003	1.4779e-016	-2.5268e-016	-1.0113e-016	-3.1475e-016
1.5204e-020	1.6485e-021	-2.0357e-020	-2.6164e-021	-6.9308e-004	-9.7589e-002	2.6500e-003	-1.5738e-003
-1.1402e-018	-3.3477e-018	3.6149e-017	-4.4234e-016	1.8750e-002	-8.3352e-003	1.3235e-002	-1.1113e-004

NAT () from Equation (4.3.18):

1.1292e+000	1.2738e+000	1.4585e+000	1.0268e+000	1.2966e+000	2.2635e+000	1.5399e+000	1.0261e+000

Table 2

Z(At)=0.3, S=100

Separation Constants:

5.9055e-001	3.9397e-001	4.2967e-001	5.6775e-001	3.4747e-001	4.2552e-001	3.9134e-001	3.4747e-001

Eigen Vector (Positive):

-1.4534e+000	-1.8369e+000	1.9349e-001	3.0838e-017	2.1924e-003	-2.3179e-015	1.6590e-016	1.0217e-014
1.3729e-002	2.1554e-001	2.4926e+000	2.3926e-016	-4.9654e-001	-2.7622e-014	-1.0983e-016	-2.3126e-012
4.1191e-017	8.8223e-016	4.3660e-015	9.8287e-003	9.0183e-013	3.1357e+000	-1.3409e-001	4.9818e-001
2.7942e-018	1.3299e-016	5.1136e-016	-1.5207e+000	-2.3721e-016	-1.4040e-001	2.0650e+000	1.3532e-003
-2.3412e+000	2.0674e+000	-1.8959e-001	7.8310e-017	-2.7794e-003	2.3467e-015	-1.8522e-016	-1.2949e-014
4.1430e-003	7.3754e-002	7.9865e-001	-2.5639e-016	2.8342e+000	-7.8948e-015	5.4042e-016	1.3198e-011
6.1034e-018	2.4712e-016	1.0182e-015	2.9756e-003	-5.1283e-012	1.0094e+000	-4.6290e-002	-2.8326e+000
1.7670e-018	1.6241e-016	5.1136e-016	-2.5611e+000	-9.0439e-016	1.2843e-001	-2.2259e+000	-1.6673e-003

Eigen Vector (Negative):

-1.7206e-001	-1.8298e-001	2.5736e-002	7.4805e-018	1.0086e-004	-3.0152e-016	1.7168e-017	4.6967e-016
3.6969e-003	1.5855e-002	2.9046e-001	-1.7159e-017	-5.4101e-003	-3.3197e-015	3.9543e-018	-2.5261e-014
-1.1092e-017	-6.4897e-017	-5.0877e-016	-2.4662e-003	-9.8260e-015	-3.5039e-001	9.4173e-003	-5.4270e-003
-5.0830e-019	7.2892e-018	-1.3091e-018	1.4471e-001	-6.7676e-017	1.7388e-002	-1.9208e-001	-5.8881e-005
-5.4699e-002	8.5913e-003	-1.1955e-002	-9.8433e-019	1.9428e-004	1.3948e-016	-1.1717e-018	9.0719e-016
1.0715e-003	4.5849e-003	8.4037e-002	7.4000e-017	-1.5634e-003	-1.0605e-015	-3.2879e-017	-7.2727e-015
-1.5785e-018	-1.5362e-017	-1.0714e-016	-7.1463e-004	-2.8289e-015	-1.0137e-001	2.7232e-003	-1.5682e-003
1.5610e-019	3.2687e-017	9.2316e-017	2.0318e-002	-1.7645e-016	-8.6340e-003	1.3019e-002	-1.1146e-004

NAT() from Equation (4.3.18):

1.1247e+000	1.2688e+000	1.4476e+000	1.3017e+000	1.0266e+000	2.2812e+000	1.5410e+000	1.0259e+000

Table 3

Z(At)=0.3, S=1000

Separation Constants:

5.9173e-001	3.9414e-001	4.3001e-001	5.6876e-001	4.2584e-001	3.9150e-001	3.4747e-001	3.4747e-001

Eigen Vector (Positive):

-1.4535e+000	-1.8363e+000	1.9334e-001	-8.6352e-018	-1.6471e-016	3.5198e-016	2.1847e-003	1.1885e-014
1.3716e-002	2.1544e-001	2.4918e+000	-5.7112e-016	-2.8214e-015	-4.3230e-016	-4.9636e-001	-2.6998e-012
-1.0743e-017	-2.2788e-016	-1.1311e-015	9.8261e-003	3.1372e+000	1.3407e-001	8.1896e-014	4.9800e-001
-6.3826e-018	-2.9319e-016	-1.1436e-015	-1.5214e+000	-1.4040e-001	-2.0648e+000	9.6310e-015	1.3484e-003
-2.3403e+000	2.0675e+000	-1.8953e-001	3.0893e-018	1.8361e-016	-3.9204e-016	-2.7703e-003	-1.5070e-014
4.1383e-003	7.3681e-002	7.9810e-001	1.2261e-015	6.5182e-016	1.5767e-015	2.8342e+000	1.5415e-011
-1.6020e-018	-6.4039e-017	-2.6510e-016	2.9744e-003	1.0095e+000	4.6255e-002	-4.7229e-013	-2.8326e+000
-2.1835e-018	-2.5181e-016	-7.5482e-016	-2.5612e+000	1.2849e-001	2.2265e+000	-6.8470e-015	-1.6618e-003

Eigen Vector (Negative):

-1.7281e-001	-1.8353e-001	2.5809e-002	-1.6216e-018	-2.3342e-017	3.6014e-017	1.0080e-004	5.4850e-016
3.7059e-003	1.5895e-002	2.9136e-001	6.1635e-017	-2.8855e-016	-6.6538e-018	-5.4083e-003	-2.9434e-014
2.9028e-018	1.6812e-017	1.3226e-016	-2.4738e-003	-3.5173e-001	-9.4431e-003	-8.9233e-016	-5.4251e-003
1.2844e-018	-9.1236e-018	2.8630e-017	1.4537e-001	1.7451e-002	1.9270e-001	-9.1858e-017	-5.8844e-005
-5.5026e-002	8.5527e-003	-1.1988e-002	1.8906e-018	1.7380e-017	-1.1886e-018	1.9434e-004	1.0575e-015
1.0741e-003	4.5964e-003	8.4296e-002	-3.1931e-016	-2.4070e-016	-1.0381e-016	-1.5629e-003	-8.4731e-015
4.1580e-019	3.9949e-018	2.8001e-017	-7.1683e-004	-1.0176e-001	-2.7307e-003	-2.6043e-016	-1.5677e-003
-5.3868e-019	-6.1362e-017	-1.8376e-016	2.0484e-002	-8.6646e-003	-1.2995e-002	2.8979e-017	-1.1149e-004

NAT() from Equation (4.3.18):

1.1242e+000	1.2683e+000	1.4465e+000	1.3023e+000	2.2830e+000	1.5411e+000	1.0266e+000	1.0259e+000

Table 4

Z(At)=0.5, S=10

Separation Constants:

5.7131e-001	3.9108e-001	4.2394e-001	3.4747e-001	5.5110e-001	4.2013e-001	3.8862e-001	3.4747e-001

Eigen Vector (Positive):

1.4506e+000	1.8476e+000	-1.9617e-001	-2.3318e-003	1.8321e-018	-1.6042e-015	7.3404e-017	9.5422e-016
-1.3953e-002	-2.1721e-001	-2.5058e+000	4.9971e-001	-1.5721e-018	-1.8833e-014	-2.6633e-017	-2.0549e-013
-1.6970e-017	-5.7646e-017	-5.7929e-015	-4.9446e-016	-9.8718e-003	3.1115e+000	1.3457e-001	-5.0146e-001
-6.4476e-017	6.1814e-017	8.5772e-016	3.6594e-016	1.5082e+000	-1.4039e-001	-2.0683e+000	-1.4399e-003
2.3562e+000	-2.0653e+000	1.9064e-001	2.9429e-003	1.9081e-018	1.6087e-015	-8.3191e-017	-1.2059e-015
-4.2219e-003	-7.5053e-002	-8.0819e-001	-2.8336e+000	2.6836e-017	-5.9659e-015	-3.0048e-016	1.1661e-012
-2.0490e-018	-1.5449e-019	-1.6843e-015	2.3501e-015	-2.9971e-003	1.0085e+000	4.6929e-002	2.8320e+000
-1.0994e-016	1.0808e-016	8.6804e-016	1.2534e-015	2.5594e+000	1.2748e-001	2.2149e+000	1.7663e-003

Eigen Vector (Negative):

1.5980e-001	1.7369e-001	-2.4511e-002	-1.0193e-004	1.6739e-019	-1.9609e-016	6.7596e-018	4.1752e-017
-3.5419e-003	-1.5182e-002	-2.7538e-001	5.4430e-003	9.1949e-019	-2.1515e-015	-1.2941e-017	-2.2096e-015
4.3077e-018	4.0291e-018	6.3664e-016	5.3858e-018	2.3389e-003	-3.2807e-001	-8.9831e-003	5.4609e-003
6.8323e-018	5.1929e-018	-1.3550e-018	8.9701e-017	-1.3389e-001	1.6348e-002	1.8167e-001	5.9579e-005
4.9400e-002	-9.1729e-003	1.1392e-002	-1.9328e-004	1.4145e-019	9.0216e-017	-2.4241e-019	7.8509e-017
-1.0264e-003	-4.3902e-003	-7.9669e-002	1.5729e-003	-5.9089e-018	-6.3540e-016	1.2376e-017	-6.3537e-016
4.9813e-019	9.0368e-021	1.6603e-016	1.3045e-018	6.7762e-004	-9.4906e-002	-2.5975e-003	1.5781e-003
2.7836e-018	1.7518e-017	1.4736e-016	2.3531e-016	-1.7684e-002	-8.1223e-003	-1.3363e-002	1.1091e-004

NAT() from Equation (4.3.18):

1.1324e+000	1.2774e+000	1.4664e+000	1.0269e+000	1.2929e+000	2.2509e+000	1.5391e+000	1.0262e+000

Table 5

Z(At)=0.5, S=100

Separation Constants:

5.8968e-001	3.9384e-001	4.2942e-001	3.4747e-001	5.6700e-001	4.2528e-001	3.9122e-001	3.4747e-001

Eigen Vector (Positive):

1.4533e+000	1.8373e+000	-1.9361e-001	-2.1981e-003	-4.9345e-019	-2.5800e-015	-1.8300e-016	2.4838e-015
-1.3739e-002	-2.1561e-001	-2.4932e+000	4.9667e-001	-1.1152e-016	-3.0245e-014	-3.7122e-017	-5.5946e-013
-4.2524e-017	-1.3579e-016	-1.3340e-014	-1.8407e-015	-9.8306e-003	3.1347e+000	-1.3411e-001	4.9832e-001
-6.4515e-017	5.7517e-017	1.1310e-015	-6.5965e-017	1.5201e+000	-1.4039e-001	2.0652e+000	1.3568e-003
2.3418e+000	-2.0673e+000	1.8964e-001	2.7862e-003	7.3207e-019	2.6029e-015	2.0802e-016	-3.1441e-015
-4.1464e-003	-7.3808e-002	-7.9905e-001	-2.8341e+000	4.2004e-016	-1.0233e-014	1.2873e-015	3.1906e-012
-5.1629e-018	-6.1161e-019	-3.8379e-015	5.5092e-015	5.5092e-015	1.0094e+000	-4.6316e-002	-2.8325e+000
-1.1033e-016	9.9684e-017	5.4033e-016	1.6952e-015	2.5610e+000	1.2839e-001	-2.2254e+000	-1.6714e-003

Eigen Vector (Negative):

1.7152e-001	1.8257e-001	-2.5683e-002	-1.0090e-004	-1.5854e-019	-3.3414e-016	-1.7948e-017	1.1376e-016
-3.6902e-003	-1.5825e-002	-2.8980e-001	5.4115e-003	1.8150e-017	-3.6813e-015	3.7311e-017	-6.1616e-015
1.1422e-017	9.9669e-018	1.5507e-015	2.0056e-017	2.4606e-003	-3.4940e-001	9.3982e-003	-5.4284e-003
7.2560e-018	5.0748e-018	-4.1685e-017	1.0781e-016	-1.4423e-001	1.7342e-002	-1.9162e-001	-5.8908e-005
5.4457e-002	-8.6195e-003	1.1931e-002	-1.9423e-004	2.8583e-019	1.5179e-016	5.9803e-019	2.2129e-016
-1.0695e-003	-4.5764e-003	-8.3846e-002	1.5638e-003	-1.0345e-016	-1.0155e-015	-6.5499e-017	-1.7748e-015
1.3317e-018	3.7922e-020	4.0272e-016	3.0398e-018	7.1301e-004	-1.0108e-001	2.7177e-003	-1.5687e-003
3.1595e-018	1.7116e-017	1.6842e-016	2.6910e-016	-2.0196e-002	-8.6114e-003	1.3037e-002	-1.1143e-004

NAT() from Equation (4.3.18)

1.1250e+000	1.2691e+000	1.4484e+000	1.0266e+000	1.3013e+000	2.2799e+000	1.5409e+000	1.0259e+000

Table 6

Z(At)=0.5, S=1000

Separation Constants:

5.9164e-001	3.9413e-001	4.2999e-001	5.6869e-001	4.2582e-001	3.9149e-001	3.4747e-001	3.4747e-001

Eigen Vector (Positive):

1.4535e+000	1.8363e+000	1.9335e-001	1.8363e-017	6.3421e-016	-9.3000e-016	-2.1853e-003	1.6675e-014
-1.3717e-002	-2.1545e-001	2.4918e+000	4.5125e-016	7.9377e-015	-2.5977e-017	4.9638e-001	-3.7865e-012
6.4943e-017	6.9742e-016	-5.2737e-015	9.8263e-003	3.1371e+000	1.3407e-001	-4.6236e-014	-4.9801e-001
3.2357e-018	4.9073e-017	-3.3301e-016	-1.5213e+000	-1.4040e-001	-2.0648e+000	-4.1155e-015	-1.3488e-003
2.3403e+000	-2.0675e+000	-1.8954e-001	2.9531e-017	-6.4555e-016	1.0549e-015	2.7710e-003	-2.1144e-014
-4.1386e-003	-7.3686e-002	7.9814e-001	-8.4051e-016	8.7895e-016	6.4216e-015	-2.8342e+000	2.1619e-011
1.2229e-017	1.8108e-016	-1.2352e-015	2.9745e-003	1.0095e+000	4.6257e-002	2.3690e-013	2.8326e+000
3.9348e-018	6.1449e-017	-4.1440e-016	-2.5612e+000	1.2849e-001	2.2264e+000	4.5722e-015	1.6622e-003

Eigen Vector (Negative):

1.7275e-001	1.8349e-001	2.5804e-002	4.0495e-018	8.3353e-017	-9.1273e-017	-1.0080e-004	7.6921e-016
-3.7052e-003	-1.5892e-002	2.9129e-001	-4.4371e-017	9.0806e-016	2.1103e-016	5.4084e-003	-4.1301e-014
-1.7543e-017	-5.1442e-017	6.1648e-016	-2.4732e-003	-3.5163e-001	-9.4412e-003	5.0378e-016	5.4252e-003
-4.6907e-019	2.7996e-018	-4.5194e-018	1.4532e-001	1.7446e-002	1.9266e-001	1.4689e-016	5.8846e-005
5.5002e-002	-8.5556e-003	-1.1986e-002	-1.5168e-018	-4.0117e-017	1.3464e-018	-1.9434e-004	1.4836e-015
-1.0739e-003	-4.5956e-003	8.4277e-002	2.2257e-016	4.3154e-016	-3.1851e-016	1.5629e-003	-1.1890e-014
-3.1731e-018	-1.1293e-017	1.3043e-016	-7.1667e-004	-1.0173e-001	-2.7301e-003	1.3063e-016	1.5677e-003
-3.8281e-021	1.2254e-017	-6.5176e-017	2.0471e-002	-8.6623e-003	-1.2996e-002	2.5807e-016	1.1149e-004

NAT() from Equation (4.3.18):

1.1242e+000	1.2683e+000	1.4466e+000	1.3022e+000	2.2829e+000	1.5411e+000	1.0266e+000	1.0259e+000

Table 7

Z(At)=0.7, S=10

Separation Constants:

5.6377e-001	3.8991e-001	4.2164e-001	5.4454e-001	4.1797e-001	3.4747e-001	3.8752e-001	3.4747e-001

Eigen Vector (Positive):

-1.4492e+000	-1.8522e+000	1.9731e-001	1.0385e-017	-3.5422e-015	-2.3942e-003	1.3340e-017	-1.3392e-014
1.4042e-002	2.1794e-001	2.5111e+000	7.2491e-016	-4.0043e-014	5.0112e-001	-5.9206e-018	2.8045e-012
-5.4855e-017	-1.2815e-015	-6.2144e-015	-9.8893e-003	-3.1017e+000	-2.5697e-013	1.3478e-001	-5.0292e-001
-1.5824e-018	-8.6703e-017	-3.2224e-016	1.5030e+000	1.4042e-001	-3.2576e-015	-2.0698e+000	-1.4787e-003
-2.3622e+000	2.0644e+000	-1.9109e-001	-1.5810e-017	3.5163e-015	3.0158e-003	-1.5213e-017	1.6873e-014
4.2540e-003	7.5627e-002	8.1226e-001	-1.5004e-015	-1.7362e-014	-2.8334e+000	-6.7979e-017	-1.5859e-011
-8.0535e-018	-3.6338e-016	-1.4662e-015	-3.0062e-003	-1.0084e+000	1.4610e-012	4.7215e-002	2.8318e+000
-1.4950e-018	-1.4363e-016	-4.5084e-016	2.5588e+000	-1.2711e-001	1.9114e-016	2.2104e+000	1.8105e-003

Eigen Vector (Negative):

-1.5492e-001	-1.6991e-001	2.4012e-002	1.0328e-018	-4.2038e-016	-1.0244e-004	1.1756e-018	-5.7312e-016
3.4772e-003	1.4908e-002	2.6924e-001	-6.9955e-017	-4.6119e-015	5.4575e-003	-2.8704e-018	3.0501e-014
1.3584e-017	8.7659e-017	6.6630e-016	2.2871e-003	3.1913e-001	2.7986e-015	-8.8067e-003	5.4760e-003
2.3459e-019	-6.6101e-018	-7.0784e-018	-1.2961e-001	-1.5932e-002	-9.6360e-017	1.7744e-001	5.9907e-005
-4.7359e-002	9.3726e-003	-1.1161e-002	-1.8248e-018	1.8621e-016	-1.9291e-004	-3.7784e-020	-1.0777e-015
1.0076e-003	4.3107e-003	7.7888e-002	3.5901e-016	-9.3648e-016	1.5771e-003	2.7245e-018	8.7764e-015
1.9075e-018	2.0713e-017	1.4059e-016	6.6258e-004	9.2317e-002	8.1318e-016	-2.5465e-003	1.5824e-003
-4.0752e-020	-2.3239e-017	-6.1132e-017	-1.6693e-002	7.9163e-003	-3.3004e-016	-1.3465e-002	1.1071e-004

NAT() from Equation (4.3.18):

1.1353e+000	1.2810e+000	1.4740e+000	1.2894e+000	2.2387e+000	1.0271e+000	1.5383e+000	1.0264e+000

Table 8

Z(At)=0.7, S=100

Separation Constants:

5.8882e-001	3.9372e-001	4.2916e-001	3.4747e-001	5.6626e-001	4.2504e-001	3.9110e-001	3.4747e-001

Eigen Vector (Positive):

