CHAPTER - IV

FLOW OF VISCOUS FLUID THROUGH CIRCULAR PIPE

PART-ONE

NOTE ON THE FLOW OF AN INCORMPRESSIBLE VISCOUS FLUID BETWEEN TWO POROUS CONCENTRIC CIRCULAR CYLINDERS SUBJECTED TO SUCTION OR INJECTION

Introduction

Surya Prakash (1961) has considered the periodic flow in annulus of two porous coaxial circular cylinders for ordinary viscous incompressible fluid. In 1964, Devi Singh has discussed the motion of a visco-elastic Maxwell fluid through two concentric circular cylinders under the presence of exponential pressure gradient. Singh (1967) has considered the motion of viscoelastic Maxwell fluid through two porous concentric circular cylinders. In his problem he has chosen the pressure gradient to be of the form $Ke^{-\alpha t} Cos\,\beta t$. In the year 1970 Gupta and Kulshrestha has studied the slow steady flow of a viscous liquid in an annulus with arbitrary suction and injection along the rough wall. In 1992 Usha Singh and G.C. Sharma has studied three dimensional MHD flow in a porous media with pressure gradient and fluid injection.

In the present note, we have considered the flow of viscous incompressible fluid through two porous concentric circular cylinders subjected to suction or injection under the influence of pressure gradient which is a function of time alone. The general solution of the problem is obtained by using Laplace Transforms and it is believed that the general solution for this problem has not been done by any investigators. By putting Suction parameter zero, we directly obtained the flow of viscous incompressible fluid through coaxial circular cylinders under the presence of pressure gradient which are functions of time.

Formulation of the Problem

Consider the flow of a viscous incompressible fluid in the annular region between two infinite concentric circular cylinders of radii a and b respectively (b > a) under the influence of pressure gradient.

Using the cylindrical polar co-ordinate (r, θ, z), let us take u, v, w to be the components of velocity in the direction of increasing r, θ and z respectively.

Nature of motion gives v=0 and since it is symmetrical about the axis of z, we have

$$\frac{\partial}{\partial \theta} = 0$$

The Navier-Stokes equations of motion for incompressible viscous liquid without body forces become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) \tag{1}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{r\partial \theta} \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3}$$

and the equation of continuity is $\quad \frac{\partial}{\partial r}(ur) = 0 \quad$ (4)

i.e. ru = constant = - S (say) (5)

where s>0is the suction parameter and s < 0 is the injection parameter. Making substitution of the value of u from equation (5) in equation (1) and (3) we have

$$\frac{\partial p}{\partial r} = \frac{\rho S^2}{r^3} \tag{6}$$

$$\frac{\partial w}{\partial t} - \frac{s}{r}\frac{\partial w}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{7}$$

Let us take the motion to be due to the pressure gradient which is a function of time alone

i.e.
$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) \tag{8}$$

Equation (7) and (8) give

$$p = k - \frac{pS^2}{2r^2} - \rho z f(t) \tag{9}$$

the value of k is determined from a known value of p at a point (r, θ, z)

Making substitution from equation (8) in equation (7) we have

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = -\frac{1}{\nu} f(t) \quad - \frac{s}{\nu r}.\frac{\partial w}{\partial r} + \frac{1}{\nu}\frac{\partial w}{\partial t} \tag{10}$$

The boundary conditions for the problem are

$$\left.\begin{array}{l} w = 0 \quad \text{on} \quad r = a, t \geq 0 \\ w = 0 \quad \text{on} \quad r = b, t \geq 0 \end{array}\right\} \tag{11}$$

Solution of the Problem

We introduce the Laplace Transform defined by

$$\overline{w}(r, \lambda) = \int_0^\infty e^{-\lambda t}\, w(r, t)\, dt, \; Re(\lambda) > 0 \tag{12}$$

Subsidiary equation corresponding to (10) becomes

$$\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\left(1+\frac{s}{\nu}\right)\frac{d\bar{w}}{dr}-\frac{\lambda}{\nu}\bar{w}=-\frac{1}{\nu}\bar{f}(\lambda) \qquad (13)$$

Writing $\bar{w}-\frac{1}{\lambda}\bar{f}(\lambda)=\bar{F}$, $\quad \frac{s}{v}=2m$ and $-\frac{\lambda}{v}=n^2$ (14)

we get

$$\frac{d^2\bar{F}}{dr^2}+\frac{1}{r}(1+2m)\frac{d\bar{F}}{dr}+n^2\bar{F}=0 \qquad (15)$$

