Chapter 3

NON-ARCHIMEDEAN EXTENSION OF REAL NUMBER SYSTEM

3.1 Introduction

The main objective of the present chapter is to study the formulation of a *scale invariant analysis* that aims at developing a coherent framework for analysis on the real line R as well as on Cantor like fractal subsets of R [13, 14]. The formulation of a scale invariant analysis was motivated by an effort in justifying the construction of the so called non-smooth (i.e. higher derivative discontinuous) solutions [15] of the simplest scale invariant Cauchy problem

$$x\frac{dX}{dx} = X, X(1) = 1 \tag{3.1}$$

in a rigorous manner. It is clear that the framework of classical analysis, because of Picard's uniqueness theorem, can not rigorously accommodate such solutions, except possibly only in an *approximate* sense. To bypass the obstacle, it becomes imperative to look for a non-archimedean extension of the classical setting, thus allowing for existence of non-trivial *infinitesimals* (and hence, by inversion, *infinities*). Robinson's original model of non-standard analysis appeared to be unsatisfactory, because (1) infinitesimals here are infinitesimals even in "values", since the value of an infinitesimal is the usual Euclidean value and (2) these are new extraneous elements in R. Although, the non-standard $*R$ is of course non-archimedean, but still an infinitesimal behaves more in a "real number like" manner; that is to say, in essence, it fails to have an identity, except for its *infinitesimal* Euclidean value. Such non-standard infinitesimals are known to generate proofs of harder theorems of mathematical analysis in a more intuitively appealing manner. Further, any new theorem proved in the non-standard approach is expected to have a classical proof, though, may be, using lengthier arguments. Justifying a higher derivative discontinuous solution of (3.1), therefore, appeared to be difficult even in the conventional non-standard analysis. To counter this problem, we develop a novel non-archimedean extension $\mathbf{R}$ of R by completing the rational number field Q under a novel *ultra metric* which treats arbitrarily small and large rational (and real) numbers [13] *separately.* The ultra metric reduces to the usual Euclidean value for finite real numbers, but, nevertheless, leads to a *new definition of scale invariant infinitesimals* in the present context.

Another motivation (as already mentioned) for this Chapter is to formulate "motion" (variation of a quantity) in a smooth differentiable sense on a zero or a positive measure Cantor like fractal sets which arise copiously in complex system studies.

3.2 Non-Archimedean model

Infinitesimals: Let $*R$ be a non-standard extension (c.f. Sec. 2.2) of the real number set R. Let $\mathbf{0}$ denote the set of infinitesimals in $*R$. Then an element of $*R$, denoted as $\mathbf{x}$, is written as $\mathbf{x}=x+\tau$, $x \in R$ and $\tau \in \mathbf{0}$. The set $\mathbf{0}$ and hence $*R$ is linearly ordered that matches with the ordering of R. The set $\mathbf{0}$ is thus of cardinality c, the continuum. The non zero elements of $\mathbf{0}$ are new numbers added to R which are constructed from the ring $\mathbf{S}$ of sequences of real numbers via a choice of an *ultra filter* to remove the zero divisors of $\mathbf{S}$. A non-standard infinitesimal is realized as an equivalence class of sequences under the ultra filter and may be considered extraneous to R. The magnitude of an element $\mathbf{x}$ of $*R$ is evaluated using the usual Euclidean absolute value $|\mathbf{x}|_e$.

We now give a *new* construction relating infinitesimals to arbitrarily small elements of R in a more *intrinsic* manner. The words "arbitrarily small elements" are made precise in a limiting sense in relation to a *scale.* The infinitesimals so defined are called *relative infinitesimals* [16, 17].

Definition 1. *Given an arbitrarily small positive real variable $x \to 0^+$, there exists a rational number $\delta > 0$ and a set $I_{\rm in}^+$ of positive reals $\tilde{x}(x) = \tilde{x}(x, \lambda)$ satisfying $0 < \tilde{x}(x) < \delta < x$ and the inversion rule*

$$\frac{\tilde{x}(x)}{\delta} = \lambda(\delta)\frac{\delta}{x}, \tag{3.2}$$

where $0 < \lambda(\delta) << 1$, is a real constant, so that $\tilde{x}$ also satisfies the scale invariant equation

$$x\frac{d\tilde{x}}{dx} = -\tilde{x} \tag{3.3}$$

The elements $\tilde{x}(x)$ so defined are called relative infinitesimals relative to the scale δ. A necessary condition for relative infinitesimals is that $0 < \tilde{x}_1 < \tilde{x}_2 < \delta$ means $0 < \tilde{x}_1 + \tilde{x}_2 < \delta$. A relative infinitesimal $\tilde{x}$ is negative if $-\tilde{x}$ is a positive relative infinitesimals. Further, the associated scale invariant infinitesimal corresponding to the relative infinitesimal $\tilde{x}$ is defined by $\tilde{X} = \lim\limits_{\delta\to 0}\frac{\tilde{x}}{\delta}$.

Now, because of linear ordering of $\mathbf{0}^+$, the set of positive infinitesimals of $*R$, that is inherited from R, and the fact that the cardinality of $\mathbf{0}^+$ equals that of R, there is a one-one correspondence between $\mathbf{0}^+$ and $(0, \delta) \subset R$, which we can write as $\tau(\tilde{x}) = \tau_0(\tilde{x}/\delta)$ for an infinitesimal $\tau_0 \in \mathbf{0}^+$ and a relative infinitesimal $0 < \tilde{x} < \delta$, $\delta \to 0^+$. This may be interpreted as by saying that for each arbitrarily small $\delta > 0$, there exists in the non-standard $*R$ an infinitesimal $\tau_0 \in \mathbf{0}^+$ so that the dimensionless equality of the form τ/τ_0=$\tilde{x}/\delta$ holds good independent of the scale δ. We, henceforth identify $\mathbf{0}^+$ with the set of relative infinitesimals $I_{\rm in}^+$ in $I_\delta^+ = (0, \delta) \subset R$ so that $I_{\rm in}^+ \subset I_\delta$. We use symbols $\mathbf{0}$ and $I_{\rm in}$ interchangeably henceforth to refer the set of (relative) infinitesimals. We remark that in this framework, a positive *real* variable x is defined relative to the scale δ by the condition $x > \delta$.

Infinitesimals , so modeled, will be assigned with a new absolute value. The real number set R equipped with this absolute value (denoted henceforth by $\mathbf{R}$) will be shown to support naturally the generalized class of solutions of eq(3.1).

Definition 2. *A relative infinitesimal $\tilde{x} \in I_{\rm in} \subset I_\delta = (-\delta, \delta)$ $(\neq 0)$ is assigned with a new absolute value, $v(\tilde{x}) := \|\tilde{x}\| = \lim\limits_{\delta\to 0} \log_{\delta^{-1}} \tilde{x}_1^{-1}$, $\tilde{x}_1 = |\tilde{x}|_e/\delta$. We also set $\|0\| = 0$.*

Remark 1. We observe that there exists a nontrivial class of infinitesimals (viz., those satisfying $|\tilde{x}|_e \leq \delta.\delta^\delta$) for which the value $|\tilde{x}|$ assigned to an infinitesimal $\tilde{x}$ is a real number, i.e., $|\tilde{x}| \geq \delta$. One of our aims here is to point out nontrivial influence of these infinitesimals in real analysis. This is to be contrasted with the conventional approach. The Euclidean value of an infinitesimal in Robinson's non-standard analysis is numerically an infinitesimal. Further, the limit $\delta \to 0^+$ is, of course, considered in the above definitions in the Euclidean metric.