-1.4532e+000	-1.8378e+000	-1.9372e-001	2.2039e-003	5.6976e-020	3.0404e-015	-3.4591e-017	-1.3024e-015
1.3749e-002	2.1568e-001	-2.4937e+000	-4.9680e-001	-7.4440e-017	3.5456e-014	1.9957e-016	2.9172e-013
4.6352e-019	6.4027e-018	-6.9142e-017	8.9034e-016	9.8324e-003	-3.1336e+000	-1.3413e-001	-4.9845e-001
1.2063e-016	-1.3796e-017	3.4228e-016	1.6993e-016	-1.5196e+000	1.4039e-001	2.0653e+000	-1.3603e-003
-2.3425e+000	2.0672e+000	1.8968e-001	-2.7929e-003	-1.1990e-018	-3.0594e-015	3.8634e-017	1.6481e-015
4.1499e-003	7.3862e-002	-7.9946e-001	2.8341e+000	4.0648e-016	1.1564e-014	-9.9252e-016	-1.6629e-012
-7.3464e-018	-3.6083e-017	3.8205e-016	-5.1005e-015	2.9775e-003	-1.0093e+000	-4.6343e-002	2.8325e+000
2.0106e-016	-3.3412e-017	6.8737e-016	-1.0950e-015	-2.5609e+000	-1.2835e-001	-2.2249e+000	1.6755e-003

Eigen Vector (Negative):

-1.7097e-001	-1.8216e-001	-2.5629e-002	1.0094e-004	-1.4226e-019	3.9207e-016	-3.3823e-018	-5.9744e-017
3.6836e-003	1.5796e-002	-2.8914e-001	-5.4129e-003	1.6400e-017	4.2896e-015	-4.8708e-018	3.2233e-015
-1.2418e-019	-4.6893e-019	8.0168e-018	-9.7006e-018	-2.4551e-003	3.4841e-001	9.3792e-003	5.4298e-003
-1.3864e-017	-1.8356e-018	2.1743e-017	-6.4811e-017	1.4374e-001	-1.7296e-002	-1.9117e-001	5.8936e-005
-5.4218e-002	8.6474e-003	1.1906e-002	1.9419e-004	1.5377e-019	-1.7558e-016	-7.6640e-020	-1.1533e-016
1.0676e-003	4.5679e-003	-8.3655e-002	-1.5642e-003	-9.7752e-017	1.2266e-015	6.0713e-017	9.2860e-016
1.8899e-018	2.2315e-018	-3.9978e-017	-2.8150e-018	-7.1139e-004	1.0080e-001	2.7122e-003	1.5691e-003
-5.3697e-018	-5.0659e-018	8.2091e-017	-1.5827e-016	2.0076e-002	8.5889e-003	1.3054e-002	1.1141e-004

NAT() from Equation (4.3.18):

1.1254e+000	1.2695e+000	1.4493e+000	1.0266e+000	1.3009e+000	2.2785e+000	1.5408e+000	1.0259e+000

Table 9

Z(At)=0.7,   S=1000

Separation Constants:

5.9155e-001	3.9412e-001	4.2996e-001	5.6861e-001	3.4747e-001	4.2579e-001	3.9148e-001	3.4747e-001

Eigen Vector (Positive):

-1.4535e+000	-1.8364e+000	1.9336e-001	-3.7135e-018	2.1858e-003	-4.9336e-015	1.0835e-016	3.2953e-015
1.3718e-002	2.1546e-001	2.4919e+000	2.7554e-016	-4.9639e-001	-5.7466e-014	4.1791e-016	-7.4566e-013
-7.3485e-017	-1.5369e-015	-7.6084e-015	9.8265e-003	-6.9346e-013	-3.1370e+000	1.3407e-001	-4.9802e-001
1.7356e-016	1.3814e-016	-1.2799e-015	-1.5213e+000	-3.8968e-015	1.4040e-001	-2.0649e+000	-1.3491e-003
-2.3404e+000	2.0675e+000	-1.8954e-001	-7.3725e-018	-2.7717e-003	4.9842e-015	-1.2507e-016	-4.1753e-015
4.1390e-003	7.3692e-002	7.9818e-001	-1.2125e-015	2.8342e+000	-1.9225e-014	-2.9821e-015	4.2549e-012
-1.1064e-017	-4.3107e-016	-1.7752e-015	2.9745e-003	3.9523e-012	-1.0095e+000	4.6260e-002	2.8326e+000
2.8600e-016	2.4422e-017	-3.0867e-015	-2.5612e+000	5.5850e-015	-1.2848e-001	2.2264e+000	1.6627e-003

Eigen Vector (Negative):

-1.7270e-001	-1.8345e-001	2.5798e-002	-2.2950e-019	1.0081e-004	-6.4017e-016	1.0755e-017	1.5162e-016
3.7046e-003	1.5889e-002	2.9122e-001	-5.1092e-017	-5.4086e-003	-7.0258e-015	-3.9983e-017	-8.2132e-015
1.9845e-017	1.1334e-016	8.8918e-016	-2.4727e-003	7.5558e-015	3.5153e-001	-9.4393e-003	5.4254e-003
-2.0334e-017	-1.9167e-018	-1.1602e-016	1.4527e-001	2.2114e-016	-1.7442e-002	1.9261e-001	5.8849e-005
-5.4978e-002	8.5584e-003	-1.1984e-002	-9.2445e-019	1.9433e-004	2.8805e-016	-9.9432e-019	2.9469e-016
1.0737e-003	4.5947e-003	8.4258e-002	2.9736e-016	-1.5629e-003	-1.9589e-015	1.7290e-016	-2.3657e-015
2.8701e-018	2.6878e-017	1.8739e-016	-7.1651e-004	2.1795e-015	1.0170e-001	-2.7295e-003	1.5678e-003
-7.5385e-018	1.9202e-017	-3.4009e-016	2.0459e-002	4.5053e-016	8.6601e-003	-1.2998e-002	1.1149e-004

NAT() from Equation (4.3.18):

1.1243e+000	1.2683e+000	1.4467e+000	1.3022e+000	1.0266e+000	2.2828e+000	1.5411e+000	1.0259e+000

Table 10

Z(At)=0.9, S=10

Separation Constants:

5.5659e-001	3.8877e-001	4.1943e-001	3.4747e-001	5.3827e-001	3.4747e-001	3.8644e-001	4.1588e-001

Eigen Vector (Positive):

-1.4478e+000	-1.8566e+000	1.9844e-001	2.4585e-003	1.2174e-019	-2.4986e-015	-7.4978e-017	1.0483e-015
1.4128e-002	2.1867e-001	2.5162e+000	-5.0256e-001	3.7217e-017	5.1082e-013	-6.9569e-017	1.2286e-014
-4.3809e-018	-2.4668e-017	-2.5344e-016	-3.2106e-015	9.9066e-003	5.0441e-001	-1.3500e-001	-3.0922e+000
0	0	0	0	-1.4980e+000	1.5187e-003	2.0713e+000	1.4047e-001
-2.3680e+000	2.0634e+000	-1.9153e-001	-3.0907e-003	-1.0754e-018	3.1422e-015	8.4671e-017	-1.0436e-015
4.2853e-003	7.6214e-002	8.1635e-001	2.8331e+000	-1.1014e-016	-2.8799e-012	8.2311e-016	3.5522e-015
0	0	0	0	3.0152e-003	-2.8315e+000	-4.7510e-002	-1.0084e+000
0	0	0	0	-2.5584e+000	-1.8559e-003	-2.2058e+000	-1.2675e-001

Eigen Vector (Negative):

-1.5023e-001	-1.6623e-001	2.3526e-002	1.0298e-004	8.6185e-021	-1.0471e-016	-6.6298e-018	1.2175e-016
3.4134e-003	1.4640e-002	2.6324e-001	-5.4724e-003	-4.5286e-018	5.5563e-015	1.5731e-017	1.3242e-015
1.0584e-018	1.6515e-018	2.6515e-017	3.4960e-017	-2.2367e-003	-5.4915e-003	8.6349e-003	3.1050e-001
0	0	0	0	1.2551e-001	-6.0255e-005	-1.7334e-001	-1.5530e-002
-4.5432e-002	9.5467e-003	-1.0935e-002	1.9257e-004	-9.2287e-020	-1.9528e-016	4.0774e-019	-5.6029e-017
9.8899e-004	4.2332e-003	7.6152e-002	-1.5814e-003	2.4789e-017	1.5986e-015	-3.9995e-017	4.2353e-016
0	0	0	0	-6.4793e-004	-1.5869e-003	2.4968e-003	8.9817e-002
0	0	0	0	1.5769e-002	-1.1053e-004	1.3545e-002	7.7168e-003

NAT() from Equation (4.3.18):

1.1380e+000	1.2846e+000	1.4814e+000	1.0272e+000	1.2860e+000	1.0265e+000	1.5376e+000	2.2269e+000

Table 11

Z(At)=0.9, S=100

Separation Constants:

5.8796e-001	3.9359e-001	4.2891e-001	3.4747e-001	5.6552e-001	4.2480e-001	3.9098e-001	3.4747e-001

Eigen Vector (Positive):

-1.4530e+00	-1.8383e+00	-1.9383e-001	2.2097e-003	2.3371e-018	7.2556e-016	-4.7931e-017	2.5227e-015
1.3759e-002	2.1575e-001	-2.4943e+00	-4.9694e-001	2.4073e-016	7.9919e-015	4.5969e-018	-5.6659e-013
4.0249e-018	2.1072e-017	-2.2346e-016	2.9664e-015	-9.8343e-003	-3.1325e+00	1.3415e-001	4.9859e-001
1.7316e-017	-7.7377e-018	1.1152e-016	-7.7427e-016	1.5190e+000	1.4039e-001	-2.0654e+00	1.3639e-003
-2.3432e+00	2.0671e+000	1.8973e-001	-2.7997e-003	-4.8986e-019	-7.1597e-016	5.3388e-017	-3.1958e-015
4.1533e-003	7.3916e-002	-7.9986e-001	2.8341e+000	-2.7773e-016	3.0773e-015	6.8995e-016	3.2307e-012
-3.4696e-020	1.1469e-020	-1.7927e-019	9.9887e-019	-2.9784e-003	-1.0093e+00	4.6369e-002	-2.8325e+00
2.9143e-017	-1.3252e-017	1.9028e-016	-1.3331e-015	2.5609e+000	-1.2831e-001	2.2244e+000	-1.6796e-003

Eigen Vector (Negative):

-1.7043e-001	-1.8176e-001	-2.5576e-002	1.0098e-004	9.0368e-019	9.2447e-017	-4.4337e-018	1.1521e-016
3.6769e-003	1.5767e-002	-2.8849e-001	-5.4142e-003	-1.7565e-017	9.8106e-016	2.4061e-017	-6.1958e-015
-1.0756e-018	-1.5399e-018	2.5844e-017	-3.2319e-017	2.4495e-003	3.4743e-001	-9.3603e-003	-5.4312e-003
-1.9665e-018	-6.6868e-019	4.8661e-018	-1.1448e-016	-1.4326e-001	-1.7251e-002	1.9071e-001	-5.8964e-005
-5.3980e-002	8.6750e-003	1.1882e-002	1.9414e-004	-1.3431e-018	-3.6884e-017	-5.8699e-019	2.2194e-016
1.0656e-003	4.5594e-003	-8.3465e-002	-1.5646e-003	7.8469e-017	2.3379e-016	-3.3224e-017	-1.7838e-015
8.9021e-021	-7.0742e-022	1.8707e-020	5.5142e-022	7.0977e-004	1.0051e-001	-2.7067e-003	-1.5695e-003
-7.9399e-019	-2.2753e-018	2.4682e-017	-3.0855e-016	-1.9956e-002	8.5664e-003	-1.3072e-002	-1.1138e-004

NAT() from Equation (4.3.18):

1.1257e+000	1.2699e+000	1.4501e+000	1.0266e+000	1.3006e+000	2.2772e+000	1.5407e+000	1.0259e+000

Table 12

Z(At)=0.9, S=1000

Separation Constants:

5.9146e-001	3.9410e-001	4.2994e-001	5.6854e-001	3.4747e-001	4.2577e-001	3.9147e-001	3.4747e-001

Eigen Vector (Positive):

-1.4535e+00	-1.8364e+00	1.9337e-001	6.7674e-018	2.1864e-003	-2.3474e-015	-3.6843e-016	-1.1443e-014
1.3719e-002	2.1546e-001	2.4920e+000	3.4225e-016	-4.9640e-001	-2.5766e-014	-3.1475e-016	2.5968e-012
5.0543e-017	1.0772e-015	5.3402e-015	9.8267e-003	1.0367e-012	3.1369e+000	-1.3407e-001	4.9804e-001
4.2158e-018	1.9704e-016	7.6278e-016	-1.5212e+00	-2.3454e-015	-1.4040e-001	2.0649e+000	1.3495e-003
-2.3405e+00	2.0675e+000	-1.8954e-001	-2.9844e-017	-2.7724e-003	2.3789e-015	4.1630e-016	1.4508e-014
4.1393e-003	7.3697e-002	7.9822e-001	-1.0779e-015	2.8342e+000	-1.1204e-014	3.6075e-015	-1.4826e-011
7.4922e-018	3.0162e-016	1.2451e-015	2.9746e-003	-5.8812e-012	1.0095e+000	-4.6263e-002	-2.8326e+00
2.2947e-018	2.2158e-016	6.8981e-016	-2.5612e+00	-2.3075e-015	1.2848e-001	-2.2263e+00	-1.6631e-003

Eigen Vector (Negative):

-1.7264e-001	-1.8341e-001	2.5793e-002	-1.5129e-018	1.0081e-004	-2.9798e-016	-3.6908e-017	-5.2842e-016
3.7039e-003	1.5886e-002	2.9116e-001	-4.8566e-017	-5.4087e-003	-3.2412e-015	7.4768e-017	2.8298e-014
-1.3646e-017	-7.9418e-017	-6.2394e-016	-2.4721e-003	-1.1296e-014	-3.5143e-001	9.4374e-003	-5.4255e-003
-7.9291e-019	9.5767e-018	-6.7780e-018	1.4522e-001	-2.3737e-016	1.7437e-002	-1.9256e-001	-5.8852e-005
-5.4953e-002	8.5613e-003	-1.1981e-002	9.2489e-019	1.9433e-004	1.2677e-016	1.5110e-018	-1.0163e-015
1.0735e-003	4.5939e-003	8.4239e-002	2.7045e-016	-1.5630e-003	-6.4444e-016	-1.9759e-016	8.1460e-015
-1.9431e-018	-1.8801e-017	-1.3140e-016	-7.1634e-004	-3.2433e-015	-1.0167e-001	2.7290e-003	-1.5678e-003
2.7218e-019	4.6594e-017	1.3366e-016	2.0447e-002	-6.5969e-016	-8.6578e-003	1.3000e-002	-1.1148e-004

NAT() from Equation (4.3.18):

1.1243e+000	1.2684e+000	1.4467e+000	1.3021e+000	1.0266e+000	2.2826e+000	1.5411e+000	1.0259e+000

Ocean Tables

TABLE-13

Z (OC) =1.0, S=10 **Wind Speed = 3m/Sec**

Separation Constants: **Incident Direction = 0.5**

1.8952e-001	2.1855e-001	5.3389e-001	6.3794e001	2.2478e-001	2.1796e-001	5.4763e-001	6.3871e-001

Eigen vector (positive):

-3.2778e+000	-5.0587e-001	2.8470e-001	-1.3960e-002	9.9754e-016	1.1312e-015	6.9030e-018	3.8343e-018
-4.7378e-001	3.6621e+000	-4.9951e-002	-5.5222e-002	-8.0609e-015	-8.1027e-015	3.9880e-018	2.1563e-017
-2.5104e-019	4.0346e-018	1.1334e-017	2.7018e-016	4.0912e+000	-3.5672e-001	-3.0182e-003	4.6591e-002
-1.6238e-018	2.6178e-017	7.4894e-017	1.7908e-015	3.6391e-001	-3.9685e+000	1.7579e-001	-1.6075e-003
-4.5378e-001	-1.6056e-001	-1.5187e+000	3.8024e-002	2.7822e-016	3.5808e-016	-3.3050e-017	-1.5782e-017
-2.3254e-002	9.8057e-002	3.7869e-002	1.5538e+000	-2.4147e-016	-3.7320e-017	-1.5337e-016	-2.5447e-016
4.7853e-019	-7.4319e-018	-1.6577e-017	-3.7784e-016	9.1654e-002	-1.0343e-002	2.9517e-003	-1.5575e+000
-1.0440e-019	1.6913e-018	4.9725e-018	1.1943e-016	3.6337e-002	-3.3385e-001	-1.5463e+000	2.9045e-003

Eigen vector (Negative):

-3.4655e-001	-4.4854e-002	-4.3776e-002	5.1910e-004	8.3133e-017	1.0062e-016	-8.6471e-019	-4.8215e-019
-3.0875e-002	1.8994e-001	1.5847e-002	1.1828e-002	-3.0393e-016	-4.3282e-016	-2.7105e-019	-7.5942e-018
-1.9978e-020	3.4243e-020	-4.6703e-018	-1.3054e-016	-1.7004e-001	1.2611e-002	-6.0765e-004	1.0345e-002
-5.8213e-020	-9.2614e-019	-3.3539e-017	-9.2422e-016	-1.4849e-002	1.3861e-001	-9.2228e-003	4.3390e-004
-1.2377e-001	-5.3023e-002	-3.3917e-002	-2.0444e-003	8.4387e-017	1.1861e-016	-3.6761e-019	9.9255e-019
-2.9964e-003	2.6824e-002	2.7852e-003	1.7352e-003	-6.8419e-017	2.8624e-017	-1.1863e-017	-2.1472e-019
2.6313e-019	-3.7027e-018	-1.6580e-018	-4.2797e-018	-2.3901e-002	.3219e-003	-6.2827e-005	1.5408e-003
3.2818e-021	-1.7309e-019	-2.4862e-018	-6.7327e-017	-8.3928e-004	-7.5497e-003	-8.3983e-003	1.0692e-004

NOC() from Equation (4.3.33):

1.3713e+000	1.6882e+000	3.9485e-001	4.0340e-001	2.0810e+000	1.9766e+000	4.0268e-001	4.0492e-00

TABLE-14

Z(OC)=1.0, S=100 **Wind Speed = 3m/Sec**

Incident Direction = 0.5

Separation Constants:

1.3713e+000	2.1581e-001	5.2739e-001	6.3761e-001	2.2246e-001	2.1504e-001	5.4079e-001	6.3841e-001

Eigen vector (Positive):

-3.2314e+000	-4.9718e-001	3.0197e-001	-1.4728e-002	-1.7861e-015	-1.4992e-015	-2.2353e-018	1.6466e-020
-4.6472e-001	3.6418e+000	-5.4384e-002	-5.9060e-002	1.4741e-014	1.0819e-014	-8.5718e-019	7.7096e-018
2.3127e-019	-3.9793e-018	-1.0584e-017	-2.5486e-016	4.1057e+000	-3.5064e-001	-3.3447e-003	5.0055e-002
1.3485e-018	-2.3280e-017	-6.3049e-017	-1.5230e-015	3.5875e-001	-3.9635e+000	1.8991e-001	-1.7211e-003
-4.7484e-001	-1.7128e-001	-1.5173e+000	3.8224e-002	-5.5916e-016	-5.1941e-016	1.0691e-017	-7.2465e-018
-2.4150e-002	1.0450e-001	3.8060e-002	1.5537e+000	4.3447e-016	2.6048e-016	1.6080e-017	1.5556e-016
-3.9376e-019	6.5265e-018	1.3715e-017	3.1450e-016	9.8871e-002	-1.0937e-002	3.0090e-003	-1.5575e+000
9.4803e-020	-1.6437e-018	-4.5559e-018	-1.1050e-016	3.8982e-002	-3.6031e-001	-1.5455e+000	2.9557e-003

Eigen vector (Negative):