The boundary conditions reduce to

$$\left.\begin{array}{l}\bar{w}=0 \text{ on } r=a\\ \bar{w}=0 \text{ on } r=b\end{array}\right\} \qquad (16)$$

The solution of equation (15) is (when m is an integer)

$$\bar{F}=\frac{1}{r^m}\left[AJ_m(nr)+BY_m(nr)\right]$$

i.e.

$$\bar{W}=\frac{1}{\lambda}\bar{f}(\lambda)+\frac{1}{r^m}\left[AJ_m(nr)+BY_m(nr)\right] \qquad (17)$$

Where A and B are given by the equations

$$0=\frac{1}{\lambda}\bar{f}(\lambda)+\frac{1}{a^m}\left[AJ_m(na)+BY_m(na)\right] \qquad (18a)$$

$$0=\frac{1}{\lambda}\bar{f}(\lambda)+\frac{1}{b^m}\left[AJ_m(bn)+BY_m(nb)\right] \qquad (18b)$$

solving we get

$$A=-\frac{\bar{f}(\lambda)}{\lambda}\left[\frac{a^m Y_m(nb)-b^m Y_m(na)}{T_m(n,a,b)}\right] \qquad (19)$$

$$B=-\frac{\bar{f}(\lambda)}{\lambda}\left[\frac{a^m J_m(nb)-b^m J_m(na)}{T_m(n,a,b)}\right] \qquad (20)$$

Making substitution the values of A and B from (19) and (20) into the equation (17) we get, when m is an integer

$$\overline{W} = -\frac{\bar{f}(\lambda)}{\lambda}\left[\frac{a^m T_m(n,r,b) + b^m T_m(n,a,r)}{r^m T_m(n,a,b)}\right] \tag{21}$$

where

$$T_m(n,x,y) = J_m(nx)\ Y_m(ny) - Y_m(nx)\ J_m(ny)$$

when m is not an integer, we have

$$\overline{W} = -\frac{\bar{f}(\lambda)}{\lambda}\left[\frac{a^m T^1{}_m(n,r,b) + b^m T^1{}_m(n,a,r)}{r^m T^1{}_m(n,a,b)} - 1\right] \tag{22}$$

where

$$T'_m(n,x,y) = J_m(nx)\ J_{-m}(ny) - J_{-m}(nx)\ J_m(ny)$$

Now applying theorem of inversion on equation (21) we get,

$$W = -\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{e^{\lambda t}\bar{f}(\lambda)}{\lambda}\left[\frac{a^m T_m(n,r,b) + b^m T_m(n,a,r)}{r^m T_m(n,a,b)} - 1\right] d\lambda$$

(23)

where γ is impulsive pressure gradient greater than the real part of all singularities of the integrand in (23)

Case I

We choose impulsive pressure gradient defined by $f(t) = c\delta(t)$ where $\delta(t)$ is the **Dirac Delta** function

Then $\bar{f}(\lambda) = c\int_0^\infty e^{-\lambda t}\,\delta(t)\,dt = c$ (24)

Making substitution the value of $\bar{f}(\lambda)$ from equation (24) into the equation (23) we get

$$W = -\frac{c}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda}\left[\frac{a^m T_m(n,r,b)+b^m T_m(n,a,r)}{r^m T_m(n,a,b)}-1\right]d\lambda \qquad (25)$$

The integrand in (25) is a single valued function of λ and has pole at $\lambda=0$ and the other poles are the zeros of $T_m(n, a, b)$. To find the zeros of $T_m(n,a,b)$ we put $\lambda=-\nu\beta^2$ in $T_m(n, a, b)$ then it becomes

$$T_m(n,a,b) = J_m(na)\ Y_m(nb) - J_m(nb)\ Y_m(na)$$

$$= J_m(a\beta)\ Y_m(b\beta) - Y_m(a\beta)\ J_m(b\beta)$$

Let $\beta_1\ \beta_2\ \beta_3 \ldots\ldots\ldots\ \beta_s \ldots\ldots\ldots\ldots$ be the roots of the equation

$$J_m(a\beta)\ Y_m(b\beta) - Y_m(a\beta)\ J_m(b\beta) = 0 \qquad (26)$$

Then $\lambda = -\nu\beta_1^{\,2},\ -\nu\beta_2^{\,2}, -\nu\beta_3^{\,2}, \ldots\ldots -\nu\beta_s^{\,2} \ldots\ldots$ are the zeroes of $T_m(n,a,b)$ which are simple poles of the integrand in (25).

