Chapter 1

INTRODUCTION

1.1 Preamble

World around us is nonlinear. Nonlinear differential equations and systems play vital role in formulating theories elucidating and explaining the origin, nature and structure of nonlinearity observed in various natural, biological, financial and other related fields. Although a systematic study of such problems were already initiated in the later half of nineteenth century by Henry Poincare in the context of the planetary three or many-body systems, this field of nonlinear dynamical system is still very active. The aim of the present thesis was originally to investigate some aspects of singularly perturbed nonlinear differential systems with the aim of formulating and developing systematically a novel non-perturbative method analogous to, and also possibly superseding some of those available in the literature, for instance, the admonian method [1], homotopy analysis [2] and variational iteration method [3], re-normalization group method [4], method of geometric singular perturbation [5] and so on, which would yield more efficiently relevant information depicted in a nonlinear differential system, for instance, determination of orbit, amplitude, frequency etc of a nonlinear periodic orbit.

The actual results reported in the thesis now transpires, however, that a more appropriate and reasonable title of the thesis should have been "**On Some Aspects of a Scale Invariant Analysis and Applications**", (inability to go with this title stems from rigid Ph.D Registration rules in the University) since a major part of the thesis deals with a formalism of a scale invariant nonlinear analysis on an extended real number system which allows a real variable to undergo changes by inversions rather than simply by linear translations, as in the conventional classical formulation of analysis of a real variable. As a consequence, the framework of the new analysis may be said to have acquired the structure of an *intrinsic nonlinearity* as compared with that of the linear classical analysis. The motivation of contemplating such a nonlinear formalism is not only to gain better insights into the existing nonlinear techniques available in the literature but also to shed hopefully new light on the meaning and structure of complex nonlinearity as a whole. The latter portion of the thesis reports some simple but nontrivial applications of this nonlinear formalism leading to *emergence* of complex nonlinear structures even from linear differential systems. These emergent nonlinear phenomena from a linear system is shown to offer, in turn, a new *nonperturbative* handle for a specific nonlinear differential system (Van der Pol system).

1.2 Introduction

The conventional treatments of nonlinear problems generally consider nonlinear (ordinary or partial) differential equations when actual nonlinearity appears as new terms with one (or more) (small) parameter(s), for instance, the pendulum equation, the Duffing equation, the Van der Pol oscillator [6] . The standard (regular) perturbation method attempts to find an approximate solution to a nonlinear problem, which cannot be solved exactly, by starting from the exact solution of a related (simpler and generally a linear) exactly solvable problem. Perturbation methods are applicable if the problem at hand can be formulated in a way when the nonlinearity term comes with a "small" parameter. The relevant dynamical quantities are expressed as a formal power series in the "small" parameter known as a perturbation series for the dynamical quantity concerned that quantifies the degree and type of deviations from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while higher order terms describe the (small) deviations in the solution, due to the nonlinear coupling. Perturbation methods are plagued with several limitations: perturbation series for a nonlinear problem are known to diverge generally. More often such a naive formal power series in the small parameter is known to have the problem of non-uniformity: a power series which is known to yield meaningful (convergent) result for small values of time $t < 1/\epsilon$, ϵ being the small parameter, will become meaningless for sufficiently large values of time i.e., when $t > 1/\epsilon$. Perturbation method also fails when the nonlinearity parameter assumes larger values, that is when, $\epsilon \geq 1$. More often, nonlinear equation may not come with any small or large parameter either.

The standard approaches in resolving some of the limitations of the perturbation theory are the method of multiple time scales [7], renormalization group method [4], homotopy analysis method [2] and so on, besides the more sophisticated phase space (plane) analysis. The geometric singular perturbation method and the method of boundary layers [5] are some approaches commonly used in singularly perturbed problems. Most of these methods rely and make use of some asymptotic matching of two or many branches of approximate solutions, obtained by solving some reduced component equations of the original equation, as the nonlinearity parameter asymptotically approaches some fixed value (may be 0, ∞ or any fixed finite number).

