On Some Non-Perturbative Methods in Nonlinear Differential Systems

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DEDICATION

To

Parampujyapad Acharyadeva Sree Sree Dada
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Abstract

In this thesis, a scale invariant analysis on the set of real number system R is developed using the concept of relative infinitesimals , scale free infinitesimals and their corresponding non-archimedean valuation (absolute value). With this valuation we first extend the real number system R to an infinite dimensional non-archimedean system $\mathbf{R}$ accommodating infinitesimally small and infinitely large numbers. Next we determine the corresponding inversion mediated metric space $\mathcal{R}$ and interpret a directed variation of a real variable in a dynamical sense. Then applying the scale free analysis on $\mathcal{R}$, we present a new elementary proof of the well known Prime Number Theorem. Next this analysis is applied on a class of differential equations. We report in particular, some simple but nontrivial applications of this nonlinear formalism leading to emergence of complex nonlinear structures even from a linear differential system. These emergent nonlinear phenomena from a linear system is argued to offer, a new non-perturbative method for computing solutions and estimate amplitude, frequency etc. for a specific nonlinear system, viz, the Van der Pol equation. It is also shown that anomalous mean square fluctuations can arise naturally from the ordinary diffusion equation interpreted scale invariantly in the present formalism endowing real numbers with a non-archimedean multiplicative structure.

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Chapter 1

INTRODUCTION

1.1 Preamble

World around us is nonlinear. Nonlinear differential equations and systems play vital role in formulating theories elucidating and explaining the origin, nature and structure of nonlinearity observed in various natural, biological, financial and other related fields. Although a systematic study of such problems were already initiated in the later half of nineteenth century by Henry Poincare in the context of the planetary three or many-body systems, this field of nonlinear dynamical system is still very active. The aim of the present thesis was originally to investigate some aspects of singularly perturbed nonlinear differential systems with the aim of formulating and developing systematically a novel non-perturbative method analogous to, and also possibly superseding some of those available in the literature, for instance, the admonian method [1], homotopy analysis [2] and variational iteration method [3], re-normalization group method [4], method of geometric singular perturbation [5] and so on, which would yield more efficiently relevant information depicted in a nonlinear differential system, for instance, determination of orbit, amplitude, frequency etc of a nonlinear periodic orbit.

The actual results reported in the thesis now transpires, however, that a more appropriate and reasonable title of the thesis should have been "**On Some Aspects of a Scale Invariant Analysis and Applications**", (inability to go with this title stems from rigid Ph.D Registration rules in the University) since a major part of the thesis deals with a formalism of a scale invariant nonlinear analysis on an extended real number system which allows a real variable to undergo changes by inversions rather than simply by linear translations, as in the conventional classical formulation of analysis of a real variable. As a consequence, the framework of the new analysis may be said to have acquired the structure of an *intrinsic nonlinearity* as compared with that of the linear classical analysis. The motivation of contemplating such a nonlinear formalism is not only to gain better insights into the existing nonlinear techniques available in the literature but also to shed hopefully new light on the meaning and structure of complex nonlinearity as a whole. The latter portion of the thesis reports some simple but nontrivial applications of this nonlinear formalism leading to *emergence* of complex nonlinear structures even from linear differential systems. These emergent nonlinear phenomena from a linear system is shown to offer, in turn, a new *nonperturbative* handle for a specific nonlinear differential system (Van der Pol system).

1.2 Introduction

The conventional treatments of nonlinear problems generally consider nonlinear (ordinary or partial) differential equations when actual nonlinearity appears as new terms with one (or more) (small) parameter(s), for instance, the pendulum equation, the Duffing equation, the Van der Pol oscillator [6] . The standard (regular) perturbation method attempts to find an approximate solution to a nonlinear problem, which cannot be solved exactly, by starting from the exact solution of a related (simpler and generally a linear) exactly solvable problem. Perturbation methods are applicable if the problem at hand can be formulated in a way when the nonlinearity term comes with a "small" parameter. The relevant dynamical quantities are expressed as a formal power series in the "small" parameter known as a perturbation series for the dynamical quantity concerned that quantifies the degree and type of deviations from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while higher order terms describe the (small) deviations in the solution, due to the nonlinear coupling. Perturbation methods are plagued with several limitations: perturbation series for a nonlinear problem are known to diverge generally. More often such a naive formal power series in the small parameter is known to have the problem of non-uniformity: a power series which is known to yield meaningful (convergent) result for small values of time $t < 1/\epsilon$, ϵ being the small parameter, will become meaningless for sufficiently large values of time i.e., when $t > 1/\epsilon$. Perturbation method also fails when the nonlinearity parameter assumes larger values, that is when, $\epsilon \geq 1$. More often, nonlinear equation may not come with any small or large parameter either.