We also notice that the inversion in Definition 1 is nontrivial in the sense that in the absence of it, the scale δ can be chosen arbitrarily close to an infinitesimal $\tilde{x}$ (say), so that letting $\delta \to \tilde{x}$, which, in turn, $\to 0^+$, one obtains $|\tilde{x}| = 0$. Thus, dropping the inversion rule, we reproduce the ordinary real number system R with zero being the only infinitesimal.

Clearly, the above absolute value is well defined and also scale invariant. For, even as $\delta \to 0^+$ the relative ratio $\eta = x/\delta$ might be a constant (or approaches zero at a slower rate) in (0,1), so that Definition 2 can yield non-trivial values.

Remark 2. An infinitesimal $\tilde{x} \in I_\delta$ has a countable number of different realizations, each for a specific choice of the scale δ, having valuation $|\tilde{x}|_\delta$. Indeed, given a decreasing sequence of (*primary*) scales δ_n so that $\delta_n \to 0$ as $n \to \infty$, the limit in the Definition-2 can instead be evaluated over a sequence of *secondary* smaller scales of the form δ_n^m, $m \to \infty$ for each fixed n.

This observation allows one to extend that definition slightly which is now restated as

Definition 3. *i) Scale free (invariant) infinitesimals* $\tilde{X}_\delta = \tilde{x}_n/\delta^n$ *satisfying* $0 < \tilde{x}_n < \delta^n$ *(and* $\tilde{x}_n/\delta^n = (\tilde{x}/\delta)^n$ *as* $n \to \infty$*) are called (positive) scale-free* δ*-infinitesimals. By inversion, elements of* $|\tilde{X}_\delta^{-1}|_e > 1$ *are scale free* δ*-infinities. (ii) A relative* (δ) *infinitesimal* $\tilde{X}_\delta (\neq 0) \in I_\delta$ *is assigned with a new* (δ *dependent) absolute value* $v(\tilde{x}) = |\tilde{x}|_\delta = \lim_{n\to\infty} \log_{\delta^{-n}} (\tilde{x}_n/\delta^n)^{-1}$. *(In this scale free notation, all the finite real numbers* *are mapped to 1. We denote this set of δ infinitesimals and infinities by R_δ.)*

Clearly, the above definition yields $|\tilde{x}|_\delta = \log_{\delta^{-1}}(\delta/\tilde{x})$ for each fixed δ, which will have important applications in the following. The Euclidean absolute value, however, is uniquely defined $|x|_e = x,\ x > 0$. We also notice that for a (non-zero) real number $x \in R$, the only meaningful scale is $|x|$. It therefore makes sense to identify all finite real numbers to 1 in R_δ.

Proposition 7. *$|.|_\delta$ defines a nonarchimedean semi-norm on* $\mathbf{O}$.

Remark 3. To simplify notations, $|.|_\delta$, is written often as $|.|$. The $\delta-$ infinitesimals are also denoted simply by $\tilde{x}$. By a semi-norm we mean that $|.|$ satisfies three properties (i) $|\tilde{x}| > 0,\ \tilde{x} \neq 0$, (ii) $|-\tilde{x}| = |\tilde{x}|$ and (iii) $|\tilde{x}_1 + \tilde{x}_2| \leq \max\{|\tilde{x}_1|, |\tilde{x}_2|\}$. Property (iii) is called the *strong ultra metric triangle* inequality. Note that this definition of semi-norm on a set differs from the semi-norm on a vector space. However, this suffices our purpose here.

Proof. The first two are obvious from the definition. For the third, let $0 < \tilde{x}_2 < \tilde{x}_1$ in $\mathbf{O}$. Then there exists $\delta > 0$ so that $0 < \eta_2 < \eta_1 < 1$ where $\eta_i = \tilde{x}_i/\delta^n \neq 1$ and $|\tilde{x}_i| = \log_{\delta^{-n}} \eta_i^{-1},\ n \to \infty$. Clearly, $|\tilde{x}_2| > |\tilde{x}_1|$. Moreover, $0 < \eta_2 < \eta_1 < \eta_1 + \eta_2 < 1$. By Definition 2, we thus have $|\tilde{x}_1 + \tilde{x}_2| = \log_{\delta^{-n}}(\eta_1 + \eta_2)^{-1} \leq \log_{\delta^{-n}} \eta_2^{-1} \leq |\tilde{x}_2|$. Moreover, $|\tilde{x}_1 - \tilde{x}_2| = |\tilde{x}_1 + (-\tilde{x}_2)| \leq \max\{|\tilde{x}_1|, |\tilde{x}_2|\} = |\tilde{x}_2|$. ■

Now, to restore the product rule, viz., $|\tilde{x}_1\tilde{x}_2| = |\tilde{x}_1||\tilde{x}_2|$, we note that given $\tilde{x}$ and δ ($0 < \tilde{x} < \delta$), there exist $0 < \sigma(\delta) < 1$ and $a : \mathbf{O} \to R$ such that

$$\frac{\tilde{x}_n}{\delta^n} = (\delta^n)^{\sigma^{a(\tilde{x})}} \tag{3.4}$$

so that, in the limit $n \to \infty$, we have

$$v(\tilde{x}) = |\tilde{x}| = \sigma^{a(\tilde{x})}$$

For definiteness, we choose $\sigma(\delta) = \delta$ (this is justified later). The function $a(\tilde{x})$ is a (discretely valued) valuation satisfying (i) $a(\tilde{x}_1\tilde{x}_2) = a(\tilde{x}_1) + a(\tilde{x}_2)$ and (ii) $a(\tilde{x}_1 + \tilde{x}_2) \geq \min\{(a(\tilde{x}_1), a(\tilde{x}_2)\}$. As a result, $|\tilde{x}_1\tilde{x}_2| = |\tilde{x}_1||\tilde{x}_2|$ and hence we have deduced.

Proposition 8. $|.|$ *defines a nonarchimedean absolute value on* $\mathbf{0}$.

Remark 4. The above definition of valuation (3.4) can be extended further to include an extra piece in the exponent, viz.,

$$\tilde{x}_n/\delta^n = (\delta^n)^{(v(\tilde{x})+\xi(\tilde{x},\delta))} \tag{3.5}$$

where ξ (>0) vanishes with δ in such a manner that $(\delta^n)^\xi = 1$ in the limit. This observation offers an alternative definition of a scale free (δ) infinitesimal, viz., $\tilde{X} = \lim \frac{\tilde{x}_n}{(\delta^n)^{(1+|\tilde{x}|_\delta)}} = O(\delta^{n\xi})$, as $n \to \infty$, which will be useful in the following.

We now recall the general topological structure of a non-archimedean space.

Definition 4. *The set* $B_r(a) = \{x|\ |x-a| = v(x-a) < r\}$ *is called an open ball in* $\mathbf{0}$. *The set* $\bar{B}_r(a) = \{x|\ |x-a| \leq r\}$ *is called closed ball in* $\mathbf{0}$.

Lemma 3. *1) Every open ball is closed and vice versa(clopen ball).*

2) Every point $b \in B_r(a)$ *is a centre of* $B_r(a)$.

3) Any two balls in $\mathbf{0}$ *are either disjoint or one is contained in another.*

4) $\mathbf{0}$ *is the union of at most a countable family of clopen balls.*

5) The set $\mathbf{0}$ *equipped with the absolute value* $|.|$ *is totally disconnected.*

The proof of these assertions follows directly from the ultra metric property (See Chapter 2, Sec. 2.3) and the fact that $\mathbf{0}$ is an open set. Because of the property (4) the set $\mathbf{0}^+$ can be covered by at most a countable family of clopen balls viz., $\mathbf{0}^+ = \cup B(t_i)$ where t_i is a bounded sequence in $\mathbf{0}$, on each of which the absolute value $|.|$ can have a constant value. With this choice of absolute value $|.|$ is discretely valued.