-3.6582e-001	-4.7657e-002	-4.3991e-002	4.1858e-004	-1.6180e-016	-1.4461e-016	3.3159e-019	-2.4466e-019
-3.2127e-002	2.0415e-001	1.6859e-002	1.2477e-002	6.0129e-016	6.2636e-016	1.9810e-019	-4.9190e-018
1.8760e-020	-1.9556e-020	4.3943e-018	1.2501e-016	-1.8475e-001	1.3339e-002	-6.2442e-004	1.0994e-002
5.1071e-020	8.9557e-019	2.8400e-017	7.9619e-016	-1.5890e-002	1.4952e-001	-1.0694e-002	4.7555e-004
-1.2949e-001	-5.6470e-002	-3.4834e-002	-2.1858e-003	-1.7081e-016	-1.7238e-016	1.1799e-019	6.1058e-019
-3.0516e-003	2.8708e-002	2.9385e-003	1.8110e-003	1.1558e-016	6.1020e-017	1.3490e-018	-1.5598e-020
-2.2025e-019	3.2936e-018	1.4859e-018	4.3615e-018	-2.5877e-002	1.3783e-003	-5.8269e-005	1.6241e-003
-1.8124e-021	1.5650e-019	2.2538e-018	6.2204e-017	-8.5913e-004	-8.6635e-003	-8.9483e-003	1.0975e-004

NOC() from Equation (4.3.33):

1.3343e+000	1.6688e+000	3.9545e-001	4.0341e-001	2.0948e+000	1.9739e+000	4.0291e-001	4.0499e-001

TABLE-15

Z(OC)=1.0, S=10 **Wind Speed = 5m/Sec**

Separation Constants:

Incident Direction = 0.5

1.6790e-001	2.0410e-001	5.0123e-001	6.3623e-001	2.1241e-001	2.0241e-001	5.1235e-001	6.3716e-001

Eigen vector (Positive):

-3.0486e+000	-4.6053e-001	3.7086e-001	-1.7622e-002	1.8363e-016	7.7478e-017	3.0059e-017	-1.5681e-017
-4.3015e-001	3.5532e+000	-7.3807e-002	-7.4860e-002	-1.6885e-015	-5.6823e-016	7.1989e-018	-7.3461e-017
-3.3484e-019	7.6447e-018	1.6897e-017	4.2721e-016	4.1694e+000	-3.2529e-001	4.9127e-003	-6.4599e-002
-1.5677e-018	3.5944e-017	8.0884e-017	2.0530e-015	3.3780e-001	-3.9269e+000	-2.5071e-001	2.1888e-003
-5.5063e-001	-2.1427e-001	-1.5126e+000	3.8645e-002	7.8203e-017	3.8550e-017	-1.1598e-016	3.4295e-017
-2.7204e-002	1.3024e-001	3.8332e-002	1.5534e+000	-3.7983e-017	-1.0430e-016	-2.7656e-016	1.4287e-015
3.8770e-019	-8.4059e-018	-1.4336e-017	-3.3889e-016	1.2987e-001	-1.3155e-002	-3.2406e-003	1.5577e+000
-1.5531e-019	3.5635e-018	8.0434e-018	2.0429e-016	5.0091e-002	-4.7159e-001	1.5424e+000	-3.1643e-003

Eigen vector (Negative):

-4.3926e-001	-5.8568e-002	-4.1549e-002	-1.7190e-004	2.3410e-017	1.0017e-017	-2.9500e-018	-2.7386e-019
-3.6326e-002	2.6361e-001	2.0672e-002	1.4849e-002	-9.1609e-017	-4.3535e-017	5.6328e-019	1.5812e-017
-2.9485e-020	-7.1408e-020	-7.2473e-018	-2.2261e-016	-2.5023e-001	1.6050e-002	6.2429e-004	-1.3500e-002
-7.0713e-020	-1.8851e-018	-3.7487e-017	-1.1328e-015	-2.0196e-002	1.9540e-001	1.8309e-002	-6.6595e-004
-1.4962e-001	-7.0114e-002	-3.7297e-002	-2.7263e-003	2.3349e-017	1.3008e-017	-1.7585e-018	-2.4274e-018
-3.1247e-003	3.6379e-002	3.4690e-003	2.0455e-003	-4.1419e-018	-4.9127e-017	-3.0265e-017	8.7250e-019
2.3301e-019	-4.4970e-018	-2.1046e-018	-9.7002e-018	-3.4496e-002	1.5515e-003	1.8889e-005	-1.9161e-003
-5.8421e-021	-2.1353e-019	-3.7750e-018	-1.1379e-016	-8.7241e-004	-1.4228e-002	1.1013e-002	-1.1101e-004

NOC() from Equation (4.3.33):

1.1934e+000	1.5856e+000	3.9911e-001	4.0352e-001	2.1559e+000	1.9495e+000	4.0456e-001	4.0528e-001

TABLE-16

Z(OC)=1.0, S=100 **Wind Speed = 5m/Sec**

Incident Direction = 0.5

Separation Constants:

1.6292e-001	6.3582e-001	4.9372e-001	2.0054e-001	2.0932e-001	1.9852e-001	5.0394e-001	6.3679e-001

Eigen vector (Positive):

-2.9980e+000	-1.8388e-002	3.9045e-001	-4.4959e-001	-7.1066e-016	-2.7138e-016	-2.0937e-017	-1.4595e-017
-4.2086e-001	-7.9487e-002	-7.9840e-002	3.5260e+000	6.4721e-015	2.0239e-015	-2.7291e-018	-7.8015e-017
1.1481e-016	-1.5768e-016	2.1883e-016	-1.3959e-014	4.1894e+000	-3.1781e-001	5.4480e-003	-6.8942e-002
-6.5905e-018	-3.3555e-015	-1.3816e-016	-8.1003e-015	3.3175e-001	-3.9108e+000	-2.6932e-001	2.3255e-003
-5.6987e-001	3.8652e-002	-1.5115e+000	-2.2645e-001	-3.6551e-016	-1.4050e-016	7.7584e-017	2.7990e-017
-2.7935e-002	1.5534e+000	3.8249e-002	1.3751e-001	2.5944e-016	2.4549e-016	1.4398e-016	1.3818e-015
2.8661e-018	7.7181e-016	4.0248e-017	-4.4829e-016	1.3936e-001	-1.3731e-002	-3.3060e-003	1.5577e+000
-6.4224e-019	-3.3656e-016	-1.3400e-017	-1.0769e-015	5.3417e-002	-5.0456e-001	1.5415e+000	-3.2246e-003

Eigen vector (Negative):

-4.5911e-001	-3.9734e-004	-3.9843e-002	-6.1557e-002	-9.3778e-017	-3.7649e-017	1.9281e-018	-6.8611e-019
-3.7292e-002	1.5444e-002	2.1656e-002	2.8128e-001	3.7682e-016	1.6961e-016	-7.4832e-019	1.7131e-017
6.9867e-018	6.3684e-017	-1.0106e-016	6.1260e-016	-2.7101e-001	1.6750e-002	5.9816e-004	-1.4173e-002
-1.5447e-019	1.8806e-015	6.4835e-017	4.5768e-016	-2.1465e-002	2.0902e-001	2.1062e-002	-7.2782e-004
-1.5457e-001	-2.8670e-003	-3.7671e-002	-7.3925e-002	-1.1277e-016	-4.6895e-017	1.2408e-018	-1.9343e-018
-3.1042e-003	2.0884e-003	3.5883e-003	3.8589e-002	6.9256e-017	1.0992e-016	1.6908e-017	8.7174e-019
-1.0498e-018	3.2433e-018	-7.1329e-019	1.2266e-016	-3.7173e-002	1.5835e-003	-6.7838e-008	-1.9836e-003
-1.4454e-019	1.9781e-016	6.5944e-018	-2.4631e-017	-8.5350e-004	-1.6165e-002	1.1534e-002	-1.0768e-004

NOC() from Equation (4.3.33):

1.1558e+000	4.0358e-001	4.0050e-001	1.5604e+000	2.1751e+000	1.9380e+000	4.0527e-001	4.0536e-001

TABLE-17

Z(OC)=1.6, S=10 **Wind Speed = 3m/Sec**

Incident Direction = 0.5

Separation Constants:

1.9228e-001	2.2029e-001	5.3811e-001	6.3815e-001	2.2625e-001	2.1980e-001	5.5200e-001	6.3890e-001

Eigen vector (Positive):

-3.3078e+000	-5.1140e-001	2.7351e-001	-1.3453e-002	6.8380e-016	1.0066e-015	-3.7253e-018	7.2653e-019
-4.7972e-001	3.6749e+000	-4.7171e-002	-5.2762e-002	-5.9687e-015	-7.1527e-015	-2.2685e-018	8.3822e-019
-1.9816e-019	3.0522e-018	8.9644e-018	2.1264e-016	4.0821e+000	-3.6061e-001	-2.8190e-003	4.4386e-002
-2.0411e-018	3.1499e-017	9.3601e-017	2.2244e-015	3.6723e-001	-3.9709e+000	1.6687e-001	-1.5348e-003
-4.3967e-001	-1.5364e-001	-1.5196e+000	3.7875e-002	2.1304e-016	2.9021e-016	2.0347e-017	-5.5899e-018
-2.2644e-002	9.3889e-002	3.7722e-002	1.5539e+000	-2.0580e-016	3.6972e-016	-2.8129e-017	-5.2074e-019
2.7179e-019	-4.0475e-018	-9.3802e-018	-2.1292e-016	8.7090e-002	-9.9519e-003	2.9149e-003	-1.5574e+000
-1.4539e-019	2.2430e-018	6.6511e-018	1.5801e-016	3.4651e-002	-3.1705e-001	-1.5468e+000	2.8715e-003

Eigen vector (Negative):

-3.3387e-001	-4.3029e-002	-4.3467e-002	5.7454e-004	5.6557e-017	8.4116e-017	7.5014e-019	-1.1640e-019
-3.0020e-002	1.8089e-001	1.5186e-002	1.1398e-002	-2.1274e-016	-3.6443e-016	1.1451e-018	-3.7905e-019
-1.2310e-020	-1.7665e-020	-3.7993e-018	-1.0444e-016	-1.6083e-001	1.2131e-002	-5.9410e-004	9.9209e-003
-6.9492e-020	-1.0683e-018	-4.1805e-017	-1.1394e-015	-1.4182e-002	1.3168e-001	-8.3503e-003	4.0810e-004
-1.1993e-001	-5.0790e-002	-3.3256e-002	-1.9523e-003	6.8558e-017	9.5049e-017	1.8236e-019	3.6569e-019
-2.9526e-003	2.5615e-002	2.6826e-003	1.6829e-003	-6.4945e-017	2.1968e-016	-2.0350e-018	-7.5812e-021
1.4896e-019	-2.0187e-018	-9.4503e-019	-3.3709e-018	-2.2658e-002	1.2829e-003	-6.4873e-005	1.4851e-003
-5.6850e-021	-6.4730e-020	-2.9436e-018	-8.0344e-017	-8.2341e-004	-6.8828e-003	-8.0388e-003	1.0466e-00

NOC() from Equation (4.3.33):

1.3955e+000	1.7005e+000	3.9453e-001	4.0340e-001	2.0723e+000	1.9777e+000	4.0256e-001	4.0487e-001

TABLE-18

Z(OC)=1.6, S=100 **Wind Speed = 3m/Sec**

Incident Direction = 0.5

Separation Constants:

1.8553e-001	2.1600e-001	5.2783e-001	6.3763e-001	2.2262e-001	2.1524e-001	5.4125e-001	6.3843e-001

Eigen vector (Positive):

-3.2345e+000	-4.9777e-001	3.0081e-001	-1.4676e-002	1.1040e-015	9.5728e-016	-9.8705e-018	-4.4368e-018
-4.6533e-001	3.6432e+000	-5.4078e-002	-5.8799e-002	-8.7976e-015	-6.9266e-015	-5.0846e-018	-2.3922e-017
-9.4569e-020	1.6195e-018	4.3217e-018	1.0398e-016	4.1047e+000	-3.5105e-001	-3.3218e-003	4.9819e-002
-6.4832e-019	1.1140e-017	3.0281e-017	7.3095e-016	3.5910e-001	-3.9639e+000	1.8894e-001	-1.7134e-003
-4.7344e-001	-1.7056e-001	-1.5174e+000	3.8212e-002	3.2361e-016	3.2902e-016	4.5324e-017	1.9561e-017
-2.4091e-002	1.0407e-001	3.8048e-002	1.5537e+000	-2.6565e-016	-8.1693e-017	1.7998e-016	3.0301e-016
1.6883e-019	-2.7860e-018	-5.8787e-018	-1.3475e-016	9.8376e-002	-1.0897e-002	3.0051e-003	-1.5575e+000
-4.6776e-020	8.0627e-019	2.2289e-018	5.3960e-017	3.8801e-002	-3.5850e-001	-1.5456e+000	2.9522e-003

Eigen vector (Negative):

-3.6453e-001	-4.7468e-002	-4.3986e-002	4.2597e-004	9.7645e-017	9.1826e-017	1.2122e-018	3.5837e-019
-3.2045e-002	2.0318e-001	1.6791e-002	1.2434e-002	-3.5725e-016	-3.9952e-016	3.4769e-019	8.0752e-018
-7.7309e-021	9.5587e-021	-1.7907e-018	-5.0894e-017	-1.8374e-001	1.3290e-002	-6.2346e-004	1.0951e-002
-2.4384e-020	-4.2758e-019	-1.3637e-017	-3.8185e-016	-1.5819e-002	1.4877e-001	-1.0590e-002	4.7266e-004
-1.2911e-001	-5.6236e-002	-3.4776e-002	-2.1763e-003	9.6664e-017	1.0912e-016	5.3564e-019	-1.3347e-018
-3.0483e-003	2.8580e-002	2.9284e-003	1.8061e-003	-7.2391e-017	2.9362e-018	1.4939e-017	2.5267e-019
9.4270e-020	-1.4037e-018	-6.3070e-019	-1.7791e-018	-2.5741e-002	1.3747e-003	-5.8634e-005	1.6186e-003
1.7467e-022	-6.4175e-020	-1.0759e-018	-2.9777e-017	-8.5797e-004	-8.5848e-003	-8.9113e-003	1.0959e-004

NOC() from Equation (4.3.33):

1.3368e+000	1.6701e+000	3.9541e-001	4.0341e-001	2.0939e+000	1.9741e+000	4.0289e-001	4.0498e-001

TABLE-19

Z(OC)=1.6, S=10 **Wind Speed = 5m/Sec**

Separation Constants: **Incident Direction = 0.5**

1.7115e-001	2.0638e-001	5.0613e-001	6.3649e-001	2.1438e-001	2.0489e-001	5.1778e-001	6.3741e-001

Eigen vector (Positive):

-3.0822e+000	-4.6758e-001	3.5803e-001	-1.7106e-002	-2.3551e-016	-1.4193e-016	2.1531e-017	-6.2158e-018
-4.3637e-001	3.5706e+000	-6.9980e-002	-7.1861e-002	2.4933e-015	1.0070e-015	5.7058e-018	-2.5783e-017
4.6196e-019	-9.9935e-018	2.2785e-017	–5.7022e-016	4.1568e+000	-3.3013e-001	-4.5853e-003	6.1805e-002
2.1295e-018	-4.6252e-017	-1.0735e-016	-2.6965e-015	3.4174e-001	-3.9360e+000	2.3884e-001	-2.1001e-003
-5.3749e-001	-2.0627e-001	-1.5134e+000	3.8613e-002	-1.3962e-016	-6.4785e-017	-8.6513e-017	2.5480e-017
-2.6695e-002	1.2546e-001	3.8345e-002	1.5535e+000	6.9785e-017	2.4250e-016	-2.0554e-016	2.5463e-016
-5.6649e-019	1.1671e-017	2.0631e-017	4.8455e-016	1.2383e-001	-1.2764e-002	3.1975e-003	-1.5577e+000
1.9701e-019	-4.2872e-018	-1.0034e-017	-2.5249e-016	4.7954e-002	-4.5027e-001	–1.5430e+000	3.1251e-003

Eigen vector (Negative):

-4.2602e-001	-5.6580e-002	-4.2417e-002	-3.8355e-005	-3.0026e-017	-1.7511e-017	-2.2456e-018	6.3370e-019
-3.5639e-002	2.5221e-001	1.9999e-002	1.4438e-002	1.2805e-016	7.2487e-017	1.4810e-019	8.8274e-018
4.0012e-020	6.7399e-020	9.7118e-018	2.9382e-016	-2.3717e-001	1.5573e-002	-6.3422e-004	1.3048e-002
9.3532e-020	2.2933e-018	4.9453e-017	1.4725e-015	-1.9376e-002	1.8660e-001	-1.6658e-002	6.2738e-004
-1.4620e-001	-6.7597e-002	-3.6978e-002	-2.6302e-003	-4.6896e-017	-2.1649e-017	-1.2407e-018	-2.2299e-018
-3.1284e-003	3.4937e-002	3.3823e-003	2.0112e-003	1.3802e-017	1.1636e-016	-2.1460e-017	2.8851e-019
-3.3587e-019	6.1722e-018	2.8695e-018	1.2309e-017	-3.2799e-002	1.5262e-003	-2.9220e-005	1.8677e-003
4.6214e-021	3.0010e-019	4.7829e-018	1.4147e-016	-8.7888e-004	-1.3048e-002	-1.0652e-002	1.1220e-004

NOC() from Equation (4.3.33):

1.2187e+000	1.6017e+000	3.9828e-001	4.0349e-001	2.1437e+000	1.9558e+000	4.0416e-001	4.0522e-001

TABLE-20

Z(OC)=1.6, S=100 **Wind Speed = 5m/Sec**

Incident Direction = 0.5

Separation Constants:

1.6325e-001	2.0079e-001	4.9422e-001	6.3585e-001	2.0953e-001	1.9879e-001	5.0451e-001	6.3682e-00

Eigen vector (Positive):

-3.0014e+000	-4.5033e-001	3.8913e-001	-1.8337e-002	1.1507e-016	4.4898e-017	-9.0115e-018	4.4282e-0
-4.2148e-001	3.5279e+000	-7.9427e-002	-7.9174e-002	-6.8434e-016	-3.3389e-016	-1.1588e-017	2.5095e-0
3.5681e-019	-8.8194e-018	-1.8811e-017	-4.8375e-016	4.1880e+000	-3.1831e-001	-5.4106e-003	6.8647e-0
2.0011e-018	-4.9611e-017	-1.0711e-016	-2.7621e-015	3.3216e-001	-3.9120e+000	2.6805e-001	-2.3162e-0
-5.6861e-001	-2.2564e-001	-1.5116e+000	3.8653e-002	3.2262e-018	2.7113e-017	3.3575e-017	-7.2754e-0
-2.7887e-002	1.3703e-001	3.8257e-002	1.5534e+000	-2.3711e-017	-1.5475e-016	7.3806e-017	-4.8072e-0
-2.2676e-019	5.2824e-018	8.4916e-018	2.0208e-016	1.3871e-001	-1.3693e-002	3.3016e-003	-1.5577e+0
2.3749e-019	-5.8583e-018	-1.2395e-017	-3.1816e-016	5.3191e-002	-5.0233e-001	-1.5415e+000	3.2205e-0

Eigen vector (Negative):

-4.5779e-001	-6.1358e-002	-3.9972e-002	-3.8135e-004	1.3902e-017	6.4602e-018	8.9351e-019	3.9627e-01
-3.7230e-002	2.8008e-001	2.1592e-002	1.5405e-002	-4.0249e-017	-2.6332e-017	3.3095e-018	-5.0396e-01
2.6859e-020	2.4740e-019	8.3706e-018	2.6132e-016	-2.6958e-001	1.6704e-002	-6.0037e-004	1.4129e-00
9.2074e-020	2.8462e-018	5.0153e-017	1.5486e-015	-2.1379e-002	2.0810e-001	-2.0867e-002	7.2354e-00
-1.5424e-001	-7.3670e-002	-3.7650e-002	-2.8578e-003	-4.6337e-018	9.4253e-018	5.3014e-019	4.1909e-01
-3.1061e-003	3.8440e-002	3.5808e-003	2.0859e-003	-5.6183e-018	-7.8013e-017	8.7258e-018	-2.8009e-01
-1.4261e-019	2.9701e-018	1.5597e-018	1.2069e-017	-3.6989e-002	1.5816e-003	-1.3363e-006	1.9793e-00
2.2417e-020	5.1565e-020	5.3224e-018	1.6748e-016	-8.5513e-004	-1.6029e-002	-1.1500e-002	1.0797e-00