The standard approaches in resolving some of the limitations of the perturbation theory are the method of multiple time scales [7], renormalization group method [4], homotopy analysis method [2] and so on, besides the more sophisticated phase space (plane) analysis. The geometric singular perturbation method and the method of boundary layers [5] are some approaches commonly used in singularly perturbed problems. Most of these methods rely and make use of some asymptotic matching of two or many branches of approximate solutions, obtained by solving some reduced component equations of the original equation, as the nonlinearity parameter asymptotically approaches some fixed value (may be 0, ∞ or any fixed finite number).

With this rich background, the present thesis aims at formulating an altogether new approach in the study of nonlinear problems. It is now well known that a nonlinear system may yield very complicated geometric as well as dynamical structures, when the control (coupling) parameter of the system is varied continuously over a well defined range of values [8, 9]. For instance, consider the continuously perturbed time dependent oscillator of the form

$$\ddot{x} + \mu(t, \dot{x}, x) \sin x = 0 \tag{1.1}$$

where $\mu(t, \dot{x}, x)$ is a time and x dependent frequency. Such a non autonomous and nonlinear pendulum equation may lead to chaotic dynamics when the perturbed frequency becomes sufficiently large ($\mu >> 1$). By chaos we mean here sensitive dependence on initial conditions and to the fact that the late time evolutionary pattern of the system is so irregular that knowing the state of the system at a particular moment does not guarantee one to predict the position of the state in the phase space at any future moment. It is also well known that the basic reason for the emergence of such an irregular dynamics (motion) of a system is the spontaneous formation of one or multidimensional Cantor set like fractal sets in a bounded region of the phase space [9]. Cantor sets are compact, perfect, totally disconnected subsets of the Euclidean space R^n. The ordinary smooth periodic, say small amplitude, oscillations of the perturbed pendulum over an initially connected portion of the phase space would experience a dramatic change to nonsmooth, irregular chaotic motion when the pendulum state is attracted toward and ultimately is arrested on a lower dimensional Cantor subset (strange attractor) in an asymptotic late time. The state of the pendulum (nonlinearly driven) is then found to execute random jump motion on the strange attractor set, the analysis of which can not simply be made meaningfully by a set of ordinary differential equations as in the case of smooth motion. More involved techniques involving geometric measure theory [10], methods of ergodic theory, nonlinear functional analysis [11], fractional differential equations [12] and others are being used by several authors to explore and understand intricate structures those appear to emerge in such a driven system in the late time. One important aspect of such investigations is to understand the precise mechanism of generation of multiple scales dynamically in, for instance, the phase trajectory of the evolving nonlinear system.

Let us recall that any natural system, for example the pendulum in equation (1.1) is not an isolated system, but essentially placed in an environment. In an idealized problem, the perturbation due to environment may be considered negligibly small. However, assuming that the pendulum is designed to execute small amplitude periodic oscillations over very (i.e. infinitely) long time scales, the original (arbitrarily small) environ induced perturbations (in the form of systematic driven force(s) and/or random noise) is likely to grow to a non negligible O(1) level and as a consequence the small amplitude simple harmonic oscillation would be driven presumably to a nonlinear irregular (fluctuating) motion. The non autonomous equation (1.1) is a classical modeling of the above scenario. In the sense of the approach presented in this thesis this might be designated as *extrinsic modeling* of nonlinearity making an idealized simple differential equation more and more complicated via coupling to higher order nonlinear terms. In the proposed *non-classical* nonlinear formalism we aim to present another level of *intrinsic nonlinearity* over and above the standard extrinsic one. According to this *intrinsic principle* one expects that *a system evolving following a simple linear equation (say) would experience a late time nontrivial scale invariant asymptotic motion induced by an a priori nonlinear fractal structure in the time variable that could be revealed as time asymptotes to ∞ utilizing a nontrivial iteration process.*

Over the past few years an approach to a scale invariant nonlinear analysis [13, 14, 15, 16, 17, 19] is being developed. One of the aims of these initiatives was to develop a scale invariant analytical framework that would be suitable to construct a rigorous analysis on Cantor like fractal subsets of R [16, 17]. Since a Cantor set C is a totally disconnected, compact, perfect subset of R, the ordinary analysis of R can not be meaningfully extended over C, i.e., when a real variable x is assumed to live and undergo changes only over the points of C. More specifically, the concept of a derivative in the sense of rate of change of a dynamic quantity, namely, a function of time when time is supposed to vary over a Cantor set, (say)[1] can not be formulated consistently on such a set. The general trend in the literature is to bypass defining derivatives directly on such sets, by taking recourse to technically more involved approaches based on geometric measure theory [10], harmonic analysis [22], functional analysis on non commutative spaces [23], probability theory [24] and so on. The present scale invariant analysis utilizing the concepts of *relative and scale invariant infinitesimals* turns out not only simpler than the other contemporary approaches but also offers an elegant avenue extending the well-known differential calculus of R over a Cantor set C in a conceptually appealing manner. Recall that ordinary measure theoretic arguments can essentially establish an analytic statement on R up

[1]The possibility of a time variation on a Cantor like fractal set is considered in Continuous Time Random Walk theories of statistical mechanics [25].

