Chapter 4

Paradox of Identically Accelerated Twins in Different Worlds and the Ordinary Twin Paradox

4.1 Introduction

As is evident from the title, in this chapter we shall discuss the desynchronization effect of the case of identically accelerated twins undergoing through the same chain of events in the relativistic and classical world. This is an extension of an earlier paper "Principle of Equivalence and the Twin Paradox" where a variation of the twin paradox is used to connect gravity with the the desynchronization effect. The paradox has been explained there using EP whose role is to provide a physical agent i.e. gravity which can supply the required extra aging to the traveller during the acceleration phase through a gravitational time offset effect. I should mention here that the paper "Principle of Equivalence and the Twin Paradox" has been reported by one of my collaborators, but still for the sake of completeness I reproduce the whole article as a background.

As discussed earlier in the canonical version of the twin paradox, of the two twins initially living on earth (assumed to be an inertial frame), one leaves the earth by a fast rocket to a distant star and then returns to meet her stay-at-home brother to discover that they age differently. This as such is not a paradox since the rocket-bound sibling, on account of her high velocity will suffer relativistic time dilation of her (biological) clock throughout her journey and will therefore return younger with respect to her brother. Indeed with respect to the inertial frame of the stay-at-home twin, the world lines of the twins in the Minkowski diagram are different (although from the description of the problem the end points of these lines i.e the time and the place of departure and that of their reunion, meet) and hence the asymmetry in the aging can be attributed to the fact that proper time is not integrable[1]. The paradox arises if one näively treats the perspectives of the twins symmetrically. For example if the traveller twin considers herself to remain stationary and relate the motion to her brother, she would (erroneously) expect her brother to stay younger by believing that the Lorentz transformation (LT) predicts reciprocal time dilation of moving clocks. Qualitatively the resolution lies in the observation that one of the twins is in an accelerated (non-inertial) frame of reference and hence the postulates of Special Relativity (SR) are not applicable to it and therefore the claim of reciprocity of time dilation between the frames of reference of the twins falls through. Indeed Einstein himself found this sort of argument preferable in dismissing the paradoxical element in the twin problem[2]. However this suggestion should not be construed as a statement that the resolution of the paradox falls outside the purview of SR . On the contrary much of the expositions found in the literature on the subject deal with the problem in the frame work of SR alone[1], although many tend to believe that the introduction of GR and a gravitational field at the point of acceleration is the right way to understand the asymmetry in the perspectives of the twins. Bohm notes in the context that " two clocks running at places of different gravitational potential will have different rates"[5]. This suggests that EP can directly be used to explain the asymmetry (difference between the experiences of the rocket-bound and the stay-at-home twin). However, as pointed out by Debs and Redhead[1] and also others[6], that since in the twin problems one deals with flat space-time, any reference of GR in this context is quite confusing.

Coming back to the issue of acceleration, one finds often that the direct role of acceleration of the rocket-bound twin in causing the differential aging has been much criticized although it is quite clear that in order to have twice intersecting trajectories of the twins (this is necessary since the clocks or ages of the twins have to be compared at the same space-time events) one cannot avoid acceleration.

[1]Very extensive treatment is available in Special Relativity Theory-Selected Reprints[3], (see also Ref.[4]). For newer expositions see for example Ref.[1] and references therein.

In an interesting article Gruber and Price[7] dispel the idea of any direct connection between acceleration and asymmetric aging by presenting a variation of the paradox where although one twin is subjected to undergo an arbitrarily large acceleration, no differential aging occurs. That the acceleration per se cannot play a role is also evident from the usual calculation of the age difference from the perspective of the inertial frame of the stay-at-home twin if one notes that the duration of the turn-around process of the rocket can be made arbitrarily small in comparison to that for the rest of the journey and hence the final age difference between the twins can then be understood in terms of the usual relativistic time dilation of the traveller twin during essentially the unaccelerated segment of her journey [2]. One is thus caught in an ambivalent situation that, on the one hand the acceleration does not play any role, on the other hand the paradox is not well posed unless there is a turn-around (acceleration) of the traveller twin[3].

In order to get out of this dichotomy it is enough to note that from the point of view of the traveller twin, the acceleration (or the change of reference frame in the abrupt turn-around scenario) is important. The consideration of this acceleration only has the ability to explain that the expectation of symmetrical time dilation of the stationary twin from the point of view of the rocket-bound twin is incorrect.

In an interesting paper A.Harpaz[10] tries to explain the twin paradox by calculating the age difference from the perspective of the traveller twin directly by applying EP i.e by introducing GSDC. From the previous discussions it may seem unnecessary (or even confusing) to invoke gravity in the essentially special rela-

[2]In such a calculation the time dilation is also calculated during the acceleration phase (assuming the clock hypothesis to be true[1]) and is shown to contribute arbitrarily small value in the age offset if the duration of the acceleration phase is assumed to tend to zero.

[3]Here we are considering the standard version of the paradox and the variation where the twins live in a cylindrical universe[8, 9] has been kept out of the present scope.

tivistic problem. However the fact is, Harpaz's approach apparently provides an alternate explanation for the differential aging from the traveller's perspective.

The author of the pedagogical article observes that although the special relativistic approach can correctly account for the age difference between the twins, " it does not manifest the 'physical agent' responsible for the creation of such a difference" [10]. It is held that EP provides such an agent and that is gravity. But how does gravity find way into the problem? Gravity enters through EP and its connection with the resolution of the paradox can symbolically be written as

$$Acceleration \stackrel{EP}{\rightarrow} Gravity \rightarrow Gravitational\ red\text{-}shift \rightarrow GSDC \rightarrow Extra\ aging$$

where the last item of the flow diagram indicates that with respect to the rocket-bound twin, GSDC provides the extra aging of the stay-at-home one, explaining the asymmetrical aging of the problem.

