Chapter 4

APPLICATION TO NUMBER THEORY: PRIME NUMBER THEOREM

4.1 A Brief History

Analytic number theory is a branch of number theory that uses methods from mathematical analysis to solve problems about natural numbers [45, 47]. The modern study of analytic number theory may be said to have begun in the eighteenth century with Euler's proof of the divergence of the series of inverse primes $\sum \frac{1}{p} = \infty$ and later with Dirichlet's introduction of Dirichlet L-functions in the first half of the nineteenth century to give the first proof of Dirichlet's theorem on arithmetic progressions [45]. Another major interest in this subject is the Prime Number Theorem.

Here we shall focus mainly on the Prime Number Theorem (PNT).

Statement of Prime Number Theorem :

Let $\Pi(x)$ be the prime counting function that gives the number of primes less than or equal to x, for any positive real number x. For example, $\Pi(10) = 4$ because there are four prime numbers (2,3,5 and 7) less than or equal to 10. The PNT then states that the limit of the quotient of the two functions $\Pi(x)$ and $x/\ln(x)$ as x approaches infinity is 1, which is expressed by the formula $\lim_{x\to\infty} \frac{\Pi(x)}{x/\ln(x)} = 1$, known as the asymptotic law of distribution of prime numbers. Using asymptotic notation this result can be restated as $\Pi(x) \sim x/\ln(x)$.

This notation (and the theorem) does not say anything about the limit of difference of the two functions as x approaches infinity. Indeed, the behavior of this difference is very complicated and related to the Riemann hypothesis (RH) [47]. Instead, the theorem states that $x/\ln(x)$ approaches $\Pi(x)$ in the sense that the relative error of this approximation approaches 0 as x approaches infinity. According to the RH , the relative correction (error) should be given by $\Pi(x) \times (x/\ln(x)) = 1 + O(x^{-\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$. So far no proof of the PNT could retrieve and substantiate the RH correction term, although all the current experimental searches on primes are known to agree with the RH value [47].

The prime number theorem is equivalent to the statement that the nth prime number p_n is approximately equal to $n\ln(n)$, again with the relative error of this approximation approaching 0 as n approaches infinity.

Based on some deep results derived on 1859 by B. Riemann on the relationship of PNT and the complex zeros of the Riemann Zeta function, the first proof of the PNT was given independently by J Hadamard and de la Vallee Poussin on 1896 using methods of advanced theory of complex analysis. The first *elementary* proof of the PNT without using Complex analysis was obtained by A. Selberg and P. Erdos on 1949.

4.2 New Elementary Proof

4.2.1 Introduction

We present a new proof of the PNT [13]. We call it elementary because the proof does not require any advanced techniques from the analytic number theory and complex analysis. Although the level of presentation is truly elementary even in the standard of the elementary calculus, except for some basic properties of non-archimedean spaces [42, 48], some of the novel analytic structures that have been uncovered here seem to have significance not only in number theory but also in other areas of mathematics, for instance the non commutative geometry [49], infinite trees [50] and network [51] and emergence of nonlinear complex structures.

The proof of the PNT is derived on the scale invariant, non-archimedean model **R** of real number system R, involving non-trivial infinitesimals and infinities which have been introduced in the previous chapter. The model **R** is realized as a completion of the field of rational numbers Q under a new non-archimedean absolute value $||.||$, which treats arbitrarily small and large numbers separately from any finite number. The model constructed is distinct from the usual non standard models of R in two ways: (1) infinitesimals arise because of our nontrivial scale invariant treatments of small and large elements and so may be regarded members of R itself and (2) it is a completion of Q under the new absolute value. The so-called scale-invariant, infinitesimals are therefore modeled as p-adic integers X_i with $|X_i|_p < 1$, $|.|_p$ being the p-adic absolute value and is given by the adelic formula $X = X_p \prod_{q>p}(1 + Xq)$. By inversion, infinities are identified with a general p-adic number X with $|X|_p > 1$. The infinitesimals considered here are said to be *active* as the definition involves an asymptotic limit of the form $x \to 0^+$, thereby letting an infinitesimal *directed* i.e. having a direction. We show that as a consequence the value of a scale invariant infinitesimal X would undergo infinitely slow variations over p-adic local fields Q_p as a scale free real variable x^{-1}, called the *internal time variable*, approaches ∞ through the sequence of primes p. We show that these p-adic infinitesimals leaving in **R** conspire, via non trivial absolute values, to have an influence over the structure of the ordinary real number system R thereby extending it into an associated infinite dimensional Euclidean space $\mathcal{R}$, so that a finite real number r gets an infinitely small correction term given by $r_{cor} = r + \epsilon(x)||X||$, where $\epsilon(x^{-1}) = \log x^{-1}/x^{-1}$ is the inverse of the asymptotic PNT formula of the prime counting function $\Pi(x^{-1}) = \sum_{p<x^{-1}} 1$. In the ordinary analysis, there is no room for such an ϵ thus making the value of r exact, viz., $r_{cor} = r$.

