Chapter 2

REVIEW OF RELEVANT CONCEPTS

In this chapter we give a somewhat detailed review of several relevant inter-related concepts such as Cantor set, Lebesgue and Hausdorff measure, the formalism of non-standard analysis and analytic and topological properties of nonarchimedean ultra metric spaces including p-adic local fields which will be useful in the latter Chapters. Resources available in the inter net web are used freely in compiling this review.

2.1 Cantor Set

2.1.1 Introduction

The Cantor set, introduced originally by George Cantor in 1883 (though however, discovered in 1875 by Henry J. S. Smith) [27, 28], is a set of points lying on a straight line segment that has a number of remarkable and deep properties and was mainly introduced as counter examples of various general topological concepts. A Cantor set is a totally disconnected compact and perfect subset of the real line. Such a set displays many paradoxical properties. Although the set is uncountable, its Lebesgue measure vanishes. The topological dimension of the set is also zero. Cantor set is an example of a self similar fractal set that arises in various fields of applications. The chaotic attractors of a number of one-dimensional maps, such as the logistic maps, turn out to be topologically equivalent to Cantor sets. Recently there have been a lot of activities developing an analysis on a Cantor-like fractal sets [29, 30]. Because of disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approach based on the fractional derivatives [35, 36] and measure theoretic harmonic analysis [37] have already been considered at length in the literature.

2.1.2 Basic Definitions

A Cantor set is defined as a countable intersection of a finite unions of closed (and bounded) subsets of R. For definiteness, let $C \subset I = [0,1]$. Then by definition, $C = \bigcap\limits_{1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{m=1}^{p^n} F_{nm}$ where $F_{nm} \subset I$ are closed with $F_{00} = I$. Equivalently, C is also defined as $C = I - \bigcup\limits_{i=1}^{\infty} O_i$ where O_i are open intervals which are deleted recursively from I. We also give an alternative definition of a Cantor set. Let us call the initial set $I = [0,1]$ as S_0. For the first iteration a fraction α is taken away from S_0 such that S_1 contains two disconnected sets of real numbers $[0, \frac{1}{2}(1-\alpha)]$ and $[1-\frac{1}{2}(1-\alpha), 1]$. Let $a := \frac{1}{2}(1-\alpha)$ and $b := 1-\frac{1}{2}(1-\alpha)$, and call $[0,a]$ as S_{1a} and $[b,1]$ as S_{1b}. For the 2nd iteration take away the fraction α from S_{1a} and S_{1b} such that S_2 contains four disconnected sets of real numbers $[0, \frac{1}{2}(a-\alpha a)]$, $[a-\frac{1}{2}(a-\alpha a), a]$, $[b, \frac{b+1}{2}(a-\alpha a)]$, and $[1-\frac{b+1}{2}(a-\alpha a), 1]$. This is continued until S_∞ is reached, and S_∞ is called the Cantor set; more specifically the middle-α Cantor set C_α . Consequently, a Cantor set is often defined as the limit set of an iterated function system (IFS) $f = \{f_i | f_i : I \to I\}$ so that $C = f(C)$ where $f_i(x) = \beta x + i(1-\beta), i = 0,1$ where the scale factor β is defined by $\alpha + 2\beta = 1$. A point $x \in C_\alpha$ has the infinite word representation $x = (1-\beta)\sum_{i=0}^{\infty} x_i \beta^i = x_0 x_1 ....,\ x_i \in \{0,1\}$. For the special value $\alpha = \beta = 1/3$ one obtains the classical one-third (triadic) Cantor set.

Some Important Properties of Cantor sets

We here list a few well-known properties of a Cantor set:

(a) Cantor sets are self-similar fractals. Cantor sets look like the same no matter the level at which they are seen. All (n+1) sections of S_n looks the same as S_0 when magnified.

(b) Cantor sets are totally disconnected (no-where dense) in R.

(c) There are no intervals within a Cantor set.

(d) Cantor sets are uncountable.

(e) Cantor sets are closed.

(f) Cantor sets are compact.

(g) Cantor sets are perfect.

(h) Cantor sets C_α have Lebesgue measure of zero.

Instead of repeatedly removing the middle α of every piece as in the Cantor set C_α, we could also keep removing any other fixed percentage (other than 0 percent and 100 percent) from the middle. The resulting sets are all homeomorphic to the Cantor set and also have Lebesgue measure 0.

But removing progressively smaller percentages of the remaining pieces in every step, one can also construct sets homeomorphic to the Cantor set that have positive Lebesgue measure, while still being no-where dense. This is called a fat Cantor set and is denoted by $\tilde{C}$.

Before proceeding further we present a short note about measure.

Measure

Lebesgue Measure

The Lebesgue measure is the standard way of assigning a measure to subsets of n-dimensional Euclidean space. For n=1,2, or 3 it coincides with the standard measure of length, area or volume. In general it is also called n-dimensional volume or simply volume. Sets those can be assigned a Lebesgue measure are called Lebesgue measurable; the measure of a Lebesgue measurable set A is denoted by $m(A)$.

The construction of Lebesgue measure proceeds as follows:

Fix $n \in N$. A box in R^n is a set of the form $B = \prod_{i=1}^{n}[a_i, b_i]$ where $b_i \geq a_i$ and the product symbol here represents a Cartesian product.

The volume vol(B) of this box is defined to be $\prod_{i=1}^{n}(b_i - a_i)$.

For any subset A of R^n we can define its outer measure $m^*(A)$ by

$m^*(A) = \inf\{\sum_{B\in C} \text{vol}(B)$: C is a countable collection of boxes whose union covers $A\}$.