to a *Lebesgue measure zero set* only. Our analysis, on the other hand, succeeds in deducing results which are valid *everywhere* in R. For instance, a Cantor function $\phi(x)$ can be defined classically as a non decreasing continuous function which satisfies $\frac{d\phi}{dx} = 0$ *almost everywhere* in [0,1]. In the present scale invariant approach a Cantor function is shown to be locally constant *everywhere* in [0,1]. Further, the global variability of a Cantor function is shown to get exposed in a double logarithmic scale $\log\log x^{-1}$. Some simple evolutionary equations are also defined and studied on such a Cantor set [17, 19]. The scale invariant approach rests on an extension of the usual ultra metric structure of a Cantor set into an inequivalent class of ultra metrics using a seemingly new concept of relative infinitesimals that are shown to exist in the gaps of infinitesimally small neighborhoods of 0, considered as an element of another Cantor set $\tilde{C} \subset [0,1]$.

The present thesis extends the above framework of scale invariant analysis to the level of ordinary classical analysis on the real number system R [13, 14, 19, 21]. To develop a meaningful scale invariant analytic framework on R one needs to proceed in steps. A real variable $t \in R$ essentially represents a dimensionless variable and may be assumed to have been measured in the unit of 1. Any change in the unit of measurement of the length of the closed interval $[0,t]$ would simply introduce a constant multiplicative factor $k > 0$ (say) transforming t to a new variable $t_1 = kt$. The framework of classical analysis is trivially scale invariant in the sense that fundamental definitions of limit, continuity, derivative etc can be stated in either of the two variables t or t_1 yielding identical results, perhaps up to an appropriate scaling constant. For example, if $\lim_{t\to a} f(t) = l$ then $\lim_{t_1\to a_1} f(t_1) = l_1$ where $a_1 = ka$ and $l_1 = kl$. Next, one observes that there does not exist any nontrivial smaller scale in R other than 0, in the sense that if one supposes existence of a nontrivial small scale $\epsilon > 0$ satisfying $0 < \epsilon < t$ and $t \to 0$ then it automatically means that $\epsilon = 0$. On the face of this obstruction i.e. non-availability of a nontrivial smaller scale (on the classical triadic Cantor set, on the other hand $\epsilon_n = 3^{-n}$ gives a countable set of nontrivial scales) construction of a scale invariant analytic framework on R requires extending the conventional (Archimedean) real number system R over an infinite dimensional non-Archimedean space **R** accommodating infinitesimally small and infinitely large scales (numbers). The reason for contemplating an infinite dimensional space arises from the obstruction offered by the Frobenius field extension theorem. The non-Archimedean extension **R** must involve new elements in the form of infinitely small and large numbers analogous to A. Robinson's nonstandard extension [26] R_{NS} of R. The present thesis introduces the concept of relative and scale invariant infinitesimally small numbers *afresh* exploiting a possible alternative definition of limit introducing new nontrivial scales. Such an infinitesimally small number is *shown* to carry a non-archimedean absolute value which would allow non-null values even when the classical Euclidean value goes to zero in a limiting problem. We report here several new analytic and dynamical results involving imprints of these *dynamically active* infinitesimals and their nontrivial values. In short, the present thesis represents a body of analytic results and their applications in a class of linear and nonlinear differential equations which are of interdisciplinary in nature involving various themes such as real analysis, measure theory, Cantor like fractal sets, theory of infinitesimals, nonarchimedean spaces, analysis on p-adic local fields and so on.

1.3 Main Results Of The Thesis

In Chapter 2, salient features of several key notions such as Cantor set, Non-Standard Analysis, Non-Archimedean Ultra metric theory and P-adic Number and Analysis which are used in the subsequent development of our results, are reviewed briefly.

In Chapter 3 we mainly study the formulation of a Scale Invariant analysis. To this end, we extend the Real Number System R to a Non-Archimedean System **R**. For this purpose we first introduce the basic concepts of relative infinitesimals, scale free infinitesimals, non-archimedean ultrameric norm (ultra metric absolute value) and study some of their properties. The real number system R equipped with this norm then defines the ultra metric space **R**. We next consider the completion of the field of rational numbers Q under this norm which yields infinite number of scale free models $\mathbf{R_p}$ of R for each non-trivial scale δ where $\delta = 1/p$ and p is prime number.

In Chapter 4 we present a new proof of Prime Number Theorem (PNT). This proof is derived on the above scale invariant, non-archimedean model $\mathbf{R}$ of real number system R, involving non-trivial infinitesimals and infinites. More specifically, here we introduce a new generalized inversion mediated metric space $\mathcal{R}$ having several branches R and R_p and also interpret a directed variation of a real variable in a dynamical sense. In ordinary real number system R, increment of a variable is possible only by means of linear translation. But in $\mathcal{R}$, increments of a variable are mediated by a combination of linear translations and inversions. Also there exists two types of inversions: (i) global or growing mode leading to an asymptotic finite order variation in the value of a dynamic variable of $\mathcal{R}$ following the asymptotic growth formula of the prime counting function and (ii) Localized inversion mode leading to an asymptotic scaling to a directed infinitesimal and the relative correction to the PNT. Finally we show in this Chapter that prime counting function is a locally constant function on $\mathcal{R}$.