However while there is as such no harm in understanding the twin problem from a different perspective (here, this is in terms of GSDC), Harpaz's approach suffer from two fold conceptual difficulties which we will elaborate in the next section. These difficulties include the fact that the calculations are only approximate. The other difficulty will be seen to be of more fundamental in nature. The aim of the present study (reported in this chapter) is to remove these difficulties and give an *accurate* account of the asymmetric aging from the perspective of the rocket-bound twin directly in terms of a time-offset between the siblings which is introduced due to the pseudo-gravity experienced by the traveller twin.

4.2 GSDC and Extra Aging

In the standard version of the twin paradox the differential aging from the perspective of the stay-at-home (inertial) observer A can easily be calculated assuming that for the most parts of the journey of the traveller twin B, the motion remains uniform except that there is a turn-around acceleration of the rocket so that finally the siblings are able to meet and compare their ages. In the Minkowski diagram the whole scenario is characterized primarily by three events: (1) Meeting of the world lines of A and B when the voyage starts taking place. (2) the turn around of B and (3) meeting of the world lines when A and B reunite. For the paradox it is not necessary that at events (1) and (2), the relative velocity between A and B has to be zero, since ages or clocks can be compared at a point even if the observers are in relative motion, therefore the analysis of the problem can be done by considering the acceleration only during the turn-around. The duration of the acceleration phase can be considered to be arbitrarily small compared to the time it takes during its forward and return journeys and hence the age difference occurs due to the usual relativistic time dilation of a clock for its uniform motion. This is clearly given by

$$Age\ \ difference = 2t_A(1-\gamma_v^{-1}) \approx 2t_A\frac{v^2}{c^2}, \tag{4.1}$$

where $2t_A$ is the time the rocket takes for its entire journey (up and down) in uniform speed v and $\gamma = (1-\frac{v^2}{c^2})^{-1/2}$ is the usual Lorentz factor.

The paradox is resolved if one can show that B also predicts the same difference in spite of the fact that the time dilation effect is reciprocal. Clearly some new considerations (that were absent in arriving at Eq.(4.1)) must offset this reciprocal time dilation and also this must provide some extra aging to A from the point of view of B so that the age difference remains independent of the two perspectives. One of these new considerations, as has already been pointed out, is the one of a synchronization gap that B discovers due to her change of inertial frame during her entire voyage. This has been clearly demonstrated by Bondi[11] in the context of

Lord Hulsbery's three brother approach[1] to understanding the twin paradox.

The other way of understanding the same thing is the consideration of pseudo-gravity experienced by B because of its turn-around. In order to demonstrate how EP plays the role in the analysis, Harpaz uses the gravitational red-shift formula, which can be obtained heuristically (using the EP) as

$$\Delta\nu = \nu_0(1 + \frac{gh}{c^2}), \tag{4.2}$$

where g is the acceleration due to (pseudo) gravity and $\Delta\nu$ represents the change of frequency of light observed from a distance h from the source where the frequency of the same light is seen to be ν_0. Interpreting this red-shift effect in terms of GSDC, the formula can be written as

$$t_1 = t_2(1 + \frac{\Delta\Phi}{c^2}), \tag{4.3}$$

where t_1 and t_2 are times measured by clocks located at two points P_1 and P_2 (say) and $\Delta\Phi = gh$, is the potential difference between these points. It has been shown that with respect to B the acceleration plays a role by providing an extra time difference between B and A, because of the integrated effect of GSDC during the (arbitrarily) short duration of B's acceleration. This time difference more than offsets the age difference calculated by B solely assuming the reciprocal time dilation so much so that finally B ages less by the correct amount. As pointed out earlier there are two conceptual difficulties in understanding the treatment. First, in an effort to find a "physical agent" responsible for the extra aging, Harpaz relies on some approximate formulae including that of the gravitational red-shift because of his assumption, $v^2/c^2 \ll 1$ inherent in the analysis, and therefore, the pseudo-gravitational effect has the ability to resolve the paradox only approximately. Clearly there is no valid reason to make any such small velocity approximation for the problem. One might of course argue that for the author's stated purpose it would be enough to show that the " physical agent" i.e. gravity is at work when B's point of view is considered. However, it will be shown that such an argument would also not hold good and the reason for it concerns the second difficulty. The explanations based on SR relies on the fact that during the direction reversing acceleration, the travelling twin changes from one reference frame to another and the lack of simultaneity of one reference frame with respect to the other provides the "missing time" which constitutes the reason for the differential aging[1]. Now the lack of agreement in simultaneity is a special relativistic concept without any classical analogue, on the other hand in many standard heuristic derivations of the gravitational red-shift formula (see for example[12, 13, 14]) which is also followed by the author of Ref.[10], one finds that no reference to SR is made. Indeed the well-known formula for the red-shift parameter $Z = gh/c^2$ is only approximate and is derived by making use of the classical Doppler effect for light between the source of light and a detector placed at a distance h along the direction of acceleration g of an Einstein elevator[10]. According to EP an observer within the elevator will " attribute his observations in the elevator, to the existence of a uniform gravitational field in a rest system of reference"[10]. Thus the equivalence of gravity and acceleration in terms of gravitational red-shift or GSDC therefore turns out to be as if a purely classical (Newtonian) concept in this approximation! How then is GSDC able to account for an effect, viz. the lack of simultaneity which is essentially a standard relativistic phenomenon?

In the next section we will show that indeed the EP can explain the twin paradox exactly provided the connection of EP and GSDC is obtained using the full machinery of SR.