We then define the set A to be Lebesgue measurable if for every subset S of R^n,

$$m^*(S) = m^*(A \cap S) + m^*(S - A).$$

and the Lebesgue measure is defined by $m(A) = m^*(A)$ for any Lebesgue measurable set A.

Examples of positive measure Cantor sets :

1. Let at each step we remove α_n portion of the length of each component of the previous closed set F_{n-1} so that $F_{n-1} = F_{n_0} \cup O_n \cup F_{n_1}$ and $|O_n| = \alpha_n |F_{n-1}|$, $|F_{n_0}| = |F_{n_1}| = \frac{1}{2}(1 - \alpha_n)|F_{n-1}|$. By induction, each of 2^n components of F_n has length

$|F_{n_i}| = 1/2^n \prod_0^n (1 - \alpha_j), i = 1, 2, ..., 2^n$.

Consequently $m(\tilde{C}) = \lim_{n\to\infty} |F_{n-1}| = \prod_0^\infty (1 - \alpha_i) > 0$ when $\sum \alpha_n < \infty$.

2. Suppose at the nth step an open interval of length $\delta/3^n, (0 < \delta < 1)$ is removed from each of the 2^n components of F_n. The length of each component of F_n is $\frac{1}{2^n}(1 - \delta/3 - \cdots - \frac{2^{n-1}}{3^n}\delta)$. The sum of lengths of all open intervals removed is $\sum 2^n \delta / 3^{n+1} = \delta$, so that $m(\tilde{C}) = 1 - \delta$.

Hausdorff measure

Let U be a nonempty subset of n-dimensional Euclidean space R^n. Diameter of U is defined as $|U| = \sup\{|\mathrm{x} - \mathrm{y}| : \mathrm{x}, \mathrm{y} \in \mathrm{U}\}$, the greatest distance between any pair of points in U. If $\{U_i\}$ be a countable (or finite) collection of sets of diameter at most δ that cover F, i.e., $F \subset \cup_{i=1}^\infty U_i$, with $0 < |U_i| \le \delta$, for each i, then we say that $\{U_i\}$ is a δ cover of F. Suppose that F is a subset of R^n and we define

$$H_\delta^s(F) = \inf\{\textstyle\sum_{i=1}^\infty |\mathrm{U_i}|^s : \{\mathrm{U_i}\} \text{ is a } \delta \text{ cover of F }\}$$

where s is a non-negative number and infimum is taken over all possible δ covers.

As, δ decreases, the class of permissible covers of F is reduced. Therefore, the infimum increases and so approaches a limit as $\delta \to 0$. We write

$$H^s(F) = \lim_{\delta \to 0} H_\delta^s(F)$$

This limit exists for any subset F of R^n, though the limiting value can be 0 or ∞. We call $H^s(F)$, the s-dimensional Hausdorff measure of F. If $\{F_i\}$ is any countable collection of disjoint Borel sets then

$$H^s(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} H^s(F)$$

Hausdorff measure generalizes the familiar ideas of length, area, volume etc.

Hausdorff dimension

$H^s_\delta(F)$ is non-increasing with s so that $H^s(F)$ is also non-increasing with s. Now if $t > s$, and $\{U_i\}$ is δ cover of F, we have $\sum |U_i|^t \leq \delta^{t-s} \sum |U_i|^s$

and so taking infimum for each fixed s, $H^t_\delta \leq \delta^{t-s} H^s_\delta(F)$.

Letting $\delta \to 0$, we see that if $H^s(F) < \infty$, then $H^t(F) = 0$ for $t > s$. Thus it shows that there is a critical value of s at which $H^s(F)$ jumps from ∞ to 0. This critical value is called the Hausdorff Dimension of F.

$H^s(F) = \infty$ if $s < dim_H F$ and $H^s(F) = 0$ if $s > dim_H F$.

Thus $H^s(F)$ jumps from ∞ to 0, at a critical value s_0. This critical value is called the Hausdorff dimension of F. It can be shown that for a totally disconnected uncountable set $F \subset R$, $0 < H^{s_0}(F) < \infty \Leftrightarrow 0 < s_0 < 1$.

2.1.3 Cantor function

The Cantor function is an example of a function that is continuous, but not absolutely continuous. It is referred to as the Devil's Staircase.

Definition

Formally, the Cantor function $\phi : [0,1] \to [0,1]$ is defined as follows:

1. Express x in base 3.
2. If x contains a 1, replace every digit after the first 1 by 0.
3. Replace all 2's with 1's.
4. Interpret the result as a binary number. The result is $\phi_(x)$.

To construct ϕ explicitly, let $\phi(0) = 0, \phi(1) = 1$. Assign $\phi(x)$ a constant value $\phi(x) = i2^{-n}$, $i = 1, 2, ...2^n - 1$, on each of the deleted open intervals, i.e. the gaps (including the end points of the deleted intervals) of C_α. Next let $x \in C_\alpha$. Then at the nth iteration, x belongs to the interior of exactly one of the 2^n remaining closed intervals each of length β^n. Let $[a_n, b_n]$ be one of such interval. Then $b_n - a_n = \beta^n$. Moreover $\phi(b_n) - \phi(a_n) = 2^{-n}$. At the next iteration, let $x \in [a_{n+1}, b_{n+1}]$, $(a_n = a_{n+1})$.Then we have $\phi(x) = \lim_{n\to\infty} \phi(a_n) = \lim_{n\to\infty} \phi(b_n)$. Then $\phi : [0,1] \to [0,1]$ is a continuous, non decreasing function. Also $\phi'(x) = 0$ for $x \in I - C_\alpha$ when it is not differentiable at any $x \in C_\alpha$.

Also we can define a sequence $\{f_n\}$ of functions on the unit interval that converges to the Cantor function.