Remark 5. To emphasize, the definition of relative infinitesimals takes note of relative position of $\tilde{x}$ with respect to δ, which could then be extended as a geometrical progression to a sequence $\tilde{x}_n$ satisfying $0 < \tilde{x}_n < \delta^n$ so that $\tilde{x}_n/\delta^n = (\tilde{x}/\delta)^n$ for the evaluation of $|\tilde{x}|_\delta$. Further, we use the symbol X to denote a scale free infinitesimal and the sequence $\tilde{x}_n$ of arbitrarily small real numbers are called the real valued realizations of the infinitesimal X. δ infinitesimals carry traces of residual influence of the scale, as reflected in the corresponding absolute values $|\tilde{x}|_\delta = \log_{\delta^{-1}} \delta/|\tilde{x}|_e$, where as a genuinely scale free one should be independent of any scale. We notice that the above absolute value awards the real number system R a novel structure, viz., for an arbitrarily small scale δ, numbers x and $\tilde{x}$ satisfying $x > \delta$ and $0 < \tilde{x} < \delta$ now are represented as

$$x = \delta.\delta^{-|\tilde{x}_0|} \text{ and } \tilde{x} = \lambda.\delta.\delta^{|\tilde{x}_0|}$$

for a λ ($0 < \lambda << 1$), so that the inversion rule is satisfied. Actually, $\tilde{x}$ belongs to an open set I^+_{in} (say) of $(0, \delta)$, the size of which is determined by λ. Here, $\tilde{x}_0$ is a special reference point in I^+_{in}, for instance, $\tilde{x}_0 = x^{-1} \in I^+_{\text{in}}$. It is also often useful to rewrite the inversion rule as an exponentiation: $\tilde{x}/\delta = (\delta/x)^\mu$ so that $\mu \log(x/\delta) = \log \lambda^{-1} + \log(x/\delta)$, for a given x and δ. It also follows that although λ is a constant, the exponent μ is actually a function both of the real variable x and the scale δ. For $x \to \delta$, and $\delta \to 0^+$, we have $\mu \to \infty$ and $|\tilde{x}_0| \to 0$ in such a manner that $|\tilde{x}|$ may have a finite value. For a δ infinitesimal, on the other hand, μ may tend to 1^+ as $\delta \to 0^+$. Indeed, in that case, we have, for a given arbitrarily small x and δ, a sequence $\tilde{x}_n$ such that $\tilde{x}_n = \delta^n \delta^{\mu|\tilde{x}_0|_\delta} = \delta^n \delta^{n(\tilde{\mu}|\tilde{x}_0|_\delta)}$ where $\tilde{\mu} = \mu/n \to 1$ for a sufficiently large n. Notice that such a sequence always exists. In the limit $\delta \to 0^+$, a δ infinitesimal should go over to a scale free infinitesimal. Letting $x_1 = x/\delta = \delta^{-|\tilde{x}_0|} \approx 1 + \eta$, so that $\eta \approx |\tilde{x}_0| \log \delta^{-1}$, we get $\tilde{x}_1 = \tilde{x}/\delta = \delta^{\mu|\tilde{x}_0|} \approx 1 - \mu\eta$, $\mu = O(1)(> 0)$. Moreover the rate of approach of a real variable x, which equals 1 in the ordinary analysis, gets slower in the presence of scale free infinitesimals. In fact x approaches 0 now as $x^{1-\alpha}$, $\alpha = |\tilde{x}|$, rather than simply as x.

Remark 6. The scale δ might correspond to the accuracy level in a computational problem. In this context, 0 in R is identified with the interval $\bar{I}_\delta$ (the closure of I_δ) and thus is raised to $\mathbf{0}$. A computation is therefore interpreted as an activity over an extended field $\tilde{R}$. By letting $\delta^n \to 0^+$ as $n \to \infty$, we consider an infinite precision computation, which is achieved progressively by increasing the accuracy level, when real numbers are represented as δ-adically, for instance, the binary or decimal representation correspond to $\delta = 1/2$ or $\delta = 1/10$ respectively. Consequently, one arrives at a class of (scale free) infinitesimals $X = \tilde{x}/\delta^n \in (-1, 1)$, $n \to \infty$, which seem to remain available even in the ordinary analysis. To avoid any conflict with the standard real analysis results (for instance, the Lebesgue measure of Cantor sets in R), the scale free infinitesimals may be assumed to live in a zero measure Cantor set. As a consequence, the topological dimension of R_δ is zero. To re-emphasize, a scale free infinitesimal X is an element of a Cantor set C_δ (with the scale factor δ) in I_δ, while the sequence of realizations $\tilde{x}_n$ corresponds to its nth iteration realization. In such a realization, $\tilde{x}_n$ (say) is an element of a closed (undeleted) subset F_{in} of I_δ, each element of which is mapped to the finite real number x by the inversion rule. The Cantor set structure of the scale free infinitesimals is consistent with the ultrametricity of $\mathbf{0}$.

Remark 7. **Relationship between $|X|$ and $|X|_\delta$:**

We have already noticed that for an arbitrarily small $\delta \to 0^+$, we have the asymptotic representations $x = \delta\delta^{-|\tilde{x}_0|}$ and $\tilde{x} = \lambda\delta\delta^{|\tilde{x}_0|}$. Accordingly, $|\tilde{x}|$ may have a finite value even as $\tilde{x}$ vanishes as $\delta \to 0$. For a δ infinitesimal $\tilde{X}$, on the other hand the analogous representations are $x = \delta\delta^{-|\tilde{x}_0|_\delta}$ and $\tilde{x}_n = \delta^n(\delta^n)^{\tilde{\mu}(\delta)|\tilde{x}_0|_\delta}$, where $n \to \infty$ but δ is kept fixed, and $\tilde{\mu}$ now depends on δ. It follows that $|\tilde{x}|_\delta = \lim_{n\to\infty} \log_{\delta^{-n}}(\delta/\tilde{x})^n = \log_{\delta^{-1}}(\delta/\tilde{x}) = \tilde{\mu}(\delta)|\tilde{x}_0|_\delta$. Recalling that $\tilde{\mu} = 1 + \sigma_\delta(x)$, where $\sigma_\delta \to 0$ with δ, we therefore write, $|\tilde{x}|_\delta = (1 + \sigma_\delta(x))|\tilde{x}_0|_\delta := \tilde{x}_1$. From the remarks following the proof of Proposition 7 it also follows that $|\tilde{x}_0|_\delta = \delta^\alpha \leq 1$ for $\alpha \geq 0$.

Example 1. Let us recall that a p-adic integer is given by $X_r = p^r(1 + \sum_0^\infty a_i p^i)$, $r > 0$, where a_i assumes values from 0,1,2,....(p-1) with p-adic norm $|X_r|_p = p^{-r}$. As an example of δ infinitesimals we now consider p-infinitesimals, which are related to the p-adic integers in $Z_p \subset Q_p$. Let $\delta = 1/p$, p being a prime. Then there exist a class of p infinitesimals X_p (actually an equivalence class of such infinitesimals) which are ordered according to the primes. Let, $x = p^{-(1-1/p^r)}$, for some positive integer r, be a given value of a real variable x relative to the scale $1/p$. Then we have a class of p-infinitesimals $0 < X_{rp} < 1/p$ given by $X_{rp} = p^{-n\mu_p(x)(1+1/p^r)}$ where $0 < r \leq n$ and $\mu_p = (1 + \sigma_p(x)), \sigma_p(x)$ being a small positive variable and goes to zero faster than $1/p$. Then we have $|X_{rp}| = p^{-r}(1 + \sigma_p)$. When $\sigma_p = 0$, one obtains a p-adic integer, realized as a p-infinitesimal, because in that case we have $|X_{rp}| = p^{-r}$. The sequence of partial sums $S_m = p^r(1 + \sum_0^n a_i p^i)$, which is divergent in the usual metric of Q and is an infinitely large element in the conventional non-standard models of Q, is realized in the present model as a p-infinitesimal X_{rp}.

