Chapter 3

The Case Of Identically Accelerated Twins and the Ordinary Twin Paradox

3.1 Introduction

In a pedagogically interesting and thought provoking article, S. P. Boughn[1] discussed a variation of the twin paradox parable where twins (P and Q say) on board two identical rockets (with equal amount of fuel), initially at rest a distance L apart in an inertial frame Σ, underwent identical accelerations for some time in the direction $\overrightarrow{PQ}$, and eventually came to rest (when all their fuels had expended) with another inertial frame Σ' moving with non-zero relative velocity v with respect to Σ. From the simple application of Lorentz transformation Boughn obtained a rather surprising result that in the new abode (Σ') the age of P became less than that of Q! Viewed differently, if the twins would carry presynchronized clocks, the outcome would have been a net time-offset effect between these clocks in Σ'.

The result is counter-intuitive by virtue of the fact that the twins of the parable throughout had identical local experiences yet their presynchronized clocks (also their own biological clocks) went out of synchrony!

Quantitatively this time-offset or desynchronization turns out to be

$$\delta t'_{desync} = -\gamma_v v L/c^2, \tag{3.1}$$

where the Lorentz factor

$$\gamma_v = (1 - v^2/c^2)^{-1/2} \tag{3.2}$$

and c is the speed of light in free space.

The result can be seen to follow from the simple application of LT:

$$\begin{aligned} x_k' &= \gamma_v(x_k - vt_k) \\ t_k' &= \gamma_v(t_k - vx_k/c^2), \end{aligned} \tag{3.3}$$

where in the current context, t_k and x_k denote the time and space coordinates of the observer k (k stands for P or Q) with respect to Σ and the prime refers to the corresponding coordinates of the observers in Σ' where they arrive and settle stationary after their acceleration phases are over.

From the time transformation of Eq.(3.3) one obtains

$$t_Q' - t_P' = \gamma_v[(t_Q - t_P) - v(x_Q - x_P)/c^2]. \tag{3.4}$$

The clocks of the observers P and Q are initially synchronized in Σ and continue (as the symmetry of the situation demands) to remain synchronized with respect to the frame during their acceleration phases and thereafter. For a pair of events simultaneous with respect to Σ, say for instance birthdays of P and Q, $t_Q - t_P = 0$. Using this and noting that the twins are separated by a distance $L = x_Q - x_P$, the last equation gives Eq.(3.1) where the desynchronization between the clocks P and Q is denoted by $\delta t'_{desync} = t'_Q - t'_P$. Obviously the above desynchronization corresponds to a differential aging of the twins in their new abode.

The apparently paradoxical result that the twins age differently in spite of their identical history of acceleration is readily explained if one notes that for spatially separated (biological) clocks the change of relative synchronization cannot have any unequivocal meaning. They can only be compared unambiguously when they are in spatial coincidence. For instance in Σ', one of the observers can slowly walk towards the other (or both of them can do the walking) and compare their ages (or their clock readings) when they meet. Since in the relativistic world the so called "slow transport synchronization" is equivalent to the Einstein synchronization[2], the calculated differential aging or time-offset between their clocks when they were spatially separated would continue to hold even when the twins meet after their slow walk. However in that case it can easily be seen[3] that they do not have symmetrical experiences, and hence the paradox gets resolved.

While the paradoxical element of the counter-intuitive outcome melts away, the fact remains that the differential aging for the "case of identically accelerated twins" given by Eq.(3.1) is *correct* and the time-offset can be verified at one spatial point as has already been pointed out. Boughn in his paper claimed that the ordinary twin paradox could be explained in terms of this effect (which hereafter will be referred to as the Boughn effect (BE))[1]. According to the parable of the ordinary twin paradox, Adam (A) stays at home on earth in a frame of reference Σ_0, while his traveller twin sister Beatrice (B) on board a fast rocket leaves earth with velocity v for a voyage to a distant star and subsequently turns around and then returns with the same speed v to meet her stay-at-home sibling to discover that they age differently. By applying time dilation formula (TDF) of special relativity (SR) on B's (biological) clock, A predicts that B should be younger on her return. The apparent paradox arises if B tries to apply the special relativistic TDF on A's clock (pretending that A is doing all the moving) and makes the contradictory claim that it is B who should be younger after the round-trip.