The proof of the PNT in the present formulation is accomplished by proving that the value $||X||$ of a scale free infinitesimal actually corresponds to the prime counting function $\Pi(x^{-1})$ as the internal time x^{-1} approaches infinity through larger and larger scales denoted by primes p. To this end we consider an equivalent (infinite dimensional) extension $\mathcal{R}$ of R, in the usual metric topology, however, with a caveat that increments of a variable are mediated by a combination of *linear translations* *and inversions.* We show that there exist two types of inversions,, viz. the *global or growing mode* leading to an asymptotic finite order variation in the value of a dynamic variable of $\mathcal{R}$ following the asymptotic growth formula of the prime counting function. On the other hand, the *localized inversion mode* is shown to lead to an asymptotic (golden ratio) scaling to a directed (dynamic) infinitesimal and the relative correction to the PNT.

4.2.2 Dynamical Properties

To recapitulate, we note that the set R, in the presence of non-trivial scales, proliferates into the above p-adically induced extensions $\mathbf{R}_p$. We now investigate the converse question, "How do these field extensions influence the standard asymptotic behaviors in R?" Let us recall that the standard asymptotic behavior of an ordinary real variable x as it approaches 0 is that x *vanishes linearly* as $x \rightarrow 0$ (at the uniform rate 1). In the presence of non trivial scales the situation is altered significantly. The point 0 is now identified with the set $I_\delta = [-\delta, \delta], \delta = 1/p^r$, for some $r > 0$ inhabiting infinitesimals as in the Definition 1 of previous chapter. Corresponding to these infinitesimals there exists scale free (p)- infinitesimals $X_p = \lim \tilde{x}_n/\delta^n, n \rightarrow \infty$ with absolute values of the form $||X_p|| = |X_p| = |X_{rp}|_p(1 + \sigma(\eta))$ when we choose $\delta = p^{-r}$. Here, η is defined by the real variable x=$\delta(1 + \eta)$ approaching 0^+, viz., δ from the right. In the ordinary real analysis, δ =0 and the limiting value of x, viz., 0 is attained exactly. For a non zero δ, this exact value is attained by the rescaled variable $x_1 = x/\delta$, although the value attained is now 1 instead, i.e., $x_1 = x/\delta = 1$ (which also means equivalently $\log x_1 = 0$). We are thus still in the framework of the real analysis (and the computational models based on this analysis). The presence of infinitesimals of the form $\tilde{x}$ (in the conventional sense) does not appear to induce any new structure beyond those already existing in the system. The definition of scale free infinitesimals and the associated nonarchimedean absolute values now provide us with a new input.

Definition 11. *The scale free (p)- infinitesimals, in association with absolute values $|.|$, are called valued (scale-free) infinitesimals. These infinitesimals are also said to be "active" (directed), when infinitesimals of conventional non-standard models are inactive (or passive, non-directed).*

Remark 16. Because of scale free infinitesimals, the exact equality of R in the ordinary analysis is replaced by an approximate equality, for instance, the equality $x = 1$ is now reinterpreted as $x = O(1)$. In an associated non-archimedean realization $\mathbf{R_p}$ the exact equality is again realized, albeit, in the ultra metric absolute value viz., $||x|| = 1$. Further, the rescalings defined by the inversion rule (so that a variable x is replaced by the rescaled variable x/δ) accommodates also a *residual rescalings* (because of the nontrivial factor of the form $\delta^{\pm|X_p|}$ in the representations $x = \delta \times \delta^{-|X_p|}$ and $\tilde{x} = \lambda\delta \times \delta^{|X_p|}$ (c.f. Remark 7) when the absolute value of a valued infinitesimal X_p at the scale $\delta = 1/p^{-n}$ is given by $|X_p| = |X_{np}|_p(1 + \sigma(x))$, $|X_{np}|_p = p^{-n}$).

Influence of infinitesimals on R**:** We consider a class of infinitesimals $\tilde{\tilde{x}}$ as defined by $0 < \tilde{\tilde{x}} < \epsilon < \tilde{x} < \delta < x$ (ϵ is determined shortly). We show that valued infinitesimals from $(0, \epsilon]$ would affect the *ordinary value* of x non-trivially. Infinitesimals in (ϵ, δ) are (relatively) passive. As stated above, in ordinary analysis, the limit $x \to 0^+$ in the presence of a scale δ is *evaluated exactly* viz., $x_1 = x/\delta = 1$ i.e., $\log x_1 = 0 := O(\delta)$. Infinitesimals, in conventional scenario, are *passive* in the sense that their values remain always infinitesimally small in any linear process (or dynamical problem). In the present case, however, the numerically small (in the ordinary Euclidean sense) infinitesimals lying closer to 0 relative to δ, are *dominantly valued*, so as to induce a *nontrivial influence* over a finite real variable (number), because of the definitions of valued infinitesimals.