Lemma 4. *A closed ball in* **0** *is both complete and compact.*

Proof. The proof follows from the following observations. Given $\epsilon > 0$, consider a closed interval $[a, b] \subset \mathbf{0}$ (in the usual topology) such that $0 < a < b < \epsilon$. The valuation v realizes this closed interval as an ultra metric (sub) space U of **0** which is an union of at most of a countable family of disjoint clopen balls (by Lemma 3).

Now we consider completeness. A sequence $\{x_n\} \subset U$ is Cauchy $\Leftrightarrow v(x_m - x_n) \rightarrow 0 \Leftrightarrow v(x_{n+1} - x_n) \rightarrow 0 \Rightarrow \exists N > 0$ such that $v(x_{n+1}) = v(x_n)$ for all $n \geq N$. Now since for a non-zero infinitesimal x_n, the associated valuation is non-zero, it follows that $x_n \rightarrow x_N \in U$ in the ultra metric in the sense that $v(x_n) = v(x_N)$ as $n \rightarrow \infty$. Compactness is a consequence of the fact that any sequence in U has a convergent subsequence. Indeed, a sequence $\{x_n\}$ in U can not be divergent in the given ultra metric since $0 \leq v(x_n) \leq 1$. ■

Next, we extend this nonarchimedean structure of **0** on the whole of R, which is already assumed as an *intrinsically* nonstandard extension $\tilde{R}$.

Definition 5. *Let* $I_\delta(r) = r + I_\delta(0)$ $I_\delta(0) = (-\delta, \delta)$, $\delta > 0$ *for a real number* $r \in R$. *For a finite* $r \in R$, *i.e., when* $r \notin I_\delta(0)$, *we have* $||r|| = |r|_e = r$. *For an* $r \in I_\delta(0)$, *on the other hand, we have* $||r|| = |r| = \lim_{\delta \to 0} \log_{\delta^{-1}}(\delta/r) = v(r)$, *while, for an arbitrarily* *large r ($\to \infty$), i.e., when $|r|_e > N,\ N > 0$, we define $||r|| = |r^{-1}|$ which is evaluated with the scale $\delta \le 1/N$.*

Proposition 9. *$||.||$ is a nonarchimedean absolute value on R. It is discretely valued over the set of infinitely small and large numbers.*

Notation 1. *The ultra metric space $\{R, ||.||\}$ is denoted as $\mathbf{R}$.*

Proof. For an infinitely small or large r, the proof follows from Proposition 8. For a finite (non-zero) value of $r \in R$, we have, on the other hand, $r = s + \tau(t),\ s = r - t, \tau(t) = t,\ t \in I_\delta^+(0)$, so that $||r|| = \max\{||s||, ||\tau(t)||\} = s = |r|_e$, by letting $t \to 0$. Discreteness on the set of infinitesimals and infinities follows from the discreteness of $|.|$. ∎

Corollary 3. *$\mathbf{R}$ is a locally compact complete (ultra-)metric space.*

The proof follows from Lemma 4 and Proposition 9.

Proposition 10. *The topology induced by $||.||$ on R is equivalent to the usual topology. Moreover, the embedding $i : R \to \mathbf{R}$ is continuous.*

Proof: It is easy to verify that an open set of R in usual topology is open in $||.||$ and conversely since $||.||$ reduces to the usual absolute value on R and the continuity also follows from Definition 5.

Definition 6. *A (δ) infinitesimal $\tilde{X}$ is an (non-archimedean) integer if $|\tilde{X}| < 1$. It is a unit if $|\tilde{X}| = 1$.*

Lemma 5. *A scale free (δ) unit $\tilde{X}_u$ on $\mathbf{R}$ has the form $\tilde{X}_u = 1 + \tilde{X}$ where $|\tilde{X}| < 1$.*

Proof. A scale free (δ) unit is defined by $||X_u|| = 1$. According to the valuation equation (3.5), we have

$$\tilde{x}_{un}/\delta^n = (\delta^n)^{1+\xi(\tau_u,\delta)} \tag{3.6}$$

since $v(X_u) = 1$ ($\tilde{x}_{un}$ are various realizations of X_u). We also assume $\xi > 0$. Thus $\tilde{x}_{un}/\delta^{2n} = O(\delta^{n\xi}) \to 1$, as $n \to \infty$ and subsequently $\delta \to 0$. Thus writing $\tilde{X}_n = \xi \log \delta^n$ (Remark 4), we have the lemma, since, as $n \to \infty$, $X_u = \lim \tilde{x}_{un}/\delta^{2n} = \lim e^{\tilde{X}_n} = 1 + \tilde{X}_0$, when we have $||\tilde{X}_n(1 + o(\tilde{X}_n)|| = ||\tilde{X}_0||$, (because of the ultrametricity of $||.||$) for a $\tilde{X}_0 = \tilde{X}_n(1 + o(\tilde{X}_n))$. ■

Lemma 6. *Let X_i be two δ_i infinitesimals, $i = 1, 2$ such that $\delta_1 > \delta_2$. Then there is a canonical decomposition $X_1 = \tilde{X}_1(1 + X_2)$ where $|X_1| = |\tilde{X}_1| < 1$.*

Proof. Recall that the positive δ-infinitesimals live in $(0, 1) \subset R_\delta$, which is covered by clopen balls $B(\tilde{X}_{1j})$, $j = 1, 2, \ldots$. Let $X_1 \in B(\tilde{X}_{1j})$ for some j. Suppose X_{1j} is defined by (3.4). For a general infinitesimal X_1 we have, on the other hand, the extended definition given by (3.5), viz., $X_1 = \tilde{x}_{1n}/\delta_1^n = (\delta_1^n)^{(|X_{1j}|_{\delta_1} + \xi(X_1, \delta_2))}$, where $\xi > 0$ goes to zero faster than $|X_{1j}| = v(X_{1j})$ such that $(\delta_1^n)^\xi = 1$ as $\delta_1^n \to 0$. Now writing $\xi = (\tilde{\xi}) \log \delta_2 / \log \delta_1$ and using Lemma 5, we obtain $\tilde{x}_{1n}/\delta_1^n = \tilde{x}_{1jn}/\delta_1^n \times \tilde{x}_{un}/\delta_2^{2n}$ and so taking limit $n \to \infty$ the desired result follows. ■

Let us recall that $dX/dx = 0$ means X=constant, on R. However, in a non-archimedean space $\mathbf{R}$, X can be a *locally* constant function, which we call here a *slowly varying* function. In a non-archimedean extension of R, λ (in Definition 1) may be a slowly varying function. Thus, the nonsmooth solutions of R (Section 3.1) are realized as smooth in the non-archimedean space $\mathbf{R}$. We recall that differentiability in a non-archimedean space is defined in the usual sense by simply replacing the usual Euclidean metric by the ultra metric $||x - y||$, $x, y \in \mathbf{R}$.