In this context Boughn observed that according to twin B, twin A would age less rapidly by a factor $1/\gamma$ during the entire trip. However, with obvious reference to the time-offset effect discussed earlier, Boughn further argued that because of acceleration at turn around, there would be a change in synchronization between the two twins' clocks. This change would overcompensate for the apparent slowdown in twin A's aging and finally twin A would be the older of the two. This was how both the twins could finally agree on their predictions.

But can one really explain the ordinary twin paradox in terms of Boughn effect?

[1]Although the problem of "identically accelerated twins" perhaps was well known to many in one form or the other, many authors[1, 2, 3, 4] have chosen to refer to Boughn's paper to describe this effect (paradox). The present paper also continues with this tradition in calling this effect after Boughn.

Commenting on Boughn's paper, Desloge and Philpott[3] noted that using Boughn's scenario "to analyse the twin paradox can be misleading since twins A and B do not start and end together whereas the twins in the conventional twin problem do." In spite of the criticism we feel that the pedagogical power of Boughn's paradox *can be used* to explain the usual twin paradox. However as outlined in the preceding paragraph, the brief account given by Boughn himself to this end is only a qualitative one. Besides, Boughn's paradox refers to the time-offset between two twins whose spatial separation has been maintained constant throughout with respect to Σ. One may therefore wonder how this can be related to the desynchronization of twins (of the ordinary twin paradox), one of whom remains stationary while the other makes the round-trip. In a recent paper, referring to the desynchronization of clocks in the case of identically accelerated twins (clocks), Styer[5] attempted to provide a resolution of the ordinary twin paradox in terms of it. However, in doing so the author in his otherwise interesting paper treated the Earth clock and the star clock as the identically accelerated ones whereas indeed the travelling twin suffered the turn-around acceleration thus rendering his arguments untenable. Besides, the resolution relied on an arbitrarily rapid jumping ahead of the Earth (remote) clock during the brief turn-around of the traveller. However, this drastic alteration in the reading of the remote clock is often difficult to accept[6]. Perhaps this is the reason why this resolution (as Styer himself had put it) "leaves most students with a gnawing pit in their guts".

The purpose of the present paper among other things is to remove this gnawing pit in the guts, and show how Boughn's paradox can be fruitfully used to resolve the usual twin paradox *quantitatively*. To our knowledge this has not been done before. Indeed the actual demonstration of unequivocal prediction for differential aging from both the twins' perspectives by employing BE will be found to be a nontrivial exercise. A careful analysis of the problem will provide a lot of additional insight into the century old counter-intuitive puzzle. It will be shown how the travelling twin in spite of her non-inertiality can make use of the special relativistic time dilation formula to obtain the proper time of the stay-at-home twin and yet predict the correct differential aging from her perspective. Interestingly, as the paper attempts to explain one paradox in terms of another, the present endeavour may be looked upon, in a lighter vein, as to justify the proverb "like cures like".

3.2 Coordinate Time, Time Dilation and Desynchronization

The relativistic time dilation effect relates times of two different nature. One concerns the rate of ticking of a moving clock at its position and the corresponding time is known as the proper time (often denoted by τ) of the clock. The other refers to readings of spatially separated coordinate clocks (at rest with respect to some inertial frame of reference), as the concerned clock moves past these coordinate clocks. Time recorded by the coordinate clocks are therefore known as coordinate time which may be denoted by t. Note that, since the coordinate clocks are spatially separated, the coordinate time for a given pair of events depends on the synchronization convention (or the standard of simultaneity) adopted to synchronize these coordinate clocks. In SR we adopt the standard synchronization or Einstein synchronization according to which the one-way-speed of light is *stipulated* to be equal to its round trip speed. The proper time τ of a clock however is independent of any synchronization convention.

The standard relativistic TDF which connects τ and t is therefore valid provided the coordinate clocks are synchronized following Einstein's convention. According to the conventionality of simultaneity thesis[2] however, other quite equally valid synchronization schemes can be adopted but in that case the relativistic TDF will not be valid. Since in the twin paradox thought experiment, τ of one twin (clock) is "calculated" from the "knowledge" of the coordinate time elapsed in the other twin's frame of reference, one must ascertain the latter with great caution.