4.3 EP and the Gravitational Time Offset

S. P. Boughn [15] in an interesting article presented a paradox in SR with a great pedagogical power. This paradox put forth by Boughn involves twins who age differently although they experience equal amount of acceleration for the same time. According to Boughn's parable twins P and Q on board two identical rockets (with equal amount of fuel), initially at rest a distance x_0 apart in an inertial frame Σ_0, underwent identical accelerations for some time in the direction $\overrightarrow{PQ}$ (x-direction say) and eventually came to rest (when all their fuel had expended) in another inertial frame Σ moving with velocity v relative to Σ_0. Using Lorentz transformation (LT) Boughn obtained a counter-intuitive result that after the completion of the acceleration phase, the age of the forwardly placed (with respect to the direction of acceleration) twin Q turned out to be more than that of her brother P!

The result is counter-intuitive owing to the fact that the twins of the parable throughout had identical local experiences yet their presynchronized (biological) clocks went out of synchrony. The amount of this desynchronization or the age difference comes out to be

$$\delta t_{desync} = -\frac{\gamma_v v x_0}{c^2}. \tag{4.4}$$

where the Lorentz factor

$$\gamma_v = (1 - v^2/c^2)^{-1/2} \tag{4.5}$$

and c is the speed of light in free space.

The above result can be obtained from the simple application of LT:

$$t_k = \gamma_v(t_{0k} - \frac{v x_{0k}}{c^2}), \tag{4.6}$$

where t_{0k} and x_{0k} denote the time and space coordinates of the observer k (k stands for P or Q) with respect to their first inertial frame Σ_0 and t_k is the time in their final frame Σ where they settle stationary after the acceleration phase is over.

From Eq.(4.6) it follows that

$$t_Q - t_P = \gamma_v[(t_{0Q} - t_{0P}) - v(x_{0Q} - x_{0P})/c^2]. \tag{4.7}$$

Since both the twins had identical experiences the ages of the twins with respect to the frame of their mom and dad, Σ_0 must be the same throughout their journey, even after the completion of their acceleration phase. In other words the clocks of the twins P and Q which were initially synchronized in Σ_0 continue to remain so in Σ_0-frame during their acceleration phases and thereafter. But what is simultaneous to mom and dad is not simultaneous with respect to their new frame Σ. For a pair of events simultaneous with respect to Σ_0, say for instance birthdays of the twins which occur at the same time in Σ_0 (i.e $t_{0Q} = t_{0P}$), does not occur at the same time in Σ. This is evident from the above Eq.(4.7) which shows that for $t_{0Q} = t_{0P}$ the time difference $t_Q - t_P \neq 0$ as the twins are separated by a constant distance $x_{0Q} - x_{0P} = x_0$, the difference is rather given by Eq.(4.4) where $\delta t_{desync} = t_Q - t_P$.

This temporal offset effect of identically accelerated clocks gives an important insight into the behaviour of clocks in a uniform gravitational field, for, according to EP "...all effects of a uniform gravitational field are identical to the effects of a uniform acceleration of the coordinate system"[13]. This suggests, as correctly remarked by Boughn that two clocks at rest in a uniform gravitational field are in effect perpetually being accelerated into the new frames and hence the clock at the higher gravitational potential (placed forward along the direction of acceleration) runs faster. With this insight we write Eq.(4.4) as

$$t - t_0 = -\frac{\gamma_v(t)v(t)x_0}{c^2} = -f(t), \tag{4.8}$$

where now t and t_0 are the readings of two clocks at higher and lower potentials respectively and also $f(t)$ stands for the right hand side of Eq.(4.4) without the minus sign

$$f(t) = \gamma_v(t)v(t)x_0/c^2. \tag{4.9}$$

In terms of differentials one may write Eq.(4.8) as

$$\delta t - \delta t_0 = -\dot{f}(t)\delta t, \tag{4.10}$$

where the time derivative $\dot{f}(t) = \frac{gx_0}{c^2}$, with $g = \frac{d}{dt}(\gamma_v v)$ is the *proper acceleration*. We may now replace δt and δt_0 by n and n_0, where the later quantities corresponds to the number of ticks (second) of the clocks at their two positions. We therefore have,

$$\frac{n - n_0}{n_0} = -\dot{f}(t), \tag{4.11}$$

or in terms of frequency of the clocks

$$-\frac{\delta\nu}{\nu_0} = \dot{f}(t), \tag{4.12}$$

where $\delta\nu$ refers to the frequency shift of an oscillator of frequency ν_0. The slowing down parameter for clocks, $-\delta\nu/\nu_0$ in Eq.(4.12) is nothing but the so called red-shift parameter Z for which we obtain the well-known formula[4]

$$Z = \frac{gx_0}{c^2}. \tag{4.13}$$

One thus observes that the time-offset relation (4.8) of Boughn's paradox can be interpreted as the accumulated time difference between two spatially separated clocks because of the pseudo-gravity experienced by the twins.[5] We shall see the importance of the time-offset relation (4.8) in accounting for the assymetrical aging of the standard twin paradox from the perspective of the traveller twin.

[4]In terms of *ordinary* acceleration $\overline{g} = \frac{dv}{dt}$, measured with respect to S the formula comes out to be $Z = \frac{\overline{g}\gamma x_0}{c^2}(1 - \frac{v^2\gamma^2}{c^2})$ which for small velocities can also be written as $Z = \frac{\overline{g}x_0}{c^2}$.

[5]The connection between gravity with this temporal offset through EP was first pointed out by Barron and Mazur[16], who derived the approximate formula for the "clock rate difference" mentioned in the previous foot-note.

Also, the above time offset effect between the identically accelerated twins calculated by Boughn can however be explained by noting that for spatially separated clocks the change of relative synchronization cannot be unequivocally determined. The clocks or the ages of the twins can be compared unambiguously only when they are at the same spatial point. For example in Σ either of the observers can slowly walk towards the other or both the observers can walk symmetrically towards each other and compare their clocks (ages) when they meet.