Definition 7. *Let $f : \mathbf{R} \to \mathbf{R}$ be a mapping from $\mathbf{R}$ to itself. Then f is differentiable at $x_0 \in \mathbf{R}$ if $\exists\ l \in \mathbf{R}$ such that given $\varepsilon > 0, \exists\ \eta > 0$ so that*

$$\left| \frac{\| f(x) - f(x_0) \|}{\| x - x_0 \|} - \| l \| \right| < \varepsilon \tag{3.7}$$

when $0 < \| x - x_0 \| < \eta$, and we continue to write the standard notation $f'(x_0) = \frac{df(x_0)}{dx} = l$.

Remark 8. The above definition is in conformity with the more conventional definition, viz.,

$$\| \frac{f(x)-f(x_0)}{x-x_0} - l \| < \varepsilon \tag{3.8}$$

since $\left| \frac{\|f(x)-f(x_0)\|}{\|x-x_0\|} - \| l \| \right|_e < \| \frac{f(x)-f(x_0)}{x-x_0} - l \| < \varepsilon$. As long as $|x-x_0|_e \to 0^+$, but $|x-x_0| \geq O(\delta)$, the above definition reduces to the ordinary differentiability. But when $|x-x_0|_e \to \delta$ the above gets extended to the logarithmic derivative $x\frac{d\log f(x_0)}{dx} = l$, when we make use of the nonarchimedean absolute value $|.|$.

So far in the above discussion the scale δ is unspecified. In the following, we introduce the nonarchimedean absolute value (Definition 10) on the field of rational numbers Q, construct its Cauchy completion and finally, because of the Ostrowski theorem, relate it to the local p-adic fields. We get a *minimal* nonarchimedean extension of R (and which is a subset of the above $\mathbf{R}$) for each given scale δ, thus sufficing our purpose of relating the (secondary) scales δ with the inverse primes viz., $\delta = 1/p$. Consequently, we have a countable number of distinct field extensions $\mathbf{R}_p$ of R depending on the scale at which the origin 0 of R is probed.

However, before proceeding further, let us collect a few more general properties of the valuation $v(\tilde{x})$ in the following subsection.

3.3 Non-archimedean valuation: A few more properties

The set of infinitesimals $\mathbf{0}$ reduces to the singleton $\{0\}$ when $\delta \to 0$ classically. However, the corresponding asymptotic expressions for the scale free (invariant) infinitesimals are non-trivial, in the sense that the associated valuations (Definition 3) can be shown to exist as finite real numbers. Below we give a definite construction indicating the exact sense how relative infinitesimals and associated values could arise in a limiting problem.

Fix a value $\delta = \delta_0$ and let $C_{\delta_0} \subset [0, \delta_0] = I^+_{\delta_0}$ be a Cantor set defined by an IFS of the form $f_1(x) = \lambda x$, $f_2(x) = \lambda x - (\lambda/\delta_0 - 1)\delta_0$ where $\lambda = \beta\delta_0$, $0 < \beta < 1$

and $\alpha + 2\beta = 1$. Thus at the first iteration an open interval O_{11} of size $\alpha\delta_0$ is removed from the interval $I^+_{\delta_0}$, at the second iteration two open intervals O_{21} and O_{22} each of size $\alpha\delta_0(\beta)$ are removed and so on, so that a family of gaps O_{ij} of size $\alpha\delta_0(\beta)^{i-1}$, $j = 1, 2, ...2^{i-1}$ are removed in subsequent iterations from each of the closed subintervals I_{ij}, $j = 1, 2, ...2^i$ of $I^+_{\delta_0}$. Consequently, $C_{\delta_0} = I^+_{\delta_0} - \cup_i O_{ij} = \cap_i \cup_j I_{ij}$. Notice that the total length removed is $\sum \alpha\delta_0(2\beta)^{i-1} = \delta_0$, so that the linear Lebesgue measure $m(C_{\delta_0}) = 0$.

Next, consider $\tilde{I}_N = [0, \beta^N]$ and let $N = n + r$ and $N \to \infty$ as $n \to \infty$ for a fixed $r \geq 0$. Choose the scale $\delta = \alpha\beta^n\delta_0$ and define $\tilde{x}_r \in [0, \alpha\beta^N\delta_0]$, a relative infinitesimal (relative to the scale δ) provided it also satisfies the inversion rule $\tilde{x}/\delta = \lambda\delta/x$, for a real constant $\lambda(\delta)$ $(0 << \lambda \leq 1)$. For each choice of x and δ, we have a unique $\tilde{x}$ for a given $\lambda \in (0, 1)$. Consequently, by varying λ in an open subinterval of (0,1), we get an open interval of relative infinitesimals in the interval $(0, \delta)$, all of which are related to x by the inversion formula. In the limit $\delta \to 0$, relative infinitesimals $\tilde{x}_r$, of course, vanish identically. However, the corresponding scale invariant infinitesimals $\tilde{X}_r = \tilde{x}_r/\delta$, $\delta \to 0$ are, nevertheless, nontrivial and are weighted with new scale invariant absolute values (norms) (Definition 3).

The set of infinitesimals are uncountable, and as already shown the above norm satisfies the stronger triangle inequality $v(x + y) \leq max\{v(x), v(y)\}$. Accordingly, the zero set $\mathbf{0}=\{0, \pm\delta\tilde{X}_r |\ \tilde{X}_r \in (0, \beta^r), r = 0, 1, 2, ..., \delta \to 0^+\}$ may be said to acquire dynamically the structure of a Cantor like ultra metric space, for each $\beta \in (0, 1/2)$. The set $\mathbf{0}$ indeed is realized as a set of nested circles $S_r = \{\tilde{x}_r |\ v(\tilde{x}_r) = \alpha_r\}$ in the ultra metric norm, when we order, with out any loss of generality, $\alpha_0 > \alpha_1 >$ ..... The ordinary 0 of R is replaced by this set of scale free infinitesimals $0 \to \bar{\mathbf{0}} = \mathbf{0}/\sim = \{0, \cup S_r\}$; $\bar{\mathbf{0}}$ being the equivalence class under the equivalence relation $\sim$, where $x \sim y$ means $v(x) = v(y)$.

Remark 9. The concept of infinitesimals and the associated absolute value considered here become significant only in a limiting problem (process), which is reflected in the explicit presence of "$\lim_{\delta\to 0}$" in the relevant definitions. For the continuous real valued function $f(x) = x$, the statement $\lim_{x\to 0} x = 0$, means that the statement $x \to 0$ essentially means that $x = 0$ (i.e. x not only tends to 0 but, in fact, assumes the value 0 "exactly"). This may be considered to be a passive evaluation of limit. The present approach is active (dynamic), in the sense that it offers not only a more refined *intermediate* stages in the evaluation of the limit, but also provides a clue how one may induce new (nonlinear) structures in the limiting (asymptotic) process. The inversion rule is one such non linear structure which may act non trivially as one investigates more carefully the motion of a real variable x (and hence of the associated scale $\delta < x$) as it goes to 0 more and more accurately. Notice that at any "instant", elements defined by $0 < \tilde{x} < \delta < x$ in a limiting process are well defined; relative infinitesimals are meaningful only in that dynamic sense (classically, these are all zero, as x itself is zero). Scale invariant infinitesimals $\tilde{X}$, however, may or may not be zero classically. $\tilde{X} = \mu(\neq 0)$, a constant, for instance, is non zero even when x and δ go to zero. On the other hand, $\tilde{X} = \delta^{\alpha}$, $0 < \alpha < 1$, of course, vanish classically, but as shown below, are non trivial in the present formalism. As a consequence, relative infinitesimals may be said to exit even as real numbers in this dynamic sense. The accompanying metric $|.|$, however is an ultra metric.