To state formally, we reinterpret *the concerned effect* of the valuation (in the sense of an absolute value), in the context of ordinary analysis, as one admitting an extension of the ordinary (positive) real line from (δ, ∞) to the larger set $(\epsilon, \delta] \cup (\delta, \infty)$ so that ordinary zero is now identified as $[0, \epsilon]$, where the value of ϵ is defined by

$\epsilon = \tilde{\tilde{x}}_1^{-1} \delta \log \delta^{-1}$ for an O(1) rescaled (renormalized) variable $\tilde{\tilde{x}}_1 = |\tilde{\tilde{x}}| = \delta^{-\alpha} > 1$ for $\alpha > 0$ (c.f. Remark 7).

Indeed, from Definitions 1,2 and Remarks 5 & 6 , a variable x approaching 0^+ is replaced by the dressed representation $x = \delta\delta^{-|\tilde{\tilde{x}}|}(1 + o(1))$ for an infinitesimal $\tilde{\tilde{x}} = \lambda\delta\delta^{|\tilde{\tilde{x}}|}(1 + o(1))$, so that the classical limit $x/\delta \approx 1$ is replaced by $x \approx \delta + |\tilde{\tilde{x}}|\delta \log \delta^{-1}$, as $\delta \to 0^+$. The presence of the extra logarithmic term now facilitates the above mentioned extension. Infact, we notice from Remark 7 that the absolute value of an infinitesimal of the form $\tilde{\tilde{x}}$ can exceed 1 for a negative valuation i.e. $|\tilde{\tilde{x}}| = \delta^{-\alpha} > 1$ for $\alpha > 0$ which is attained by a sufficiently small infinitesimal $\tilde{\tilde{x}} << \delta$ relative to the scale δ, so that the size of the linear neighborhood $[0, \delta]$ gets extended to the level $[\delta, \tilde{\epsilon}]$ for an $\tilde{\epsilon} = |\tilde{\tilde{x}}|\delta \log \delta^{-1}$. Defining $\epsilon = \tilde{\tilde{x}}_1^{-1} \delta \log \delta^{-1} < \delta$ proves the desired assertion when α is sufficiently large.

As a consequence, we shall now have $\log x_1 = O(\epsilon)$ improving the classical value $\log x_1 = O(\delta)$. In the language of a *computational model, the accuracy of the model is therefore increased to the level denoted by* ϵ. So, from Remark 7 we can write

$$x\tilde{\tilde{x}}_1 = x(|X_r|_p(1 + \sigma_p(\eta)) + o(1)) = O(\delta^2)$$

since both x and $|\tilde{\tilde{x}}| \sim O(\delta)$. Thus, fixing $r = n$, so that $|X_n|_p = p^{-n} = \delta$, we get the first correction

$$(1 + \eta)(1 + \sigma_p(\eta)) = O(1) \tag{4.1}$$

to the ordinary (classical) value of $x_1 = 1 + \eta$ from (p) infinitesimals, even for a fixed value of η.

We note that $\tilde{x}_{1p} := 1 + \sigma_p(\eta)$ (c.f. notation in Remark 7) and σ_p >0 must be of higher degree in the real variable η for a $\tilde{x}_p \in \mathbf{R}_p$. Eq.(4.1) is *interpreted as one encoding the influence of the (first order) (p) infinitesimals.* Taking into account successively the higher order (p) infinitesimals, and iterating the above steps on each rescaled variables $\tilde{x}_{1p} = O(1)$, one obtains $\log(1 + \sigma_p(\eta)) = O(\tilde{\epsilon})$, where $\tilde{\epsilon} = \tilde{x}_2 \times \tilde{\delta} \log \tilde{\delta}^{-1}$, $\tilde{\delta} = q^{-n}$, q being the immediate successor to the prime p, and so on, so that we get finally an extended version of the equality $x_1 = O(1) \in R$ when the effective influence of infinitesimals $\tilde{x}$ living in $\mathbf{R}$ on R is encoded as

$$\mathcal{X}(\eta) := (1+\eta)\prod_{q\geq p}(1+\sigma_q(\eta))(=O(1)) \tag{4.2}$$

The variable $x=(1+\eta)\in R$ is thus replaced by the modified variable $\mathcal{X}\in\mathcal{R}$ (where $\mathcal{R}$ is the infinite dimensional Euclidean (Archimedean) extension of R) and hence, in this extended framework, a solution of

$$x d\tilde{x}/dx = -\tilde{x} \tag{4.3}$$

is written, for a $x>1$, as $0<\tilde{x}(x) = (\mathcal{X}(\eta))^{-1} < 1$, which belongs to the class of nonsmooth solutions of Eq. (3.1) [15, 18]. Here, $\sigma_p' s$ take care of the residual rescalings of Ref.[15], and thereby introduce small scale variations in the value of η.