NOC() from Equation (4.3.33):

1.1583e+000	1.5622e+000	4.0040e-001	4.0357e-001	2.1738e+000	1.9389e+000	4.0522e-001	4.0536e-001

TABLE-21

Z(OC)=1.0, S=100 **Wind Speed = 3m/Sec**

Separation Constants:

Incident Direction = 0.866

2.1727e-001	2.3502e-001	5.7747e-001	6.4003e-001	2.3853e-001	2.3510e-001	5.9033e-001	6.4051e-001

Eigen vector (Positive):

-3.5778e+000	-5.5858e-001	1.6888e-001	-8.4828e-003	-3.5171e-015	1.6111e-014	1.2318e-017	-1.2952e-018
-5.3623e-001	3.7790e+000	-2.4637e-002	-3.0891e-002	2.7557e-014	-1.0941e-013	9.3708e-018	-1.0337e-017
3.1598e-020	-3.3153e-019	-1.5392e-018	-3.4930e-017	4.0071e+000	-3.9447e-001	1.3522e-003	-2.5297e-002
5.1884e-019	-5.4544e-018	-2.5868e-017	-5.8827e-016	3.9711e-001	-3.9694e+000	-9.2126e-002	8.9113e-004
-2.9091e-001	-9.0740e-002	-1.5303e+000	3.5805e-002	-5.0368e-016	2.6279e-015	-9.7735e-017	1.8711e-017
-1.5726e-002	5.5787e-002	3.5602e-002	1.5546e+000	4.7513e-016	-2.2710e-015	-5.2557e-016	-1.6545e-016
-1.4975e-019	1.5451e-018	5.8486e-018	1.2973e-016	4.8575e-002	-6.1510e-003	-2.5897e-003	1.5570e+000
1.5500e-020	-1.6397e-019	-8.2978e-019	-1.8985e-017	1.9968e-002	-1.7467e-001	1.5510e+000	-2.5761e-003

Eigen vector (Negative):

-2.0963e-001	-2.5945e-002	-3.4549e-002	7.5130e-004	-1.5452e-016	7.4784e-016	-2.1685e-018	4.6233e-019
-2.0435e-002	1.0259e-001	8.9776e-003	7.1372e-003	5.4120e-016	-2.9471e-015	-1.6321e-018	5.2701e-018
2.2611e-021	-1.0748e-020	6.0716e-019	1.5264e-017	-8.6284e-002	7.4712e-003	4.0007e-004	-5.9575e-003
1.1087e-020	9.7493e-020	1.1281e-017	2.8047e-016	-8.2816e-003	7.2855e-002	2.7491e-003	-2.0745e-004
-7.9022e-002	-3.0250e-002	-2.4452e-002	-1.1127e-003	-1.5781e-016	8.7669e-016	-5.3830e-019	-6.9246e-019
-2.2322e-003	1.4847e-002	1.6530e-003	1.1054e-003	1.3624e-016	-7.3330e-016	-2.1615e-017	-8.7895e-021
-7.4105e-020	7.1789e-019	3.1381e-019	2.9612e-019	-1.2378e-002	8.4607e-004	5.8494e-005	-9.2566e-004
-2.1670e-021	2.9197e-020	4.6276e-019	1.1241e-017	-5.8448e-004	-2.4552e-003	4.7026e-003	-7.1951e-005

NOC: () from Equation (4.3.33):

1.6238e+000	1.8022e+000	3.9421e-001	4.0349e-001	2.0016e+000	1.9692e+000	4.0235e-001	4.0448e-001

TABLE-22

Z(OC)=1.0, S=100 **Wind Speed = 3m/Sec**

Incident Direction = 0.866

Separation Constants:

2.1442e-001	2.3342e-001	5.7283e-001	6.3982e-001	2.3721e-001	2.3347e-001	5.8606e-001	6.4033e-001

Eigen vector (Positive):

-3.5478e+000	-5.5345e-001	1.8119e-001	-9.0858e-003	-4.9031e-016	3.3496e-015	-1.1798e-017	-4.2804e-018
-5.2960e-001	3.7682e+000	-2.6975e-002	-3.3357e-002	4.2364e-015	-2.2940e-014	-8.1109e-018	-1.7391e-017
-3.9076e-020	4.2784e-019	1.8547e-018	-4.2257e 017	-4.0150e+000	3.9071e-001	-1.4930e-003	2.7401e-002
-3.2226e-019	3.5356e-018	1.5642e-017	3.5716e-016	-3.9371e-001	3.9714e+000	1.0015e-001	-9.6330e-004
-3.0994e-001	-9.7927e-002	-1.5288e+000	3.6107e-002	-1.0001e-016	5.8183e-016	8.6903e-017	1.7279e-017
-1.6657e-002	6.0163e-002	3.5910e-002	1.5545e+000	1.5817e-017	1.4228e-016	4.6403e-016	7.0876e-016
1.5264e-019	-1.6408e-018	-5.7832e-018	-1.2852e-016	-5.2737e-002	6.6061e-003	2.6259e-003	-1.5570e+000
-7.6949e-021	8.5769e-020	4.3802e-019	1.0141e-017	-2.1598e-002	1.9002e-001	-1.5505e+000	2.6096e-003

Eigen vector (Negative):

-2.2470e-001	-2.7938e-002	-3.6136e-002	7.6022e-004	-2.4489e-017	1.6876e-016	2.0269e-018	3.9867e-019
-2.1707e-002	1.1121e-001	9.6969e-003	7.6539e-003	9.0551e-017	-6.7675e-016	1.1803e-018	4.4621e-018
-2.7869e-021	1.2398e-020	-7.3601e-019	-1.8694e-017	9.4075e-002	-8.0280e-003	-4.2719e-004	6.4188e-003
-7.4770e-021	-6.8049e-020	-6.8307e-018	-1.7156e-016	8.9449e-003	-7.9212e-002	-3.2049e-003	2.2763e-004
-8.4264e-002	-3.2618e-002	-2.5738e-002	-1.2075e-003	-3.3909e-017	1.9268e-016	5.2586e-019	-5.8839e-019
-2.3458e-003	1.6057e-002	1.7776e-003	1.1800e-003	-6.4055e-018	1.3920e-016	2.0711e-017	2.5886e-019
7.6345e-020	-7.6730e-019	-3.3607e-019	-3.6984e-019	1.3470e-002	-9.0288e-004	-6.1077e-005	9.9371e-004
2.7104e-021	-3.3351e-020	-3.0482e-019	-7.3355e-018	6.1954e-004	2.8281e-003	-5.0851e-003	7.6604e-005

NOC: () from Equation (4.3.33):

1.5974e+000	1.7915e+000	3.9399e-001	4.0347e-001	2.0091e+000	1.9716e+000	4.0230e-001	4.0452e-001

TABLE-23

Z(OC)=1.0, S=10 **Wind Speed = 5m/Sec**

Separation Constants: **Incident Direction = 0.866**

2.1345e-001	2.3287e-001	5.7126e-001	6.3974e-001	2.3676e-001	2.3291e-001	5.8460e-001	6.4027e-001

Eigen vector (Positive):

-3.5375e+000	-5.5169e-001	1.8535e-001	-9.2886e-003	-1.6310e-015	1.9276e-014	-3.2651e-018	1.2563e-018
-5.2735e-001	3.7645e+000	-2.7783e-002	-3.4196e-002	1.2508e-014	-1.3171e-013	-2.1374e-018	4.7042e-018
1.8828e-019	-2.0916e-018	-8.8690e-018	-2.0235e-016	4.0178e+000	-3.8943e-001	1.5422e-003	-2.8120e-002
2.1474e-018	-2.3906e-017	-1.0344e-016	-2.3652e-015	3.9255e-001	-3.9720e+000	-1.0290e-001	9.8789e-004
-3.1628e-001	-1.0037e-001	-1.5283e+000	3.6206e-002	-2.6252e-016	3.4961e-015	2.4541e-017	-1.6900e-017
-1.6965e-002	6.1648e-002	3.6012e-002	1.5544e+000	1.8328e-016	-1.8696e-015	9.1212e-017	2.0798e-016
-7.3296e-019	7.9895e-018	2.7520e-017	6.1203e-016	5.4164e-002	-6.7596e-003	-2.6382e-003	1.5570e+000
6.8056e-020	-7.6388e-019	-3.5644e-018	-8.2136e-017	2.2154e-002	-1.9528e-001	1.5504e+000	-2.6210e-003

Eigen vector (Negative):

-2.2977e-001	-2.8614e-002	-3.6641e-002	7.6157e-004	-7.8345e-017	9.9887e-016	5.8891e-019	-5.2892e-019
-2.2129e-002	1.1415e-001	9.9411e-003	7.8279e-003	2.7500e-016	-3.9849e-015	4.4339e-019	-3.0734e-018
1.3443e-020	-5.8494e-020	3.5256e-018	8.9863e-017	-9.6761e-002	8.2159e-003	4.3618e-004	-6.5751e-003
5.0585e-020	4.7776e-019	4.5215e-017	1.1395e-015	-9.1708e-003	8.1392e-002	3.3694e-003	-2.3464e-004
-8.6010e-002	-3.3421e-002	-2.6157e-002	-1.2398e-003	-8.2052e-017	1.1628e-015	1.4782e-019	7.0323e-019
-2.3825e-003	1.6470e-002	1.8196e-003	1.2048e-003	4.3054e-017	-4.4198e-016	4.1843e-018	3.8174e-020
-3.6790e-019	3.7443e-018	1.6402e-018	1.8707e-018	-1.3845e-002	9.2181e-004	6.1863e-005	-1.0166e-003
-1.1287e-020	1.5879e-019	2.0606e-018	5.0478e-017	-6.3103e-004	-2.9620e-003	5.2153e-003	-7.8139e-005

NOC() from Equation (4.3.33):

1.5884e+000	1.7878e+000	3.9394e-001	4.0347e-001	2.0116e+000	1.9724e+000	4.0229e-001	4.0454e-001

TABLE-24

Z(OC)=1.0, S=100 **Wind Speed = 5m/Sec**

Incident Direction = 0.866

Separation Constants:

2.1036e-001	2.3111e-001	5.6630e-001	6.3952e-001	2.3530e-001	2.3110e-001	5.7994e-001	6.4008e-001

Eigen vector (Positive):

-3.5044e+000	-5.4605e-001	1.9852e-001	-9.9276e-003	6.5528e-016	1.7889e-014	1.0107e-017	1.2120e-018
-5.2021e-001	3.7524e+000	-3.0403e-002	-3.6876e-002	-5.2295e-015	-1.2283e-013	6.3867e-018	1.2518e-017
-8.9307e-020	1.0392e-018	4.1146e-018	9.4293e-017	4.0266e+000	-3.8532e-001	1.7038e-003	-3.0425e-002
-1.0619e-018	1.2386e-017	5.0103e-017	1.1511e-015	3.8887e-001	-3.9735e+000	-1.1177e-001	1.0665e-003
-3.3606e-001	-1.0815e-001	-1.5269e+000	3.6508e-002	1.5199e-016	3.5440e-015	-6.8232e-017	-9.0906e-018
-1.7914e-002	6.6377e-002	3.6321e-002	1.5544e+000	-7.6443e-017	-2.2921e-015	-3.4675e-016	-1.0651e-016
3.4187e-019	-3.8972e-018	-1.2524e-017	-2.7918e-016	5.8755e-002	-7.2447e-003	-2.6778e-003	1.5571e+000
-3.8576e-020	4.5321e-019	1.9494e-018	4.5092e-017	2.3935e-002	-2.1225e-001	1.5499e+000	-2.6573e-003

Eigen vector (Negative):

-2.4574e-001	-3.0757e-002	-3.8136e-002	7.6013e-004	3.4122e-017	1.0078e-015	-1.6781e-018	-1.5981e-019
-2.3435e-002	1.2360e-001	1.0718e-002	8.3763e-003	-1.2487e-016	-4.0491e-015	-8.0292e-019	-5.0128e-018
-6.7933e-021	3.0303e-020	-1.6295e-018	-4.2058e-017	-1.0545e-001	8.8100e-003	4.6398e-004	-7.0718e-003
-2.6770e-020	-2.7034e-019	-2.1953e-017	-5.5936e-016	-9.8924e-003	8.8413e-002	3.9272e-003	-2.5747e-004
-9.1461e-002	-3.5977e-002	-2.7433e-002	-1.3433e-003	5.1575e-017	1.1791e-015	-4.6459e-019	3.8550e-019
-2.4928e-003	1.7788e-002	1.9523e-003	1.2824e-003	-1.6481e-017	-6.3837e-016	-1.7235e-017	-8.0455e-020
1.7350e-019	-1.8384e-018	-8.0280e-019	-9.4820e-019	-1.5057e-002	9.8081e-004	6.4044e-005	-1.0889e-003
4.9337e-021	-7.8795e-020	-1.0795e-018	-2.6806e-017	-6.6620e-004	-3.4130e-003	5.6300e-003	-8.2870e-005

NOC() from Equation (4.3.33):

1.5598e+000	1.7758e+000	3.9380e-001	4.0345e-001	2.0199e+000	1.9745e+000	4.0227e-001	4.0459e-001

TABLE-25

Z=1.6, S=10 **Wind Speed = 3m/Sec**

Incident Direction = 0.866

Separation Constants:

2.1908e-001	2.3602e-001	5.8044e-001	6.4016e-001	2.3935e-001	2.3612e-001	5.9302e-001	6.4062e-001

Eigen vector (Positive):

-3.5966e+000	-5.6179e-001	1.6103e-001	-8.0964e-003	1.0796e-015	-3.6277e-015	-9.5038e-019	2.0503e-018
-5.4044e-001	3.7857e+000	-2.3187e-002	-2.9334e-002	-8.1407e-015	2.4579e-014	-6.5865e-019	6.9658e-018
1.4518e-019	-1.4838e-018	-7.2638e-018	-1.6455e-016	-4.0021e+000	3.9683e-001	-1.2662e-003	2.3974e-002
2.3909e-018	-2.4469e-017	-1.2165e-016	-2.7596e-015	-3.9926e-001	3.9680e+000	8.7113e-002	-8.4561e-004
-2.7857e-001	-8.6193e-002	-1.5312e+000	3.5605e-002	1.3270e-016	-5.4401e-016	7.5886e-018	-8.1251e-018
-1.5116e-002	5.3015e-002	3.5399e-002	1.5546e+000	-8.6861e-017	-3.9294e-017	6.7955e-017	-4.4100e-016
-5.3816e-019	5.4107e-018	2.1486e-017	4.7603e-016	-4.5969e-002	5.8604e-003	2.5669e-003	-1.5569e+000
7.3668e-020	-7.5709e-019	-3.9409e-018	-8.9773e-017	-1.8941e-002	1.6508e-001	-1.5513e+000	2.5550e-003

Eigen vector (Negative):

-1.9996e-001	-2.4678e-002	-3.3465e-002	7.4183e-004	4.3886e-017	-1.5877e-016	1.3984e-019	-1.3732e-019
-1.9605e-002	9.7179e-002	8.5221e-003	6.8064e-003	-1.5147e-016	6.2684e-016	-8.1530e-023	-1.3342e-018
8.2303e-021	-2.8963e-020	2.9467e-018	7.3346e-017	8.1438e-002	-7.1159e-003	-3.8243e-004	5.6646e-003
4.8533e-020	4.1414e-019	5.2997e-017	1.3094e-015	7.8628e-003	-6.8880e-002	-2.4818e-003	1.9500e-004
-7.5627e-002	-2.8750e-002	-2.3596e-002	-1.0530e-003	4.0151e-017	-1.8011e-016	4.1211e-020	2.3588e-019
-2.1558e-003	1.4084e-002	1.5734e-003	1.0572e-003	-1.7064e-017	-8.5650e-017	2.6429e-018	-1.3560e-019
-2.6490e-019	2.5079e-018	1.1128e-018	1.3675e-018	1.1696e-002	-8.0928e-004	-5.6662e-005	8.8213e-004
-6.8133e-021	9.7856e-020	2.0595e-018	5.0029e-017	5.6136e-004	2.2349e-003	-4.4609e-003	6.8908e-005

NOC() from Equation (4.3.33):

1.6404e+000	1.8089e+000	3.9438e-001	4.0351e-001	1.9970e+000	1.9675e+000	4.0238e-001	4.0445e-001

TABLE-26

Z(OC)=1.6, S=100 **Wind Speed = 3m/Sec**

Incident Direction = 0.866

Separation Constants:

2.1461e-001	2.3353e-001	5.7314e-001	6.3983e-001	2.3730e-001	2.3358e-001	5.8635e-001	6.4035e-001

Eigen vector (Positive):

-3.5498e+000	-5.5381e-001	1.8035e-001	-9.0448e-003	-3.1221e-015	2.5262e-014	-2.6726e-018	-2.9931e-018
-5.3005e-001	3.7690e+000	-2.6812e-002	-3.3187e-002	2.4698e-014	-1.7230e-013	-1.5495e-018	-1.0747e-017
2.1733e-020	-2.3729e-019	-1.0348e-018	-2.3573e-017	4.0145e+000	-3.9097e-001	1.4831e-003	-2.7256e-002
2.8152e-019	-3.0796e-018	-1.3688e-017	-3.1246e-016	3.9394e-001	-3.9713e+000	-9.9592e-002	9.5834e-004
-3.0865e-001	-9.7434e-002	-1.5289e+000	3.6087e-002	-5.0575e-016	4.4444e-015	1.9891e-017	1.7782e-017
-1.6595e-002	5.9863e-002	3.5890e-002	1.5545e+000	4.1548e-016	-2.8998e-015	1.0554e-016	3.3805e-016
-8.3843e-020	8.9868e-019	3.1823e-018	7.0711e-017	5.2450e-002	-6.5750e-003	-2.6234e-003	1.5570e+000
9.1603e-021	-1.0086e-019	-4.7700e-019	-1.0957e-017	2.1486e-002	-1.8896e-001	1.5506e+000	-2.6073e-003

Eigen vector (Negative):

-2.2368e-001	-2.7802e-002	-3.6032e-002	7.5985e-004	-1.4839e-016	1.2673e-015	4.5653e-019	4.0694e-019
-2.1622e-002	1.1061e-001	9.6476e-003	7.6187e-003	5.2468e-016	-5.0363e-015	1.5148e-019	3.2269e-018
1.4782e-021	-6.1417e-021	4.1337e-019	1.0483e-017	-9.3536e-002	7.9900e-003	4.2536e-004	-6.3872e-00
6.4345e-021	5.9655e-020	5.9793e-018	1.5006e-016	-8.8993e-003	7.8773e-002	3.1723e-003	-2.2623e-004
-8.3909e-002	-3.2456e-002	-2.5652e-002	-1.2010e-003	-1.6047e-016	1.4800e-015	1.1933e-019	-6.5117e-019
-2.3383e-003	1.5974e-002	1.7691e-003	1.1749e-003	1.1195e-016	-8.0587e-016	4.6842e-018	1.3313e-019
-4.1915e-020	4.2021e-019	1.8458e-019	2.1454e-019	-1.3394e-002	8.9904e-004	6.0912e-005	-9.8908e-004
-1.2035e-021	1.7411e-020	2.6398e-019	6.4690e-018	-6.1719e-004	-2.8016e-003	5.0589e-003	-7.6291e-005

NOC() from Equation (4.3.33):

1.5992e+000	1.7923e+000	3.9401e-001	4.0347e-001	2.0086e+000	1.9715e+000	4.0230e-001	4.0452e-001