In Chapter 5 we present an application of this scale invariant analysis on a Cantor set C and show that a real variable $x \in C$ and approaching 0 on C is extended to a sub linear variation $x \log x^{-1} \to 0$ in R. Here, we also derive a differential measure on the Cantor set C.

In Chapter 6, a few interesting applications of the above non-linear analysis are presented in the context of some selected topics of differential equations. First we consider ordinary first order differential equation $\frac{dx}{dt} = 1$ in the extended space $\mathcal{R}$ and we find a generalized class of solution of the equation. For $t \sim O(1)$, the new extended solution reduces to the standard solution in R. However as t grows to an asymptotically large value, the time t is extended to the deformed time $T(t)$ with an O(1) directed multiplicative component acquired from the directed infinitesimals of $\mathcal{R}$.

Next we consider harmonic oscillation. We find that admitting non-trivial small scale structures in real number system, the classical sinusoidal orbits of a harmonic oscillator would undergo a nonlinear late time evolution. The original linear oscillation would also experience nonlinear late time perturbations. The simple harmonic oscillation may be deformed into a driven Lineard type system as $t \to \infty$. Then we also give a derivation of the Van der Pol oscillator like variations when the late time variation is modeled as a special amplitude variation for the original harmonic oscillator. Next we consider the reversed problem. That is, beginning from Van der Pol equation, we reproduce the harmonic oscillator equation in an infinitesimal time scale. Finally we point out a possible future application of homotopy type analysis method in the above type of linear to nonlinear transitions and vice versa.

Next we consider linear diffusion equation. Anomalous diffusion is known to occur in diverse complex systems enjoying fine structures such as in disordered or fractal media. The hallmark of such a diffusion process is the occurrence of an anomalous law for the mean square displacement viz.$< \Delta x^2(t) >= t^\nu$ with $\nu \neq 1$. Sub-diffusive ($\nu < 1$) behavior is usually predominant in disordered systems. Super-diffusion ($\nu > 1$), on the other hand, may arise from long range correlations in velocity fields of turbulent flows. Here we offer a potentially new insight into the actual mechanism of the dynamics of anomalous motion and show that the anomalous mean square fluctuations can arise naturally from the ordinary diffusion equation interpreted scale invariantly in the formalism endowing real numbers with a non-archimedean multiplicative structure.

In the concluding chapter, we summarize our main results and also indicate briefly possible future applications of the formalism developed here.

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1. D.P.Datta and A.Ray Chaudhuri, Scale Free Analysis and Prime Number Theorem, Fractals, 18, (2010), 171-184.

2. D.P.Datta, S.Raut and A.Ray Chaudhuri, Ultra metric Cantor sets and Growth of measure, P-adic Numbers, ultra metric Analysis and Applications, (2011), Vol.3, No.1, 7-22.

3. D.P.Datta, S.Raut and A.Ray Chaudhuri, Diffusion in a Class of Fractal sets, International Journal of Applied Mathematics and Statistics, Vol.30 (2012), 34-50.

4. A.Ray Chaudhuri, D.P.Datta, Rescaling Symmetry, Ultrametricity and Emergent Nonlinearity, communicated (2013).

Chapter 2

REVIEW OF RELEVANT CONCEPTS

In this chapter we give a somewhat detailed review of several relevant inter-related concepts such as Cantor set, Lebesgue and Hausdorff measure, the formalism of non-standard analysis and analytic and topological properties of nonarchimedean ultra metric spaces including p-adic local fields which will be useful in the latter Chapters. Resources available in the inter net web are used freely in compiling this review.

2.1 Cantor Set

2.1.1 Introduction

The Cantor set, introduced originally by George Cantor in 1883 (though however, discovered in 1875 by Henry J. S. Smith) [27, 28], is a set of points lying on a straight line segment that has a number of remarkable and deep properties and was mainly introduced as counter examples of various general topological concepts. A Cantor set is a totally disconnected compact and perfect subset of the real line. Such a set displays many paradoxical properties. Although the set is uncountable, its Lebesgue measure vanishes. The topological dimension of the set is also zero. Cantor set is an example of a self similar fractal set that arises in various fields of applications. The chaotic attractors of a number of one-dimensional maps, such as the logistic maps, turn out to be topologically equivalent to Cantor sets. Recently there have been a lot of activities developing an analysis on a Cantor-like fractal sets [29, 30]. Because of disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approach based on the fractional derivatives [35, 36] and measure theoretic harmonic analysis [37] have already been considered at length in the literature.