Remark 17. It is important to note that the above derivation is performed purely in the framework of the ordinary analysis, except for the fact that we make use of the special representations of x and $\tilde{x}$ as induced from the definitions of valued infinitesimals. Consequently, (i) the transitions between real and infinitesimals are interpreted as being facilitated by inversions (for instance, either as $x_- \mapsto x_-^{-1} = \tilde{x}_+$, or as $x_+ \mapsto \tilde{x}_- = x_+^{-1}$, as the case may be) as opposed to linear shifting operations only, and (ii) the non-archimedean p-adic absolute value $|X_r|_p$ generates a scale factor in the smaller scale logarithmic variables. In fact, this correspondence could be made more precise.

Proposition 12. 1. **[43, pages 14,15]** *Let $X_p \in Q_p$, so that $X_p = p^r(1+\sum_1^\infty a_i p^i)$, where a_i assumes values from $1,2,\ldots(p-1)$ and $r\in Z$. Then there exists a one to one continuous mapping $\phi: Q_p \to R_+$ given by $\phi(X_p) = p^{-r}(1+\sum a_i p^{-2i})$.*

2. **[44, pages 63-65]** *The set of p-adic integers Z_p is homeomorphic to a Cantor set C_p under the homeomorphism $\psi: Z_p \to C_p$ defined by*

$$\psi(X_p) = (2p-1)^{-r}(1+\textstyle\sum_1^\infty \frac{2a_i}{(2p-1)^{i+1}})$$

where $r>0$.

We denote $R_p = \phi(Q_p)$. It then follows that any bounded subset of R_p is a zero (Lebesgue) measure Cantor set C_p in R. Accordingly, the treatments of ordinary analysis can be extended in a scale free manner (though remaining in the framework of the usual topology of R) over a *more general metric space* $\mathcal{R}$ accommodating the above new structure. In view of Theorem 4 of previous chapter, $\mathcal{R}$ is locally a Cartesian product, viz.,$\mathcal{R}$=$R \times \prod_p R_p$.

The product space is interpreted as an hierarchical sense. An ordinary real variable x is extended over $\mathcal{R}$ as $\mathcal{X}$=$x \prod x_P$ where $x_p = 1 + \epsilon_p X_p$, $X_p \in R_P$ and $\epsilon_P \approx \delta p^{-1}\log(p\delta^{-1})$ denotes the enhanced level of accuracy because of valued infinitesimals $\tilde{x} \in \mathbf{R}$ at the scale $\tilde{\delta} = \delta p^{-1}, \delta \to 0^+$. Noting that $\epsilon_q = p/q \times \epsilon_p \to 0$, as $\delta \to 0^+$, for $q > p$, one may re-express $\mathcal{X}$ as $\mathcal{X}$=$x(1 + \epsilon X)$, where $\epsilon \approx \delta \log \delta^{-1}$ when the scale is identified with $\delta = 2^{-(n-1)}$ so that $p = 2$, and the scale free infinitesimal X now resides and varies in $\prod R_p$ in an orderly (hierarchical) manner as detailed below, as $x \to 0^+ \equiv O(\delta)$.

Let us first recall that the concept of relative infinitesimals is introduced originally in the context of a Cantor set [16, 17]. Because of the existence of relative infinitesimals, each element of a Cantor set, denoted here as C_p, for each class of (p) infinitesimals, is effectively identified with a closed and bounded interval of R_p at every level of the scale δ, as $\delta = 1/p \to 0$. In the presence of infinitesimals a Cantor set is thus realized as a compact subset of R_p except for the fact that the *motion* on C_p is now *visualized as a combination of linear shift (along a compact and connected line segment) together with an inversion in the vicinity of a Cantor point.* As a consequence, the generalized, inversion mediated metric space $\mathcal{R}$ is denoted locally as $\mathcal{R}$=$R \times \prod R_p$. A *generalized motion on* $\mathcal{R}$ *now is represented as follows.*

Definition 12. *The set* $\mathcal{R}$ *is interpreted as having several branches* R *and* R_p*,* p *being a prime. The branches* R_p *accommodating scale free Euclidean (Archimedean) infinitesimals and infinites are thought to be knotted at (the scale free number) 1. A real variable* $x \in R$ *approaching* 0 $(\equiv O(\delta))$ *from 1(say) is replaced, because of the* *scale invariance, by the scale free variable $x_1 = x/\delta$. For simplicity of notation we continue to denote x_1 by x and call it a scale free variable. So, as the scale free $x \to 1^+$, the unique linear motion is replaced by two* **inversion mediated modes***: (i) Local or vertical mode: x_+ is replaced by $x_+ \mapsto x_+^{-1} = \tilde{x}_-$ which takes note of the* **localized** *effects of (Euclidean) infinities on an (Euclidean) infinitesimal and (ii) global or horizontal (growing) mode: a scale free (Euclidean) infinitesimal $0 < \tilde{x} \in R_p$ grows infinitely slowly following the linear law until it shifts to a O(1) variable living possibly in another branch R_q by inversion $\tilde{x}_- \mapsto \tilde{x}_-^{-1} = x_+ = 1 + \tilde{\tilde{x}}$ where $0 \leq \tilde{\tilde{x}} \in R_q$.*

Remark 18. We disregard henceforth distinguishing Euclidean (Archimedean) and Non-Archimedean infinitesimals and infinities explicitly in notations. We indicate the difference whenever there is a room for confusion. The basic characteristic of an Euclidean infinitesimal is the explicit presence of a logarithmic factor.