One may therefore define without much ado the reality of the temporal offset effect, provided the clocks are finally compared when they are brought together. We shall hereafter refer to the desynchronization effect between the identically accelerated twins when they are in spatial separation as the apparent Boughn effect (ABE) and that which persists (if any) when those clocks are brought together by slow transport as real Boughn effect (RBE). To check the reality of the time offset effect in the relativistic world, we consider any two arbitrary reference frames Σ_1 and Σ_2 moving with relative velocity w. LT between these two frames is

$$\begin{aligned} x_1 &= \gamma_w(x_2 + wt_2) \\ t_1 &= \gamma_w(t_2 + wx_2/c^2). \end{aligned} \tag{4.14}$$

The above time transformation gives the time dilation suffered by a clock stationary in Σ_2 as

$$\Delta t_1 = \gamma_w \Delta t_2, \tag{4.15}$$

where Δt_2 denotes the proper time between two events at the same point in Σ_2 and Δt_1 is the corresponding time measured by the observers in Σ_1.

Now if two synchronized clocks are spatially separated by a distance x in Σ_1 and a third clock attached to Σ_2 covers this distance slowly, then the time taken by this clock to cover the distance in Σ_1 is

$$\Delta t_1 = \frac{x}{w}. \tag{4.16}$$

The corresponding time measured in the Σ_2-clock (using Eq.(4.15)) is

$$\Delta t_2 = {\gamma_w}^{-1}\frac{x}{w}. \tag{4.17}$$

The difference of these times is therefore

$$\delta t = \Delta t_2 - \Delta t_1 = ({\gamma_w}^{-1} - 1)\frac{x}{w}. \tag{4.18}$$

This δt represents the integrated effect of time dilation suffered by the clock when it is being transported. Clearly this effect is dependent on the velocity of clock transport. Indeed, from the above equation one can see that for very small velocity i. e for $w \longrightarrow 0$ the above time reduces to zero

$$\underset{w \longrightarrow 0}{Lt}\, \delta t = \underset{w \longrightarrow 0}{Lt}\, \frac{({\gamma_w}^{-1} - 1)x}{w} = 0. \tag{4.19}$$

In other words one can say that the contribution of the time dilation effect to the time offset can be reduced to zero by transporting the clocks very slowly. In the relativistic world therefore, if two spatially separated clocks are desynchronized by some amount then on their reunion (after slow transport) they will carry the same amount of desynchronization. The time offset due to slow transport of the clocks is thus zero, viewed differently, the effect of slow transport is null i.e slow transport effect, $STE = 0$.

If the two clocks stationary in Σ_1 are considered as two Boughn observers then the desynchronization between these clocks (as in Boughn's thought experiment) will continue to be there when the clocks are brought together slowly for comparison. In other words, the apparent Boughn effect (as defined earlier) is equivalent to the real Boughn effect. In the above example involving the twins P and Q one can therefore see that the age of the forwardly placed twin Q turns out to be more than that of P even after they are reunited (after slow walk) at the same location, showing that the time offset effect suggested by Boughn has an absolute meaning[17], or the effect is a real one (according to our definition) in the relativistic world. The conclusion for the relativistic world can therefore be symbolically represented as $ABE \equiv RBE$ and $STE = 0$.

In order to understand the issue more clearly we will provide some counter examples. As the first example we consider Boughn's scenario in the classical (Galilean) world. In this world, it is well-known that mysteries and paradoxes do not generally exist. However quite surprisingly we shall see that the counter-intuitive effect discussed by Boughn can be reinvented in the Galilean world too. In the next section (Sec.(4.4)) we study the question of reality of Boughn's time offset effect in such a world. We shall indeed see that there can be ABE but not RBE in the Galilean world.

In another example we consider the relativistic world with non-standard synchronization of coordinate clocks in the moving frame. We shall see that although there will not be any desynchronization of clocks, the RBE will pop up during their slow walk reunion.

4.4 Galilean World: Reality of Boughn Effect

Classical or Galilean world is a kinematical world endowed with a preferred frame (of ether) Σ_0 with respect to which the speed of light c is isotropic and moving rods and clocks do not show any length contraction and time dilation effects. However in any arbitrary frame Σ (other than the preferred frame) the speed of light changes and depends on direction. As the clocks do not experience any time dilation effect so they can be transported freely and hence all clocks can be synchronized at one spatial point and then may be transported with arbitrary speed to different locations[6]. In the classical world one generally uses the Galilean transformation (GT) to compare events in different inertial frames, which is

$$\begin{aligned} x &= x_0 - vt_0, \\ t &= t_0, \end{aligned} \tag{4.20}$$

From GT one can obtain the two way speed (TWS) of light $\overleftrightarrow{c}(\theta)$ in Σ along any direction θ with respect to the x-axis (direction of relative velocity between Σ_0 and Σ) as

$$\overleftrightarrow{c}(\theta) = \frac{c(1-\beta^2)}{(1-\beta\sin^2\theta)^{1/2}}. \tag{4.21}$$

This TWS of light is not the same as the one-way speed (OWS) since, for example, along the x-axis the TWS is $c(1-v^2/c^2)$, while the OWS is $c-v$ and $c+v$ in the positive and negative x-directions respectively. However, one may playfully choose to synchronize the clocks in Σ such that the OWS (to and fro) along a given direction θ are the same as $\overleftrightarrow{c}(\theta)$, this is nothing but Einstein's stipulation in SR, commonly known as the standard synchrony. This synchrony is somewhat awkward in the Galilean world but there is nothing wrong in adopting such a method. In the Galilean world for this synchrony the transformation equations are given by

$$\begin{aligned} x &= (x_0 - vt_0), \\ t &= \gamma_v^2(t_0 - vx_0/c^2). \end{aligned} \tag{4.22}$$

These equations were first obtained by E. Zahar and are therefore known as the Zahar transformations (ZT)[18, 19, 20, 21]. The phase term and γ_v^2 in the time

[6] In the relativistic world due to the time dilation effect this process is generally forbidden.

transformation distinguishes ZT from GT. ZT has been successfully used in clarifying some recently posed paradoxes of SR[22, 23]. An interesting feature of ZT is the existence of apparent time dilation and length contraction effects as observed from an arbitrary reference frame Σ whereas with respect to the preferred frame Σ_0 there are no such effects.