Let, $f_o(x) = x$

Then for every integer $n \geq 0$, the next function $f_{n+1}(x)$ will be defined in terms of $f_n(x)$ as follows:

$$f_{n+1}(x) = \begin{cases} 0.5 f_n(3x), & 0 \leq x \leq 1/3; \\ 0.5, & 1/3 \leq x \leq 2/3; \\ 0.5 + 0.5 f_n(3x - 2), & 2/3 \leq x \leq 1; \end{cases} \tag{2.1}$$

The three definitions are compatible at the end points 1/3 and 2/3, because $f_n(0) = 0$ and $f_n(1) = 1$ for every n. f_n converges point wise to the Cantor function defined above.

Although a Cantor function is continuous everywhere and has zero derivative almost everywhere, ϕ goes from 0 to 1 as x goes from 0 to 1, and takes on every value in between. Moreover a Cantor function is actually uniformly continuous but not absolutely. It is constant on intervals of the form $(0.x_1x_2 \ldots x_n02222\ldots, 0.x_1x_2 \ldots x_n20000\ldots)$, and every point not in the Cantor set is in one of these intervals, so its derivative is zero outside of the Cantor set. On the other hand, it has no derivative at any point in an uncountable subset of the Cantor set containing the interval end points described above.

2.2 Non Standard Analysis

2.2.1 Introduction

Non-Standard analysis is a branch of classical analysis that formulates analysis using a rigorous notion of an infinitesimal number. Non-Standard analysis was introduced in the early of 1960s by the mathematician Abraham Robinson. He wrote:

" the idea of infinitely small or infinitesimal quantities seems to appeal naturally to our intuition. At any rate, the use of infinitesimals was widespread during the formative stages of the Differential and Integral calculus. As for the objection that the distance between two distinct real numbers can not be infinitely small, G.W.Leibnitz argued that the theory of infinitesimals implies the introduction of ideal numbers which might be infinitely small or infinitely large compared with the real numbers but which were to possess the same properties as the latter."

Much of the earliest development of infinitesimal calculus by Newton and Leibnitz was formulated using expressions such as infinitesimal number and vanishing quantity. These formulation were widely criticized by George Berkely and others. It was a challenge to develop a consistent theory of analysis using infinitesimals and the first person to do this in a satisfactory way was Abraham Robinson [26]. In 1958 Schmieden and Laugwitz proposed a construction of a ring containing infinitesimals [38]. The ring was constructed from sequences of real numbers. Two sequences were considered equivalent if they differed only in a finite number of elements. Arithmetic operations were defined element wise. However, the ring constructed in this way contains zero divisors and thus can not be a field.

2.2.2 Approach to Non-Standard Analysis

There are two different approaches to non-standard analysis: the semantic or model-theoretic approach and the syntactic approach. Both of these approaches apply to other areas of mathematics beyond analysis, including number theory, algebra, topology and etc.

Robinson's original formulation of non-standard analysis falls into the category of semantic approach. As developed by him in his papers, it is based on studying models (in particular saturated models) of a theory. Since Robinson's work first appeared, a simpler semantic approach (due to Elias Zakon) has been developed using purely set-theoretic object called super structures. In this approach a model of a theory is replaced by an object called a super structure $V(S)$ over a set S. Starting from a super structure $V(S)$ one construct another object $*V(S)$ using the ultra power construction together with a mapping $V(S) \rightarrow *V(S)$ which satisfies the transfer principles. The map * relates formal properties of $V(S)$ and $*V(S)$. Moreover it is possible to consider a simplified form of saturation called countable saturation. This simplified approach is also more suitable for use by mathematicians who are not specialist in model theory or logic.

Let us briefly recall the ultra power construction of Robinson. Though less direct than the axiomatic approach, it allows one to get a more intuitive contact with the origin of the new structure . Indeed the new infinite and infinitesimal numbers are formulated as equivalence classes of sequences of real numbers, in a way quite similar to the construction of the set of real numbers R from rationals.

Let N be the set of natural numbers. A non-principal (free) ultra filter U on N is defined as follows:

U is a non empty set of subsets of N $[P(N) \supset U \supset \phi]$, such that

i) $\phi \in U$

ii) $A \in U$ and $B \in U \Rightarrow A \bigcap B \in U$

iii) $A \in U$ and $B \in P(N)$ and $B \supset A \Rightarrow B \in U$

iv) $B \in P(N) \Rightarrow$ either $B \in U$ or $\{j \in N : j \notin B\} \in U$, but not both.

v) $B \in P(N)$ and B is finite $\Rightarrow B \notin U$.

Then the set $*R$ is defined as the set of equivalence classes of all sequences of real numbers modulo the equivalence relation: $a \equiv b$, provided $\{j : a_j = b_j\} \in U$, a and b being two sequences $\{a_j\}$ and $\{b_j\}$.

Similarly, a given relation is said to hold between elements of $*R$ if it holds term wise for a set of indices which belongs to the ultra filter. For example $a < b \Rightarrow \{j : a_j < b_j\} \in U$.

R is isomorphic to a subset of $*R$, since one can identify any real $r \in R$ with the class of sequences $\{r, r, ...\}$. Moreover $*R$ is an ordered field.

The syntactic approach requires much less logic and model theory to understand and use. This approach was developed in the mid 1970s by the mathematician Edward Nelson. Nelson introduced an entirely axiomatic formulation of non-standard analysis that he called Internal Set Theory (IST) [39]. IST is an extension of Zermelo Fraenkel Set Theory (ZST). Along with the basic binary membership relation, it introduces a new unary predicate standard which can be applied to elements of the mathematical universe together with some axioms for reasoning with this new predicate.