The genesis of the twin paradox lies in the failure to do so in the frame of reference attached to the traveller twin. Let us now clarify this. Consider the abrupt turn around scenario of the standard twin parable. Assume that the turn around of B takes place when the distance between the twins (with respect to Σ) measures L say. Now, just before the deceleration phase starts, one may consider another observer Alfred ($\bar{A}$) of the same age as that of Beatrice (i.e it is assumed that $\bar{A}$'s clock is synchronized with B's in Σ) and at the same location of A comoving with respect to B such that, like in Boughn's scenario, $\bar{A}$ and B both undergo the same but arbitrarily large negative acceleration with respect to Σ, which moves with constant velocity v with respect to Σ_0. From Σ frame, B and $\bar{A}$ may be considered as Boughn's twins accelerating from rest along the negative x-direction (i.e now $\bar{A}$ is forwardly placed with respect to B) and settles in some inertial frame Σ' moving with velocity $-w$ (say) with respect to Σ (and $-v$ with respect to Σ_0).

BE therefore tells us that with respect to Einstein synchronized clocks in Σ', there is a desynchronization effect between the clocks (or ages) of Alfred and Beatrice,

$$t'_B - t'_{\bar{A}} = \delta t'_{desync} = \gamma_w wL/c^2, \tag{3.5}$$

which has been obtained from Eq.(3.1) replacing γ_v and v by γ_w and $-w$ respec-

[2]See for example[4, 7, 8]. For a more comprehensive review of the thesis see a recent paper by Anderson, Vetharaniam and Stedman[9].

tively. Note that here

$$\gamma_w = (1 - w^2/c^2)^{-1/2} = (1 + v^2/c^2)/(1 - v^2/c^2), \tag{3.6}$$

where we have used

$$w = 2v/(1 + v^2/c^2). \tag{3.7}$$

The last relation of course follows from the relevant relativistic velocity addition law. Using the last two expressions in Eq.(3.5) one obtains

$$\delta t'_{desync} = 2v\gamma_v^2 L/c^2. \tag{3.8}$$

The above desynchronization also corresponds to a synchronization gap between the Einstein synchronized reference frames Σ and Σ'. The presence of this synchronization gap between instantaneously comoving inertial frames for an accelerated observer is the reason why such frames cannot be meshed together.

Owing to the (instantaneous) turn-around, Beatrice switches her inertial frame and because of desynchronization, the clocks of Alfred and Beatrice no longer represent the Einstein synchronized coordinate clocks of Σ'. Now consider the following scenario: Instead of not turning around if Beatrice would continue to move forward covering the same length of journey with uniform speed as she would do after the turn-around, coordinate clocks (Einstein synchronized) of Σ frame of Beatrice could be used to measure the lapse of coordinate time for A's trip and connect the same with the proper time of Adam through TDF for A's entire one-way trip. However if during the second phase of the trip someone would playfully tamper with the synchronization, any coordinate time measurement following it would then be erroneous and hence a calculation to obtain the proper time τ_A of A from this measurement (by applying TDF on it) would give wrong result. Clearly, in order to get the correct answer the remedy is to first undo the playful tampering of synchronization by getting back to the Einstein synchronization that was adopted before and then one would be free to use TDF in order to obtain the proper time from the coordinate time. Let us now see what the corresponding situation is if we consider Beatrice's turn-around. In this case the second leg of Beatrice's journey corresponds to the inertial frame Σ'. The adoption of Einstein synchronization in this frame can be equated with the deliberate alteration of synchronization just discussed in connection with the uniform motion scenario of Beatrice, since the standard of simultaneity in Σ' is thus made different from that in Σ which corresponds to the earlier leg of Beatrice's trip.

It is clear that the proper time and coordinate time of a clock are connected by TDF of special relativity provided the coordinate clocks are standardly synchronized and they remain so during measurements. We then ask if there is any way so that one can continue with the standard of simultaneity (synchrony) of Σ in Σ'. The answer is in the affirmative and is provided by Boughn's thought experiment. From the symmetry of the problem it is evident that clocks of Alfred and Beatrice initially synchronized in Σ continue to remain synchronized with respect to Σ even when they arrive stationary in Σ' after the turn-around acceleration[3]. From B's perspective one can easily obtain the round-trip time $\Delta t_B(B)$ recorded in B's clock for A's journey (see later), but this does not correspond to the coordinate time for the same in Σ. Clearly a correction term $\delta t'_{desync}$, is to be added to $\Delta t_B(B)$ (see below) in order to obtain the said coordinate time. This correction is equivalent to the process of restoration of the synchronization mentioned in Beatrice's non

[3] In other words as if the clocks $\bar{A}$ and B behave in an obstinate manner and refuse to be synchronized in the new frame according to the standard synchrony *automatically*. The clock readings are to be tampered with in order to resynchronize them in Σ' according to the Einstein synchrony. If instead the clocks are left alone, these coordinate clocks then define absolute synchronization which Selleri refers to as "nature's choice"[10, 11].

turn-around example.