Remark 10. A genuine (nontrivial) scale free infinitesimal $\tilde{X}$ can not be a constant. Let, $\tilde{x} = \mu\delta$, $0 < \mu < 1, \mu$ being a constant. Then $v(\tilde{x}_0) = \lim_{\delta\to 0} \log_{\delta^{-1}} \mu^{-1} = 0$, so that $\tilde{x}_0$ is essentially the trivial infinitesimal 0. More precisely, such a relative infinitesimal belongs to the equivalence class of 0.

Example 2. *Let,* $x_n = \epsilon^{n(1-l)}$, $0 < l < 1$, $0 < \epsilon < 1$. *Then scale invariant infinitesimals are* $\tilde{X}_{n\lambda} = \lambda\epsilon^{nl}$, $0 < \lambda < 1$, *when* $\delta = \epsilon^{n}$, *for a sufficiently large* n, *is chosen as a scale. Analogously, for a continuous variable* x *approaching* 0^{+}, *say, and considered as a scale, a class of relative infinitesimals are represented as* $\tilde{x} = x^{1+l}(1 + o(x))$, $0 < l < 1$, *so that the corresponding scale invariant infinitesimals* *are defined by the asymptotic formula* $\tilde{X} = \lambda x^l + o(x^m)$, *$m > l$. The corresponding scale invariant absolute value has the non-trivial value* $|\tilde{x}| = \lim_{x\to 0} \log_{x^{-1}} \frac{x}{\tilde{x}} = l$. *Notice that a scale invariant infinitesimals goes to zero at a smaller (ultra metric) rate* $l : \tilde{X} = \lambda x^l \Rightarrow d\log \tilde{X}/d\log x = l$

Remark 11. The scale free infinitesimals of the form $\tilde{X}_m \approx \delta^{\alpha_m} + o(\delta^\beta), \beta > \alpha$ goes to 0 at a slower rate compared to the linear motion of the scale δ. The associated non trivial absolute value $v(\tilde{x}_m) = \alpha_m$ essentially quantifies this decelerated motion.

Theorem 3. *The norm v has the following properties*

1) v is an ultra metric, and hence $\mathbf{0}$ *equipped with v is an ultra metric space (non-archimedean space).*

2) v is a locally constant Cantor function. Conversely, given a Cantor function $\phi(\tilde{x})$, there exists a class of scale invariant infinitesimals determined by $\phi(\tilde{x})$, those live on the extended ultra metric neighborhood $\mathbf{0}$ *of 0.*

Remark 12. The part 1 of the theorem is already proved in proposition 7 and Proposition 8. We present here a slightly improved concise proof of the same.

Proof. 1)(a) v is well defined. Indeed, the open set $\mathbf{0}$ is written as a countable union of disjoint open intervals I_{δ_i} of relative infinitesimals, i.e. $\mathbf{0}=\cup I_{\delta_i}$. Let $v(\tilde{x}_i) = \alpha_i$, a constant for all $\tilde{x}_i (= \lambda\delta\delta^{\alpha_i}) \in \bar{I}_{\delta_i}$, the closure of I_{δ_i}. Thus v exists and well defined.

(b) Let $0 < \tilde{x}_2 < \tilde{x}_1 < \tilde{x}_1 + \tilde{x}_2 < \delta$ be two relative infinitesimals. We have, $0 < \tilde{X}_2 < \tilde{X}_1 < \tilde{X}_1 + \tilde{X}_2 < 1$ and $v(\tilde{x}_2) > v(\tilde{x}_1) > v(\tilde{x}_1 + \tilde{x}_2)$, thus proving the strong triangle inequality $v(\tilde{x}_1 + \tilde{x}_2) \leq max\{v(\tilde{x}_1), v(\tilde{x}_2)\}$. Next, given $0 < \tilde{x} < \delta$, there exists a constant $0 < \sigma(\delta) < 1$ and $a : \mathbf{0} \to \mathrm{R}$, such that $\tilde{X} = \lambda\delta^{v(\tilde{x})}$ and $v(\tilde{x}) = \sigma^{a(\tilde{x})}$. Accordingly, $a(\tilde{x})$ is a discrete valuation satisfying (i) $a(\tilde{x}_1\tilde{x}_2) = a(\tilde{x}_1) + a(\tilde{x}_2)$, (ii) $a(\tilde{x}_1 + \tilde{x}_2) \geq min\{a(\tilde{x}_1), a(\tilde{x}_2)\}$. As a result, $v(\tilde{x}_1\tilde{x}_2) = v(\tilde{x}_1)v(\tilde{x}_2)$. Hence $\{\mathbf{0}, v\}$ is an ultra metric space.

2) Let $\bar{\mathbf{0}} = (\cup\bar{I}_{\delta i}) \bigcup (\cup J_k)$, the closure of $\mathbf{0}$. The open intervals J_k are gaps between two consecutive closed intervals $\bar{I}_{\delta i}$. J_k's actually contain new points those arise as

the limit points of sequences of the end points of the open intervals $I_{\delta i}$. Clearly, $\bar{\mathbf{0}}$ is connected in usual topology. However, in the ultra metric topology, both $I_{\delta i}$ and J_k are clopen sets and $\bar{\mathbf{0}}$ is totally disconnected. Since, it is bounded and also is perfect, $\mathbf{0}$ is equivalent to an ultra metric Cantor set.

Now, the local constancy of v in the ultra metric $\bar{\mathbf{0}}$ follows from the definition:

$$\frac{dv(\tilde{x})}{dx} = \lim_{\delta\to 0^+}\frac{d}{dx}\left(\frac{\log x}{\log\delta}+1\right) = 0 \tag{3.9}$$

The vanishing derivative above arises from a logarithmic divergence arising from the nontrivial finer scales. This is unlike the ordinary analysis, when one interprets $\bar{\mathbf{0}}$ as a connected subset of R, thereby forcing v to vanish uniquely, so as to recover the usual structure of R. The above vanishing derivative can be interpreted non trivially as a LCF [43] when $x \in R$ is supposed to belong to a Cantor subset of R.

Eq(3.9) also reveals the *reparameterization invariance* of a locally constant valuation $v(x)$. As a consequence, v may be a function of any reparametrized monotonic variable $\tilde{x} = \tilde{x}(x)$ with $\tilde{x}'(x) > 0$, instead being simply a function of the original real variable x.

Now to construct a general class of locally constant functions in the ultra metric space, let us proceed as in 1a) above, with the supposition that the constants α_i's are arranged in ascending order. Thus, $v(\tilde{x}_i) = \alpha_i$, $\alpha_i \leq \alpha_j \Leftrightarrow i \leq j$ for all $\tilde{x}_i \in I_i$(we drop the suffix δ for simplicity). Clearly, Definition 5 holds over for all I_i. On the other hand, for an $\tilde{x} \in J_k$, where J_k separates two consecutive I_i and I_{i+1}, say, so that $\tilde{x}_i < \tilde{x} < \tilde{x}_{i+1}$, where $\tilde{x}_i$ is the right end point of I_i and $\tilde{x}_{i+1}$ is the left end point of I_{i+1}, we have $v(\tilde{x}_{i+1})-v(\tilde{x}_i) = (\alpha_{i+1}-\alpha_i)$. Because of ultrametricity, one can always choose $\alpha_i = \beta_{ij_i}\sigma(i)^s$, for $\beta_{ij_i} >0$ ascending and $\sigma(i) \to 0$ as $i \to \infty$ and $j_i = 0, 1, 2, ...k(i)$ for some i dependent constant $k(i)$. Consequently, $v(\tilde{x}_{i+1})-v(\tilde{x}_i) = (\beta_{(i+1)j_{i+1}}-\beta_{ij_i})\sigma(i)^s$. It follows that the sequence $v(\tilde{x}_{i+1})$ is decreasing and $v(\tilde{x}_i)$ is increasing. Thus, $v(\tilde{x}) := \lim v(\tilde{x}_i)$ as $i \to \infty$. Hence, $v : \mathbf{0} \to \mathbf{I}^+$ is indeed a Cantor function.