Syntactic non-standard analysis requires a great deal of care in applying the principle of set formation which mathematicians usually take for granted. As Nelson pointed out, a common fallacy in reasoning in IST is that of illegal set formation. For instance, there is no set in IST whose elements are precisely the standard integers (here standard is understood in the sense of new predicate). To avoid illegal set formation, one must only use predicates of Zermelo-Frankel-Choice (ZFC) to define subsets [39].

2.2.3 Basic Definitions and Constructions of Extended Number Systems

An infinitesimals is a number that is smaller than every positive real number and is larger than every negative real number, or, equivalently, in absolute value it is smaller than $1/m$ for all $m \in N = \{1, 2, 3, ...\}$. Zero is the only real number that at the same time is an infinitesimal, so that the non zero infinitesimals do not occur in standard analysis. Yet, they can be treated in much the same way as can be for ordinary numbers. For example, each non zero infinitesimal ϵ can be inverted and the result is the number $\omega = 1/\epsilon$. It follows that $|\omega| > m$ for all $m \in N$, for which reason ω is called hyper large or infinitely large. Hyper large numbers too do not occur in ordinary analysis, but nevertheless can be treated like ordinary numbers. If, for example, ω is positive hyper large, we can compute $\omega/2, \omega - 1, \omega + 1, 2\omega, \omega^2$ etc. The positive hyper large numbers must not be confused with ∞, which should not be regarded a number at all.

If ϵ is hyper small, if δ too is hyper small but non zero and if ω is positive hyper large, so that $-\omega$ is negative hyper large, we write $\epsilon \cong 0, \delta \approx 0$, $\omega \approx \infty, -\omega \approx -\infty$ respectively. It would be wrong of course, to deduce from $\omega \approx \infty$ that the difference between ω and ∞ would be hyper small.

Given any $x \in R$, $x \neq 0$ and any $\delta \cong 0$, let $t = x + \delta$, then $\epsilon < |t| < \omega$, for all $\epsilon \approx 0$ and all $\omega \approx \infty$. The number t is called *finite* (or appreciable/moderately small or large) number (as it is not too small and not too large).

Three non overlapping sets of numbers (old or new) can now be formed.

a) the set of all infinitesimals, to which zero belongs,

b) the set of all finite numbers, to which all non zero reals belong, and

c) the set of all hyper large numbers, containing no ordinary numbers at all.

Together these three sets, constitute the set of all numbers of "Real Non-Standard Analysis". This set, which clearly an extension of R, is indicated by $*R$ and is called the * transform of R. The elements of $*R$ are called hyper real.

If a number is not hyper large it is called finite or limited. Clearly, $t \in *R$ is finite iff $t = x + \epsilon$ for some $x \in R$ and $\epsilon \cong 0$. Given such a t, both x and ϵ are unique, for,

$x + \epsilon = y + \delta$, $x, y \in R$, $\epsilon, \delta \cong 0$, we have $x = y$ (as $x - y \in R$)and $\epsilon \cong \delta$.

By definition x is called the standard part of t, and this is written as $x = st(t)$.

The standard part function st provides an important bridge between the finite numbers of non-standard analysis and the ordinary real numbers. Trivially, if t is itself an ordinary real number, then $st(t) = t$.

The * transform not only can be obtained for R but also for N, Z, Q, and in fact for any set X of classical mathematics. Their * transforms are indicated by $*N, *Z, *Q, *X$ respectively.

Selecting all finite numbers from $*N$ and $*Z$ we obtain again N and Z, but this is not true for $*Q$, simply because $*Q$ (just as $*R$) contains finite non-standard numbers. But again there is a distinct difference between $*Q$ and $*R$ in this respect; there are finite elements t of $*Q$ that can not be written as $t = x+\epsilon$, with $x \in Q$, $\epsilon \in *Q$, $\epsilon \approx 0$. For let c be any irrational number, say $c = \sqrt{2}$, and let $\{r_1, r_2, ...\}$be some Cauchy sequence of rationals converging to c. The sequence $\{r_1 - c, r_2 - c, ...\}$ generates an infinitesimals δ in $*R$ (because this sequence converges to zero). On the other hand $\{r_1, r_2, ...\}$ generates an element $r \in *Q \subset *R$ and r is finite, but it has no standard part in Q, for otherwise $r = x + \epsilon$ for some $x \in Q$ and $\epsilon \in *Q$, $\epsilon \cong 0$. But $\{r_1 - c, r_2 - c, ...\}$ also generates the finite number $r - c \in R$, so that $r - c = \delta \cong 0$. It follows that $x - c = \delta - \epsilon \cong 0$, hence $x - c = 0$ (as $x - c$ is ordinary real), which would mean that $c \in Q$, a contradiction.

There are various ways to introduce new numbers. One way is done by means of infinite sequences of real numbers. In particular, the elements of $*R$ will be generated by means of infinite sequences of reals and it will be necessary to consider all such sequences. (Recall that the elements of R can be generated by means of rather special infinite sequence of rationals, i.e, the Cauchy sequences). More generally, given any set X the elements of its * transform $*X$ will be generated by means of infinite sequence of elements of X, quotionted by the equivalence class generated by the chosen ultra filter and again all such sequences must be taken into account. For example $\{1, 2, 3, ...\}$ generates a hyper large element of $*N$, and $\{3/2, 5/4, 9/8, ...\}$ generates a finite element of $*Q$, and an infinitesimal, generated by $\{1/2, 1/4, 1/8, ...\}$. Different sequences may generate the same elements of $*X$. In fact, given any $x \in *X$ there are many (uncountably many) different sequences which form an equivalence class under the ultra filter which represents the element $x \in *X$. As a consequence, changing finitely many terms of a generating sequence has no effect on the element generated.