Remark 19. The unidirectional motion of a real variable $x \in R$ approaching 0^+ thus bifurcates into two possible modes: a variable x approaching 0 ($\equiv O(\delta)$) from above will experience, as it were, a bounce at $x \approx \delta$ and so would get replaced by the inverted rescaled infinitesimal variable $\tilde{x} = \delta/x$ living in a scale free branch R_p (say). A fraction of the asymptotic limit $x \to 0^+ \equiv O(\delta)$ of R (viz., $x \mapsto x_+ = x/\delta = 1 + \eta, \eta \to 0^+$) is therefore replaced by a growing mode of the rescaled variable $\tilde{x}_-^{-1} = x_{p^+} = 1 + \eta_p, x_{p^+}, \eta_p \in R_P$ and $\eta_p \geq 0$ initially but subsequently growing to $\to 1^-$ in the branch R_p. Besides this growing mode, another fraction of the decreasing (decaying) mode of the flow in R, viz., $x \mapsto x_+ = x/\delta = 1+\eta, \eta \to 0^+$, is also available as a localized mode in another branch R_q (say) in the form $x_+^{-1} = (1 + \eta_q)^{-1} = \tilde{x}_{q^-}$, where again $\eta_q \approx 0$ initially, but grows subsequently to $\eta_q \to 1^-$ slowly. As a consequence, the limiting value 1 of the rescaled variable in R, now, gets a dynamic (multiplicative) partitioning of the form $\tilde{x}_{q^-} x_{p^+} \approx 1 \Rightarrow (1 - \mu(\eta_q)\eta_q)(1 + \eta_p) = O(1), 0 \leq \eta_p \in R_p,\ 0 \leq \eta_q \in R_q$, which equivalently can also be written more conveniently as $(1 - x^{\eta_q})(1 + \eta_p) = O(1)$. The local and global modes in $\mathcal{R}$ therefore induce, as it were, a competition between the effects generated by infinitesimals and infinities living in $\mathbf{R}$, leading to this dynamic partitioning of the unity. The localized factor $(1 - x^{\tilde{X}})$ arising from active infinitesimals $\tilde{X}$ living in a branch of $\mathcal{R}$ will lead to an asymptotic scaling of any scale free (locally constant) variable $\tilde{\mathcal{X}} = \mathcal{X}/x$ in $\mathcal{R}$ (see Sec. Scaling(4.2.4)). Accordingly, $d\tilde{\mathcal{X}}/dx = 0$ and hence $\tilde{X} \in \prod R_p$ satisfies the scale free equation

$$\log x \frac{d\tilde{X}}{d \log x} = -\tilde{X} \tag{4.4}$$

Consequently, asymptotic limits either of the forms $x \to 0^+$ or $x \to \infty$, in R would ultimately behave as a *directed* (monotonically increasing) variable in $\mathcal{R}$. Moreover, as $x^{-1} \to \infty$, the ordinary linear motion of $x \to 0$ will undergo small scale mutations, because of zigzag motion of the inverted variables $\tilde{x}_p (\sim O(1))$, living successively in the rescaled branches R_p and mapping recursively the smaller and smaller neighborhoods of 0 to the smaller and smaller neighborhoods of 1. As a consequence, extended real numbers $\mathcal{X}$ of $\mathcal{R}$ are *directed*, since each of the (p) infinitesimals are, by definition, directed.

Definition 13. Intrinsic (Internal) Time*: A continuous monotonically increasing variable* $\tilde{x}$ *living in the product space* $\prod R_p$*, from the initial value 1, is called an internal evolutionary time. The rate of variations of* $\tilde{x}$ *is infinitely small because of the presence of scale factors of the form* $\delta p^{-1}, \delta \to 0^+$.

Any variable $X \in \prod R_p$ *is called dynamic since it undergoes spontaneous changes (mutations) relative to the (scale free) internal time* $\tilde{x}$.

With this dynamic interpretation of $\mathcal{R}$, it now follows that the new solution constructed in Eq.(4.2) is indeed smooth in $\mathcal{R}$ (as it is evident from the derivation). However, because of the presence of the irreducible O(1) correction factors this solution can not be accommodated in the ordinary analysis (i.e., even in the context of $\mathcal{R}$) in an exact sense. In the non-archimedean extensions $\mathbf{R}_p$, such a solution is not only admissible and smooth but also exact, in the sense of absolute values, viz., $||X|| = 1$, since $||x|| = ||\tilde{x}_i|| = 1$ for each i, thus retrieving the ordinary equality $|x_i|_e = 1$ in the ultra metric sense.

We restate the above deductions as the following Lemma.