2.1.2 Basic Definitions

A Cantor set is defined as a countable intersection of a finite unions of closed (and bounded) subsets of R. For definiteness, let $C \subset I = [0,1]$. Then by definition, $C = \bigcap\limits_{1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{m=1}^{p^n} F_{nm}$ where $F_{nm} \subset I$ are closed with $F_{00} = I$. Equivalently, C is also defined as $C = I - \bigcup\limits_{i=1}^{\infty} O_i$ where O_i are open intervals which are deleted recursively from I. We also give an alternative definition of a Cantor set. Let us call the initial set $I = [0,1]$ as S_0. For the first iteration a fraction α is taken away from S_0 such that S_1 contains two disconnected sets of real numbers $[0, \frac{1}{2}(1-\alpha)]$ and $[1 - \frac{1}{2}(1-\alpha), 1]$. Let $a := \frac{1}{2}(1-\alpha)$ and $b := 1 - \frac{1}{2}(1-\alpha)$, and call $[0,a]$ as S_{1a} and $[b,1]$ as S_{1b}. For the 2nd iteration take away the fraction α from S_{1a} and S_{1b} such that S_2 contains four disconnected sets of real numbers $[0, \frac{1}{2}(a-\alpha a)]$, $[a - \frac{1}{2}(a-\alpha a), a]$, $[b, \frac{b+1}{2}(a-\alpha a)]$, and $[1-\frac{b+1}{2}(a-\alpha a), 1]$. This is continued until S_∞ is reached, and S_∞ is called the Cantor set; more specifically the middle-α Cantor set C_α . Consequently, a Cantor set is often defined as the limit set of an iterated function system (IFS) $f = \{f_i | f_i : I \to I\}$ so that $C = f(C)$ where $f_i(x) = \beta x + i(1-\beta), i = 0, 1$ where the scale factor β is defined by $\alpha + 2\beta = 1$. A point $x \in C_\alpha$ has the infinite word representation $x = (1-\beta)\sum_{i=0}^{\infty} x_i \beta^i = x_0 x_1 ....,\ x_i \in \{0,1\}$. For the special value $\alpha = \beta = 1/3$ one obtains the classical one-third (triadic) Cantor set.

Some Important Properties of Cantor sets

We here list a few well-known properties of a Cantor set:

(a) Cantor sets are self-similar fractals. Cantor sets look like the same no matter the level at which they are seen. All (n+1) sections of S_n looks the same as S_0 when magnified.

(b) Cantor sets are totally disconnected (no-where dense) in R.

(c) There are no intervals within a Cantor set.

(d) Cantor sets are uncountable.

(e) Cantor sets are closed.

(f) Cantor sets are compact.

(g) Cantor sets are perfect.

(h) Cantor sets C_α have Lebesgue measure of zero.

Instead of repeatedly removing the middle α of every piece as in the Cantor set C_α, we could also keep removing any other fixed percentage (other than 0 percent and 100 percent) from the middle. The resulting sets are all homeomorphic to the Cantor set and also have Lebesgue measure 0.

But removing progressively smaller percentages of the remaining pieces in every step, one can also construct sets homeomorphic to the Cantor set that have positive Lebesgue measure, while still being no-where dense. This is called a fat Cantor set and is denoted by $\tilde{C}$.

Before proceeding further we present a short note about measure.

Measure

Lebesgue Measure

The Lebesgue measure is the standard way of assigning a measure to subsets of n-dimensional Euclidean space. For n=1,2, or 3 it coincides with the standard measure of length, area or volume. In general it is also called n-dimensional volume or simply volume. Sets those can be assigned a Lebesgue measure are called Lebesgue measurable; the measure of a Lebesgue measurable set A is denoted by $m(A)$.

The construction of Lebesgue measure proceeds as follows:

Fix $n \in N$. A box in R^n is a set of the form $B = \prod_{i=1}^{n}[a_i, b_i]$ where $b_i \geq a_i$ and the product symbol here represents a Cartesian product.

The volume vol(B) of this box is defined to be $\prod_{i=1}^{n}(b_i - a_i)$.

For any subset A of R^n we can define its outer measure $m^*(A)$ by

$m^*(A) = \inf\{\sum_{B\in C} \text{vol}(B)$: C is a countable collection of boxes whose union covers $A\}$.

We then define the set A to be Lebesgue measurable if for every subset S of R^n,

$$m^*(S) = m^*(A \cap S) + m^*(S - A).$$

and the Lebesgue measure is defined by $m(A) = m^*(A)$ for any Lebesgue measurable set A.

Examples of positive measure Cantor sets :

1. Let at each step we remove α_n portion of the length of each component of the previous closed set F_{n-1} so that $F_{n-1} = F_{n0} \cup O_n \cup F_{n1}$ and $|O_n| = \alpha_n |F_{n-1}|$, $|F_{n0}| = |F_{n1}| = \frac{1}{2}(1 - \alpha_n)|F_{n-1}|$. By induction, each of 2^n components of F_n has length

$|F_{n_i}| = 1/2^n \prod_0^n (1 - \alpha_j), i = 1, 2, ..., 2^n$.

Consequently $m(\tilde{C}) = \lim_{n\to\infty} |F_{n-1}| = \prod_0^\infty (1 - \alpha_i) > 0$ when $\sum \alpha_n < \infty$.