3.3 Resolution

Before we proceed to provide the quantitative resolution of the twin paradox using BE, let us for convenience, remove the inconsequential initial and final accelerations from the problem. We thus assume that B makes a flying start and also after the return trip she flies past A. The only unavoidable acceleration that we keep is the one associated with B's turn-around without which A and B cannot compare their clocks (or ages) at one spatial point after the latter's round-trip. The resolution can now be laid down in the following steps:

Perspective of A:

Step 1:

The reciprocity of the relativistic TDF from the perspective of A and B can symbolically be expressed as,

$$TDF1: \qquad \Delta\tau_B(A) = \gamma_v^{-1}\Delta t_A(A), \tag{3.9}$$

$$TDF2: \qquad \Delta\tau_A(B) = \gamma_v^{-1}\Delta t_B(B). \tag{3.10}$$

In the above we follow a notation scheme, where $\Delta\tau_B(A)$ $[\Delta\tau_A(B)]$ denotes the B $[A]$-clock reading for a time interval between two events occurred at its position as inferred by the observer A $[B]$. The inference of course is drawn from its own coordinate clocks' records for the interval, $\Delta t_A(A)$ $[\Delta t_B(B)]$ and its knowledge of the relevant time dilation effect. Indeed the time intervals $\Delta\tau_B(A)$ or $\Delta\tau_A(B)$ are based on one clock measurements and hence they refer to proper times of B and A respectively.

Regarding the notations $\Delta t_B(B)$ or $\Delta t_A(A)$, a clarification is needed. While, for example $\Delta\tau_A(B)$ refers to the difference between one clock (A) reading for two events, $\Delta t_B(B)$ refers to in general, the observed difference in readings (for the same pair of events) recorded in two spatially separated (synchronized) clocks stationary with respect to the frame of reference attached to B. However when $\Delta t_B(B)$ concerns measurement of the round trip time of an object or a clock (A say), it also refers to a single clock (B) measurement. Although τ-symbol would have been more appropriate in the later case but we shall continue to use the symbol 't' to emphasize that the corresponding time is supposed to be the coordinate time.

We now quote the relevant length contraction formula (LCF),

$$LCF: \qquad L = \gamma_v^{-1} L_0, \tag{3.11}$$

where L_0 is the distance of the distant star from the earth (measured in Σ_0) and L is the corresponding distance measured in Σ.

Step 2:

A-clock time for B's up and down travel of distance $2L_0$ is

$$\Delta t_A(A) = 2L_0/v, \tag{3.12}$$

and using the above result, the B-clock time for the same as calculated by A using TDF 1 (Eq.(3.9)) is

$$\Delta\tau_B(A) = \gamma_v^{-1} 2L_0/v. \tag{3.13}$$

Step 3:

Differential aging with respect to A is therefore given by

$$\delta t(A) = \Delta t_A(A) - \Delta\tau_B(A) = (1 - \gamma_v^{-1}) 2L_0/v. \tag{3.14}$$

Perspective of B:

Step 4:

From B's point of view, A makes the round trip and B measures the time for this trip as $\Delta t_B(B)$. This is nothing but the B-clock time as calculated by A, $\Delta\tau_B(A)$ which is given by Eq.(3.13). Hence

$$\Delta t_B(B) = \gamma_v^{-1} 2L_0/v \tag{3.15}$$

This can also be seen in the following way. According to B, A travels a distance $\gamma_v^{-1}2L_0$ (using LCF Eq.(3.11)) for the round trip . The speed of A with respect to B is also v as LT honours the reciprocity of relative velocity. Hence the travel time $\Delta t_B(B)$ is again calculated as $\gamma_v^{-1}2L_0/v$.