The notion of relativity of simultaneity can thus be imported to the classical world, as can be seen from the time transformation of Eq.(4.22) that two events that are simultaneous in Σ_0 frame are not necessarily so in Σ. By adopting Einstein's mode of convention one should therefore be able to recast the effect linked with relativity of simultaneity or the Boughn effect (BE) in the Galilean world too. Using the above transformation and extending the arguments leading to Eq.(4.4), here too one can obtain a time offset or desynchronization between identically accelerated twins,

$$\delta t_{desync} = -\frac{\gamma_v^2 v x_0}{c^2}. \tag{4.23}$$

By simple alteration of the mode of synchronizing distant clocks, one is thus able to recast the Boughn's paradox even in the Galilean world. The above time offset effect or the differential aging between two spatially separated twins is therefore an artifact of the adopted synchronization scheme.

Although BE can be artificially created in the Galilean world the question here arises whether the effect persists even when the spatially separated clocks are brought together for comparison (as it does in the relativistic world). As mentioned earlier one may define the reality of BE provided the clocks are finally compared when they are brought together. Below we check the reality of BE in the classical world by calculating the effect of clock transport from ZT.

From Eq.(4.22) one can readily obtain the transformation equation of coordinates between any two arbitrary inertial frames Σ_1 and Σ_2 as,

$$x_1 = {\gamma_2}^2(1 - \frac{v_1 v_2}{c^2})x_2 - (v_1 - v_2)t_2, \tag{4.24}$$

$$t_1 = {\gamma_1}^2[(1 - \frac{v_1 v_2}{c^2})t_2 - \frac{\gamma_2^2}{c^2}(v_1 - v_2)x_2], \tag{4.25}$$

where v_1 and v_2 refer to the velocities of the concerned frames, i.e. Σ_1 and Σ_2 respectively with respect to the preferred frame Σ_0 . Also $\gamma_n = (1 - \frac{v_n^2}{c^2})^{-1/2}$ with $n = 1, 2$.

From the above time transformation the time dilation suffered by a clock stationary with respect to Σ_2 is

$$\Delta t_1 = \frac{1 - v_1 v_2/c^2}{1 - {v_1}^2/c^2}\Delta t_2, \tag{4.26}$$

where Δt_2 refers to the proper time between two events at the same point of Σ_2 and Δt_1 is the corresponding time measured by observers in Σ_1.

Now suppose two synchronized clocks are spatially separated by a distance x in Σ_1 and a third clock attached to Σ_2 slowly covers the distance, then the time taken by this clock to cover this distance in Σ_1 is given by

$$\Delta t_1 = \frac{x}{w}, \tag{4.27}$$

where

$$w = \frac{(1 - \frac{v_1^2}{c^2})(v_2 - v_1)}{1 - \frac{v_1 v_2}{c^2}}. \tag{4.28}$$

is the relative velocity of Σ_2 with respect to Σ_1.

The corresponding time measured by the third clock (Σ_2 - clock) may be obtained from Eq.(4.26) as

$$\Delta t_2 = \frac{1 - v_1^2/c^2}{1 - v_1 v_2/c^2} . \frac{x}{w}, \tag{4.29}$$

The difference of these two times

$$\delta t = \Delta t_2 - \Delta t_1 = \frac{v_1 x}{c^2}\gamma_1^2. \tag{4.30}$$

The above equation shows that the integrated effect of time dilation in the classical world due to clock transport is independent of the speed (v_2) at which the clock is transported. The effect is thus non-vanishing even for very slow speed of clock transport. This is just the contrary to the scenario in the relativistic world where we have seen that the result for clock transport is very much dependent on the speed of the clock and in particular vanishes for very small speed.

If as before the two clocks (stationary in Σ_1) are assumed to represent the two Boughn's observers, they have precisely the same amount (δt given by Eq.(4.30)) of desynchronization with a negative sign. The two effects thus nullify each other, or in other words, on comparison of these clocks at one spatial point (i.e. if one of the observer walk towards the other, no matter whether slow or fast) the result will be zero time difference (differential aging). This observation thus demonstrates that although the time-offset effect can be recast in the Galilean world, it is just an artifact and not real according to our definition of "reality" of the effect. That the effect has no absolute meaning is also evident from the fact that it is dependent on the synchronization convention, as, if instead of ZT one uses GT (Eq.(4.20)) to synchronize distant clocks then clearly one cannot find any ABE or RBE. However with ZT one can create ABE in the classical world but cannot obtain RBE since ABE is just the negative of STE. As in the relativistic world here too one can symbolically summarize the conclusion as $ABE \neq RBE$ and $ABE = -STE$.

Thus GSDC cannot be obtained from this Boughn's effect in the classical world via EP. Conversely Boughn's temporal offset may be regarded as an integrated effect of GSDC while in the classical world if it exists is just an artifact of the synchrony. This proves that the connection of the time-offset and GSDC is purely relativistic in nature.

4.5 Absence of BE in the Relativistic World!

In order to further clarify the matter let us consider the other extreme where in the pure relativistic world, apparently there may not be any time offset effect but there will still be differential aging in "real" terms. In order to understand this we will have to incorporate the conventionality of simultaneity (CS) thesis[7] in the relativistic world (we have already considered this thesis in the context of the classical world, although we did not mention it, while we used ZT to describe the world).