2. Suppose at the nth step an open interval of length $\delta/3^n, (0 < \delta < 1)$ is removed from each of the 2^n components of F_n. The length of each component of F_n is $\frac{1}{2^n}(1 - \delta/3 - \cdots - \frac{2^{n-1}}{3^n}\delta)$. The sum of lengths of all open intervals removed is $\sum 2^n \delta/3^{n+1} = \delta$, so that $m(\tilde{C}) = 1 - \delta$.

Hausdorff measure

Let U be a nonempty subset of n-dimensional Euclidean space R^n. Diameter of U is defined as $|U| = \sup\{|\mathrm{x} - \mathrm{y}| : \mathrm{x}, \mathrm{y} \in \mathrm{U}\}$, the greatest distance between any pair of points in U. If $\{U_i\}$ be a countable (or finite) collection of sets of diameter at most δ that cover F, i.e., $F \subset \cup_{i=1}^\infty U_i$, with $0 < |U_i| \leq \delta$, for each i, then we say that $\{U_i\}$ is a δ cover of F. Suppose that F is a subset of R^n and we define

$$H_\delta^s(F) = \inf\{\textstyle\sum_{i=1}^\infty |\mathrm{U_i}|^s : \{\mathrm{U_i}\} \text{ is a } \delta \text{ cover of F }\}$$

where s is a non-negative number and infimum is taken over all possible δ covers.

As, δ decreases, the class of permissible covers of F is reduced. Therefore, the infimum increases and so approaches a limit as $\delta \to 0$. We write

$$H^s(F) = \lim_{\delta\to 0} H_\delta^s(F)$$

This limit exists for any subset F of R^n, though the limiting value can be 0 or ∞. We call $H^s(F)$, the s-dimensional Hausdorff measure of F. If $\{F_i\}$ is any countable collection of disjoint Borel sets then

$$H^s(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} H^s(F)$$

Hausdorff measure generalizes the familiar ideas of length, area, volume etc.

Hausdorff dimension

$H^s_\delta(F)$ is non-increasing with s so that $H^s(F)$ is also non-increasing with s. Now if $t > s$, and $\{U_i\}$ is δ cover of F, we have $\sum |U_i|^t \leq \delta^{t-s} \sum |U_i|^s$

and so taking infimum for each fixed s, $H^t_\delta \leq \delta^{t-s} H^s_\delta(F)$.

Letting $\delta \to 0$, we see that if $H^s(F) < \infty$, then $H^t(F) = 0$ for $t > s$. Thus it shows that there is a critical value of s at which $H^s(F)$ jumps from ∞ to 0. This critical value is called the Hausdorff Dimension of F.

$H^s(F) = \infty$ if $s < dim_H F$ and $H^s(F) = 0$ if $s > dim_H F$.

Thus $H^s(F)$ jumps from ∞ to 0, at a critical value s_0. This critical value is called the Hausdorff dimension of F. It can be shown that for a totally disconnected uncountable set $F \subset R$, $0 < H^{s_0}(F) < \infty \Leftrightarrow 0 < s_0 < 1$.

2.1.3 Cantor function

The Cantor function is an example of a function that is continuous, but not absolutely continuous. It is referred to as the Devil's Staircase.

Definition

Formally, the Cantor function $\phi : [0,1] \to [0,1]$ is defined as follows:

1. Express x in base 3.
2. If x contains a 1, replace every digit after the first 1 by 0.
3. Replace all 2's with 1's.
4. Interpret the result as a binary number. The result is $\phi_(x)$.

To construct ϕ explicitly, let $\phi(0) = 0, \phi(1) = 1$. Assign $\phi(x)$ a constant value $\phi(x) = i2^{-n}$, $i = 1, 2, ...2^n - 1$, on each of the deleted open intervals, i.e. the gaps (including the end points of the deleted intervals) of C_α. Next let $x \in C_\alpha$. Then at the nth iteration, x belongs to the interior of exactly one of the 2^n remaining closed intervals each of length β^n. Let $[a_n, b_n]$ be one of such interval. Then $b_n - a_n = \beta^n$. Moreover $\phi(b_n) - \phi(a_n) = 2^{-n}$. At the next iteration, let $x \in [a_{n+1}, b_{n+1}]$, $(a_n = a_{n+1})$.Then we have $\phi(x) = \lim_{n \to \infty} \phi(a_n) = \lim_{n \to \infty} \phi(b_n)$. Then $\phi : [0, 1] \to [0, 1]$ is a continuous, non decreasing function. Also $\phi'(x) = 0$ for $x \in I - C_\alpha$ when it is not differentiable at any $x \in C_\alpha$.

Also we can define a sequence $\{f_n\}$ of functions on the unit interval that converges to the Cantor function.