TABLE-27

Z(OC)=1.6, S=10 **Wind Speed = 5m/Sec**

Incident Direction = 0.866

Separation Constants:

2.1541e-001	2.3398e-001	5.7443e-001	6.3989e-001	2.3767e-001	2.3404e-001	5.8754e-001	6.4040e-001

Eigen vector (Positive):

-3.5582e+000	-5.5524e-001	1.7693e-001	-8.8776e-003	2.1435e-015	-1.4037e-014	-8.0510e-018	5.8931e-019
-5.3190e-001	3.7720e+000	-2.6156e-002	-3.2500e-002	-1.7212e-014	9.5643e-014	-7.0937e-018	6.4391e-018
-3.2730e-020	3.5321e-019	1.5731e-018	3.5805e-017	4.0122e+000	-3.9202e-001	1.4434e-003	-2.6668e-002
-2.6478e-019	2.8621e-018	1.2963e-017	2.9557e-016	3.9489e-001	-3.9708e+000	-9.7348e-002	9.3821e-004
-3.0340e-001	-9.5432e-002	-1.5293e+000	3.6004e-002	3.7566e-016	-2.4185e-015	6.0332e-017	-4.5139e-018
-1.6339e-002	5.8644e-002	3.5805e-002	1.5545e+000	-2.9160e-016	1.6963e-015	3.5990e-016	-5.5274e-017
1.0859e-019	-1.1507e-018	-4.1535e-018	-9.2239e-017	5.1286e-002	-6.4486e-003	-2.6133e-003	1.5570e+000
-6.8824e-021	7.5285e-020	3.8171e-019	8.7979e-018	2.1031e-002	-1.8466e-001	1.5507e+000	-2.5980e-003

Eigen vector (Negative):

-2.1949e-001	-2.7247e-002	-3.5602e-002	7.5797e-004	9.9975e-017	-6.8838e-016	1.3549e-018	-9.4610e-020
-2.1271e-002	1.0821e-001	9.4473e-003	7.4753e-003	-3.5718e-016	2.7296e-015	1.2913e-018	-2.6053e-018
-2.0437e-021	7.4810e-021	-6.3484e-019	-1.6031e-017	-9.1352e-002	7.8353e-003	4.1789e-004	-6.2588e-003
-5.9772e-021	-5.3692e-020	-5.6579e-018	-1.4160e-016	-8.7144e-003	7.6994e-002	3.0419e-003	-2.2055e-004
-8.2461e-002	-3.1796e-002	-2.5300e-002	-1.1745e-003	1.2325e-016	-8.0610e-016	3.5681e-019	1.6659e-019
-2.3073e-003	1.5637e-002	1.7345e-003	1.1543e-003	-8.0050e-017	4.9358e-016	1.5625e-017	-3.7118e-020
5.4166e-020	-5.3752e-019	-2.3785e-019	-3.1090e-019	-1.3088e-002	8.8335e-004	6.0225e-005	-9.7019e-004
1.8351e-021	-2.2932e-020	-2.4553e-019	-5.9278e-018	-6.0757e-004	-2.6952e-003	4.9523e-003	-7.5011e-005

NOC() from Equation (4.3.33):

1.6065e+000	1.7953e+000	3.9406e-001	4.0348e-001	2.0065e+000	1.9708e+000	4.0232e-001	4.0451e-001

TABLE-28

Z(OC)=1.6, S=100 **Wind Speed = 5m/Sec**

Incident Direction = 0.866

Separation Constants:

2.1057e-001	2.3124e-001	5.6664e-001	6.3953e-001	2.3540e-001	2.3122e-001	5.8026e-001	6.4009e-001

Eigen vector (Positive):

-3.5067e+000	-5.4644e-001	1.9762e-001	9.8842e-003	1.0084e-015	-4.4667e-014	4.6141e-019	-2.3260e-018
-5.2070e001	3.7532e+000	-3.0221e-002	3.6692e-002	-7.2581e-015	3.0686e-013	1.2113e-017	-1.3644e-018
3.8378e-019	-4.4612e-018	-1.8091e-017	4.1539e-016	4.0260e+000	3.8561e-001	-1.6925e-003	3.0267e-002
-2.1536e022	2.6242e-021	1.5008e-020	-3.5608e-019	3.8912e-001	3.9735e+000	1.1116e-001	-1.0611e-003
-3.3473e001	-1.0762e-001	1.5270e+000	-3.6488e-002	9.9926e-017	-8.8140e-015	-4.6749e-018	1.4706e-017
1.7850e-002	6.6053e-002	3.6301e-002	1.5544e+000	-1.1109e-016	5.2293e-015	4.7784e-018	1.7228e-016
2.5849e-021	-3.1497e-020	-1.8013e-019	4.2739e-018	5.8439e-002	7.2117e-003	2.6751e-003	1.5571e+000
1.3355e-022	-1.6273e-021	-9.3067e-021	2.2081e-019	2.3812e-002	2.1108e-001	1.5499e+000	2.6548e-003

Eigen vector (Negative):

-2.4466e-001	-3.0611e-002	-3.8039e-002	-7.6051e-004	5.1728e-017	-2.5028e-015	-1.2466e-019	3.4931e-019
-2.3347e-002	1.2295e-001	1.0665e-002	-8.3390e-003	-1.7283e-016	1.0060e-014	-4.9158e-018	-2.6706e-019
1.1451e-020	7.5651e-020	7.8557e-018	-2.0023e-016	-1.0485e-001	-8.7696e-003	-4.6212e-004	7.0379e-003
2.1536e-022	-2.6242e-021	-1.5008e-020	3.5608e-019	-9.8429e-003	-8.7929e-002	-3.8874e-003	2.5588e-004
-9.1093e-002	-3.5802e-002	-2.7349e-002	1.3362e-003	2.2541e-017	-2.9341e-015	-2.3842e-020	-6.1133e-019
-2.4856e-003	1.7697e-002	1.9433e-003	-1.2772e-003	-2.5458e-017	1.3667e-015	1.8772e-019	5.8364e-020
-2.5849e-021	3.1497e-020	1.8013e-019	-4.2739e-018	-1.4973e-002	-9.7684e-004	-6.3911e-005	1.0840e-003
-1.3355e-022	1.6273e-021	9.3067e-021	-2.2081e-019	-6.6387e-004	3.3809e-003	-5.6016e-003	8.2554e-005

NOC() from Equation (4.3.33):

1.5618e+000	1.7767e+000	3.9381e-001	4.0345e-001	2.0193e+000	1.9743e+000	4.0227e-001	4.0458e-001

Following tables represents the values of individual stokes parameters for directions given in the first column of the table number (29-33). The corresponding equation for this calculation is (4.8.9). However the integrals involved in the above mentioned equation is evaluated from expression (4.11.1) and using individual expressions from (4.11.2-4.11.10). The source functions for atmosphere and ocean are calculated from equations (2.12.15a-2.12.15d) and (2.12.19a-2.12.19d). Specular Reflection and Transmission functions are calculated from equations (2.10.6) and (2.10.7) with specific forms given by equations (2.10.8-2.10.11) for two quadrature angles and refractive index of water as 1.34. In bottom boundary condition (4.7.49) the functions $g(\mu)$ has been calculated from (4.7.49a). We have chosen initial direction of incident beam (μ_0) as 0.74176. Optical depth corresponding to the bottom of the ocean surface is set as Z1=2.0.The interface optical depth is set at Zw as 1.0.

Table 29

Z(OC)=1.1

Direction	L	Q	U	V
0.043633	3.9736	-0.45999	7.007	0.40979
0.07854	4.0324	-0.45999	7.2183	0.41113
0.11345	4.1354	-0.50868	7.4286	0.41501
0.14835	4.2339	-0.54337	7.6068	0.4173
0.18326	4.3054	-0.57454	7.7131	0.41521
0.21817	4.3379	-0.59883	7.7253	0.40729
0.25307	4.3292	-0.61483	7.641	0.39357
0.28798	4.2832	-0.62245	7.4705	0.37481
0.32289	4.206	-0.62223	7.2286	0.35208
0.3927	3.9849	-0.60159	6.5916	0.29856
0.42761	3.8523	-0.5828	6.2222	0.2693
0.46251	3.7112	-0.55932	5.8327	0.2391
0.53233	3.4162	-0.50053	5.0219	0.17724
0.60214	3.1194	-0.42882	4.2024	0.11469
0.63705	2.9741	-0.38889	3.7978	0.083393
0.67195	2.8322	-0.34648	3.3996	0.052125
0.70686	2.6954	-0.30118	3.0091	0.02087
0.74176	2.5441	-0.2454	2.6281	-0.010292
0.77667	2.4288	-0.21483	2.552	-0.041794
0.81158	2.3024	-0.16345	1.8935	-0.073176

Table 30

Z(OC)=1.3

Mu1	L	Q	U	V
0.043633	11.622	-3.2546	-9.2975	-0.66569
0.07854	5.0627	-0.78343	-9.564	-0.71276
0.11345	4.9832	-0.74425	-9.7951	-0.75858
0.14835	5.0282	-0.76437	-9.9092	-0.7965
0.18326	5.0323	-0.78273	-9.8658	-0.82258
0.21817	4.9778	-0.79077	-9.6713	-0.83578
0.25307	4.8724	-0.78793	-9.3553	-0.83715
0.28798	4.7289	-0.77563	-8.9503	-0.82844
0.32289	4.5592	-0.75559	-8.4851	-0.81137
0.3927	4.1785	-0.69821	-7.4582	-0.75806
0.42761	3.9802	-0.66309	-6.9253	-0.72408
0.46251	3.7824	-0.62476	-6.3924	-0.68636
0.53233	3.398	-0.54055	-5.3478	-0.60206
0.60214	3.0384	-0.4486	-4.3528	-0.50878
0.63705	2.8695	-0.40025	-3.8778	-0.45952
0.67195	2.7086	-0.35036	-3.4189	-0.4088
0.70686	2.5571	-0.29825	-2.9759	-0.35676
0.74176	2.3872	-0.23409	-2.5485	-0.30342
0.77667	2.2685	-0.20431	-2.1387	-0.24941
0.81158	2.1352	-0.14805	-1.7427	-0.19411

Table 31

Z(OC)=1.5

Mu1	L	Q	U	V
0.043633	676.09	-264.26	10.672	0.8293
0.07854	9.6494	-2.5026	10.977	0.88013
0.11345	6.4641	-1.2009	11.098	0.91797
0.14835	6.0925	-1.0776	10.951	0.93523
0.18326	5.8555	-1.0345	10.584	0.93419
0.21817	5.5968	-0.9965	10.074	0.91993
0.25307	5.3145	-0.95435	9.4857	0.89697
0.28798	5.021	-0.90797	8.8628	0.86855
0.32289	4.7271	-0.85858	8.236	0.83686
0.3927	4.1638	-0.75471	7.0099	0.76888
0.42761	3.9009	-0.70149	6.4317	0.73407
0.46251	3.6525	-0.6479	5.8799	0.69928
0.53233	3.1996	-0.54013	4.8567	0.6305
0.60214	2.8029	-0.43194	3.9369	0.56341
0.63705	2.6241	-0.37758	3.5128	0.53055
0.67195	2.4575	-0.32279	3.1108	0.49815
0.70686	2.3043	-0.26649	2.7294	0.46617
0.74176	2.1264	-0.19623	2.3683	0.43477
0.77667	2.0183	-0.17129	2.0226	0.40317
0.81158	1.8897	-0.11268	1.6961	0.37231

Table 32

Z(OC)=1.7

Mu1	L	Q	U	V
0.043633	64171	-26077	15.26	0.113
0.07854	54.235	-21.313	15.22	0.27993
0.11345	9.7642	-2.4835	14.489	0.41584
0.14835	7.1221	-1.4572	13.408	0.5167
0.18326	6.2353	-1.2076	12.237	0.58802
0.21817	5.6214	-1.0727	11.097	0.63624
0.25307	5.1083	-0.97014	10.037	0.66673
0.28798	4.5407	-0.85873	8.8063	0.68636
0.32289	4.2616	-0.80304	8.1875	0.68986
0.3927	3.5898	-0.6625	6.6687	0.67922
0.42761	3.3049	-0.59861	6.0123	0.66535
0.46251	3.0484	-0.53795	5.4141	0.64728
0.53233	2.6071	-0.42397	4.3649	0.60156
0.60214	2.244	-0.31703	3.4763	0.54695
0.63705	2.0865	-0.26528	3.0813	0.51738
0.67195	1.9432	-0.21405	2.7147	0.48678
0.70686	1.8151	-0.16181	2.3734	0.45544
0.74176	1.6508	-0.093047	2.056	0.42386
0.77667	1.5782	-0.082573	1.7557	0.39138
0.81158	1.4747	-0.029244	1.4771	0.35963

Table 33

Z(OC)=1.9

Mu1	L	Q	U	V
0.043633	6.1442e+006	-25633e+006	16.322	1.2049
0.07854	592.75	-258.05	12.939	0.99231
0.11345	21.152	-8.2225	10.337	0.82686
0.14835	6.6629	-1.8781	8.4648	0.70872
0.18326	4.3801	-0.99815	7.0815	0.6227
0.21817	3.4801	-0.71372	6.0246	0.55819
0.25307	2.9401	-0.56604	5.1926	0.50848
0.28798	2.5545	-0.46742	4.5203	0.46925
0.32289	2.2565	-0.3923	3.9657	0.4377
0.3927	1.8175	-0.2777	3.103	0.3906
0.42761	1.6499	-0.23075	2.7604	0.37274
0.46251	1.507	-0.18819	2.4615	0.35766
0.53233	1.2765	-0.11213	1.9644	0.3338
0.60214	1.1005	-0.04395	1.5674	0.31608
0.63705	1.0281	-0.011615	1.3973	0.30893
0.67195	0.96448	-0.020319	1.2428	0.30269
0.70686	0.91184	-0.053804	1.1017	0.29725
0.74176	0.80206	-0.10771	0.97378	0.29278
0.77667	0.80857	-0.088427	0.85139	0.28792
0.81158	0.76599	-0.12412	0.74219	0.28429

CHAPTER VI:

Bibliography

[1] Gao, B. C., Montes, M. J., Ahmed, Z., Davis, C. O., Atmospheric correction algorithm for hyperspectral remote sensing of ocean colour from space. Applied Optics, 39(6), 887–896, (2000).

[2] Hamilton, M. K., Davis, C.O., Rhea, W. J., Pilorz, S. H., Carder, K. L., Estimating Chlorophyll content and bathymetry of Lake Tahoe using AVIRIS data, Remote Sensing of Environment, 44: 217–230, (1993).

[3] Houghton, J. T., Meira Filho, L. G., Callander, B. A. Harris, N., Kattenberg, A., and Maskell, K., Eds.: Climate Change 1995: The Science of Climate Change, Cambridge University Press, 572 pp, (1996).

[4] Kinne, S., Bergstrom, R., Toon, O. B., Dutton, E., and Shiobara, M., Clear-sky atmospheric solar transmission: An analysis based on FIRE 1991 field experiment data, J. Geophys. Res., 103, 19 709–19 720, (1998).

[5] Conant, W. C., Ramanathan, V, Valero, F. P. J., Meywerk, J., An examination of the cleat sky solar absorption over the central equatorial pacific, observations versus models. Conant., Climate, 10, 1874–1884, (1997).

[6] Chou, M. D. and Zhao, W., Estimation and model validation of surface solar radiation and cloud radiative forcing using TOGA COARE measurements, J. Climate, 10, 610–620, (1997).

[7] Charlock, T. P., Alberta, T. L., The CERES/ARM/GEWEX Experiment (CGAGEX) for the retrieval of radiative fluxes with satellite data, Bull. Amer. Meteor, Soc., 77, 2673–2683, (1996).

[8] Kato, S., Ackerman, T. P., Clothiaux, E. E., Murcray, F., and Michalsky, J., Uncertainties in modeled and measured clear sky surface shortwave irradiances, J. Geophys. Res., 102, 25 881–25 898, (1997).

[9] Goloub, P., Tanre, D., Deuze, J. L., Herman, M., Marchand, A., and Breon, F. M., Validation of the first algorithm applied for deriving aerosol properties over the ocean using the POLDER/ADEOS measurements, IEEE Trans. Geosci. Remote Sens., 37, 1586–1596, (1999).

[10] Chowdhury, J., Cairns, B., and Travis, L. D., Case studies of aerosols retrievals over the ocean from multiangle, multispectral photopolarimetric remote sensing data, J. Atmos. Sci., 59, 383–397, (2002).

[11] Li, Z., Goloub, P., Devaux, C., Gu, X., Deuze, J. L., Xiao, Y., and Zhao, F., Retrieval of aerosol optical and Physical properties from ground based spectral, multi-angular and polarized sun-photometer measurements, Remote Sens., Environ., 101, 519–533, (2006).

[12] Waterman, T. H., Polarization of scattered sunlight in deep waters, Deep Sea Res., 3, Suppl., 426–434, (1955).

[13] Waterman, T. H., Polarization patterns in submarine illumination, Science, 120, 927–932, (1954).

[14] Ivanoff, A., and Waterman, T. H., Factors, mainly depth and wave length, affecting the degree of underwater light polarization, J. Mar. Res., 16, 283–307, (1958).

[15] Beardsley, G. F., Mueller Scattering matrix of sea water, J. Opt. Soc. Am., 58, 52–57, (1968).

[16] Lundgren, B., and Hojerslev, N. K., Daylight measurements in the Sargasso Sea: Results from the "DANA" expedition January–April (1966), Rep. 14, Dept. Of Phys., Oceanogr. Univ. of Copenhagen, Copenhagen, Denmark, (1971).

[17] Voss, K. J., and Fry, E. S., Measurements of the Mueller matrix for ocean water, Appl. Opt., 23(23), 4427–4439, (1984).

[18] Wing, G. M., Invariant imbedding and transport theory – A unified approach, J. Math. Anal. Appl. 2, 277, (1961).

[19] Chami, M., Santer, R., and Dilligeard, E., Radiative transfer model for the Computation of radiance and polarization in an ocean atmosphere system: polarization properties of suspended matter for remote sensing, Appl. Opt., 40(15), 2398–2416, (2001).

[20] Bohren, C. F., "Multiple scattering of light and some of its observable consequences," Am. J. Physics, 55(6), 524 – 33, (1987).

[21] Waterman T.H, Polarization of marine light fields and animal Orientation, SPIE 925, 431 – 437, (1988).

[22] Waterman, T. H., and Westell W. F., Quantitative effects of the Sun's position on submarine light polarization, J. Mar. Res., 15, 149 – 169, (1956).

[23] Tyler J. E., Estimation of percent polarization in deep oceanic water, J. Mar., Res., 21, 102 – 109, (1963).

[24] Timofeyeva, V. A., Plane of vibrations of polarized light in turbid media, I Atmos. Ocean Physics 5, 603 – 607, (1969).

[25] Timofeeva V. A., The degree of polarization of light in turbid media, Izvestiya Atmos. Ocean. Physics 5, 513–522, (1970).

[26] Novales – Flamarique, I., and Hawryshyn, C. W., Is the use of underwater polarized light by fish restricted to crespuscular time periods? Vision Res. 37, 975 – 989, (1997).

[27] Cronin, T. W., and Shashar. N., The linearly polarized light field in Clear, tropical marine waters, Spatial & temporal variation of light intensity, degree of polarization & e-vector angle, Exp. Biol. 204, 2461 – 2467, (2001).

[28] Waterman T. H., Polarization patterns in sublime illumination, Science 120, 927–932, (1954).

[29] Waterman, T. H., Polarization Sensitivity, In comparative Physiology & Evolution of vision in Invertebrates (ed. H. Autrum), pp 281 – 463, Berlin Springen-verlag (1981).

[30] Eddington, A. S., "On the radiative equilibrium of the stars," Monthly Notices of the Royal Astronomical Society, 77, 16 – 35, (1916).