In relativity theory the spatially separated clocks in a given inertial frame are synchronized by light signal whose OWS must be known beforehand. But to measure the OWS of the signal from one point to another one requires to have two presynchronized clocks and hence the whole process ends up in a logical circularity. To get out of this problem a convention is adopted in assuming the OWS of the signal within certain bounds. To break the circularity Einstein stipulated the OWS of light to be equal to c which is the same as its TWS. This convention is known as Einstein synchrony or the standard synchrony. Later Reichenbach and Grünbaum[25, 26] first suggested that other synchronization conventions can as well be adopted in synchronizing the clocks. They claimed that the simultaneity between events in an inertial frame is a matter of convention and the conventionality lies in the assumption regarding the OWS of light. This fact that the synchronization procedure in SR has an element of convention is known as conventionality of distant simultaneity or the CS-thesis.

The CS-thesis asserts that Einstein's convention is just one among the various possible choices of the OWS of light. This particular convention leads to a set of

[7] A comprehensive review of the thesis is available in a recent paper by Anderson, Vetharaniam and Stedman[24]. See also Ref.[1, 19, 21].

transformation equations in the relativistic world which are known as the Lorentz transformation (LT). Different choices for the values of the OWS yield different sets of transformation equations with varied structural features. For example it is known that the relativistic world can also be described by the so called Tangherlini transformations (TT) by adopting absolute synchrony[19, 21, 20, 27, 28, 29]

$$\begin{aligned} x &= \gamma_v(x_0 - vt_0), \\ t &= \gamma_v^{-1}t_0.. \end{aligned} \tag{4.31}$$

Note that the absence of spatial coordinate in the time transformation above, shows that the distant simultaneity is absolute. In the light of CS thesis, "relativity of simultaneity" (often considered to be one of the fundamental imports of SR), loses its meaning in this absolute synchrony set-up since there is no lack of synchrony between spatially separated events as observed from different inertial frames. It is quite evident therefore that in the relativistic world with absolute synchrony the time offset effect suggested by Boughn does not exist, but still then, as we will see later, one can obtain a temporal offset effect.

We repeat the calculations done in Sec.(4.4) using TT in the place of ZT. As before from Eq.(4.31), one may obtain the transformation equation between any two arbitrary frames Σ_1 and Σ_2 moving with velocities v_1 and v_2 respectively relative to Σ_0 as,

$$x_1 = \gamma_1[{\gamma_2}^{-1}x_2 + \gamma_2(v_2 - v_1)t_2], \tag{4.32}$$

$$t_1 = {\gamma_1}^{-1}\gamma_2 t_2, \tag{4.33}$$

where $\gamma_1 = (1 - \frac{v_1^2}{c^2})^{-1/2}$ and $\gamma_2 = (1 - \frac{v_2^2}{c^2})^{-1/2}$.

The above time transformation relation gives the time dilation formula with respect to Σ_1-frame, which is

$$\Delta t_1 = {\gamma_1}^{-1}\gamma_2 \Delta t_2, \tag{4.34}$$

where Δt_2 refers to the proper time between two events at the same point in Σ_2-frame and Δt_1 is the corresponding time (coordinate time) measured in Σ_1.

As discussed earlier (in the context of ZT), let us here too consider two synchronized clocks (or twins) in Σ_1, spatially separated by a distance x and another clock attached to Σ_2-frame slowly covers this distance with speed w which is given as

$$w = {\gamma_1}^2(v_2 - v_1). \tag{4.35}$$

The time taken by the Σ_2-clock to cover this distance in Σ_1 is given by

$$\Delta t_1 = \frac{x}{w}. \tag{4.36}$$

The corresponding time in Σ_2, taking into account the time dilation effect is

$$\Delta t_2 = \gamma_1 {\gamma_2}^{-1} \frac{x}{w}. \tag{4.37}$$

The difference between these times measured in the two frames is thus

$$\delta t = \Delta t_2 - \Delta t_1 = (\gamma_1 {\gamma_2}^{-1} - 1)\frac{x}{w} = \frac{x(\gamma_1 {\gamma_2}^{-1} - 1)}{{\gamma_1}^2(v_2 - v_1)}. \tag{4.38}$$

Clearly δtdepends not only on v_1 but also on v_2. The result is thus qualitatively different from that one obtains in the Galilean world(with Einstein synchrony). However, if now the Σ_2-clock is transported very slowly such that $w \longrightarrow 0$ or in other words $v_2 \longrightarrow v_1$ then the above time difference reduces to

$$w. \overset{Lt}{\longrightarrow} 0\ (\delta t) = -xv_1/c^2. \tag{4.39}$$

As in the classical world, here too we can see that the above value does not vanish even if the clock is transported very slowly. Thus when the spatially separated synchronized clocks (or twins) are brought together by slow transport, this non-vanishing integrated effect of time dilation gives rise to a time offset or a non-null differential aging between the twin's (biological) clocks. Hence even in this absolute synchrony set-up where there is no question of relativity of simultaneity and therefore of BE, the differential aging or the temporal offset pop up as a time dilation effect when the clocks are brought together by slow transport. In the notation form one can write here $ABE = 0$ and $RBE = STE$.