Let, $f_o(x) = x$

Then for every integer $n \geq 0$, the next function $f_{n+1}(x)$ will be defined in terms of $f_n(x)$ as follows:

$$f_{n+1}(x) = \begin{cases} 0.5 f_n(3x), & 0 \leq x \leq 1/3; \\ 0.5, & 1/3 \leq x \leq 2/3; \\ 0.5 + 0.5 f_n(3x - 2), & 2/3 \leq x \leq 1; \end{cases} \tag{2.1}$$

The three definitions are compatible at the end points 1/3 and 2/3, because $f_n(0) = 0$ and $f_n(1) = 1$ for every n. f_n converges point wise to the Cantor function defined above.

Although a Cantor function is continuous everywhere and has zero derivative almost everywhere, ϕ goes from 0 to 1 as x goes from 0 to 1, and takes on every value in between. Moreover a Cantor function is actually uniformly continuous but not absolutely. It is constant on intervals of the form $(0.x_1x_2 \ldots x_n02222\ldots, 0.x_1x_2 \ldots x_n20000\ldots)$, and every point not in the Cantor set is in one of these intervals, so its derivative is zero outside of the Cantor set. On the other hand, it has no derivative at any point in an uncountable subset of the Cantor set containing the interval end points described above.

2.2 Non Standard Analysis

2.2.1 Introduction

Non-Standard analysis is a branch of classical analysis that formulates analysis using a rigorous notion of an infinitesimal number. Non-Standard analysis was introduced in the early of 1960s by the mathematician Abraham Robinson. He wrote:

" the idea of infinitely small or infinitesimal quantities seems to appeal naturally to our intuition. At any rate, the use of infinitesimals was widespread during the formative stages of the Differential and Integral calculus. As for the objection that the distance between two distinct real numbers can not be infinitely small, G.W.Leibnitz argued that the theory of infinitesimals implies the introduction of ideal numbers which might be infinitely small or infinitely large compared with the real numbers but which were to possess the same properties as the latter."

Much of the earliest development of infinitesimal calculus by Newton and Leibnitz was formulated using expressions such as infinitesimal number and vanishing quantity. These formulation were widely criticized by George Berkely and others. It was a challenge to develop a consistent theory of analysis using infinitesimals and the first person to do this in a satisfactory way was Abraham Robinson [26]. In 1958 Schmieden and Laugwitz proposed a construction of a ring containing infinitesimals [38]. The ring was constructed from sequences of real numbers. Two sequences were considered equivalent if they differed only in a finite number of elements. Arithmetic operations were defined element wise. However, the ring constructed in this way contains zero divisors and thus can not be a field.

2.2.2 Approach to Non-Standard Analysis

There are two different approaches to non-standard analysis: the semantic or model-theoretic approach and the syntactic approach. Both of these approaches apply to other areas of mathematics beyond analysis, including number theory, algebra, topology and etc.

Robinson's original formulation of non-standard analysis falls into the category of semantic approach. As developed by him in his papers, it is based on studying models (in particular saturated models) of a theory. Since Robinson's work first appeared, a simpler semantic approach (due to Elias Zakon) has been developed using purely set-theoretic object called super structures. In this approach a model of a theory is replaced by an object called a super structure $V(S)$ over a set S. Starting from a super structure $V(S)$ one construct another object $*V(S)$ using the ultra power construction together with a mapping $V(S) \rightarrow *V(S)$ which satisfies the transfer principles. The map * relates formal properties of $V(S)$ and $*V(S)$. Moreover it is possible to consider a simplified form of saturation called countable saturation. This simplified approach is also more suitable for use by mathematicians who are not specialist in model theory or logic.

Let us briefly recall the ultra power construction of Robinson. Though less direct than the axiomatic approach, it allows one to get a more intuitive contact with the origin of the new structure . Indeed the new infinite and infinitesimal numbers are formulated as equivalence classes of sequences of real numbers, in a way quite similar to the construction of the set of real numbers R from rationals.

Let N be the set of natural numbers. A non-principal (free) ultra filter U on N is defined as follows:

U is a non empty set of subsets of N $[P(N) \supset U \supset \phi]$, such that

i) $\phi \in U$

ii) $A \in U$ and $B \in U \Rightarrow A \bigcap B \in U$

iii) $A \in U$ and $B \in P(N)$ and $B \supset A \Rightarrow B \in U$

iv) $B \in P(N) \Rightarrow$ either $B \in U$ or $\{j \in N : j \notin B\} \in U$, but not both.

v) $B \in P(N)$ and B is finite $\Rightarrow B \notin U$.

Then the set $*R$ is defined as the set of equivalence classes of all sequences of real numbers modulo the equivalence relation: $a \equiv b$, provided $\{j : a_j = b_j\} \in U$, a and b being two sequences $\{a_j\}$ and $\{b_j\}$.

Similarly, a given relation is said to hold between elements of $*R$ if it holds term wise for a set of indices which belongs to the ultra filter. For example $a < b \Rightarrow \{j : a_j < b_j\} \in U$.

R is isomorphic to a subset of $*R$, since one can identify any real $r \in R$ with the class of sequences $\{r, r, ...\}$. Moreover $*R$ is an ordered field.