Conversely, given a Cantor function $\phi(x)$, $x \in I^+$, one can define a class of infinitesimals $\tilde{x} \approx \delta\delta^{\phi(\tilde{x}/\delta)}$ belonging to the extended set $\mathbf{0}$ for $\delta \to 0^+$. This completes the proof. ■

Definition 8. *Besides the usual Euclidean value, a real variable* $x \neq 0$*, but* $x \to 0^+$ *gets a deformed ultra metric value given by* $v(x) := \lim_{\delta \to 0^+} \log_{\delta^{-1}}(x/\delta)$.

Lemma 7. $v(x) = v(\tilde{x})$.

Because of inversion rule, $x/\delta = \lambda(\delta/\tilde{x})$, $0 < \lambda < 1$, and hence $v(x) = v(\tilde{x})$ since $\lim \log_{\delta^{-1}} \lambda^{-1}$=0. □

Lemma 8. *Let,*$0 < |x| < |x'|$ *be two arbitrarily small real variables and* δ *be a scale such that* $0 < \delta < |x - x'| < |x| < |x'|$. *Then* $v(x') = v(x)$.

From Definition 8, $v(x - x') < v(x) < v(x')$. But $x' = x + (x' - x)$. So, by ultra metric inequality, $v(x') \leq max\{v(x), v(x' - x)\} \leq v(x)$ and hence the result. □

Lemma 9. *Let* $0 < |x| < |x'|$ *be two arbitrarily small real variables and* δ *and* δ' *be two scales such that* $0 < \delta < |x| < \delta' < |x'|$. *The corresponding scale invariant infinitesimals are* $\tilde{X}$ *and* $\tilde{X}'$ *with associated valuations* $v(x)$ *and* $v(x')$. *Then* $v(x') = (\alpha/s)v(x)$, *where* $\alpha = \lim \log_{\tilde{X}} \tilde{X}'$, *determines the gap size between* $\tilde{X}$ *and* $\tilde{X}'$ *and* $s = \lim \log \delta' / \log \delta$ *is the Hausdorff dimension of the Cantor set of infinitesimals as* $x, x' \to 0$.

Proof. The proof follows from

$$\frac{v(x')}{v(x)} = \lim \frac{\log(x'/\delta')}{\log(x/\delta)} \times \lim \frac{\log \delta}{\log \delta'} \tag{3.10}$$

so that $\alpha = \lim \log_{x/\delta}(x'/\delta') = \lim \log_{\tilde{X}} \tilde{X}' \Rightarrow \tilde{X}' = X^{\alpha}(1 + O(\beta(x, x')))$, $\beta \to 0$ faster than the linear approach $x \to 0$. The exponent α gives a measure of the said gap size.

■

Corollary 4. *Let* $0 < \delta < \delta' < x$ *be two scales in association with an arbitrarily small real variable and* $\tilde{X} = (x/\delta)^{-1}$ *and* $\tilde{X}' = (x/\delta')^{-1}$ *be the corresponding scale* *invariant infinitesimals. Then* $v(x') = (\alpha/s)v(x)$*, where* $\alpha = \lim \log_{\tilde{X}} \tilde{X}'$*, determines the gap size between* $\tilde{X}$ *and* $\tilde{X}'$ *and* $s = \lim \log \delta / \log \delta'$ *is the Hausdorff dimension of the Cantor set of infinitesimals as* $x, x' \to 0$*. The exponent* α *gives a measure of the said gap size.*

Definition 9. *A scale invariant jump is defined by the pure inversion* $\tilde{X}' = X^{-\alpha}$ *with the scale invariant minimal jump size* $\alpha = 1$*. The (scale invariant) jump size* α *thus runs over the set of natural numbers N.*

Remark 13. Lemma 8 characterizes the equivalence classes of infinitesimals with identical valuations. Subsequent lemma (and corollary) tells that the valuation v changes only when an infinitesimal from one equivalence class switches over to another class.

Summing up the above observations, we now state a general representation of relative infinitesimals and corresponding valuation.

Lemma 10. *A relative infinitesimal* $\tilde{x}$ *relative to the scale* δ *has the asymptotic form*

$$\tilde{x} = \delta \times \delta^{l} \times \delta^{\phi(\tilde{x}/\delta)}(1 + o(1)) \tag{3.11}$$

with associated valuation $v(\tilde{x}) = l + \phi(\tilde{x}/\delta)$ *where* $l \geq 0$ *is a constant and* ϕ *is a nontrivial Cantor function.*

The locally constant $v = v_0 + v_1$ solves $dv/dx = 0$ and so the above ansatz is the more general solution, with the trivial ultra metric valuation $v_0 = l$ and the nontrivial valuation $v_1 = \phi$. The representation for $\tilde{x}$ now follows from definition. □

Remark 14. As a real variable x and the associated scale $\delta < x$ approach 0, the corresponding infinitesimals $0 < \tilde{x} < \delta$ may also live (in contrast to measure zero Cantor sets considered so far) in a positive measure Cantor set C_p. Such a possibility is already considered in [14] in relation to an interesting phenomenon of growth of measure. In such a case $v_0(\tilde{x}) = m(C_p) = l$, the Lebesgue measure of C_p. The nontrivial component v_1 then relates to the uncertainty (fatness) exponent of the positive measure 1-set. In this extended model, the valuation quantifies the presence of nontrivial motion in a limiting process: v_0 gives the uniform scale invariant motion when v_1 arises from the associated non-uniformity stemming out from measure zero Cantor sets. We, however, do not consider this aspect of the analysis any further in this thesis.

3.4 Completion of the Field of Rational Numbers

We have already shown that the real number system R gets extended over a locally compact, complete ultra metric space $\mathbf{R}$ under the nonarchimedean norm $||.||$. However, we haven't yet shown that the ultra metric space $\mathbf{R}$ is a field. In the present Section, we present yet another route extending R *minimally* as an ultra metric field completion $\mathbf{R}$ of the field of rationals Q (with a slight abuse of notation, we use same notation for both type of ultra metric extensions). Because of the Frobenius field extension theorem, this field extension must be of infinite dimensional. In the next chapter, a new elementary proof of the Prime Number Theorem will be presented. Applications to differential equations on the extended field will be studied in the subsequent latter chapter.

On the field of rationals Q, we introduce the definition of the valuation $||.|| : Q \to R_+$ as follows.