Lemma 12. *The ordinary analysis on R is extended over $\mathcal{R}$ with new structures as detailed above (Definition 12, Remark 19). In this extended formalism accommodating dynamic infinitesimals, $0 \in R^+$ (the set of positive reals) is identified with $[0, \epsilon], \epsilon = x_1^{-1}.\delta \log \delta^{-1}$, where $x_1 = x/\delta$ and $x \rightarrow \delta^+$. As a consequence, a constant in R becomes a variable over infinitesimals of $\mathcal{R}$.*

Proof. We have already seen in the above that $0 \in R^+$ is extended over $[0, \epsilon]$ in $\mathcal{R}$. We justify it further by showing that an equation of the form

$$\frac{d\phi}{dx} = 0 \tag{4.5}$$

for finite real values of x is transformed into

$$\frac{d\log\phi}{d\log x} = O(1) \tag{4.6}$$

for a relative infinitesimal $\tilde{x}$ satisfying $x/\delta = \lambda\delta/\tilde{x} = \delta^{-||\tilde{x}||}, 0 < \tilde{x} < \delta \leq x, x \rightarrow 0^+, \lambda > 0$ and $||\tilde{x}|| = \tilde{x}_1 (= |\tilde{x}|_p(1+\sigma(\eta)))$, when one interprets 0 in relation to the scale δ as $O(\frac{\delta^2}{x}\log\delta^{-1})$. Indeed, we first notice that Eq.(4.5) means $d\phi = 0 = O(\delta), dx \neq 0$ for an ordinary real variable. However, as $x \rightarrow \delta$, that is, as $dx = \eta \rightarrow 0 = O(\epsilon)$ (i.e. η is an infinitesimal in the present sense lying in $[0, \epsilon]$), the ordinary variable x gets replaced by the above extended variable, so that $d\log_{\delta^{-1}}(x/\delta) = d\tilde{x}_1 = O(\epsilon)$. As a consequence, in the infinitesimal neighborhood of 0, Eq. (4.5) is transformed into an equation of the form Eq. (4.6); since in that neighborhood, x and ϕ, are both represented as $x{=}\delta.\delta^{-\tilde{x}_1}$ and $\phi = \phi_0\delta^{k\tilde{x}_1}$ for a real constant k, whence we get Eq.(4.6).

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Before proving the PNT in the present dynamic extension of R, we need two more ingredients: viz., the origin of the prime counting function and the asymptotic scaling of active infinitesimals.

4.2.3 Prime Counting Function

The prime counting function arises in connection with the growing mode of a dynamic variable in $\mathcal{R}$, when we *assume* that a scale free variable x varies over all possible prime-adic branches R_p in an *orderly manner following the order of the primes.*

Recall that the usual $\epsilon - \delta$ definition of limit (in R) does not characterize explicitly the actual motion of a real variable x (in fact, it is taken in granted that x varies in uniform rate 1) approaching a fixed number, 0, say. In the present formalism infinitesimals are defined by *intermediate* asymptotic scaling formulas (which would ordinarily correspond simply to zero), and so may be considered to carry *an evolutionary arrow.* A limit of the form $x \to 0$ in $\mathcal{R}$ would be interpreted in the context of a dynamical problem, so that x^{-1} may be identified with the (physical) time. More precisely, when the ordinary R component of $\mathcal{R}$=R$\times \prod R_p$ is free of any arrow, the *non-trivial components* R_p do *carry an evolutionary arrow.* A problem involving the asymptotic limit $x \to 0^+$ (equivalently $x \to \infty$) in ordinary analysis is raised in the present context over $\mathcal{R}$ as the asymptotic limit of the extended variable $\mathcal{X}=x(1+\epsilon X)$ where X is an O(1) growing dynamic variable which lives hierarchically in the sets R_p, $p = 2, 3, 5, \ldots$ as explained in Definition 12 and Remark 19 (a growing dynamic infinitesimal η is represented now as $\eta = \epsilon X$).

Thus *the ordinary limit of* $\mathcal{X}$ *as* $x \to 0^+$, *that is,* $\mathcal{X} = 0$, *is interpreted in the present context as* $\log(\mathcal{X}/x) = \log(1 + \epsilon X) = O(\epsilon \Pi(x^{-1}))$ *as* $x \to 0^+$ *hierarchically through scales* δp^{-1}.

In fact, as pointed out above (in Remark 19), as $x \to \delta^+$, it changes over to various branches R_p by assuming the guise of several variables $\tilde{x}_i's$ (all of which are different R_p valued realizations of X) having the forms $\tilde{x}_i = 1 + \eta_i, x_0 \equiv x$ and i runs over the primes. Consequently, as $\tilde{x}_2 = (\lambda_1)x_1^{-1} \in R_2$, $x_1 = x/\delta$ approaches $1/2^-$, we get the next level variable $\tilde{x}_3 = (2\lambda_2)\tilde{x}_2^{-1} \in R_3$ and so on and so forth, adding one unit to the prime counting function Π at every change of the prime-adic scale. Indeed, infinitesimal η_i grows linearly (and spontaneously) to O(1) whence it undergoes inversion mediated transition of the form $\eta_{i-} \mapsto \eta_{i-}^{-1} = 1 + \eta_j$, where j being the next prime. As a consequence, we have