With this rich background, the present thesis aims at formulating an altogether new approach in the study of nonlinear problems. It is now well known that a nonlinear system may yield very complicated geometric as well as dynamical structures, when the control (coupling) parameter of the system is varied continuously over a well defined range of values [8, 9]. For instance, consider the continuously perturbed time dependent oscillator of the form

$$\ddot{x} + \mu(t, \dot{x}, x) \sin x = 0 \tag{1.1}$$

where $\mu(t, \dot{x}, x)$ is a time and x dependent frequency. Such a non autonomous and nonlinear pendulum equation may lead to chaotic dynamics when the perturbed frequency becomes sufficiently large ($\mu >> 1$). By chaos we mean here sensitive dependence on initial conditions and to the fact that the late time evolutionary pattern of the system is so irregular that knowing the state of the system at a particular moment does not guarantee one to predict the position of the state in the phase space at any future moment. It is also well known that the basic reason for the emergence of such an irregular dynamics (motion) of a system is the spontaneous formation of one or multidimensional Cantor set like fractal sets in a bounded region of the phase space [9]. Cantor sets are compact, perfect, totally disconnected subsets of the Euclidean space R^n. The ordinary smooth periodic, say small amplitude, oscillations of the perturbed pendulum over an initially connected portion of the phase space would experience a dramatic change to nonsmooth, irregular chaotic motion when the pendulum state is attracted toward and ultimately is arrested on a lower dimensional Cantor subset (strange attractor) in an asymptotic late time. The state of the pendulum (nonlinearly driven) is then found to execute random jump motion on the strange attractor set, the analysis of which can not simply be made meaningfully by a set of ordinary differential equations as in the case of smooth motion. More involved techniques involving geometric measure theory [10], methods of ergodic theory, nonlinear functional analysis [11], fractional differential equations [12] and others are being used by several authors to explore and understand intricate structures those appear to emerge in such a driven system in the late time. One important aspect of such investigations is to understand the precise mechanism of generation of multiple scales dynamically in, for instance, the phase trajectory of the evolving nonlinear system.

Let us recall that any natural system, for example the pendulum in equation (1.1) is not an isolated system, but essentially placed in an environment. In an idealized problem, the perturbation due to environment may be considered negligibly small. However, assuming that the pendulum is designed to execute small amplitude periodic oscillations over very (i.e. infinitely) long time scales, the original (arbitrarily small) environ induced perturbations (in the form of systematic driven force(s) and/or random noise) is likely to grow to a non negligible O(1) level and as a consequence the small amplitude simple harmonic oscillation would be driven presumably to a nonlinear irregular (fluctuating) motion. The non autonomous equation (1.1) is a classical modeling of the above scenario. In the sense of the approach presented in this thesis this might be designated as *extrinsic modeling* of nonlinearity making an idealized simple differential equation more and more complicated via coupling to higher order nonlinear terms. In the proposed *non-classical* nonlinear formalism we aim to present another level of *intrinsic nonlinearity* over and above the standard extrinsic one. According to this *intrinsic principle* one expects that *a system evolving following a simple linear equation (say) would experience a late time nontrivial scale invariant asymptotic motion induced by an a priori nonlinear fractal structure in the time variable that could be revealed as time asymptotes to ∞ utilizing a nontrivial iteration process.*

Over the past few years an approach to a scale invariant nonlinear analysis [13, 14, 15, 16, 17, 19] is being developed. One of the aims of these initiatives was to develop a scale invariant analytical framework that would be suitable to construct a rigorous analysis on Cantor like fractal subsets of R [16, 17]. Since a Cantor set C is a totally disconnected, compact, perfect subset of R, the ordinary analysis of R can not be meaningfully extended over C, i.e., when a real variable x is assumed to live and undergo changes only over the points of C. More specifically, the concept of a derivative in the sense of rate of change of a dynamic quantity, namely, a function of time when time is supposed to vary over a Cantor set, (say)[1] can not be formulated consistently on such a set. The general trend in the literature is to bypass defining derivatives directly on such sets, by taking recourse to technically more involved approaches based on geometric measure theory [10], harmonic analysis [22], functional analysis on non commutative spaces [23], probability theory [24] and so on. The present scale invariant analysis utilizing the concepts of *relative and scale invariant infinitesimals* turns out not only simpler than the other contemporary approaches but also offers an elegant avenue extending the well-known differential calculus of R over a Cantor set C in a conceptually appealing manner. Recall that ordinary measure theoretic arguments can essentially establish an analytic statement on R up

[1]The possibility of a time variation on a Cantor like fractal set is considered in Continuous Time Random Walk theories of statistical mechanics [25].