Since β_s is a root of (26) we have

$$\frac{J_m(a\beta_s)}{J_m(b\beta_s)} = \frac{Y_m(a\beta_s)}{Y_m(b\beta_s)} = K \quad \text{say,}$$

Then

$$\frac{d}{d\beta}T_m(\beta,a,b)_{\beta=\beta_s} = \frac{2}{\pi\beta_s}\frac{J^2{}_m(a\beta_s) - J^2{}_m(b\beta_s)}{J_m(a\beta_s)\ J_m(b\beta_s)} \quad \text{[Watson]}$$

Applying Cauchy's residue theorem to equation (25) we have

$$W = -2c\sum_{s=1}^{\infty}\frac{e^{-\nu\beta_s^2 t}}{\beta_s}\left[\frac{a^m T_m(\beta_s,r,b)+b^m T_m(\beta_s,a,r)}{r^2\frac{d}{d\beta}T_m(\beta,a,b)_{\beta=\beta_s}}+1-1\right]$$

$$= -\pi c\sum_{s=1}^{\infty}e^{-\nu\beta_s^2 t}\left[\frac{a^m T_m(\beta_s,r,b)+b^m T_m(\beta_s,a,r)}{r^m[J^2{}_m(a\beta_s)-J^2{}_m(b\beta_s)]}\times J_m(a\beta_s)\ J_m(b\beta_s)\right] \quad (27)$$

Putting m = 0 in equation (27), we get the flow of viscous incompressible fluid between two impervious circular cylinders under the presence of impulsive pressure gradient.

Thus

$$W = -\pi c \sum_{s=1}^{\infty} e^{-\nu\beta_s^2 t} \frac{J_0(a\beta_s)\, J_0(b\beta_s)}{J_0^2(a\beta_s) - J_0^2(b\beta_s)} \times$$

$$\{J_0(r\beta_s)[Y_0(b\beta_s) - Y_0(a\beta_s)] - Y_0(r\beta_s)[J_0(b\beta_s) - J_0(a\beta_s)]\} \qquad (28)$$

From equation (27) and (28) it follows that motion dies out in both the cases as t approaches infinity.

Case – II

In case of periodic pressure gradient we choose $f(t) = c\, Cos\, \omega t$

For our simplicity we use the complex notation $f(t) = \text{Re}\,C$.

Therefore $\bar{f}(\lambda) = \text{Re}\,C \int_0^{\infty} e^{-\lambda t}\, e^{i\omega t}\, dt$

$$= \text{Re}\frac{C}{\lambda - i\omega} \qquad \text{.........................} (29)$$

Making substitution from (29) in (23) we have,

$$w = -\text{Re}\frac{C}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{e^{\lambda t}}{\lambda(\lambda - i\omega)}\left[\frac{\omega^m T_m(n,r,b) + b^m T_m(n,a,r)}{r^m T_m(n,a,b)} - 1\right]\lambda d$$

$$= \text{Re}\, C\pi \sum_{s=1}^{\infty} \frac{e^{-\upsilon\beta_s^2 b}}{\upsilon\beta_s^2 + i\omega} \frac{a^m T_m(\beta_s, r, b) + b^m T_m(\beta_s, a, r)}{r^m\left[J_m^2(a\beta_s) - J_m^2(b\beta_s)\right]} \times J_m(a\beta_s) J_m(b\beta_s)$$

$$+ R_e \frac{iC}{\omega} e^{i\omega t}\left[\frac{a^m T_m(\alpha, r, b) + b^m T_m(\alpha, a, r)}{r^m T_m(\alpha, a, b)} - 1\right]$$

Where $\alpha = \sqrt{\frac{-i\omega}{\nu}}$ .... (30)

Now

$$w=\frac{ic}{\omega}e^{i\omega t}\left[\frac{a^{m}T_{m}(\alpha,r,b)+b^{m}T_{m}(\alpha,a,r)}{r^{m}T_{m}\,(\alpha,a,b)}-1\right] \qquad \text{............ (31)}$$

is a particular solution for this problem and this solution is derived by Surja Prakash.