2.2.4 The Purpose of Non-Standard Analysis

Starting from N, the sets Z, Q and R have been introduced in classical analysis (mathematics) in order to enrich analysis with more tools and to refine existing tools. The introduction of negative numbers, of fractions, and of irrational numbers is felt as a strong necessity, and without it mathematics would only be a small portion of what it actually is. The introduction of $*N, *Z, *Q$ and $*R$, however was not meant at all to enrich mathematical analysis (at least not when it all started), but only to simplifying it. In fact, definitions and theorems of classical analysis generally are greatly simplified in the context of non-standard analysis. Non-standard analysis has also been applied later in a more traditional way, namely to introduce new mathematical notions and models. Examples can be found in probability theory, asymptotic analysis, mathematical physics, economics etc.

As an example of a simpler definition, consider *continuity*. A function f from R to R is continuous at $c \in R$ if statement (i) holds:

$$\forall \epsilon \in R,\ \epsilon > 0 : \exists \delta \in R, \delta > 0 : \forall x \in R,$$
$$|x - c| < \delta : \ |f(x) - f(c)| < \epsilon \text{ ....(i)}$$

Now to f there corresponds a unique function $*f$, called the * transform of f, that is a function from $*R$ to $*R$, such that $*f(x) = f(x)$ if $x \in R$ and (i) is true iff (ii), which is the * transform of (i) is true:

$$\forall \epsilon \in *R,\ \epsilon > 0 : \exists \delta \in *R,\ \delta > 0 : \forall x \in *R,$$
$$|x - c| < \delta : | *f(x) - *f(c)| < \epsilon \text{ ....(ii)}$$

Moreover (i) is equivalent to much simpler statement (iii)

$$\forall \delta \in *R,\ \delta \cong 0 : *f(c+\delta) - *f(c) \cong 0 \text{....................(iii)}$$

An illustration of a simpler proof is that of the *Intermediate Value Theorem*:

If $f : R \to R$ is continuous in the closed interval $[a, b]$, $a < b$, a and b both finite, and $f(a) < 0,\ f(b) > 0$, then $f(c) = 0$ for some $c \in [a, b]$.

A non-standard proof of this theorem proceeds as follows:

Let, $m \in *N$ be hyper large. Divide $[a, b]$ into m equal subintervals, each of length $\delta = (b-a)/m$. Then $\delta \approx 0$. Let, n be the smallest element of $*N$ such that $*f(a+n\delta) > 0$, then $*f(a+(n-1)\delta) \leq 0$.

Let $c = st(a+n\delta)$, then, by continuity,

$*f(a+n\delta) - *f(c) = \epsilon_1$ and $*f(c) - *f(a+(n-1)\delta) = \epsilon_2$ for certain infinitesimals ϵ_1 and ϵ_2. Hence $-\epsilon_1 < f(c) = *f(c) \leq \epsilon_2$. But $f(c) \in R$, so $f(c) = 0$.

Terrence Tao, one of the most brilliant contemporary mathematicians, has been advocating strongly the use of nonstandard analysis as *soft* analysis rather than using only the classical *hard* analysis in partial differential equations and various other fields of applications in his blog page 'What's New'.

2.3 Non-Archimedean Ultra metric Theory

2.3.1 Absolute value on a Field

Definition 1. An absolute value on K is a function $|.| : K \to R_+$ that satisfies the following conditions :

1) $|x| = 0$ iff $x = 0$,

2) $|xy| = |x||y|$ for all $x, y \in K$,

3) $|x+y| \leq |x| + |y|$ for all $x, y \in K$.

We shall say an absolute value of K is non-archimedean if it satisfies the additional condition :

4) $|x+y| \leq max(|x|, |y|)$ for all $x, y \in K$.

Otherwise, the absolute value is archimedean.

Example- If we take,

$$|x| = \begin{cases} 1, & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases} \tag{2.2}$$

for any field K, then it is trivially a non-archimedean absolute value.

Theorem 1. *Let, $A \subset K$ be the image of Z in K. An absolute value $|.|$ on K is non-archimedean iff $|a| \leq 1$ for all $a \in A$. In particular, an absolute value on Q is non-archimedean iff $|n| \leq 1$ for any $n \in Z$.*

Proof. We have $|\pm 1| = 1$; hence, if $|.|$ is non-archimedean, we get that $|a \pm 1| \leq max\{|a|, 1\}$. By induction $|a| \leq 1$ for every $a \in A$.

For converse part, suppose that $|a| \leq 1$ for all $a \in A$. We want to prove that for any two elements $x, y \in K$, we have $|x + y| \leq max\{|x|, |y|\}$. If $y = 0$, this is obvious. If not, we can divide through by $|y|$, and we see that this is equivalent to the inequality $|x/y + 1| \leq max\{|x/y|, 1\}$.

This means that we need only to prove the inequality for the case when the second summand is 1, and the general fact will then follow. In other words, we want to prove that for any $x \in K$ we have $|x + 1| \leq max\{|x|, 1\}$.

Now let m be any positive integer. Then we have

$$\begin{aligned}
& |x+1|^m \\
= \ & |\sum_k ({}^mC_k x^k)| \\
\leq \ & |{}^mC_k||x^k| \\
\leq \ & \sum_k |x^k| \quad [\text{since } |({}^mC_k)| \leq 1] \\
= \ & \sum_k |x|^k \\
\leq \ & (m+1)\text{max}\{1, |x|^m\}
\end{aligned}$$

(for the last step, notice that the largest value of $|x|^k$ for $k = 0, 1, 2, ...m$ is equal to $|x|^m$ if $|x| > 1$ and equal to 1 otherwise).