to a *Lebesgue measure zero set* only. Our analysis, on the other hand, succeeds in deducing results which are valid *everywhere* in R. For instance, a Cantor function $\phi(x)$ can be defined classically as a non decreasing continuous function which satisfies $\frac{d\phi}{dx} = 0$ *almost everywhere* in [0,1]. In the present scale invariant approach a Cantor function is shown to be locally constant *everywhere* in [0,1]. Further, the global variability of a Cantor function is shown to get exposed in a double logarithmic scale $\log\log x^{-1}$. Some simple evolutionary equations are also defined and studied on such a Cantor set [17, 19]. The scale invariant approach rests on an extension of the usual ultra metric structure of a Cantor set into an inequivalent class of ultra metrics using a seemingly new concept of relative infinitesimals that are shown to exist in the gaps of infinitesimally small neighborhoods of 0, considered as an element of another Cantor set $\tilde{C} \subset [0, 1]$.

The present thesis extends the above framework of scale invariant analysis to the level of ordinary classical analysis on the real number system R [13, 14, 19, 21]. To develop a meaningful scale invariant analytic framework on R one needs to proceed in steps. A real variable $t \in R$ essentially represents a dimensionless variable and may be assumed to have been measured in the unit of 1. Any change in the unit of measurement of the length of the closed interval $[0, t]$ would simply introduce a constant multiplicative factor $k > 0$ (say) transforming t to a new variable $t_1 = kt$. The framework of classical analysis is trivially scale invariant in the sense that fundamental definitions of limit, continuity, derivative etc can be stated in either of the two variables t or t_1 yielding identical results, perhaps up to an appropriate scaling constant. For example, if $\lim_{t\to a} f(t) = l$ then $\lim_{t_1\to a_1} f(t_1) = l_1$ where $a_1 = ka$ and $l_1 = kl$. Next, one observes that there does not exist any nontrivial smaller scale in R other than 0, in the sense that if one supposes existence of a nontrivial small scale $\epsilon > 0$ satisfying $0 < \epsilon < t$ and $t \to 0$ then it automatically means that $\epsilon = 0$. On the face of this obstruction i.e. non-availability of a nontrivial smaller scale (on the classical triadic Cantor set, on the other hand $\epsilon_n = 3^{-n}$ gives a countable set of nontrivial scales) construction of a scale invariant analytic framework on R requires extending the conventional (Archimedean) real number system R over an infinite dimensional non-Archimedean space $\mathbf{R}$ accommodating infinitesimally small and infinitely large scales (numbers). The reason for contemplating an infinite dimensional space arises from the obstruction offered by the Frobenius field extension theorem. The non-Archimedean extension $\mathbf{R}$ must involve new elements in the form of infinitely small and large numbers analogous to A. Robinson's nonstandard extension [26] R_{NS} of R. The present thesis introduces the concept of relative and scale invariant infinitesimally small numbers *afresh* exploiting a possible alternative definition of limit introducing new nontrivial scales. Such an infinitesimally small number is *shown* to carry a non-archimedean absolute value which would allow non-null values even when the classical Euclidean value goes to zero in a limiting problem. We report here several new analytic and dynamical results involving imprints of these *dynamically active* infinitesimals and their nontrivial values. In short, the present thesis represents a body of analytic results and their applications in a class of linear and nonlinear differential equations which are of interdisciplinary in nature involving various themes such as real analysis, measure theory, Cantor like fractal sets, theory of infinitesimals, nonarchimedean spaces, analysis on p-adic local fields and so on.

1.3 Main Results Of The Thesis

In Chapter 2, salient features of several key notions such as Cantor set, Non-Standard Analysis, Non-Archimedean Ultra metric theory and P-adic Number and Analysis which are used in the subsequent development of our results, are reviewed briefly.

In Chapter 3 we mainly study the formulation of a Scale Invariant analysis. To this end, we extend the Real Number System R to a Non-Archimedean System $\mathbf{R}$. For this purpose we first introduce the basic concepts of relative infinitesimals, scale free infinitesimals, non-archimedean ultrameric norm (ultra metric absolute value) and study some of their properties. The real number system R equipped with this norm then defines the ultra metric space $\mathbf{R}$. We next consider the completion of the field of rational numbers Q under this norm which yields infinite number of scale free models $\mathbf{R_p}$ of R for each non-trivial scale δ where $\delta = 1/p$ and p is prime number.