Step 5:

The same time interval in A-clock as calculated by B by the *näive* application of TDF2 (Eq.(10)) on $\Delta t_B(B)$) is obtained as,

$$\Delta\bar{\tau}_A(B) = \gamma_v^{-2} 2L_0/v. \tag{3.16}$$

This is however incorrect since desynchronization of distant clocks due to BE has not been taken into account and hence we have put a bar sign on τ, to be removed later after correction.

Step 6:

The above expression must be corrected by taking into account the BE. To calculate this effect we first split the frame of reference (K) attached to B into two inertial frames Σ and Σ' which move with velocities v and $-v$ respectively with respect to Σ_0. As discussed in Sec.3.2, $\bar{A}$ and B separated by a length L in Σ after deceleration arrives in the final frame of reference Σ' producing a temporal offset (desynchronization) between their clocks which is given by Eq.(3.8)

$$\delta t'_{desync} = 2v\gamma_v^2 L/c^2 = 2v\gamma_v L_0/c^2, \tag{3.17}$$

where for the last equality we have made use of Eq.(3.11). Going back to Eq.(3.15), leading to Eq.(3.16) one now discovers that the application of Eq.(3.10) on $\Delta t_B(B)$ to obtain $\Delta\tau_A(B)$ is a mistake since, as has been explained in Sec.2, the former does not represent the coordinate time as the coordinate clocks in K fail to remain synchronized according to the standard synchronization scheme as B changes her inertial frame from Σ to Σ' for her turn around acceleration. This is the lesson we learn from BE. One therefore needs to add[4] this desynchronization effect (Eq.(3.17)) to $\Delta t_B(B)$ before the application of TDF2 (See Appendix for clarification).

Adding $\delta t'_{desync}$ to $\Delta t_B(B)$ will undo the "resynchronization" of clocks (see Sec.3.2 and footnote(4)) in Σ'[5] and hence the standard synchronization of coordinate clocks in Σ will be carried over in Σ' as well. This is indeed the precondition that ensures the applicability of the relativistic TDF.

Therefore the true coordinate time is obtained as

$$\Delta t_B^{coord}(B) = \Delta t_B(B) + \delta t'_{desync} = 2\gamma_v^{-1}L_0/v + 2v\gamma_v L_0/c^2. \tag{3.18}$$

Now applying TDF2 on the true coordinate time $\Delta t_B^{coord}(B)$, B calculates the round-trip time (proper) measured in A-clock as

$$\Delta\tau_A(B) = \gamma_v^{-1}(2\gamma_v^{-1}L_0/v + 2v\gamma_v L_0/c^2). \tag{3.19}$$

Step 7:

Thus the differential aging from the perspective of B turns out to be,

$$\delta t(B) = \Delta\tau_A(B) - \Delta t_B(B) = (1-\gamma_v^{-1})2L_0/v, \tag{3.20}$$

[4]Whether one should add or subtract this desynchronization effect depends on its definition, as $\delta t'_{desync}$ could have been defined as $t'_{\bar{A}} - t'_B$, in which case one would need to subtract the effect for undoing the resynchronization.

[5]Resynchronization has been tacitly assumed in calculating $\Delta t_B(B)$ (see arguments following Eq.(3.15)) when reciprocity of relative velocity and LCF has been assumed to be valid in the frame of reference of B in her return journey.

which agrees with Eq.(3.14).

3.4 Summary

Boughn has shown that two identically accelerated twins initially at rest with some inertial frame ages differently when they arrived stationary in another inertial frame (after their acceleration phases are over). Although the outcome is counter-intuitive (since in spite of the twins' accelerations being symmetric in every respect, they age differently), the effect is an undeniable fact. It has been remarked in the literature that ordinary twin paradox can be explained in terms of the paradox of the identically accelerated twins. In this paper we have taken up the issue and solved the usual twin paradox *quantitatively* using the Boughn paradox. We have considered the abrupt turn around scenario and the essence of the present approach to resolve the issue lies in recognizing the fact that the coordinate clocks (that of Alfred and Beatrice say) of Σ no longer represent the Einstein-synchronized coordinate clocks in Σ' after the turn around. Indeed, these coordinate clocks of Σ carry over their synchronization convention in Σ', a lesson we learn from Boughn paradox. With these clocks the standard of simultaneity in Σ, according to Einstein convention is preserved in spite of their acceleration; in Σ' though, these clocks are not Einstein-synchronized. This departure from Einstein synchronization of the clocks is reflected in the Boughn effect.