4.6 Resolution of the Ordinary Twin Paradox Using BE

Let us now move on to the details of the arguments leading to Eq.(4.1). The outward trip of the traveler twin B from the point of view of the earth twin is composed of two phases. In the first phase, the rocket moves a distance L_A in time t_{A1} with uniform velocity v which is given by

$$t_{A1} = \frac{L_A}{v}, \tag{4.40}$$

and in the second phase, which corresponds to the deceleration phase of the rocket which finally stops before it takes the turn-around, the time t_{A2} taken by B is given by

$$t_{A2} = \frac{\gamma v}{g}, \tag{4.41}$$

where the proper acceleration g has been assumed to be uniform with respect to the earth frame. In the present analysis this term does not contribute since we consider the abrupt turn-around scenario where t_{A2} tends to zero as $g \to \infty$; however for the time being we keep it. Therefore the total time elapsed in S for the entire journey is given by

$$T_A = \frac{2L_A}{v} + 2t_{A2}. \tag{4.42}$$

Now we compute this time as measured in B's clock by taking the time dilation effect from the point of view of A. For phase 1 this time t_{B1} may be computed as

$$t_{B1} = \gamma^{-1} t_{A1} = \frac{\gamma^{-1} L_A}{v}, \tag{4.43}$$

where we have applied the simple time dilation formula. For phase 2 however this time-dilation formula is differentially true as the speed is not a constant i. e one may write

$$dt_{B2} = (1 - \frac{v^2}{c^2})^{1/2} dt_{A2} = (1 - \frac{v^2}{c^2})^{1/2} \frac{1}{g} d(\gamma v). \tag{4.44}$$

Hence after integration one obtains[30]

$$t_{B2} = \frac{c}{2g} ln(\frac{1 + v/c}{1 - v/c}). \tag{4.45}$$

However once again this tends to zero as $g \to \infty$. In any case we shall however not need this expression any more. Therefore the total elapsed time measured in B's clock for the complete journey is given by

$$T_B = \frac{2\gamma^{-1} L_A}{v} + 2t_{B2}. \tag{4.46}$$

The differential aging from the point of view of A is thus

$$\delta T_A = T_A - T_B = \frac{2L_A}{v}(1 - \gamma^{-1}) + 2(t_{A2} - t_{B2}). \tag{4.47}$$

From the point of view of B the stay-at-home observer A is moving in the opposite direction and as before one may divide the relative motion of A into two phases, phase I and phase II, where the later corresponds to the acceleration phase. The phase II may be interpreted as turning on of a gravitational field. When this field is switched off (marking the end of the acceleration phase), the phase I starts where the stay-at-home observer A moves with a velocity $-v$ up to a distance L_B which on account of the Lorentz contraction of L_A is given by,

$$L_B = L_A(1 - \frac{v^2}{c^2})^{\frac{1}{2}}, \tag{4.48}$$

and the corresponding elapsed time t_{B1} is given by,

$$t_{BI} = \frac{L_B}{v} = \frac{\gamma^{-1} L_A}{v}. \tag{4.49}$$

This obviously comes out to be the same as t_{B1} since the result is obtained from considerations with respect to the inertial observer A. Similarly t_{BII} i.e. B-clock's time during phase II should be the same as t_{B2} during which the gravitational field is turned on, i.e

$$t_{BII} = t_{B2}, \tag{4.50}$$

and hence the total time

$$\tau_B = 2t_{BI} + 2t_{BII} = \frac{2\gamma^{-1} L_A}{v} + 2t_{BII} = T_B. \tag{4.51}$$

The corresponding time of A's clock by taking into account the time dilation effect is

$$t_{AI} = \gamma^{-1} t_{BI} = \frac{\gamma^{-2} L_A}{v}. \tag{4.52}$$

Writing A-clock's time during phase II from B's perspective as t_{AII}, one may write for A's clock time for the entire journey as

$$\tau_A = 2t_{AI} + 2t_{AII} = \frac{2\gamma^{-2} L_A}{v} + 2t_{AII}. \tag{4.53}$$

The difference of these times of clocks A and B as interpreted by the observer B, is given by,

$$\delta T_B = \tau_A - \tau_B = \frac{2\gamma^{-1} L_A}{v}(\gamma^{-1} - 1) + 2(t_{AII} - t_{BII}). \tag{4.54}$$

Note that at the moment we do not know the value of t_{AII}, since it refers to the time measured by A as interpreted by B when it is in its acceleration phase. The paradox is resolved if

$$\delta T_A = \delta T_B. \tag{4.55}$$

In other words using Eqs.(4.47) and (4.54) one is required to have,

$$t_{AII} = \frac{L_A}{v}(1-\gamma^{-2}) + t_{A2} = \frac{L_A v}{c^2} + t_{A2}. \tag{4.56}$$

In the abrupt turn-around scenario, as we have already observed $t_{A2} = 0$, one therefore must have

$$t_{AII} = \frac{L_A v}{c^2} = \frac{\gamma L_B v}{c^2}. \tag{4.57}$$

The resolution of the twin paradox therefore lies in accounting for this term. It is interesting to note that the term is independent of the acceleration in phase II. This is possibly the implicit reason why the role of acceleration in the explanation of the twin paradox is often criticized in the literature. However we shall now see how, we can interpret this term as an effect of the direction reversing acceleration (or the pseudo-gravity) experienced by the traveller twin.

Now recall the Boughn-effect of temporal offset between two identically accelerated observers. To be specific, consider an inertial frame of reference S attached to the observer B when it is in the uniform motion phase (phase I). Suppose now there is another observer B' at rest in S at a distance L_B behind B and both of them get identical deceleration and eventually come to rest with respect to A in the frame of reference S', which is moving with velocity $-v$ in the x-direction with respect to S. According to Boughn-effect then the clocks of these two observers get desynchronized and the amount of this desynchronization is given by the expression (4.4) only with the sign changed, that means

$$desync = \frac{\gamma v L_B}{c^2}, \tag{4.58}$$

which is nothing but t_{AII}. It has already been pointed out that this Boughn-effect may be interpreted as the effect of pseudo-gravity (in this case as experienced by the observer B) according to EP. In terms of the pseudo acceleration due to gravity the above expression can also be obtained as

$$desync = \frac{g\Delta t_B L_B}{c^2}. \tag{4.59}$$

Note that $g\Delta t_B$ is finite (equal to γv) even if $g \to \infty$.