[31] Horvath, G., and Varju, D., Underwater refraction-polarization patterns of sky light perceived by aquatic animals through Snell's window of a flat water surface, Vision Res. 35, 1651 – 1666, (1995).

[32] Sehunck, H., On the focusing of Sunlight by ocean waves, J. opt. Soc. Am. 47, 653 – 657, (1957).

[33] Snyder, R. L and Dera. J., Wave-induced light field fluctuations in the sea. J. Opt. Soc. Am. 6, 1072 – 1079, (1970).

[34] Stramska, M., and Dickey, T. D., Short-term variability of the underwater light field in the oligotrophic ocean in response to surface waves and clouds, Deep Sea Res. I 45, 1393 – 1410, (1998).

[35] Maximov V. V., Environment factors which may have led to the appearance of colour vision. Environ. Biol. Fish. 8, 173 – 184, (2000).

[36] Kuscer, I., and Ribaric, M., Optica Acta 6, 42, (1959)

[37] Ivanoff, A., and Waterman, J. H., Factors mainly depth and wavelength affecting the degree of underwater light polarization, J. Mar. Res. 16, 283 – 307, (1958).

[38] Schwind, R., Daphnia pulex swims towards the most strongly polarized light a response that leads to 'Shore flight', J. Exp, Biol. 202, 3631 – 3635, (1999).

[39] Bellman, R. E., Kalaba, R. E., and Wing, G. M., Invariant imbedding and mathematical physics: I. Particle processes, J. Math. Phys, 1, 280, (1960).

[40] Timofeyeva, V. A., Optics of turbid waters, In optical Aspects of oceanography (ed. N. Jerlov & E. Steeman –Nielsen), pp. 177 – 218, New York, Academic Press (1974).

[41] Schuster, A., "Radiation through a foggy atmosphere," Astrophysical Journal, 21, 1 – 22, (1905).

[42] Hawryshyn, C. W, Polarization vision in fish, Am. Sci 80, 164 – 175, (1992).

[43] Schwarzschild, K., “On the equilibrium of the Sun's atmosphere,” Nachrichten von der Koniglichen Gesellschaft der Wissenschaften zu Gottingen, Math. Phys. Klasse, 195, 41 – 53, (1905).

[44] Chai, J. C., Lee, H. S., and Patankar, S. V., “Ray effect and false scattering in the discrete ordinates method,” Numer. Heat Transfer Part B 24, 373–389, (1993).

[45] Lenoble, J., C. R. Acad. Sci. Paris 252, 3562, (1961).

[46] Barichello, L. B., Camargo, M., Rodrigues, P., and Siewert, C. E., Unified solutions to classical flow problems based on the BGK model, ZAMP, 52, 517, (2001).

[47] Stokes, G. G., Trans. Camb. Philos. Soc. 9, 399; also 1901 Mathematical and physical papers of Sir George Stokes (Cambridge, England: Cambridge Univ. Press), 3, 233, (1852).

[48] Bailey, P. B., and Wing, G. M., Some recent developments in invariant imbedding with applications, J. Math Phys. 6, 453, (1965).

[49] McCormick., N. J., Transport theory for Optical Oceanography, University of Washington, Mechanical Engineering Department; Seattle; Washington, (2004).

[50] Eddington, A. S., The Internal Constitution of Stars, Cambridge University Press, Cambridge, (1926).

[51] Milne, E. A., Handbuch der Astrophysik, 3(1), 114, pp. 65 – 173, (1930).

[51a]Milne, E. A., Z. f. Ap., 1, 90, (1930a).

[52] Hopf, E, Mathematical Problems of Radiative Equilibrium, Cambridge Mathematical Tract 31, Cambridge, (1934).

[53] Unsold, A., Physik der Sternatmospheren, pp-89–104, (1938).

[54] Chandrasekhar, S, Radiative Transfer (R.T), Dover, (1950).

[55] Bailey, P. B., and Wing, G. M., A rigorous derivation of some invariant equations of transport theory, J. Math. Anal Appl. 8, 144, (1964).

[56] Liou, K.N, An Introduction to Atmospheric Radiation, (Academic Press, Orlando, FL, 392pp. 1980).

[57] Liou, K.N, An Introduction to Atmospheric Radiation, 2d ed. Academic Press, 583 pp. (2002).

[58] Lenoble, J., Radiative Transfer in Scattering and Absorbing Atmospheres: Standard Computational Procedures. A. Deepak, Hampton, VA, 300pp. (1985).

[59] Stamnes, K., "The Theory of Multiple Scattering of Radiation in Plane Parallel atmospheres", Rev. Geophys. 24, 299–310, (1986).

[60] Hovenier, J. W., and Van der Mee, C. V. M., Astron. Astrophys. 128, 1, (1983).

[61] Jin, Z., and Stamnes, K., Radiative transfer in non-uniformly refracting layered media: atmosphere-ocean system, Applied Optics, Optical society of America, vol. 33, No. 3, (1994).

[62] Ambartsumian, V. A., C. R. (Doklady), Acad, URSS, 38, 257, (1943).

[63] Sobolev, V. V., A Treatise on Radiative Transfer, translated by S.I. Gaposchkin, Van Nostrand, New York, (1963).

[64] Kourganoff. V., Basic Methods in Transfer Problems, (Dover, New York), (1963).

[65] Wing, G. M., "An Introduction to Transport Theory," John Wiley, New York, (1962).

[66] Preisendorfer, R. W., Radiative Transfer on Discrete Spaces, Pergomon Press, Oxford (1965).

[67] Ambarzumian, V. A., J. Phys., Acad. Sci., USSR, 8, 65, (1944).

[68] Grant & Hunt, Proc. Roy, Soc. Loud A, 313, 183, (1969a).

[69] Hottel, H. C, and Sarofim, A. F., McGraw Hill, Radiative Transfer, New York, (1967).

[70] Barichello, L. B., Garcia, RDM, Siewert, C.E., JQSRT, in press.

[71] Redheffer, R.M, J. Math. Phys., 41, 1, (1962).

[72] Andrew Fischer, Department of Earth and Atmospheric Sciences, Cornell University, (internet).

[73] Grant, I. P., & Hunt, G. E., Proc. Roy, Soc. Loud A 313, 199, (1969b).

[74] Schwarzschild, Gottingen Nachrichten, Math-Phys. Klasse, p.41 (1906).

[75] Preisendorfer, R. W., Hydrologic optics, vol. 5, NTIS PB-259 797/9ST. NTIS, Springfield, (1976).

[76] Bellman, R. E., and Kalaba, R. E., On the principle of invariant imbedding and propagation through inhomogeneous media, Proc. Nat. Acad. Sci. USA, 42, 629, (1956).

[77] Bellman, R. E., Dynamic Programming, Princeton University Press, Princeton, N. J., (1957).

[78] Bronstein, Zs.f.Physik, 58, 696 (1929); see also Zs.f.Physik, 59, 144 (1929)

[79] Rosescu, T., Invariant imbedding in neutron transport theory, Academia Republicii Socialiste Romania, Institutul de Fizica Atomica, FR-57, (1966).

[80] Dodson, D. S., and Mingle, J. O., Escape probabilities by the method of invariant imbedding, Trans. Amer. Nucl. Soc. 7, 26, (1964).

[81] Mingle, J. O., Reflection and transmission intensities from gray slabs utilizing invariant imbedding methods, C.E.P. Symposium Series 62, No. 64, 250, (1966).

[82] Mingle, J. O., Application of the invariant imbedding method to mono-energetic neutron transport theory in slab geometry, Nucl. Sci. Eng. 28, 177, (1967).

[83] Timmons, D. H., and Mingle, J. O., The application of invariant imbedding to Green's function, Trans. Amer. Nucl. Soc. 9, 437, (1967).

[84] Kaiser, R. E., and Mingle, J. O., "Reflection of high-energy photons from semi-infinite homogeneous slabs by invariant imbedding techniques," Trans. Amer. Nucl. Soc. 10, 733, (1967).

[85] Shimizu, A., and Mizuta, H., Application of invariant imbedding to the reflection and transmission problem of gamma rays, J. Nucl. Sci. Techn. (Japan) 3, 57, (1966).

[86] Bellman, K., and Kalaba, R., and Prestrud, M., "Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness," American Elsevier, New York, (1963).

[87] Bellman, R. E., Kagiwada, H. H., and Kalaba, R. E., Invariant imbedding and radiative transfer in spherical shells, J. Comput. Phys. 1, 245, (1966).

[88] Schwarzschild, K., Sitzungsberichte d. Preuss. Akad. Berlin 1183, (1914)

[89] Kagiwada, H. H, Kalaba, R. E., and Segerblom, R., RM- 5537 PR, R and Corporation Report, [1968].

[90] Kagiwada, H. H., and Kalaba, R. E., J. Quant. Spectr. Rad. Transfer 7, 295, (1967).

[91] Chandrashekhar. S, Astrophys. J. 100, 76 (1944)

[92] Mingle, (1973).

[93] Chandrashekhar. S, Astrophys. J. 103, 165 (1946)

[94] Bailey, P. B., and Wing, G. M., A correction to some invariant imbedding equations of transport theory obtained by 'Particle Counting', J. Math. Anal. Appl. 8, 170, (1964A).

[95] Bellman, R. E., "Introduction to the Mathematical Theory of control Processes," Vol. 1, Academic Press, New York., (1967).

[96] Denman, E. D., "Coupled Modes in Plasmas, Elastic Media and Parametric Amplifiers." American Elsevier, New York, (1970).

[97] Jeans J.H, Monthly Notices Roy. Astron. Soc. London, 78, 28 (1917).

[98] Wiener, N., and Hopf, E., Berliner Ber., Math. Phys, Klasse, P. 696, (1931).

[99] Vestrucci, P., and Siewert, C. E., A numerical evaluation of an analytical representation of the components in a Fourier decomposition of the phase matrix for the scattering of polarized light, JQSRT, Vol. 31, no. 2, pp. 177 – 183, (1984).

[100]Munch, G., Astrophys. J., 108, 116, 1948.

[101]Stamens, K. Tsay, S.C Wiscombe, W., and Jayaweera, K., "Numerically stable algorithm for discrete ordinate method radiative transfer in multiple scattering and emitting layered media," Appl. Opt. 27, 2502 – 2509, (1988).

[102]Thomas, G. E., Stamnes, K., Radiative Transfer in the Atmosphere and Ocean, Cambridge University Press, (1st edition 1999).

[103]Liou, K. N., Analytic two Stream and four stream solutions for radiative transfer, J. Atmos, Sci, 31: 1473 – 1475, (1974).

[104]Stamnes, K., and Swanson, R. A., "A new look at the discrete-ordinate method for radiative transfer calculations in an isotropically scattering atmospheres," J. Atmos. Sci, 38: 387-399, (1981).

[105]Stamnes, K., and Dale, H., "A new look at the discrete ordinate method for radiative transfer calculations in an isotropically scattering atmospheres II: Intensity Computations," J. Atmos. Sci. 38, 2696 – 2706, (1981).

[106]Stamnes, K., On the Computation of angular distributions of radiation in planetary atmospheres, J. Quant. Spectrose. Radiat. Transfer. 28: 47, (1982).

[107]Stamnes, K., Reflection and transmission by a vertically inhomogeneous planetary atmosphere, Planet. Space, Sci, 30: 727, (1982).

[108]Stamnes, K., and Conklin, P., "A new multilayer discrete ordinate approach to radiative transfer in vertically inhomogeneous atmospheres," J. Quant. Spectrose. Radiat Transfer, 31: 273, (1984).

[109]Stamnes, K., The theory of multiple scattering of radiation in plane parallel atmospheres, Rev. Geophys, 24: 299, (1962).

[110]Stamnes, K., and Dahlback, A., A new spherical model for computing the radiative field available for photolysis and heating at twilight, Planet. Space Sci, 39: 671 – 683, (1991).

[111]Briegleb, B. P., Minnis, P., Ramanathan, V. and Harrison, E., Comparison of regional clear-sky albedos inferred from satellite observations and model computations. J. Climate Appl. Meteor., 25, 214–226, (1986).

[112]Nakajima, T., and Tanaka, M., Algorithms for radiative intensity calculations in moderately thick atmospheres using a truncation approximation, J, Quant. Spectrose. Radiat. Transfer, 40: 51 – 69, (1988).

[113]Jin, Z and Stamnes, J. K., Radiative transfer in non-uniformly refracting layered media: atmosphere ocean system, Applied Optics, 33: 431– 442, (1994).

[114] Modest, M. F., Radiative Heat Transfer, (McGraw Hill, New York), Chap. 15, (1993).

[115] Fireland, W.A., "The selection of discrete ordinate quadrature sets for anisotropic scattering" American Society of Mechanical Engineers Heat Transfer Division 72, 89 – 96, (1991).

[116] Lathrop, K. D.,"Spatial differencing of the transport equation: positive vs. accuracy", J. Comput., Phys. 4, 475–498, (1968).

[117]Lathrop, K. D, and Brinkley, F. W., "TWOTRAN – II Code", Los Alamos Scientific Laboratory Rep. #LA – 4848 – MS (Los Alamos Scientific Laboratory, Los Alamos, N. Mex, 1973).

[118]Kumar, S., and Mitra, K., "Microscale aspects of thermal radiation transport and laser applications", Adv. Heat, Transfer 33, 187 – 294, (1998).

[119]Mitra, K & Kumar, "Development and Comparison of models for light-pulse transport through scattering- absorbing media", S Appl. Opt. 38, 188 – 196, (1999).

[120]Guo, Z., and Kumar, S., Discrete-ordinates solution of short pulsed layer transport in two dimensional turbid media, Appl. Opt. 40, 3156 – 3163, (2001).

[121]Guo, Z & Kumar, S, "There-dimensional discrete ordinates method in transient radiant transfer", J. Therophysc. Heat Transfer 16, 289 – 296, (2002).

[122]Klose, A. D., Netz, U, Beuthan, J and Hielsher, A. H., "Optical tomography using the time independent equation of radiative transfer 1. Forward model", J. Quant Spectroce Radiat. Transfer, 72, 691 – 713, (2002).

[123]Zhixiong Guo & Kyunghan Kim, Optical society of America, (2003).

[124]Van de Hulst, H.C.: Light Scattering by small particles, Wiley, New York; also Dover, New York, 1981 (1957).

[125]Van der Mee, C. V. M., and Hovenier, J. W., Expansion coefficients in polarized light transfer, Free University, De Boelelaan 1081, NL-1081 HV Amsterdam, The Netherlands, Astron. Astrophys. 228, 559 – 568, (1990).

[126]Taylor, J. P., Edwards, J. M., Glew, M. D., Hignett, P., and Slingo, A., Studies with a flexible new radiation code. II: Comparisons with aircraft short-wave observations. Quart. J. Roy. Meteor. Soc., 122, 839–861, (1996).

[127]Carlson, B.G., Math. Computational Phys., 1, 1. (1963).

[128]Lathrop, K., Carlson, B. G., J. Comp. Phys., 2, 173, (1967).

[129]Ueno, S., J. Math. Anal. Applics., 11,11 (1965).

[130]Rybicki, G. B., and Usher, P. D., Astrophys. J. 146, 871, (1966).

[131]Van de Hulst, Multiple Light Scattering; Tables, Formulas and Applications Vol. 1, Academic Press, New York, (1980).

[132]Kattawar, G. W., Selected Papers on Multiple Scattering in Plane Parallel Atmospheres and Oceans: Methods, SPIE Milestone Series, vol. MS42, (1991).

[133]Lyzenga, D. R., "Note on the modified two-stream approximation of Sagan and Pollack," Icarus, 19, 240 – 3, (1973).

[134]Van Kampen, N G., Physkica, 21 949, (1955).

[135]Case, K. M., Zweifel, P. F., Linear Transport Theory, (Reading Massachusetts: Addison-Wesley), (1967).

[136]Case, K. M., Elementary solutions of the transport equation and their applications, Ann. Phys. 9, 1 – 23, (1960).

[137]Siewert, C. E., and McCormick, N. J., On an exact solution in the theory of line formation in stellar atmospheres, Astrophys. J. 149, 649 – 653, (1967).

[138]McCormick, N. J., and Mendelson, M. R., Transport solutions of the one-speed slab albedo problem, Nucl. Sci. Engng 20, 462 – 467, (1964).

[139]Ferziger, J. H., and Simmons, G. M., Application of Case's method to plane parallel radiative transfer, Int. J. Heat Mass Transfer 9, 987 – 992 (1966).

[140]Heaslet, M. A., and Warming, R. F., Radiative source function predictions for finite and semi-infinite non-conservative atmospheres, Astrophys. Space Sci. 1, 460 – 498 (1968).

[141]Siewert, C. E. and Benoist, P.: Nuclear Sci. Eng. (in Press), (1978).

[142]Grandjean, P. and Siewert, C. E.: Nuclear Sci. Eng. (in press), (1978).

[143]Devaux, C. and Grandjean, P.: Private Communication, (1978).

[144]Siewert, C. E., Astrophys. Space Sci. in press, (1979).

[145]Grandjean, P., and Siewert, C. E., Nucl. Sci. Eng., 69, 161 (1979).

[146]Lewis, E. E., and Miller Jr., W. F. Computational Methods of Neutron Transport, John Wiley & Sons, New York, USA, (1984).

[147]Barichello, L. B., and Siewert, C. E. A discrete-ordinates solution for a non-grey model with complete frequency redistribution. J. Quant. Spectros. Radiat. Transfer 62, 665 – 675, (1999a).

[148]Barichello, L. B., and Siewert, C. E. A discrete-ordinates solution for a polarization model with complete frequency redistribution. The Astrophysical J. 513, 370 – 382, (1999c).

[149]Barichello, L. B., and Siewert, C. E. A discrete-ordinates solution for Poiseuille flow in a plane channel. Z. Angew. Math. Phys. 50, 972 – 981, (1999c).

[150]Siewert, C. E., J. Quant. Spectrosc. Radiat. Transfer Vol. 21, pp. 35 – 39 (1979).

[151]Siewert, C. E., and P. B. Benoist, Nucl. Sci. Engng in press (1979).

[152]Garcia, R. D. M., and Siewert, C. E., The transport of neutral hydrogen atoms in a hydrogen plasma. Nuclear Sci. Eng. 136, 140–149, (2000).

[153]Siewert, C. E., and Valougeorgis, D., An analytical discrete-ordinates solution of the S-model kinetic equations for flow in a cylindrical tube. J. Quant. Spectrosc. Radiat. Transfer, 72, 531-550, (2002).

[154]Siewert, C. E., and Valougeorgis, D., The temperature-jump problem for a mixture of two gases. J. Quant. Spectrosc. Radiat. Transfer 70, 307 – 319, (2001b).

[155]Barichello, L. B., and Siewert, C. E., The search light problem for radiative transfer in a finite slab. J. Computational Phys. 157, 707 – 726, (2000a).

[156]Barichello, L. B., and Siewert, C. E. The temperature-jump problem in rarefied gas dynamics. Euro. J. Applied Math. 11, 353 – 364, (2000b).

[157]Siewert, C. E. Poiseuille and thermal-creep flow in a cylindrical tube. J. Computational Phys. 160, 470 – 480, (2000d).

[158]Siewert, C. E., Couette flow for a binary gas mixture. J. Quant. Spectrosc. Radiat. Transfer 70, 321–332, (2001a).

[159]Siewert; C. E., Generalized boundary conditions for the S-model kinetic equations basic to flow in a plane channel, J. Quant. Spectrosc. Radiat. Transfer in Press, (2001b).

[160]Siewert, C. E., Kramer's problem for a variable collision frequency model. Euro. J Applied Math. 12, 179 – 191, (2001c).

[161]Siewert, C. E. A discrete-ordinates solution for multi-group transport theory with upscattering. J. Quant. Spectrosc. Radiat. Transfer 64, 255 – 273, (2000b).

[162]Siewert, C. E. A concise and accurate solution to Chandrasekhar's basic problem in radiative transfer. J. Quant. Spectrosc. Radiat. Transfer 64, 109 – 130, (2000a).