Theorem 5. *The ordinary (linear) limiting behavior of a real variable* $x \to 0^+$ *in the real number set* R *is raised, in the inversion mediated set* $\mathcal{R}$*, to the asymptotic limit of the extended variable* $\mathcal{X} = x(1+\epsilon X)$*, where the O(1) dynamic variable* X *is realized in relation to every secondary (prime-adic) scale* $1/p$ *as a variable of the form* $x_p \geq 1$*. As a consequence, the asymptotic limit of* $\mathcal{X} = X/x$ *as* X $(\equiv x_p) \to 1/p, p \to \infty$ *is given by* $\log(\mathcal{X}/x) = O(\epsilon\Pi(x^{-1}))$*, for a locally constant infinitesimal* $\epsilon = O(\delta \log \delta^{-1}) = O(x \log x^{-1})$*, when the real variable* $x^{-1} \to \infty$

In the next section we consider the scaling of $\mathcal{X}(x) \in \mathcal{R}$ as $x \to 0$ in R.

4.2.4 Scaling

Let us begin by recalling that the main characteristic of both the inverted motions is the inherent directed sense. That is to say, although $x_{1+} = 1 + \eta, \eta \downarrow 0^+$ in R, in either of the inverted motions, we have however, $x_{1+} = 1 + \tilde{\eta}, \tilde{\eta} \approx 0$, initially, but $\tilde{\eta} \uparrow 1^-$, slowly, when $x_{1+} \in R_p$. As shown in the above sections, the growing mode induces the global evolutionary sense leading to the prime counting function. Here we study the local motion leading to the asymptotic scaling for a small scale variable $\tilde{\mathcal{X}} \in \mathcal{R}$.

Because of the valued infinitesimals in **R** that contribute non-trivially to the ordinary value of an arbitrarily small $x \in R$, the scaling behavior of the corresponding extended variable $\mathcal{X} \in \mathcal{R}$ is also nontrivial. As explained above, an ordinary, arbitrarily small $x \in R$ is extended in $\mathcal{R}$ as $\mathcal{X}/x = (1 - O(x^{\tilde{X}(x^{-1})}))\phi(x^{-1})$, for a class of (Euclidean) infinitesimals $\tilde{X}(x^{-1})$. Our aim here is to estimate $\lim \tilde{X}$ as $x \to 0$. Notice that the (-) sign in the first factor makes it a true dynamic infinitesimal living in R_p. The second factor ϕ corresponds to the growing mode of a dynamic infinitesimal and is considered in Theorem 5.

We recall that the above limit may have a constant (non zero)(ultra metric) value. Indeed, as x approaches 0, following δ, the ordinary variable x gets extended to the rescaled variable $\tilde{\mathcal{X}}_- = \mathcal{X}_-/x = 1 - O(x^{\tilde{X}})$, which now approaches 0^+, via a combination of inversions and translations. Indeed, as $x \to 0$ in R, $\tilde{\mathcal{X}}_-$ in $\mathcal{R}$ is realized as a locally constant function satisfying $d\tilde{\mathcal{X}}_-/dx = 0$ so that $\tilde{X} \in \prod R_p$ now satisfies the Eq.(4.4), i.e.

$$\log x \frac{d\tilde{X}}{d\log x} = -\tilde{X} \tag{4.7}$$

and changes from one copy of R_p to another near the scale $1/p$ by inversions via a sequence of distinct realizations $\tilde{x}_i$, i being a prime. To see in detail, let $\tilde{\eta}_- = x^{X(x^{-1})}$. As $x \to 0 \equiv O(\delta)$ linearly and the motion should have terminated at δ in R, now, instead is picked up by the rescaled variable which shifts by inversion to $\tilde{\eta}_{2-} = x^{2\tilde{x}_2}, \tilde{x}_2 \approx 0$. The limiting motion is now transmitted over to the next generation variable $\tilde{x}_2 \in R_2$, which grows to $1/2^-$ linearly, until the motion is again transferred to the next level by inversion viz., $2\tilde{x}_2 = 1/(1+3\tilde{x}_3)$, where $\tilde{x}_3(\approx 0) \in R_3$. Recall that this (and the following) local inversions essentially inject into an infinitesimal *higher order influences from infinities.* The new rescaled variable $\tilde{x}_3$ now grows to $1/3^-$ and transmits its motion to $\tilde{x}_5 \in R_5$ near $\tilde{x}_5 \to 1/5^-$ by inversion, and so on successively over all the higher prime-adic scales. The exponent in $\tilde{\eta}_-$ now asymptotically assumes the form of the *golden ratio continued fraction*, i.e., $\tilde{\eta}_{\infty-} = x^{\frac{1}{1+\frac{1}{1+\dots}}}$, so that the exponent has the value $\nu = \frac{1}{1+\frac{1}{1+\dots}} = \frac{\sqrt{5}-1}{2}$ and therefore $\tilde{\eta}_{\infty-} = x^{\nu}$. As a consequence, the asymptotic small scale variations (mutations) in the dynamic infinitesimal follow a generic golden ratio scaling exponent [20].