Putting m = 0 in (30), we get the flow of viscous incompressible fluid between two co-axial circular cylinders due to the presence periodic pressure gradient.

Thus we have

$$w=\operatorname{Re}C\pi\sum_{s=1}^{\infty}\frac{e^{-\upsilon\beta_s^2 t}}{\upsilon\beta_s^2+i\omega}\cdot\frac{J_0(r\beta_s)[Y_0(b\beta_s)-Y_0(a\beta_s)]-Y_0(r\beta_s[J_0(b\beta_s)-J_0(a\beta_s)])}{J_0^2(a\beta_s)-J_0^2(b\beta_s)}\times J_0(a\beta_s)\,J_0(b\beta_s)$$

$$+\operatorname{Re}\frac{ic}{\omega}e^{i\omega t}\left[\frac{J_0(\alpha,r)[Y_0(b\alpha)-Y_0(a\alpha)]-Y_0(ra)[J_0(b\alpha)-J_0(a\alpha)]}{J_0(\alpha a)\,Y_0(\alpha b)-Y_0(\alpha a)\,J_0(\alpha b)}-1\right] \qquad \text{...... (32)}$$

Where $\alpha=\sqrt{\dfrac{i\omega}{\upsilon}}$

Case III

If we take $f(t)=c(1-e^{-\alpha t})$

Then $\bar{f}\,(\lambda)=\dfrac{c\,\alpha}{\lambda(\lambda+\alpha)}$ ....(33)

making substitution of the value of $\bar{f}\,(\lambda)$ in (23) we get

$$W=-\frac{c\,\alpha}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda^2(\lambda+\alpha)}\left[\frac{a^m\,T_m(n,r,b)+b^m\,T_m(n,a,r)}{a^m\,T_m\,(n,a,b)}-1\right]d\lambda \qquad \text{........(34)}$$

$$= -\pi c\alpha\left[\sum_{s=1}^{\infty}\frac{e^{-\nu\beta^2 t}}{\nu\beta_s^{\,2}(\alpha-\nu\beta_s^{\,2})}\,\frac{a^m T_m(\beta_s, r, b)+b^m T_m(\beta_s, a, r)}{J^2{}_m(a\beta_s)-J^2{}_m(b\beta_s)]}\times J_m(a\beta_s)\,J_m(b\beta_s)\right]$$

$$+\frac{e^{-\alpha t}}{\alpha^2}\left\{\frac{a^m T_m(2,r,b)+a^m T_m(2,a,r)}{r^m T_m(2,a,b)}\right\}$$

$$+\frac{1}{4\nu\alpha(m+1)}\left\{r^2+\frac{(ab)^{2m}(b^2-a^2)}{(b^{2m}-a^{2m})r^{2m}}-\frac{b^{2m+2}-a^{2m+2})}{b^{2m}-a^{2m}}\right\} \quad ....(35)$$

It is unsuitable to explain the physical feature due to the presence of Bessel function. To avoid it we study the behaviour of W for large viscosity. For large viscosity we assume $\frac{1}{\nu}<<1$, when expanding Bessel function in power series and retaining upto these terms containing 1/v, we find that the equation (35) reduces to the following form

$$W\Big]_{l\arg e\ \nu} = -c\alpha\left[\frac{-e^{-\alpha t}}{4\nu\alpha(m+1)}\left\{r^2+\frac{(ab)^{2m}(b^2-a^2)}{r^{2m}(b^{2m}-a^{2m})}-\frac{b^{2m+2}-a^{2m+2})}{b^{2m}-a^{2m}}\right\}\right.$$

$$\left.+\frac{1}{4\nu\alpha(m+1)}\left\{r^2-\frac{(ab)^{2m}(b^2-a^2)}{r^{2m}(b^{2m}-a^{2m})}-\frac{b^{2m+2}-a^{2m+2})}{b^{2m}-a^{2m}}\right\}\right]$$

$$=\frac{1}{4\mu(m+1)}\left[\frac{\partial p}{\partial z}\left\{r^2+\frac{(ab)^{2m}(b^2-a^2)}{a^{2m}(b^{2m}-a^{2m})}-\frac{b^{2m+2}-a^{2m+2})}{b^{2m}-a^{2m}}\right\}\right] \quad ....(36)$$