[163]Siewert, C. E. A discrete-ordinates solution for radiative-transfer models that include polarization effects. J. Quant. Spectrosc. Radiat. Transfer 64, 227 – 254, (2000c).

[164]Barichello, L. B., and Siewert, C. E. On the equivalence between the discrete-ordinates and the spherical-harmonics methods in radiative transfer. Nuclear Sci. Eng. 130, 79 – 84, (1998).

[165]Siewert, C. E. A discrete-ordinates solution for heat transfer in a plane channel. J Computational Phys. 152, 251 – 263, (1999b)

[166]Barichello, L. B., Rodrigues, P., and Siewert, C. E, A discrete-ordinates solution for dual-mode heat transfer in a cylinder. J. Quant. Spectros. Radiat. Transfer in press, Vo. 73, pp. 583- 602, (2002).

[167]Bellman, R. E., Wing, G. M., An Introduction to Invariant Imbedding, (New York: Wiley) (1975).

[168]Wick, G. C., Zs. f. Phys., 121,702, (1943).

[169]Siewert, C. E., On the Phase Matrix Basic to the Scattering of Polarized Light, Department of Physics and Astronomy, Free University, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands, Astron. Astrophys. 109, 195 – 200, (1982).

[170]McFarlane, N. A., Boer, G. J., Blanchet, J.-P., and Lazare, M., The Canadian Climate Centre second-generation general circulation model and its equilibrium climate. J. Climate, 5, 1013–1044, (1992).

[171]McFarlane, N. A., Scinocca, J. F., Lazare, M., Harvey, R., Verseghy, D., and Li, J., The CCCma third generation atmospheric general circulation model. CCCma Internal Rep., 25 pp, (2005).

[172]Hansen, J., Russell, G., Rind, D., Stone, P., Lacis, A., Lebedeff, S., Ruedy, R., and Travis, L., Efficient three-dimensional global models for climate studies: Models I and II. Mon. Wea. Rev., 111, 609–662, (1983).

[173]Mihalas, D., Freeman, Steller atmosphere, San Francisco., (1978).

[174]Cannon, C.J.,The transfer of spectral line radiation, Cambridge University Press, Cambridge, (1984).

[175]Wehrse, R., Beckman, Crivellari. L., Progress in Steller spectral line formation theory, eds. Reidel, Dordrecht, (1985).

[176]Wehrse, R and Kalkophen, W, Astron. Astrophys, 147, 71, (1985).

[177]Schmidt, M and Wehrse, R., Discrete ordinate matrix method, Numerical radiative transfer, edited by W. Kolkophen Cambridge University Press, Cambridge, (1987).

[178]Wiscombe, W. J., and Joseph, J. H., "The range of validity of the Eddington approximation," Icarus, 32, 362 – 77, (1977).

[179]Mobley, C., "A numerical model for the computation of radiance distributions in natural waters with wind-roughened surfaces," Limnol. Oceanogr. 34, 1473 – 1483, (1989).

[180]Gordon, H., and Wang, M., "Surface-roughness considerations for atmospheric correction of ocean color sensor. I. The Rayleigh-scattering component," Appl. Opt. 31, 4247 – 4260, (1992).

[181]Kattawar, G., and Adams, C., "Stokes vector calculations of the submarine light field in an atmosphere-ocean with scattering according to a Rayleigh phase matrix: effect of interface refractive index on radiance and polarization," Limnol. Oceanogr. 34, 1453-1472, (1989).

[182]Morel, A., and Gentili, A., "Diffuse reflectance of oceanic waters: its dependence on Sun angle as influenced by the molecular-scattering contribution," Appl. Opt. 30, 4427 – 4438, (1991).

[183]Kirk, J., "Monte Carlo procedure for simulating the penetration of light into natural waters," Division of Plant Industry Tech. Paper 36 (Commonwealth Scientific and Industrial Research Organization, Canberra, Australia), p. 16, (1981).

[184]Nakajima, T., and Tanaka, M., "Effect of wind-generated waves on the transfer of solar radiation in the atmosphere ocean system," J. Quant. Spectrosc. Radiat. Transfer 29, 521-537, (1983).

[185] Denman, K. L., A time-dependent model of the upper ocean. J. Phys. Oceanogr., 3, 173–184, (1973).

[186] Simpson, and T. D. Dickey, The relationship between downward irradiance and upper ocean structure. J. Phys. Oceanogr., 11, 309–323, (1981).

[187]Charlock, T. P., Mid-latitude model analysis of solar radiation, the upper layers of the sea, and seasonal climate. J. Geophys. Res., 87, 8923–8930, (1982).

[188]Kantha, L. H., and Clayson, C. A., An improved mixed-layer model for geophysical applications. J. Geophys. Res., 99, 25 235–25 266, (1994).

[189]Schneider, N., Barnett, T., Latif, M., and Stockdale, T., Warm pool physics in a coupled GCM. J. Climate, 9, 219–239, (1996).

[190]Brainerd, K. E., and Gregg, M. C., Turbulence and stratification on the Tropical Ocean-Global Atmosphere–Coupled Ocean Atmosphere Response Experiment microstructure pilot cruise. J. Geophys. Res., 102, 10 437–10 455, (1997).

[191]Ohlmann, J. C., Siegel, D. A., and Washburn, L., Siegel, D. A., Radiant heating of the western equatorial Pacific during TOGA-COARE. J. Geophys. Res., 103, 5379–5395, (1998).

[192]Jerlov, N. G., Marine Optics. Elsevier, 229 pp., (1976).

[193]Kraus, E. B., Atmosphere–Ocean Interaction. Clarendon, 275 pp. (1972).

[194]Paulson, C. A., and Simpson, J. J., Irradiance measurements in the upper ocean. J. Phys. Oceanogr., 7, 953–956, (1977).

[195]Zaneveld, J. R., and Spinrad, R. W., An arc tangent model of irradiance in the sea. J. Geophys. Res., 85, 4919 – 4922, (1980).

[196]Paulson, C. A., and Simpson, J. J., The temperature difference across the cool skin of the ocean. J. Geophys. Res., 86, 11 044–11 054, (1981).

[197]Woods, J. D., Barkmann, W., and Horch, A., Solar heating of the oceans—Diurnal, seasonal and meridional variation. Quart. J. Roy. Meteor., Soc., 110, 633–656, (1984).

[198]Simonot, J. Y., and Treut, H. Le, A climatological field of mean optical properties of the world ocean. J. Geophys. Res., 91, 6642– 6646, (1986).

[199]Morel, A., Optical modeling of the upper ocean in relation to its biogenous matter content (case I waters). J. Geophys. Res., 93, 10 749–10 768, (1988).

[200]Morel, A., and Antoine, D., Heating rate within the upper ocean in relation to its bio-optical state. J. Phys. Oceanogr., 24, 1652– 1665, (1994).

[201]Ohlmann, J. C., Siegel, D. A., and Gautier, C., Ocean mixed layer radiant heating and solar penetration: A global analysis. J. Climate, 9, 2265–2280, (1996).

[202]Smith, R. C., and K. S. Baker, Optical classification of natural waters. Limnol. Oceanogr., 23, 260 – 267, (1978).

[203]Siegel, D. A., and Dickey, T. D., On the parameterization of irradiance for open ocean photoprocesses. J. Geophys. Res., 92, 14 648 –14 662, (1987).

[204]Lewis, M. R., Carr, M. –E., Feldman, G. C., Esaias, W., and McClain, C., Influence of penetrating solar radiation on the heat budget of the equatorial Pacific Ocean. Nature, 347, 543 – 545, (1990).

[205]Siegel, J. C. Ohlmann, L. Washburn, R. R. Bidigare, C. Nosse, E. Fields, and Y. Zhou, Solar radiation, phytoplankton pigments and radiant heating of the equatorial Pacific warm pool. J. Geophys. Res., 100, 4885 – 4891, (1995).

[206]Nann, S., and Riordan, C., Solar spectral irradiance under clear and cloudy skies: measurements and a semi-empirical model. J. Appl. Meteor., 30, 447–462 (1991).

[207]Siegel, D. A., Westberry, T., and Ohlmann, J. C., Cloud color and ocean radiant heating. J. Climate, 12, 1101–1116, (1999).

[208]Liou, K. N., An Introduction to Atmospheric Radiation. Academic Press, 392 pp., (1980).

[209]Kirk, J. T. O., Dependence of relationship between inherent and apparent optical properties of water on solar altitude. Limnol. Oceanogr., 29, 350–356, (1984).

[210]Gordon, H., “Dependence of the diffuse reflectance of natural waters on the sun angle,” Limnol. Oceanogr. 34, 1484–1489, (1989).

[211]Payne, R. E., Albedo of the sea surface. J. Atmos. Sci., 29, 959–970, (1972).

[212]Katsaros, K. B., McMurdie, L. A., Lind, R. J., and DeVault, J. E., Albedo of a water surface, spectral variation, effects of atmospheric transmittance, sun angle and wind speed. J. Geophys. Res., 90, 7313–7321, (1985).

[213]Preisendorfer, R. W., and Mobley, C. D., Albedos and glitter patterns of a wind roughened sea surface. J. Phys. Oceanogr., 16, 1293 –1316, (1986).

[214]Simpson, J. J., and Paulson, C. A., Mid-ocean observations of atmospheric radiation. Quart. J. Roy. Meteor. Soc., 105, 487– 502, (1979).

[215]Cox, C., and Munk, W., "Measurement of the roughness of the sea surface from photographs of the sun's glitter," J. Opt. Soc. Amer., 44, 838–850, (1954).

[216]Plass, G. N., Kattawar, G. W., and Guinn, J. Jr., "Radiative transfer in the earth's atmosphere and ocean: influence of ocean waves," Appl. Opt. 14, 1924 – 1936, (1975).

[217]Kirk, J. T. O., "The upwelling light stream in natural waters," Limnol. Oceanogr. 34, 1410 – 1425, (1989).

[218]Gordon, H., and Wang, M., "Surface-roughness considerations for atmospheric correction of ocean color sensors. I. The Rayleigh-scattering component," Appl. Opt. 31, 4247 – 4260, (1992).

[219]Barker, H. W., and Li, Z., Improved simulation of clear-sky shortwave radiative transfer in the CCC-GCM. J. Climate, 8, 2213–2223, (1995).

[220]Deuze, J. L., Herman, M., and Santer, R., "Fourier series expansion of the transfer equation in the atmosphere-ocean system," J. Quant. Spectrosc. Radiat. Transfer 41, 483–494, (1989).

[221]Mobley, C. D., A numerical model for the computation of radiance distributions in natural waters with wind roughened surfaces. Limnol. Oceanogr. 34, 1473–1483, (1989).

[222]Mobley, C. D., Light and Water: Radiative Transfer in Natural Waters. Academic Press, 592 pp., (1994).

[223]Mobley, C. D., Gentili, B., Gordon, H. R., Jin, Z., Kattawar, G. W., Morel, A., Reinersman, P., Stamnes, K., and Stavn, R. H., "Comparison of numerical models for computing underwater light fields," Appl. Opt. 32, 7484–7504, (1993).

[224]Mobley, C., "A numerical model for the computation of radiance distributions in natural waters with wind-roughened surface, Part-II: users' guide and code listing," NOAA Tech. Memo, ERL PMEL-81 (NTIS PB88-246871) (Pacific Marine Environmental Laboratory, Seattle, Wash.), (1988).

[225]Liou, K. N., Analytic two-stream and four-stream solutions for radiative transfer, J. Atoms. Sci., 31, 1473 – 1475, (1974).

[226]Liou, K. N., Fu, Q., and Ackerman, T. P., A simple formulation of the delta-four-stream approximation for radiative transfer parameterizations, J. Atoms. Sci., 45, 1940-1948, (1988).

[227]Fraser, R. S., and Walker, W. H., JOSA 58, 636, (1968).

[228]Dave, J. V., Tech. Report, Contract No. NAS5-21680, NASA-Goddard Space Flight Center, Greenbelt, MD, (1972).

[229]Kattawar, G. W., Plass, G. N., and Guinn, J. A., Jr., J. Phys. Oceanogr. 3, 353 (1973).

[230]Raschke, E., Beitr. Phys. Atmos. 45, 1 (1972).

[231]Busbridge, I. W., The mathematics of radiative transfer, Cambridge University, Press, Cambridge, (1960).

[232]Quenzel, H., and Kaestner, M., Appl. Opt. 19, 1338 (1980).

[233]Ahmad, Z., and Fraser, R. S., J. Atmos. Sci. 39, 656, (1982).

[234]Takashima, T., and Masuda, K., Appl. Opt. 24, 2423, (1985).

[235]Liu, Q, and Weng, F., A microwave polarimetric two-steam radiative transfer model, Journal of Atmospheric Sciences, Vol. 59, 2396–2220, (2002).

[236] Van der Mee, C. V. M. Integral Eqs. Oper. Th. 5, 573, (1982).

[237]Weng, F., A multi-layer discrete-ordinate method for vector radiative transfer in a vertically-inhomogeneous, emitting and scattering atmosphere I: Theory. J. Quant. Spec. Radiat. Transfer, 47, 19–33, (1992).

[238]Gordon, H. R., O. B. Brown, R. H. Evans, J. W. Brown, r. C. Smith, K. S. Baker, and D. K. Clark, A semi-analytic radiance model of ocean color. J. Geophys. Res., 93, 10-909 – 10-924, (1988).

[239] Hovenier, J. W., J. Atmospheric Sci. 26, 488, (1969).

[240]Seigel, D. A., J. C. Ohlmann, L. Washburn, R. R. Bidigare, C. Nosse, E. Field, and Y. Zhou, Solar radiation, phytoplankton pigments, and radiant heating of the equatorial Pacific warm pool. J. Geophys. Res., 100, 4885 – 4891, (1995).

[241]Schultz, F. M., Stamnes, K., and Weng, F., An improved and generalized discrete ordinate radiative transfer model for polarized (vector) radiative transfer computations. Quant. Spectrosc. Radiat. Transfer, 61, 105–122, (1999).

[242] Evans, K. F., and G. L. Stephens, A new polarized atmospheric radiative transfer model. J. Quant. Spectrosc. Radiat. Transfer, 46, 413–423, (1991).

[243]Waterman, P. C., Matrix-exponential description of radiative transfer. J. Opt. Amer., 27, 410–422, (1981).

[244]Liu, Q., and Ruprecht, E., A radiative transfer model: Matrix operator method. Appl. Opt., 35, 4229–4237, (1996).

[245]Coakley, J., and Chylek, P., The two-stream approximation in radiative transfer: Including the angle of the incident radiation. J. Atmos. Sci., 32, 409 – 418, (1975).

[246]Kerschgens, M., Pilz, U., and Raschke, E., A modified two-stream approximation for computations of the solar radiation budget in the cloudy atmosphere. Tellus, 30, 429–435, (1978).

[247]Schmetz, J., Operational calibration of the METEOSAT water vapor channel by calculated radiances. Appl. Opt., 28, 3030–3038, (1989).

[248]Weng, F., and Grody, N. C., Retrieval of ice cloud parameters using a microwave imaging radiometer. J. Atmos. Sci., 57, 1069 – 1081, (2000).

[249]O'Brien, D. M., "Accelerated quasi Monte Carlo integration of the radiative transfer equation," J. Quant. Spectrosc. Radiat. Transfer 48, 41 – 59, (1992).

[250]Evans, K. F., "The spherical harmonic discrete ordinate method for three-dimensional atmospheric radiative transfer," J. Atmos. Sci. 55, 429 – 446, (1998).

[251]Sanchez, A., Smith, T. F., and Krajewski, W. F., "A three-dimensional atmospheric radiative transfer model based on the discrete-ordinates method," Atmos. Res. 33, 283 – 308, (1994).

[252]Haferman, J. L., Smith, T. F., and Krajewski, W. F., "A multidimensional discrete-ordinates method for polarized radiative transfer. Part I: validation for randomly oriented axisymmetric particles," J. Quant. Spectrosc. Radiat. Transfer 58, 379 – 398, (1997).

[253]Gu, Y., and Liou, K. N., "Radiation parameterization for three-dimensional inhomogeneous cirrus clouds: application to climate models," J. Clim. 14, 2443 – 2457, (2001).

[254]Marshak, A., and Davis, A. B., eds., 3D Radiative Transfer in Cloudy Atmospheres, Springer, (2005).

[255]Chen, Y., Liou, K. N., and Gu, Y., "A efficient diffusion approximation for 3D radiative transfer parameterization: application to cloudy atmospheres," J. Quant. Spectrosc. Radiat. Transfer 92, 189 – 200, (2005).

[256]Stenholm, L. G., Storzer, H., and Wehrse, R, "An efficient method for the solution of 3-D radiative transfer problems," J. Quant. Spectrosc. Radiat. Transfer 45, 47 – 56, (1991).

[257]I3RC group, "I3RC Monte Carlo community model of 3D radiative transfer," http://i3rc.gsfc.nasa.gov/I3RC-intro.html.

[258]Pincus, R., Barker, H. W., and Morcrette, J. J., "A fast, flexible, approximate technique for computing radiative transfer in inhomogeneous cloud fields," J. Geophys. Res. 108, 4376 – 4381, (2003).

[259]Jin, Z., Charlok, P. Rutledge, K. Stamnes, K. Wang, Y., Analytical solution of radiative transfer in the coupled atmosphere-ocean system with rough Surface, Applied optics, vol. 45, no. 28, 7443 – 7455, (2006).

[260]Haltrin, V. I., One Parameter Sea Water Light Scattering Phase Function in the Form of Legendre Polynomial Series. OCEANS 2003 MTS-IEEE Conference Proceedings on CD-ROM,pp. 2842-2844,san Diego, California, September 22-26, (2003).

[261]Gordon, H. R., "Ship perturbation of irradiance measurements at sea. 1: Monte Carlo simulations," Appl. Opt. 24, 4172 – 4182, (1985).

[262]Reinersman, P. N., and Carder, K. L., "Monte Carlo simulation of the atmospheric point-spread function with an application to correction for the adjacency effect," Appl. Opt. 34, 4453 – 4471, (1995).

[263]Peng-Wang Zhai, Kattawar, G. W., and Yang, P., "Impulse response solution to the three-dimensional vector radiative transfer equation in atmosphere–ocean systems. II., The hybrid matrix operator—Monte Carlo method," Vol. 47, No. 8, Appl. Opt., p – 1063, (2008).

[264]Minnaert, M., "The Nature of Light and Colour in the Open Air," Dover, New York, (1954).

[265]Jerlov, N. G., Optical Oceanography, Elsevier, New York, (1968).

[266]Jerlov, N. G., and Nielsen, E. S., Eds., Optical Aspects of Oceanography, Academic, New York, (1974).

[267]Plass, G. N., and Kattawar, G. W., Appl. Opt. 7, 415, (1968).

[268]Plass, G. N., and Kattawar, G. W., J. Atmos. Sci. 28, 1187, (1971).

[269]Plass, G. N., and Kattawar, G. W., Appl. Opt. 8, 455, (1969).

[270]Plass, G. N., and Kattawar, G. W., J. Phys. Oceanogr. 2, 249, (1972).

[271]Kattawar, G. W., and Plass, G. N., J. Phys. Oceanogr. 2, 146, (1972).

[272] Feldman, I.A. Matem. Issled. Kishinev 10(3), 226, (1975)

[273]Raschke, E., Beitr. Phys. Atmos. 45, 1, (1972).

[274]Gordon, H. R., and Brown, "Irradiance reflectivity of a flat ocean as a function of its optical properties," Appl. Opt. 12, 1549–1551,(1973).

[275]Haltrin. V. I., "One-parameter two-term Henyey-Greenstein phase function for light scattering in sea water," Applied Optics, 41, 1022-1028, (2002).

[276]Karp, A. H., Greenstadt, J., and J. A. Fillmore, Radiative Transfer through an Arbitrarily Thick, Scattering Atmosphere, J. Quant. Spectrosc. Radiat. Transfer, 24, pp. 391 – 406, (1980).