Taking m-th root on both sides gives, $|x + 1| \leq (m + 1)^{1/m} max(1, |x|)$.

Now this inequality holds for every positive integer m, and we know that $\lim_{m \to \infty}(m + 1)^{1/m} = 1$.

Therefore, if we let $m \to \infty$ we get $|x+1| \leq max\{|x|, 1\}$ which is what we wanted to prove.

■

Archimedean property

Given $x, y \in K, x \neq 0$, there exists a positive integer n such that $|nx| > |y|$.

The archimedean property is equivalent to the assertion that $\sup\{|n| : n \in Z\} = +\infty$.

Corollary 1. *An absolute value* $|.|$ *is non-archimedean iff* $\sup\{|n| : n \in Z\} = 1$.

2.3.2 Topology

The essential point of an absolute value is that it provides us with a notion of "size". In other words, once we have an absolute value, we can use it to measure distances between numbers, that is, to put a metric on our field. Having the metric, we can define open and closed sets, and in general investigate the (metric) topology of our field.

Definition 2. Let, K be a field and $|.|$ an absolute value on K. We define the distance $d(x, y)$ between two elements $x, y \in K$ by $d(x, y) = |x - y|$.

The function $d(x, y)$ is called the metric induced by the absolute value and the metric $d(x, y)$ has the following properties:

1) for any $x, y \in K$, $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$

2) for any $x, y \in K$, $d(x, y) = d(y, x)$

3) for any $x, y, z \in K$, $d(x, z) \leq d(x, y) + d(y, z)$

The last inequality is called triangle inequality.

Lemma 1. *Let* $|.|$ *be an absolute value on a field K, and define a metric by* $d(x, y) = |x - y|$. *Then* $|.|$ *is non-archimedean iff for any* $x, y, z \in K$, *we have* $d(x, y) \leq max\{d(x, z), d(z, y)\}$.

Proof. To get one way, apply the non-archimedean property to the equation $(x-y) = (x-z)+(z-y)$.

For the converse, take $y = -y$ and $z = 0$ in the inequality satisfied by $d(.,.)$. ∎

This inequality is known as the "Ultra metric inequality" and a metric for which it is true is called an "Ultra metric". A space with an ultra metric is called an "Ultra metric Space".

Proposition 1. *Let K be a field and let $|.|$ be a non-archimedean absolute value on K. If $x, y \in K$ and $x \neq y$, then $|x+y| = max\{|x|, |y|\}$.*

Proof. Exchanging x and y if necessary, we may suppose $|x| > |y|$. Then $|x+y| \leq |x| = max\{|x|, |y|\}$.

On the other hand $x = (x+y)-y$, so that $|x| \leq max\{|x+y|, |y|\} = |x+y|$.

So, $|x| = |x+y| \Rightarrow |x+y| = max\{|x|, |y|\}$. ∎

Corollary 2. *In an ultra metric space, all triangles are isocels.*

Definition 3. Let, K be a field with an absolute value $|.|$. Let $a \in K$ be an element and let $r \in R_+$ be a positive real number. The open ball of radius r and center a is the set $B(a,r) = \{x \in K : d(x,a) < r\} = \{x \in k : |x-a| < r\}$

The closed ball of radius r and centre a is the set $\bar{B}(a,r) = \{x \in k : d(x,a) \leq r\} = \{x \in k : |x-a| \leq r\}$.

Proposition 2. *Let K be a field with a non-archimedean absolute value.*

1) If $b \in B(a,r)$, then $B(a,r) = B(b,r)$; in other words, every point that is contained in an open ball is a centre of that ball.

2) If $b \in \bar{B}(a,r)$ then $\bar{B}(a,r) = \bar{B}(b,r)$; in other words, every point that is contained in a closed ball is a centre of that ball.

3) The set $B(a,r)[\bar{B}(a,r)]$ is both open and closed.

4) any two open balls (closed balls) are either disjoint or contained in one another.

Proof. 1) By definition $b \in B(a,r)$ iff $|b-a| < r$. Now, taking any x for which $|x-a| < r$, the non-archimedean property says that $|x-b| \leq max\{|x-a|, |b-a|\} < r$. So, $x \in B(b,r)$. That is $B(a,r) \subset B(b,r)$. Switching a and b, we get the opposite inclusion, so that the two balls are equal.

2) Similar as (1).

3) The open ball $B(a,r)$ is always an open set in any metric space. What we need to show is that in non-archimedean case, it is also closed. So, take an x in the boundary of $B(a,r)$; this means that any open ball centred in x must contain points those are in $B(a,r)$. Choose a number $s \leq r$. Now, since x is a boundary point, $B(a,r) \cap B(x,s) \neq \phi$; so that there exists an element $y \in B(a,r) \cap B(x,s)$.

This means that $|y-a| < r, |y-x| < s \leq r$

Now $|x-a| \leq max\{|x-y|, |y-a|\} < max\{s,r\} \leq r$, so $x \in B(a,r)$. This shows that boundary point of $B(a,r)$ belongs to $B(a,r)$, which means that $B(a,r)$ is closed.

Converse part is similar.

4) Let, $B(a,r)$ and $B(b,s)$ are two open balls such that $B(a,r) \cap B(b,s) \neq \phi$.

We can assume that $r \leq s$. If the intersection is not empty, there exists a $c \in B(a,r) \cap B(b,s)$. Then from (1) we have, $B(a,r) = B(c,r)$ and $B(b,s) = B(c,s)$. Hence $B(a,r) = B(c,r) \subset B(c,s) = B(b,s)$. ∎

Definition 4. Let, K be a field with an absolute value $|.|$. We say a set $S \subset K$ is clopen if it is both an open and a closed set.