Definition 10. *Let $I_\delta(r) = r + I_\delta, I_\delta = (-\delta, \delta)$, $\delta > 0$ for rational numbers r and $\delta \in Q$. For a finite $r \in Q$, i.e., when $r \notin I_\delta(0)$, we have $||r|| = |r|_e = r$. For an $r \in I_\delta(0)$, on the other hand, we have $||r|| = |r| = \lim\limits_{\delta \to 0} \log_{\delta^{-1}}(\delta/r) = v(r)$, while, for an arbitrarily large $r(\to \infty)$, i.e., when $|r|_e > N, N > 0$, we define $||r|| = |r^{-1}|$ with a scale $\delta \leq 1/N$.*

Proposition 11. $||.||$ *is a non-archimedean absolute value over Q.*

Proof is similar to that on R which is given in the previous Section. □

Now by the Ostrowski theorem [42], any non-trivial absolute value on Q must be equivalent to any of the p-adic absolute values $|.|_p$, $p > 1$ being a prime and $|.|_\infty = |.|_e$ is the usual Euclidean absolute value. From Definition 10, finite rationals of Q get the Euclidean value, while $|.|$, on the arbitrarily small and large values, must be related to the p-adic valuations. Consequently, the set of *primary scales* are represented uniquely by the inverse primes $\delta = p^{-1}$.

To construct the completion of Q under $||.||$ we first consider the ring $\mathcal{S}$ of all sequences of Q. The zero divisors in $\mathcal{S}$ are removed by the choice of an ultra filter, as in the usual non-standard models of R [52]. The quotient set of the Cauchy sequences C ($\subset$S), under the usual absolute value, modulo the maximal ideal $\mathcal{N}$ consisting of sequences converging to 0, gives rise to the ordinary real number set R=$\mathcal{C} - \mathcal{N}$. The elements of diverging sequences in $\mathcal{S}_{div} = \mathcal{S} - \mathcal{C}$ correspond to the infinitely large elements, when the inverse $\{a_n^{-1}\} \in \mathcal{S}_{div}$ of a divergent sequence $\{a_n\}$ leads to an infinitesimal in the conventional approaches of non-standard analysis. In our approach, this realization is, however, somewhat reversed.

Notice that the set of divergent sequences $\mathcal{S}_{div}$ is quite a large set. Now among the all possible divergent sequences, there exists a subset $\mathcal{S}_p$ of sequences which are nevertheless p (-adically) convergent ($\mathcal{S}_p \subset \mathcal{S}_{div}$, since the sequence $\{n\}$ is p-adically divergent for each p). For each fixed p, let us consider the Cauchy completion of p convergent sequences $\{a_n^p\}$ (say) (modulo the sequences p-adically converging to zero), viz., the local field Q_p. We identify, by definition, the p-adic integers $X \in Z_p \subset Q_p$ with $|X|_p \leq 1$ as the p infinitesimals. On the other hand, the elements $\tilde{X}$ of Q_p with $|\tilde{X}|_p > 1$ are identified with infinitely large numbers of *type p*. In other words, $\tilde{X}$ denotes the $p-$adic limit of an inverse sequence of the form $\{(a_n^p)^{-1}\}$, leading to the inversion symmetry $\tilde{X} = X^{-1}$ which is valid for a suitable p infinitesimal X. The absolute value $||.||$ when restricted to $\mathcal{S}_p$ thus relates an infinitesimal X, i.e. an element of $*R$ (a non-standard model of R), to a countable number of p-adic realizations $X_p \in Z_p$ with valuations $||X|| = \mu_p|X_p|_p$, p=2,3,....; $p \neq \infty, \mu_p$ being a constant for each p, as the neighborhood of 0 in R is probed deeper and deeper by letting $\delta = p^{-1} \to 0$ as $p \to \infty$. In the computational model (c.f Sec.3.2, Remark 6), this might be interpreted as (in equivalence classes of) higher precision models of a computation. Consequently, equipped with $||.||$, the set $\mathcal{S}_p$ decomposes into (a countably infinite Cartesian product of) local fields Q_p in a hierarchical sense as detailed in the Lemma 11. We note that any element X of $\mathcal{S}_p$ is an equivalence class of sequences of rational numbers under the chosen ultra filter. In each of such a class there exists a unique sequence $\{a_n^p\}$, say, converging to a p-adic integer or its inverse X_p. A scale free infinitesimal X then relates to X_p, and that X indeed is an infinitesimal tells that $||X|| = \mu_p|X_p|_p \leq 1$. We say that 0 of R is probed at the depth of the (secondary) scale $1/p$ when a scale free (δ)- infinitesimal X is related to a p-adic infinitesimal X_p. We, henceforth, denote infinitesimals, as usual, by $\mathbf{0}$, when p- infinitesimals are denoted as $\mathbf{0}_p$ ($\mathbf{0}$ is identified with $\mathbf{0_p}$) at the level of the (secondary) scale $\tilde{\delta} = 1/p$.

Example 3. *Let $a_{pn} = 1+\sum_1^n \alpha_i p^i$, where α_i assumes values from 0,1,.....(p-1). The sequence a_{pn} is divergent in R for each prime p. In the non-standard set $*R, \{a_{pn}\}$ denote a distinct infinitely large number for each p. p-adically, however, a_{pn} converges to the unity X_{pu} ($|X_{pu}|_p = 1$). The scale free unity $X_u \in \mathcal{S}_p$ now denotes the larger sequence $\{a_{pn}, \forall p\}$. At the level of secondary scale $\tilde{\delta} = 1/p$, unity X_u is realized as X_{pu}.*

Remark 15. By "hierarchical" we mean that as a scale free real variable $\tilde{x} = x/\delta^n, n \to \infty$, approaches 0 from the initial value 1 through the secondary scales $1/p$, $p \to \infty$ and $\delta = p^{-1}$, the ordinary real variable $x \in$R would experience changes over various local fields Q_p successively by inversions.

Lemma 11. *Let $X_p \in Z_p$ and $X_q \in Z_q$, q being the immediate successor of the prime p. Then an infinitesimal $X \in \mathbf{0}$ when realized as a p- infinitesimal has the* *representation*

$$X = X_p(1 + X_q) \tag{3.12}$$

Further, a p−unit is given as $X_{pu} = 1 + X_p, |X_p| < 1$.

Let us fix the scale at $\delta = 1/p^n$. Then the proof follows from Lemma 5 and 6 . □

Corollary 5. *One also has the following adelic extension*

$$X = X_p \prod_{q>p}(1 + X_q) \tag{3.13}$$

where the product is over all the primes q greater than p.

Proof. This follows from the above Lemma when the valuation formula (3.5) is extended further

$$X/\delta^n = (\delta^n)^{(|X_i| + \sum_m \xi_m(X,\delta_m))} \tag{3.14}$$

where $|X| = |X_i|$, δ_m^{-1} $(m > 1)$ are primes greater than p, $\delta = 1/p$ and each of the indeterminate functions ξ_m satisfies conditions analogous to that in the formula in Lemma 6 . Further, ξ_q goes to zero faster than ξ_p if $q > p$. ■

Collecting together the above results, we have

Theorem 4. *The completion of Q under the absolute value* $||.||$ *yields a countable number of complete scale free models* $\mathbf{R}_p$ *of R, such that each element* $x \in \mathbf{R}_p$ *has the form* $x_p = x(1 + X_p \prod_{q>p}(1 + X_q))$, $x \in R$, $X_p \in Q_p$, *where* X_p *is given by the asymptotic expression* $X_p = (p^{-n})^{(1+|X_p|_p(1+\sigma(\eta)))}$ *and* $\eta = O(\delta)$ *is a real variable. Finally,* $\mathbf{R}_p$ *locally has the Cartesian product form* $\mathbf{R}_p = R \times Q_p \times \prod_{q>p} Q_q$.

The only missing element in the proof of the above is the completeness. Let us first fix a scale $\delta = 1/p^r, r > 0$. Let $\{a_n\}$ be a Cauchy sequence in $\mathbf{R}_p$. Then it is Cauchy either in p-adic metric or in the usual metric, finishing the proof. □

In the following chapter we discuss the nature of influences that the scale free non-archimedean extensions of R would have on the basic structure of R itself.