The observer B' which is L_B distance away from B is spatially coincident with A, hence, in calculating the clock time of A from B's perspective this time-offset due to Boughn-effect must be taken into account. This effect is ignored when the twin paradox is posed by näively asserting the reciprocal time-dilation effect for the stay-at-home and the rocket-bound observers. Clearly the paradox is resolved if the Boughn-effect or the pseudo gravitational effect is taken into consideration.

4.7 Test of Boughn-Effect

We have seen that the Boughn-effect can be interpreted as the integrated effect of GSDC. The experimental test of GSDC or the gravitational red-shift is therefore a test of a differential Boughn-effect in a way. On the contrary one may directly measure the integrated effect by the following means:

First two atomic clocks may be compared (synchronized) at the sea level, then one of the clocks may be slowly transported to a hill station of altitude h and then kept there for some time T. In this time these two atomic clocks according to Boughn scenario are perpetually accelerated from a rest frame S to a hypothetical inertial frame S' moving with velocity v, with proper acceleration g so that $\gamma v = gT$. Boughn-effect therefore predicts a temporal offset (see Eqs.(4.58) and (4.59)),

$$\Delta t_{offset} = \frac{ghT}{c^2}. \tag{4.60}$$

This offset can be checked by bringing the hill station clock down and then comparing its time with the sea level one. Any error introduced in the measurement due to transport of clocks can be made arbitrarily small compared to Δt_{offset} by increasing T. As a realistic example for h =7000ft (altitude of a typical hill station in India), and T =1 year and taking the average g to be about $9.8m/sec^2$, the Boughn-effect comes out to be in the micro-second order:

$$\Delta t_{offset} = 7.3\mu s, \tag{4.61}$$

which is easily measurable without requiring sophisticated equipments, such as those used in Pound-Rebka type experiments.

It is interesting to note that from the empirical point of view the effect is not entirely unknown. For example Rindler[12], in seeking to cite an evidence for the GSDC effect, remarks: "Indeed, owing to this effect, the US standard atomic clock kept since 1969 at the National Bureau of standards at Boulder, Colorado, at an altitude of 5400ft. gains about five microseconds each year relative to a similar clock kept at the Royal Greenwich Observatory, England, ...". However one can consciously undertake the project with all seriousness, for the accurate determination of the time-offset (with the error bars and all that), not merely to prove GSDC but to verify the Boughn-effect of SR. It is worth while to note that the empirical verification of this time-offset as a function of T would not only test the Boughn-effect and the integral effect of GSDC but it would also provide empirical support for the relativity of simultaneity of SR. So far no experimental test has been claimed to be the one verifying the relativity of simultaneity. Indeed SR is applicable in the weak gravity condition of the earth so that gravity can be thought of as a field operating in the flat (Minkowskian) background of the space-time[31]. Clearly because of EP, the earth with its weak gravity has the ability to provide a convenient Laboratory to test some special relativistic effects like the relativity of simultaneity or the Boughn-effect.

4.8 Concluding Remarks

In a paper by Price and Gruber[17] an extension of the Boughn's twin paradox is presented. It is shown there that there is no meaning to the question as to where does the differential aging occur in the twin problem. To prove the above statement, the Boughn's scenario is studied from the point of view of a new observer, the uncle of the twins who got separated from the family before the birth of the twins and settled in the final frame of the twins after experiencing some acceleration. The uncle saw the twins born at the same place and then drift apart very slowly to start their rocket trip. The uncle being in a frame different from that of mom and dad did not not see the twins start their journey simultaneously, he rather saw the forwardly placed twin to start the trip first and hence he concluded that at the end of the trip the twins aged differently. Although the result seems to be straight forward apparently, but as pointed out in this paper this result inferred by the uncle is quite strange, as the uncle concludes that the aging has taken place before the twins enter the rocket. The question here is, to the uncle the twins were born at the same place at the same time, then what had happened to the twins during separating when they did not even suffer any acceleration or even move fast. The answer to the question, as provided in this paper is 'the twins "...lose simultaneity during the *arbitrary slow* motion of their separation!..." Although strange, the result can be obtained from the simple application of LT between the frame of mom and dad and that of the uncle.

The time difference between the clocks of the twin during separating from each other as seen by the uncle ($\Delta t'$ say) can be connected to that measured in the frame of mom and dad (Δt say) through the time transformation of LT as

$$\Delta t' = \gamma_v(\Delta t - \frac{v\Delta x}{c^2}), \tag{4.62}$$

Let us consider that each of the twin move symmetrically in the opposite directions with a very small speed v to cover a distance $L/2$. In this case with respect to the frame of mom and dad the time measured in the clocks of both the twins will be the same owing to their symmetric motion, or $\Delta t = 0$. Using this and also noting that $\Delta x = L$, the above equation gives

$$\Delta t' = \gamma_v v L/c^2. \tag{4.63}$$

Thus the uncle observes the twins age differently before the initiation of the rocket-trip. In Sec.1, however, we have seen that the differential aging of the twins occurred during the rocket-trip. One therefore cannot find a definite answer as to where the differential aging occurs, showing that the question is meaning less. In an effort to further justify this point we try to study the paradox of the uncle in the absolute synchrony set-up, i.e with TT (Eq.(4.31)).

It is evident from the time transformation of Eq.(4.31) that if as in the above example the twins separate symmetrically in the frame of their mom and dad then, even in the frame of the uncle no differential aging occurs. Hence with respect to the uncle too the twins age the same when they on board the rocket. When the twins settle at rest in their final frame (i.e the frame of the uncle) their clocks are seen to be synchronized with respect to the uncle. But the scenario changes when the twins walk to meet each other. As we have seen in the previous section during the slow walk meeting of the twins a differential aging crops up between them due to the time dilation effect. In short with absolute synchrony set-up one concludes that the differential aging occurs during the slow walk reunion of the twins.

The above example shows that answer to *where* the differential aging occurs depend not only on the perspective of the observer but also on the adopted mode of synchronization.

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