Combining this local asymptotic scaling together with the global asymptotic of Theorem 5, one finally arrives at the asymptotic law.

Theorem 6. *The generic asymptotic behavior of a dynamic variable $\mathcal{X} \in \mathcal{R}$, extending the ordinary real variable x, is given by*

$$\log \mathcal{X}/x = \epsilon O(\Pi(x^{-1}))(1 - O(x^{\nu})) \tag{4.8}$$

as $x^{-1} \to \infty$.

The above asymptotic formula is the main result of this Chapter. Over any finite (time) x scale, the right hand side effectively reduces to zero, recovering the standard variable $\mathcal{X} = x$. However, in any dynamic process which persists over many (infinitely large) (time) scales, the correction factor may become significant leading to a finite observable correction to the evolving quantity $\mathcal{X} = xe^{O(1)}$ which may arise from the annihilation (cancellation) of the infinitesimal (locally constant variable) ϵ by the growing mode of the prime counting function. The proof of the PNT now follows as a corollary to the Theorem 6.

4.2.5 Prime Number Theorem

The locally constant infinitesimal $\epsilon(x^{-1}) = O(x \log x^{-1})$ clearly corresponds to the inverse of the PNT asymptotic formula for the prime counting function $\Pi(x^{-1})$. The O(1) correction to any dynamic variable $\mathcal{X} \in \mathcal{R}$ is realized for a sufficiently large value of x^{-1} provided

$$\epsilon(x^{-1})\Pi(x^{-1}) = (1 + O(x^{\nu})), x^{-1} \to \infty \tag{4.9}$$

with the relative correction (error) $\Pi(x^{-1})\epsilon(x^{-1}) - 1 = O(x^{\nu})$, which clearly respects the Riemann's hypothesis since $x^{\nu} \leq Mx^{(1/2-\sigma)}$ for a suitable M >0 and for any $\sigma > 0$, $x \to 0^{+}$. $\square$

This completes the derivation of the PNT on a deformed real number system $\mathcal{R}$ accommodating scale invariant infinitesimals and the inversion induced nonlinear jump modes for infinitesimal increments. We close this section with another application of the scale invariant formalism to the prime counting function $\Pi(x)$. We recall that $\Pi(x)$ has by definition the structure of an irregular step function (i.e. a devil's straicase function).

Proposition 13. *The prime counting function* $\Pi(x)$ *is a locally constant function on* $\mathcal{R}$.

To prove this, let us first consider the step function

$$f(x) = \begin{cases} a, & 0 < x < p, \\ b, & x > p \end{cases} \tag{4.10}$$

with a finite discontinuity at $x = p$ in the usual sense. In the present scale invariant formalism with inversion mode for increments, we now show that f solves $x\frac{df}{dx} = 0$ every where, that is, even at $x = p$. As x increases toward p from the left linearly, the graph of f is a straight line parallel to the x-axis. In the left neighborhood of p, $x = p - \eta = px_-$, $x_- = 1 - \eta/p$. Analogously, in the right neighborhood, we have $x = p + \tilde{\eta} = px_+$, $x_+ = 1 + \tilde{\eta}/p$, so that $x_+ = x_-^{-1}$. Let us assume that η and $\tilde{\eta}$ are sufficiently small, so that the point set $\{p\}$ is identified with the closed interval $I_p = [1 - \eta/p, 1 + \tilde{\eta}/p]$, and so defines the accuracy level of a given computational problem. The interval I_p corresponds to an infinitesimally small neighborhood of p. At the level of this infinitesimal scale, the function f is interpolated by the scale invariant formula

$$\tilde{f}(x) = \begin{cases} a, & 0 < x < px_-, \\ a + (b-a)\phi_p(x), & x \in pI_p, \\ b, & px_+ < x. \end{cases} \tag{4.11}$$

where $x = (p - \eta) + (\tilde{\eta} + \eta)\tilde{x}$, $0 \leq \tilde{x} \leq 1$. Clearly, $\tilde{f}(x) = f(x)$, in the limit η, $\tilde{\eta} \to 0$. Moreover, $x\frac{d\tilde{f}}{dx} = 0$ everywhere, including $x = p$, since the locally constant Cantor function $\phi_p(x)$ on I_p does. It follows, therefore, that as x approaches to p from left and arrives at a point of the form $x = px_-$, it switches smoothly to $x = px_+$ at the right of p by inversion $x_-^{-1} = x_+$. The associated value of the function f i.e. a, however, changes over to b by a cascade of smaller scale self similar smooth jumps as represented by the Cantor function $\phi_p(x)$. The cost of this smoothness, however, is the arbitrariness in the formalism that is introduced via the arbitrariness of the choice of the Cantor function.

The prime counting function $\Pi(x)$ is a step function in the neighborhood of every prime. Hence the result. $\square$