[276a] Karp, A. H., The recent advances in spherical-harmonics methods, Personal Communication, (1997).

[277]Karp, A. H., Computing the Angular Dependence of the Radiation of a Planetary Atmosphere, J. Quant. Spectrosc. Radiat. Transfer, 25, pp. 403 – 412, (1981).

[278]Benassi, M., Garcia, R. D. M., Karp, A. H., and Siewert, C. E., A High-Order Spherical harmonics Solution to the Standard Problem in Radiative Tranfer, Astrophys. J., 280, pp. 853 – 864, (1984).

[279]McCormick, N. J., and Siewert, C. E., Particular Solutions for the Radiative Transfer Equation, J. Quant. Spectrosc. Radiat. Transfer, 46, pp. 519 – 522, (1991).

[280]Barichello, L. B., Garcia, R. D. M., and Siewert, C. E., A Spherical Harmonics Solution for Radiative Transfer Problems with Reflecting Boundaries and Internal Sources. Preprint, (1996).

[281]Siewert, C. E., and McCormick, N. J., A Particular Solution for Polarization Calculations in Radiative Transfer, J. Quant. Spectrosc. Radiat. Transfer, 50, pp. 531 – 540, (1993).

[282]Siewert, C. E., and Thomas, J. R. Jr., On Coupled Conductive – Radiative Heat Transfer in a Sphere, J. Quant. Spectrosc. Radiat. Transfer, 45, pp. 63 – 72, (1991).

[283] Wells, W. H., and Sidorowich, J. J., Computational Techniques for Radiative Transfer by Spherical Harmonics, J. Quant. Spectrosc. Radiat. Transfer, 33, pp. 347 – 363, (1985).

[284]Takeuchi, Y., Use of Spherical Harmonics in the Solution of the Radiative Transfer Problem, J. Quant. Spectrosc. Radiat. Transfer, 39, pp. 237 – 245, (1988).

[285]Dave, J. V., A Direct Solution of the Spherical Harmonics Approximation to the Transfer Equation for a Plane Parallel, Non-homogeneous Atmosphere, J. Atmos. Sci., 32, (1975).

[286]Settle, J. J., Some Properties of an Integral that Occurs in Problems of Radiative Transfer, J. Quant. Spectrosc. Radiat. Transfer, 52, pp. 195 – 206, (1994).

[287]Gander, M. J., Numerical Evaluation of a Particular Integral. Unpublished, (January 1996).

[288]Gautschi, W., ORTHPOL – A Package of Routines for Generating Orthogonal Polynomials and Gauss-type Quadrature Rules, ACM Trans. Math. Software, 20 (1994).

[289]Dehesa, J. S., Zarzo, A., Yanez, R. J., Germano, B., and Ricci, P. E., Orthogonal Polynomials and Differential Equations in Neutron Transport and Radiative Transfer Theories, J. Computational and Applied Math., 50, pp. 197 – 206, (1994).

[290]Siewert, C. E., and McCormick, N. J., Some Identities for Chandrasekhar Polynomials. preprint, (1996).

[291]Garcia, R. D. M., and Siewert, C. E., On Computing the Chandrasekhar Polynomials in High Order and High Degree, J. Quant. Spectrosc. Radiat. Transfer, 43, pp. 201 – 205, (1990).

[292]Siewert, C. E., A Spherical Harmonics Method for Multi-group or Non-gray Radiation Transport, J. Quant. Spectrosc. Radiat. Transfer, 49, pp. 95 – 106, (1993).

[293]Garcia, R. D. M., and Siewert, C. E., A Generalized Spherical Harmonics Solution for Radiative Tranfer Models the Include Polarization Effects, J. Quant. Spectrosc. Radiat. Transfer, 36, pp. 401 – 423, (1986).

[294]Li, W., and Tong, W., Radiative Heat Transfer in Isothermal Spherical Media, J. Quant. Spectrosc. Radiat. Transfer, 43, pp. 239 – 251, (1990).

[295]Siewert, C. E., and J. R. Thomas, Jr., A Particular Solution for the $\mathbf{P_n}$ Method in Spherical Geometry, J. Quant. Spectrosc. Radiat. Transfer, 46, pp. 459 – 462, (1991).

[296]Tine, S., Aiello, S., Bellini, A., and Pestellini, C. C., A Solution to the Problem of Radiative Transfer in Inhomogeneous Media Using the SHM, J. Quant. Spectrosc. Radiat. Transfer, 47, pp. 95 – 102, (1992).

[297]Evans, K. F., Two Dimensional Radiative Transfer in Cloudy Atmospheres: The Spherical Harmonic Spatial Grid Method, J. Atmos. Sci., 50, pp. 3111 – 3124, (1993).

[298]Evans, K. F., The Spherical Harmonics Discrete Ordinate Method for Three Dimensional Atmospheric Radiative Transfer. Submitted to J. Atmos. Sci., (July 1996).

[299]Gordon, H., Brown, O., and Jacobs, M., "Computed relationships between the inherent and apparent optical properties of a flat homogeneous ocean," Appl. Opt. 14, 417–427, (1975).

[300]Gordon, H., "A bio-optical model describing the distribution of irradiance at the sea surface resulting from a point source embedded in the ocean," Appl. Opt. 26, 4133–4148, (1987).

[301]Gordon, H., "Can the Lambert-Beer law be applied to the diffuse attenuation coefficient of ocean water?" Limnol. Oceanogr. 34, 1389–1409, (1989).

[302]Gordon, H., "Diffuse reflectance of the ocean: influence of non-uniform phytoplankton pigment profile," Appl. Opt. 31, 2116–2129, (1992).

[303]Ge, Y., Gordon, H., and Voss, K., "Simulation of inelastic-scattering contributions to the irradiance field in the oceanic variation in Fraunhofer line depths," Appl. Opt. 32, 4028–4036, (1993).

[304]Cox, C., and Munk, M., "Statistics of the sea surface derived from sun glitter," J. Mar. Res. 13, 198–227, (1954).

[305]Elterman, L., "UV, visible, and IR attenuation for altitudes to 50 km," Rep. AFCRL-68-0153 (U. S. Air Force Cambridge Research Laboratory, Bedford, Mass.), (1968).

[306]Mie, G., "Beitrage zur Optik truber Medien, speziell Kolloidalen Metall-Losungen," Ann. Phys. 25, 377 – 445, (1908).

[307]Deirmendjian, D., "Scattering and polarization properties of polydisperse suspensions with partial absorption," in Electromagnetic Scattering, M. Kerker, ed. (Pergamon, New York), pp. 171 – 189, (1963).

[308]Kattawar, G and Xu, X., "Filling-in of Fraunhofer lines in the ocean by Raman scattering," Appl. Opt. 31, 1055–1065, (1992).

[309 Kattawar, G., and Adams, C., "Errors in radiance calculations induced by using scalar rather than Stokes vector theory in a realistic atmosphere-ocean system," in Ocean Optics X, R. W. Spinrad, ed., Proc. Soc. Photo-Opt. Instrum. Eng. 1302, 2 – 2, (1990).

[310]Adams, C., and Kattawar, G., "Effect of volume scattering function on the errors induced when polarization is neglected in radiance calculations in an atmosphere-ocean system," Appl. Opt. 20, 4610–4617, (1993).

[311]Kattawar, G., and Adams, C., "Errors induced when polarization is neglected in radiance calculations for an atmosphere-ocean system," in Optics of the Air-Sea Interface: Theory and Measurement, L. Estep, ed., Proc. Soc. Photo-Opt. Instrum. Eng. 1749, 2 – 22, (1992).

[312]Morel, A., and Gentili, B., "Diffuse reflectance of oceanic waters II. Bidirectional aspects," Appl. Opt. 32, 6864 – 6879, (1993).

[313]Tanre, D., Herman, M., Deschamps, R., and deLeffe, A., "Atmospheric modeling for space measurements of ground reflectances including bidirectional properties," Appl. Opt. 18, 3587 – 3594, (1979).

[314]Baker, K., and Frouin, R., "Relation between photosynthetically available radiation and total insolation at the ocean surface under clear skies," Limnol. Oceanogr. 32, 1370 – 1377, (1987).

[315]Gordon, H., and Castano, D., "Coastal zone color scanner atmospheric correction algorithm: multiple scattering effects," Appl. Opt. 26, 2111, (1987).

[316]Blattern, W., Horak, H., Collins, D., and Wells, M., "Monte Carlo studies of the sky radiation at twilight," Appl. Opt. 13, 534, (1974).

[317]Stavn, R., and Weidemann, A., "Optical modeling of clear oceanlight fields: Raman scattering effects," Appl. Opt. 27, 4002 – 4011, (1988).

[318]Stavn, R., and Weidemann, A., "Raman scattering in ocean optics: quantitative assessment of internal radiant emission," Appl. Opt. 31, 1294–1303, (1992).

[319]Brine, D., and Iqbal, M., "Diffuse and global solar spectral irradiance under cloudless skies," Sol. Energy 30, 447–453, (1983).

[320]Harrison, A., Coombes, C., "Angular distribution of clear sky short wavelength radiance," Sol. Energy 40, 57–69, (1988).

[321]Bukata, R. P., Jerome, J. H., Kondratyev, K. Ya., and Pozdnyakov, D. V.: Optical Properties and Remote Sensing of Inland and Coastal Waters CRC Press, Boca Raton, (1995).

[322]Gordon, H. R.: Inverse methods in hydrologic optics. Oceanologia 44, 9 – 58, (2002).

[323]Kirk, J.T.O.: Light & Photosynthesis in Aquatic Ecosystems, 2nd ed. Cambridge, Cambridge, (1994).

[324]Petzold, T. J.,: Volume scattering functions for selected ocean waters, SIO Ref. 71 – 78, Scripps Institution of Oceanography, San Diego, Calif. (1972).

[325]Walker, R. E.: Marine Light Field Statistics, Wiley, New York, (1994).

[326]Shettle, E. P., and Weinman, J. A., "The transfer of solar irradiance through inhomogeneous turbid atmospheres evaluated by Eddington's approximation," J. Atmos. Sci., 27, 1048 – 55, (1970).

[327]Toon, O. B., McKay, C. P., Ackerman, T. P., and Santhanam, K., "Rapid calculation of radiative heating rates and photo dissociation rates in inhomogeneous multiple scattering atmospheres," J. Geophys. Res., 94, 16287 – 301, (1989).

[328]Kylling, A., Stamnes, K., and Tsay, S. C., "A reliable and efficient two-stream algorithm for radiative transfer: Documentation of accuracy in realistic layered media," J. Atmos. Chem., 21, 115 – 50, (1995).

[329]Meador, W. E. and Weaver, W. R., "Two-stream approximations to radiative transfer in planetary atmospheres: A unified description," J. Atmos. Sci., 37, 630 – 43, (1980).

[330]Zdunkowski, W. G., Welch, R. M., and Korb, G., "An investigation of the structure of typical two-stream methods for the calculation of solar fluxes and heating rates in clouds," Contrib. Atmos. Phys., 53, 147 – 66, (1980).

[331]King, M. D., and Harshvardhan, "Comparative accuracy of selected multiple scattering approximations," J. Atmos. Sci., 43, 784 – 801, (1986).

[332]Harshvardhan and King, M. D., "Comparative accuracy of diffuse radiative properties computed using selected multiple scattering approximations," J. Atmos. Sci., 50, 247 – 59, (1993).

[333]Stokes, G., "On the intensity of the light reflected from or transmitted through a pile of plates," Proc. R. Soc. London, 11, 545 – 56, (1862).

[334] Twomey, S., Jacobowitz, H., and Howell, J., "Matrix methods for multiple scattering problems," J. Atoms. Sci., 23, 289 – 96, (1966).

[335]Van de Hulst, H. C., and Grossman, K., "Multiple light scattering in planetary atmospheres," in The Atmospheres of Venus and Mars, edited by J. C. Brandt and M. V. McElroy, Gordon and Breach, New York, (1968).

[336]Hunt, G. E., and Grant, I. P., "Discrete space theory of radiative transfer and its application to problems in planetary atmospheres," J. Atmos. Sci., 26, 963 – 72, (1969).

[337]Wiscombe, W. J., "Atmospheric radiation: 1975–1983," Rev. Geophys., 21, 997 – 1021, (1983).

[338]Feautrier, P., "Sur la resolution numerique de 1'equation de transfert," C. R. Acad. Sci Paris, 258, 3189 – 91, (1964).

[339]Mihalas, D., "The computation of radiation transport using Feautrier variables, 1, Static media," J. Comput. Phys., 57, 1 – 25, (1985).

[340]Kalkofen, W. (ed.), Methods in Transfer Problems, Cambridge University Press, New York, (1984).

[341]Cheyney, H., and Arking, A., "A new formulation for anisotropic radiative transfer problems, I, Solution with a variational technique," Astrophys. J., 207, 808 – 19, (1976).

[342] Anderson, D. E., "The troposphere-stratosphere radiation field at twilight: A spherical model." Planet. Space Sci., 31, 1517 – 23, (1983).

[343]Rybicki, G. B., and Hummer, D. G., Astrophys. J. 150, 607, (1967).

[344]Strickland, D. J., "The transport of the resonance radiation in a non-isothermal medium: The effect of a varying Doppler width," J. Geophys. Res., 84, 5890 – 96, (1979).

[345]Strickland, D. J., and Donahue, T. M., "Excitation and radiative transport of OI 1304 Å radiation, 1, the dayglow," Planet. Space Sci., 18, 661 – 89, (1970).

[346]Chandrasekhar, S., Radiative Transfer. Dover, 393 pp, (1960).

[347]Dera, J., Marine Physics, Elsevier, Amsterdam, (1992).

[348]Preisendorfer, R. W., Direct and inverse irradiance models in hydrologic optics. Limnol. Oceanogr. 29: 903 – 929, (1984).

[349]Prieur, L., Transfert radiatif dans les eaux de mer. D. S. thesis, 3rd cycle. Univ. Pierre-et-Marie-Curie, Paris. 243 p., (1976).

[350]Tyler, J. E., and Smith, R. C., Measurements of spectral irradiance underwater, Gordon and Breach, (1970).

[351]Spitzer, D., and Wernand, M. R., In situ measurements of absorption spectra in the sea, Deep-Sea Res. 28: 165 – 174, (1981).

[352]Preisendorfer, R. W., Empirical determinations of the local and global optical properties of irradiance fields in the sea, p. 56 – 59, In Ocean Optics 6. Proc. Soc. Photo-opt. Instr. Eng. (SPIE), V. 208, (1979).

[353]Kishino, M., Sugihara, S., and Okami, N., Influence of fluorescence of chlorophyll **a** on underwater upward irradiance spectrum. La Mer 22: 224 – 232, (1984a).

[354]Spitzer, D., and Wernand, M. R., Multi-spectral remote sensing of fluorescent tracers: Theory and experiments. Oceanol. Acta 6: 201 – 210, (1983).

[355]Dirks, R. W. J., and Spitzer, D., On the radiative transfer in the sea, including fluorescence and stratification effects, Limnol. Oceanogr, 32(4), 942 – 953, (1987).

[356]Haltrin, V. I., "Theoretical and empirical phase functions for Monte Carlo calculations of light scattering in seawater," in Proceedings of the Fourth International Conference Remote Sensing for Marine and Coastal Environments, I, pp.509-518, (Envir. Res. Inst. Of Michigan, Ann Aebor, MI, USA, (1997).

[357]Kattawar, G. W., "A three-parameter analytic phase function for multiple scattering calculations," J. Quant. Spectrose. Radiat. Transfer 15, 839 – 849, (1975).

[358]Timofeyeva. V. A., "Relation between light-fieldparameters and between scattering phase function characteristics of turbid media, including seawater,"Izv. Acad. Sci. USSR Atmos. Ocean Phys. 14, 843 – 848, (1978)

[359]Fournier, G. R., and Forand, J. L., "Analytic phase function for ocean water," SPIE, Vol. 2258, Ocean Optics XII, J. S. Jaffe, ed., pp. 194 – 201, (1994).

[360]Born, M., and Wolf, E., Principles of optics. Pergamon, (1975).

[361]Mobley, C., and Preisendorfer, R., "A numerical model for the computation of radiance distributions in natural waters with wind-roughened surfaces," NOAA Tech. Memo. ERL PMEL – 75 (NTIS PB88- 192703) (Pacific Marine Environmental Laboratory, Seattle, Wash.), (1988).

[362]Fireland, W. A., "The selection of discrete ordinate quadrature sets for anisotropic scattering," American Society of Mechanical Engineers, Heat Transfer Division 72, 89–96, (1991).

[363]Lathrop, K. D., "Spatial differencing of the transport equation: positive vs. accuracy," J. Comput. Phys. 4, 475–498, (1968).

[364]Lathrop, K. D., and Brinkley, F. W., "TWOTRAN-II code," Los Alamos Scientific Laboratory Rep. LA-4848-MS _Los Alamos Scientific Laboratory, Los Alamos, N. Mex., (1973).

[365]Wiscombe, W. U., NCAR Tech. Note NCAR/TN-121+STR, (1979).

[366] Lee, W. L., and Liou, K. N., A Coupled Atmosphere-Ocean Radiative Transfer System Using the Analytic Four-Stream Approximation Department of Atmospheric and oceanic Sciences, University of California, Los Angeles, Los Angeles, California, (2007)

[367]Chami, M., Importance of the polarization in the retrieval of oceanic constituents from the remote sensing reflectance, Journal of Geophysical Research. Vol. 112, C05026, doi: 10. 1029 / 2006 JC003843, (2007).

[368]Gordon, H. A., "Dependence of the diffuse reflectance of natural waters on the sun angle", Limnol, Oceanogr.34, 1484-1489, (1989).

[369]Haltrin, V. I., Analytical one-parameter sea water light scattering phase function with integral properties of experimental phase functions, Proceedings of the II International Conference "Current Problems in Optics of Natural Waters,"ONW'2003, Eds. Iosif Levin and Gary Gilbert, St. Petersburg, Russia, pp. 383, (2003).

[370]Morel, A., and Gentili, B., Diffuse reflectance of oceanic waters, III, Implication of bidirectionality for the remote sensing problem, Appl. Opt, 35, 4850-4862.(1996).

[371]Haltrin, V. I., An analytic Fournier Forand scattering phase function was derived by Haltrin as an alternative to the HG phase functions in hydrologic optics, IEEE International Geoscience and Remote Sensing Symposium Proceedings, Vol. II, T.I.Sein, Ed., IEEE cat. 98CH36174, Libr. Of Congr. 97-80873, IEEE, Picataway, NJ, USA, (1998).

[372]Jin, Z., Charlock, T. P., and Rutledge, K., Analysis of broadband solar radiation and albedo over the ocean surface at COVE. J. Atmos. Oceanic Technol., 19, 1585–1601, (2002).

[373]L. B. Barichello, R. D. M. Garcia and C. E. Siewert, "Particular Solutions for the Discrete-Ordinates Method," Journal of Quantitative Spectroscopy and Radiative Transfer, 64, 219-226, (2000).

[374]Siewert, C. E., "A Concise and Accurate Solution for a Polarization Model in Radiative Transfer," Journal of Quantitative Spectroscopy and Radiative Transfer, 62, 677-684, (1999).

[375]Barichello, L. B., and Siewert, C. E., "A New Version of the Discrete-Ordinates Method," Proceedings: Computational Heat and Mass Transfer-2001, Rio de Janeiro, I, 340 – 347, (2002).

[376]Wauben, W. M. F., and Hovenier, J. W., JQSRT, 47: 491, (1992).

[377]Sobolev. V., Soviet Astron. 1, 678, (1957).

[378]Li, J, Scinocca, J, Lazare. M., McFarland, N., Von Salzen and Solhelnm. L; Ocean surface albedo and its impact on radiation balance in climate models; Canadian centre for climate modeling and analysis, Meteorological service of Canada, Victoria, British Columbia, Canada, (2006).