Relativistic TDF can be used to calculate the proper time of A-clock from the coordinate time in the frame of reference attached to B provided the coordinate clocks represent *uniform* synchronization according to Einstein's scheme. It has been shown that the round-trip time $\Delta t_B(B)$ of Adam as recorded by Beatrice's clock cannot represent the readings of the coordinate clocks of Beatrice's frame of reference having constant synchronization (because of the frame's turn-around) and it has been explained how this can be corrected using the Boughn effect. Thus the genesis of the paradox lies in the mistake in the reasoning by Beatrice who näively use the TDF on $\Delta t_B(B)$ to draw inference regarding the proper time of Adam's clock. Once this is recognized the problem gets dissolved fully in the context of SR.

3.5 Appendix

Correction to coordinate time:

A careful analysis is required to understand the precise role of BE quantitatively in estimating the (coordinate) time of A's trip with respect to B.

Rewriting Eq.(3.5) as

$$t'_{\bar{A}} = t'_B - \delta t'_{desync}, \tag{3.21}$$

where $\delta t'_{desync}$ is given by Eq.(3.17),

one can interpret the equation by saying that with respect to standardly synchronised clocks in Σ', $\bar{A}$-clock as if, has been put behind the B-clock by the amount $\delta t'_{desync}$ when these clocks arrive stationary in Σ'. In reality nothing happens to these clocks during the brief period of acceleration. Indeed if one wishes to treat these clocks as Einstein-synchronized coordinate clocks in Σ' one requires to resynchronize them by hand.

During the return trip, A leaves the clock $\bar{A}$ at some time and meets B at a later time. The elapsed coordinate time for this trip is the difference between these times. If for any pair of events the time difference (B-clock time minus A-clock time) with respect to spatially separated Einstein-synchronized clocks is X', the same with respect to these clocks before resynchronization (that is if one continues with the synchronization in Σ) will clearly be more as $\bar{A}$-clock carries an offset $-\delta t'_{desync}$.

This is given by

$$X = X' + \delta t'_{desync}. \tag{3.22}$$

For the outward trip of A the coordinate time of travel with respect to Σ is

$$\Delta t_1 = L/v, \tag{3.23}$$

for the return trip the same in Σ' (with respect to resynchronized clocks) is again given by

$$\Delta t'_2 = L/v, \tag{3.24}$$

but this is not the lapse of coordinate time with respect to Σ.

Clearly after undoing the resynchronization, i.e, when $\bar{A}$ and B clocks are left alone and these untampered clocks are used as coordinate clocks, the coordinate time for A's return trip then should read, following Eq.(3.22) as $\Delta t_2 = \Delta t'_2 + \delta t'_{desync}$, and the same for the entire trip of A is given by

$$\Delta t_B^{coord}(B) = \Delta t_1 + \Delta t_2 = \Delta t_1 + \Delta t'_2 + \delta t'_{desync} = \Delta t_B(B) + \delta t'_{desync}, \tag{3.18}$$

where we have put $\Delta t_B(B) = \Delta t_1 + \Delta t_2$.

References

[1] S. P. Boughn, *Am. J. Phys.* **57**(9), 791-793 (1989).

[2] R. H. Price and R. P. Gruber, *Am. J. Phys.* **64**(8), 1006-1008 (1996).

[3] E. A. Desloge and R. J. Philpott, *Am. J. Phys.* **59**, 280-281 (1991).

[4] M. Redhead and T. A. Debs, *Am. J. Phys.* **64**(4), 384-392 (1996).

[5] D . F. Styer, *Am. J. Phys.* **75**(9), 805-814 (2007).

[6] C. S. Unnikrishnan, *Current Science.* **89**(12), 2009-2015 (2005).

[7] S. K. Ghosal, D. Mukhopadhyay, and Papia Chakraborty, *Eur. J. Phys.* **15**, 21-28 (1994).

[8] T. Sjödin, *Il Nuovo Cimento B* **51**, 229-245 (1979).

[9] R. Anderson, I. Vetharaniam and G. E. Stedman, *Phys. Rep.* **295**, 93-180 (1998).

[10] F. Selleri, *Found. Phys.* **26**, 641-664 (1996).

[11] S. K. Ghosal, Biplab Raychaudhuri, Anjan Kumar Chowdhuri and Minakshi Sarker, *Found. Phys. Lett.* **17**(5), 457-478 (2004).