as $t\to\infty$, the expression for velocity distribution for large viscous fluid and that for ordinary viscous fluid becomes equal and are represented by the equation

$$W = -\frac{c}{4\nu(m+1)}\left[r^2+\frac{(ab)^{2m}(b^2-a^2)}{r^{2m}(b^{2m}-a^{2m})}-\frac{b^{2m+2}-a^{2m+2})}{b^{2m}-a^{2m}}\right] \quad ......(37)$$

when there is no suction or injection this reduces to

$$W = -\frac{c}{4\nu}\left[a^2 - r^2 + (b^2 - a^2)\frac{\log\frac{r}{a}}{\log\frac{b}{a}}\right]$$

which is flow between two non-porous co-axial cylinders under constant pressure gradient.

Discussion of the result :

When there is no suction or injection for large t, the distribution of velocity with the variation of r $(a \le r \le b)$ is parabolic in nature. Maximum velocity occurs at the region where $r = \frac{b-a}{2}$. But when there is suction, the velocity near the inner cylinder is greater than that when there is no suction and smaller than that near the outer cylinder and in case of injection, the velocity near the inner cylinder is smaller than that when there is no suction and greater than that near the outer cylinder as depicted from the *figure 1.*

r	w/c (m=1)	w/c (m=2)	w/c(m=0)	w/c(m=-2)
1	0	0	0	0
1.1	0.61	0.68	0.51	0.3
1.2	0.98	1.02	0.87	0.57
1.3	1.18	1.16	1.11	0.85
1.4	1.25	1.18	1.24	1.06
1.5	1.21	1.1	1.26	1.23
1.6	1.1	0.96	1.19	1.24
1.7	0.91	0.78	1.02	1.1
1.8	0.66	0.55	0.76	0.85
1.9	0.35	0.3	0.42	0.53
2	0	0	0	0

Distribution of velocity with the variation of r where $1 \le r \le 2$
Taking a=1, b=2, v=0.1

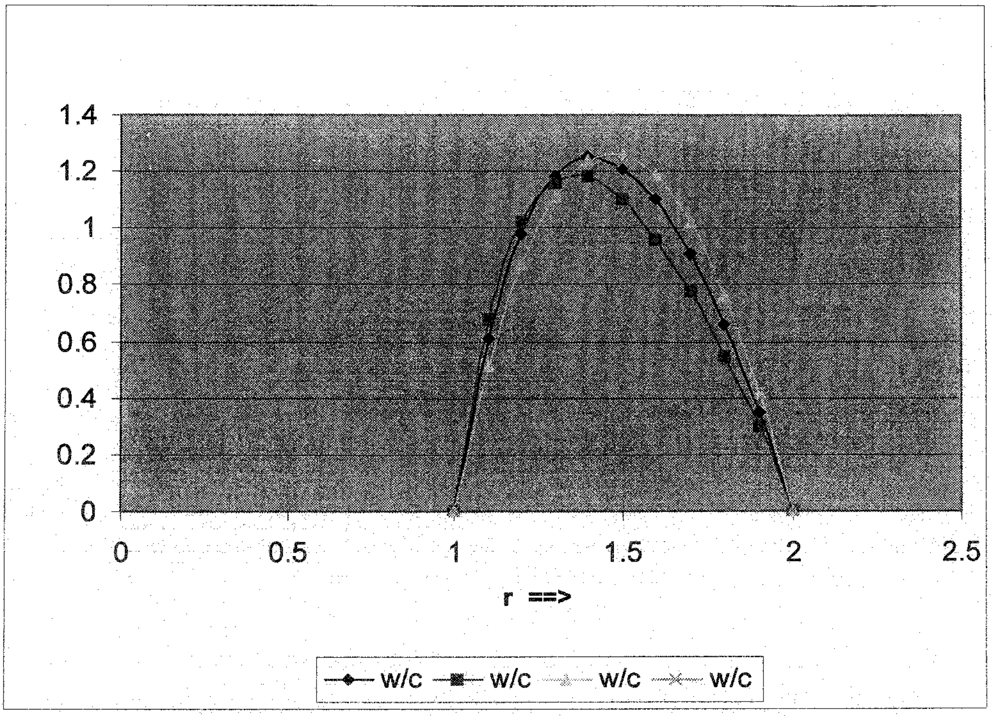

Fig. 1