Thus it is clear that a non-archimedean ball does not have well defined centres, because every point of the ball can be called its centre. Also every open (closed) ball is clopen.

A Cauchy sequence in an ultra metric space X is a sequence $\{x_n\}$ such that for any $\epsilon > 0$, $\exists n_0 \in N$ such that for all $m, n \geq n_0$, $d(x_m, x_n) < \epsilon$. This implies $d(x_{n+1}, x_n) < \epsilon$ (by strong triangle inequality) for all $n \geq n_0$.

A space X is complete if every Cauchy sequence converges to a limit in X.

2.4 p-adic Number and Analysis

2.4.1 Introduction

In mathematics and chiefly in number theory, the p-adic number system [42] for any prime number p, extends the ordinary arithmetic of the rational numbers in a way different from the extension of the rational number system to the real and complex number systems. The extension is achieved by an alternative interpretation of the concept of absolute value. The p-adic numbers were motivated primarily by an attempt to bring the ideas and techniques of power series methods into number theory. Their influence now extends far beyond this. For example, the field of p-adic analysis essentially provides an alternative form of calculus.

More formally, for a given prime p, the field Q_p of p-adic numbers is a completion of rational numbers. The field Q_p is also given a topology derived from a metric, which is itself derived from an alternative valuation on the rational numbers. This metric space is complete in the sense that every Cauchy sequence converges to a point in Q_p. This is what allows the development of calculus on Q_p, and it is the interaction of this analytic and algebraic structure which gives p-adic number systems their power and utility.

2.4.2 p-adic expansions

If p is a fixed prime number, then any positive integer can be written in a base p expansion in the form $\sum_{i=1}^{n} a_i p^i$ where the a_i are integers in $\{0, 1, ...p-1\}$. For example, the binary expansion of 27 is $1.2^4 + 1.2^3 + 0.2^2 + 1.2^1 + 1.2^0$, often written in the short hand notation $(11011)_2$ The familiar approach to extending this description to the larger domain of the rationals (and, ultimately to the reals) is to use sums of the form: $\sum_{i=-m}^{n} a_i p^i$.

A definite meaning is given to these sums based on Cauchy sequences, using the absolute value as metric. Thus for example, 1/3 can be expressed in base 5 as the limit of the sequence $(0.13131313....)_5$. In this formulation, the integers are precisely those numbers for which $a_i = 0$ for all $i < 0$.

As an alternative, if we extend the base p expansions by allowing infinite sums of the form $\sum_{i=k}^{\infty} a_i p^i$ where k is some (not necessarily positive) integer, we obtain the p-adic expansions defining the field Q_p of p-adic numbers. Those p-adic numbers for which $a_i = 0$ for all $i < 0$ are called the p-adic integers. The p-adic integers form a subring of Q_p, denoted by Z_p.

2.4.3 The field of p-adic numbers Q_p

The basic example of a norm on the field Q of rational numbers is the absolute value $|.|$. The induced metric $d(x,y) = |x-y|$ is the ordinary Euclidean distance on the real line and the field of real numbers R is the completion of Q with respect to this norm. Now the question arises: Is the Euclidean distance between rational numbers really the most "natural" one? Is there any other way to describe the "closeness" between rationals? The new ways of measuring distance between rational numbers come from the following "arithmetical" construction.

Definition 1. Let, $p \in N$ be any prime number. Define a map $|.|_p$ on Q as follows:

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p x}, & \mathrm{if} x \neq 0 \\ 0, & \mathrm{if} x = 0 \end{cases} \tag{2.3}$$

where $\mathrm{ord}_p x =$ the highest power of p which divides x, if $x \in Z$ and $\mathrm{ord}_p x = \mathrm{ord}_p a - \mathrm{ord}_p b$, if $x = a/b$, $a, b \in Z$, $b \neq 0$ is the p-adic order or p-adic valuation of x.

Proposition 3. *$|.|_p$ is a non-archimedean norm on Q.*

Property (1) of non-archimedean norm follows from definition of $|.|_p$.

Again $\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$.

Therefore, $|xy|_p = p^{-\mathrm{ord}_p(xy)} = p^{-[\mathrm{ord}_p(x)+\mathrm{ord}_p(y)]} = p^{-\mathrm{ord}_p(x)}.p^{-\mathrm{ord}_p(y)} = |x|_p.|y|_p$.

So, property (2)of non-archimedean norm is satisfied.

Let us verify property (3). If $x = 0$, $y = 0$, (3) is trivial, so assume $x, y \neq 0$. Let, $x = a/b, y = c/d$.

Then we have

$x + y = \frac{ad+bc}{bd}$ and

$$\begin{aligned} & \mathrm{ord}_p(x+y) \\ = \; & \mathrm{ord}_p(ad+bc) - \mathrm{ord}_p(bd) \\ \geq \; & min(\mathrm{ord}_p(ad), \mathrm{ord}_p(bc)) - \mathrm{ord}_p(b) - \mathrm{ord}_p(d) \\ = \; & min(\mathrm{ord}_p a - \mathrm{ord}_p b, \mathrm{ord}_p c - \mathrm{ord}_p d) \\ = \; & min(\mathrm{ord}_p x, \mathrm{ord}_p y) \end{aligned}$$

Therefore $|x+y|_p = p^{-\mathrm{ord}_p(x+y)} \leq max(p^{-\mathrm{ord}_p x}, p^{-\mathrm{ord}_p y}) = max(|x|_p, |y|_p)$.

So, the strong ultra metric triangle inequality is satisfied by this norm. So, $|.|_p$ is non-archimedean.

Note that the usual absolute value is denoted by $|.|_\infty$ and is associated to the real numbers.