In Chapter 4 we present a new proof of Prime Number Theorem (PNT). This proof is derived on the above scale invariant, non-archimedean model $\mathbf{R}$ of real number system R, involving non-trivial infinitesimals and infinites. More specifically, here we introduce a new generalized inversion mediated metric space $\mathcal{R}$ having several branches R and R_p and also interpret a directed variation of a real variable in a dynamical sense. In ordinary real number system R, increment of a variable is possible only by means of linear translation. But in $\mathcal{R}$, increments of a variable are mediated by a combination of linear translations and inversions. Also there exists two types of inversions: (i) global or growing mode leading to an asymptotic finite order variation in the value of a dynamic variable of $\mathcal{R}$ following the asymptotic growth formula of the prime counting function and (ii) Localized inversion mode leading to an asymptotic scaling to a directed infinitesimal and the relative correction to the PNT. Finally we show in this Chapter that prime counting function is a locally constant function on $\mathcal{R}$.

In Chapter 5 we present an application of this scale invariant analysis on a Cantor set C and show that a real variable $x \in C$ and approaching 0 on C is extended to a sub linear variation $x \log x^{-1} \to 0$ in R. Here, we also derive a differential measure on the Cantor set C.

In Chapter 6, a few interesting applications of the above non-linear analysis are presented in the context of some selected topics of differential equations. First we consider ordinary first order differential equation $\frac{dx}{dt} = 1$ in the extended space $\mathcal{R}$ and we find a generalized class of solution of the equation. For $t \sim O(1)$, the new extended solution reduces to the standard solution in R. However as t grows to an asymptotically large value, the time t is extended to the deformed time $T(t)$ with an O(1) directed multiplicative component acquired from the directed infinitesimals of $\mathcal{R}$.

Next we consider harmonic oscillation. We find that admitting non-trivial small scale structures in real number system, the classical sinusoidal orbits of a harmonic oscillator would undergo a nonlinear late time evolution. The original linear oscillation would also experience nonlinear late time perturbations. The simple harmonic oscillation may be deformed into a driven Lineard type system as $t \to \infty$. Then we also give a derivation of the Van der Pol oscillator like variations when the late time variation is modeled as a special amplitude variation for the original harmonic oscillator. Next we consider the reversed problem. That is, beginning from Van der Pol equation, we reproduce the harmonic oscillator equation in an infinitesimal time scale. Finally we point out a possible future application of homotopy type analysis method in the above type of linear to nonlinear transitions and vice versa.

Next we consider linear diffusion equation. Anomalous diffusion is known to occur in diverse complex systems enjoying fine structures such as in disordered or fractal media. The hallmark of such a diffusion process is the occurrence of an anomalous law for the mean square displacement viz.$< \Delta x^2(t) >= t^\nu$ with $\nu \neq 1$. Sub-diffusive ($\nu < 1$) behavior is usually predominant in disordered systems. Super-diffusion ($\nu > 1$), on the other hand, may arise from long range correlations in velocity fields of turbulent flows. Here we offer a potentially new insight into the actual mechanism of the dynamics of anomalous motion and show that the anomalous mean square fluctuations can arise naturally from the ordinary diffusion equation interpreted scale invariantly in the formalism endowing real numbers with a non-archimedean multiplicative structure.

In the concluding chapter, we summarize our main results and also indicate briefly possible future applications of the formalism developed here.

This work is based on the following published and communicated papers:

1. D.P.Datta and A.Ray Chaudhuri, Scale Free Analysis and Prime Number Theorem, Fractals, 18, (2010), 171-184.

2. D.P.Datta, S.Raut and A.Ray Chaudhuri, Ultra metric Cantor sets and Growth of measure, P-adic Numbers, ultra metric Analysis and Applications, (2011), Vol.3, No.1, 7-22.

3. D.P.Datta, S.Raut and A.Ray Chaudhuri, Diffusion in a Class of Fractal sets, International Journal of Applied Mathematics and Statistics, Vol.30 (2012), 34-50.

4. A.Ray Chaudhuri, D.P.Datta, Rescaling Symmetry, Ultrametricity and Emergent Nonlinearity, communicated (2013).