Definition 2. Two absolute values $|.|_1$ and $|.|_2$ on a field K are called equivalent if they define the same topology on K, that is, if every set that is open with respect to one is also open with respect to the other.

Lemma 2. *Let $|.|_1$ and $|.|_2$ be absolute values on a field K. The following statements are equivalent.*

1) $|.|_1$ and $|.|_2$ are equivalent absolute values;

2) for any $x \in K$ we have $|x|_1 < 1$ iff $|x|_2 < 1$;

3) there exists a positive real number α such that every $x \in K$ we have $|x|_1 = |x|_2^\alpha$.

Theorem 2. *(Ostrowski:) Every non-trivial absolute value on Q is equivalent to one of the absolute values $|.|_p$, where p is a prime number or $p = \infty$.*

Proposition 4. *For any $x \in Q - \{0\}$, we have $\prod_{p \leq \infty} |x|_p = 1$.*

Proof. It is easy to see that we only need to prove the formula when x is a positive integer, and that the general case will then follow. So, let x be a positive integer, which we can factor as $x = p_1^{\alpha_1}.p_2^{\alpha_2}.....p_k^{\alpha_k}$.

Then we have $|x|_q$ =1 if $q \neq p_i$

$|x|_{p_i} = p_i^{-\alpha_i}$ for i=1,2,....k

$|x|_\infty = p_1^{\alpha_1}.p_2^{\alpha_2}......p_k^{\alpha_k}$. The result then follows.

■

The field Q is not complete with respect to the usual absolute value $|.|_\infty$ and the set of real numbers R is the extension of Q which is complete field with respect to $|\cdot|_\infty$.

Proposition 5. *The field Q of rational numbers is not complete with respect to* $|.|_p$ *for any prime p.*

Let, p be a fixed prime. We define Q_p to be the completion of Q with respect to the p-adic absolute value $|.|_p$. Therefore $(Q_p, |.|_p)$is a complete normed field and this is called the field of p-adic numbers. The elements of Q_p are the equivalent classes of Cauchy sequences in Q with respect to the extension of the p-adic norm. So, Q can be identified with the subfield of Q_p consisting of equivalence classes of constant Cauchy sequences.

For some $a \in Q_p$, let $\{a_n\}$ be a Cauchy sequence of rational numbers representing a. Then by definition $|a|_p = \lim_{n\to\infty} |a_n|_p$.

Definition 3. A p-adic number $a \in Q_p$ is said to be a p-adic integer if its canonical expansion contains only non-negative powers of p.

The set of p-adic integers is denoted by Z_p, so $Z_p = \{\sum_{i=0}^{\infty} a_i p^i\}$

It is easy to see that $Z_p = \{a \in Q_p | |a|_p \leq 1\}$.

So, we can summarize the results as follows:

- there is an absolute value $|.| = |.|_p$ on Q_p and Q_p is complete with respect to this absolute value;

- there is an inclusion $Q \to Q_p$ whose image is dense in Q_p, and the restriction of the absolute value $|.|_p$ to (the image of) Q coincides with the p-adic absolute value;

- the set of values of Q and Q_p under $|.|_p$ is the same; specially, the two sets $\{x \in R_+ : x = |\lambda|_p \text{ for some } \lambda \in Q\}$ and $\{x \in R_+ : x = |\lambda|_p \text{ for some } \lambda \in Q_p\}$ are both equal to the set $\{p^n : n \in Z\} \cup \{0\}$ of powers of p, together with 0.

Proposition 6. *For each $x \in Q_p$, $x \neq 0$ there exists an integer $n \in Z$ such that $|x|_p = p^{-n}$.*

The topology of Q_p

1) The open balls in Q_p are both open and closed.

2) If $b \in B(a,r)$, then $B(b,r) = B(a,r)$. in other words, every point of a ball is its centre.

3) Two balls in Q_p have a non-empty intersection iff one is contained in other.

4) The set of all balls in Q_p is countable.

5) The space Q_p is locally compact.

6) The space Q_p is totally disconnected.

Proof. (1), (2), (3) follows from the fact that Q_p is an ultra metric space.

4) Write the centre of the ball $B(a,p^{-s})$ in its canonical form $a = \sum_{n=-m}^{\infty} a_n p^n$ and let $a_0 = \sum_{n=-m}^{s} a_n p^n$.

Clearly a_0 is a rational number and $|a - a_0|_p < p^{-s}$

i.e. $a_0 \in B(a,p^{-s})$. Therefore, $B(a_0,p^{-s}) = B(a,p^{-s})$.

Here both the centres and radii come from countable sets. Therefore, the product set of all pairs (a_0, s) is also countable and so the set of all balls in Q_p is countable.

To prove (5) we recall that

a subset K in a metric space is called sequentially compact if every infinite sequence of points in K contains a subsequence converging to a point in K.

Further, every infinite sequence of p-adic integers has a convergent subsequence. Therefore Z_p is sequentially compact. Therefore Z_p is compact, and so is any ball in Q_p. So, Q_p is locally compact.

6) For any $a \in Q_p$ and each $n \in N$, the set $U_n(a) = \{x \in Q_p \| x - a|_p \leq p^{-n}\}$=$\{x \in Q_p \| x - a|_p < p^{-(n+1)}\}$ is an open and closed neighborhood of a. Suppose $a \in A$ so that $A \neq \{a\}$. Then there is an $n \in N$ such that $U_n(a) \cap A \neq A$. Therefore, $A = (U_n(a) \cap A) \cup (Q_P - U_n(a) \cap A)$, where both $U_n(a)$ and its complement $Q_p - U_n(a)$ are open and non-empty; this implies A is not connected and hence the result follows. ■