The syntactic approach requires much less logic and model theory to understand and use. This approach was developed in the mid 1970s by the mathematician Edward Nelson. Nelson introduced an entirely axiomatic formulation of non-standard analysis that he called Internal Set Theory (IST) [39]. IST is an extension of Zermelo Fraenkel Set Theory (ZST). Along with the basic binary membership relation, it introduces a new unary predicate standard which can be applied to elements of the mathematical universe together with some axioms for reasoning with this new predicate.

Syntactic non-standard analysis requires a great deal of care in applying the principle of set formation which mathematicians usually take for granted. As Nelson pointed out, a common fallacy in reasoning in IST is that of illegal set formation. For instance, there is no set in IST whose elements are precisely the standard integers (here standard is understood in the sense of new predicate). To avoid illegal set formation, one must only use predicates of Zermelo-Frankel-Choice (ZFC) to define subsets [39].

2.2.3 Basic Definitions and Constructions of Extended Number Systems

An infinitesimals is a number that is smaller than every positive real number and is larger than every negative real number, or, equivalently, in absolute value it is smaller than $1/m$ for all $m \in N = \{1, 2, 3, ...\}$. Zero is the only real number that at the same time is an infinitesimal, so that the non zero infinitesimals do not occur in standard analysis. Yet, they can be treated in much the same way as can be for ordinary numbers. For example, each non zero infinitesimal ϵ can be inverted and the result is the number $\omega = 1/\epsilon$. It follows that $|\omega| > m$ for all $m \in N$, for which reason ω is called hyper large or infinitely large. Hyper large numbers too do not occur in ordinary analysis, but nevertheless can be treated like ordinary numbers. If, for example, ω is positive hyper large, we can compute $\omega/2, \omega - 1, \omega + 1, 2\omega, \omega^2$ etc. The positive hyper large numbers must not be confused with ∞, which should not be regarded a number at all.

If ϵ is hyper small, if δ too is hyper small but non zero and if ω is positive hyper large, so that $-\omega$ is negative hyper large, we write $\epsilon \cong 0, \delta \approx 0$, $\omega \approx \infty, -\omega \approx -\infty$ respectively. It would be wrong of course, to deduce from $\omega \approx \infty$ that the difference between ω and ∞ would be hyper small.

Given any $x \in R$, $x \neq 0$ and any $\delta \cong 0$, let $t = x + \delta$, then $\epsilon < |t| < \omega$, for all $\epsilon \approx 0$ and all $\omega \approx \infty$. The number t is called *finite* (or appreciable/moderately small or large) number (as it is not too small and not too large).

Three non overlapping sets of numbers (old or new) can now be formed.

a) the set of all infinitesimals, to which zero belongs,

b) the set of all finite numbers, to which all non zero reals belong, and

c) the set of all hyper large numbers, containing no ordinary numbers at all.

Together these three sets, constitute the set of all numbers of "Real Non-Standard Analysis". This set, which clearly an extension of R, is indicated by $*R$ and is called the * transform of R. The elements of $*R$ are called hyper real.

If a number is not hyper large it is called finite or limited. Clearly, $t \in *R$ is finite iff $t = x + \epsilon$ for some $x \in R$ and $\epsilon \cong 0$. Given such a t, both x and ϵ are unique, for, $x + \epsilon = y + \delta$, $x, y \in R$, $\epsilon, \delta \cong 0$, we have $x = y$ (as $x - y \in R$)and $\epsilon \cong \delta$.

By definition x is called the standard part of t, and this is written as $x = st(t)$.

The standard part function st provides an important bridge between the finite numbers of non-standard analysis and the ordinary real numbers. Trivially, if t is itself an ordinary real number, then $st(t) = t$.

The * transform not only can be obtained for R but also for N, Z, Q, and in fact for any set X of classical mathematics. Their * transforms are indicated by $*N, *Z, *Q, *X$ respectively.

Selecting all finite numbers from $*N$ and $*Z$ we obtain again N and Z, but this is not true for $*Q$, simply because $*Q$ (just as $*R$) contains finite non-standard numbers. But again there is a distinct difference between $*Q$ and $*R$ in this respect; there are finite elements t of $*Q$ that can not be written as $t = x+\epsilon$, with $x \in Q$, $\epsilon \in *Q$, $\epsilon \approx 0$. For let c be any irrational number, say $c = \sqrt{2}$, and let $\{r_1, r_2, ...\}$be some Cauchy sequence of rationals converging to c. The sequence $\{r_1 - c, r_2 - c, ...\}$ generates an infinitesimals δ in $*R$ (because this sequence converges to zero). On the other hand $\{r_1, r_2, ...\}$ generates an element $r \in *Q \subset *R$ and r is finite, but it has no standard part in Q, for otherwise $r = x + \epsilon$ for some $x \in Q$ and $\epsilon \in *Q$, $\epsilon \cong 0$. But $\{r_1 - c, r_2 - c, ...\}$ also generates the finite number $r - c \in R$, so that $r - c = \delta \cong 0$. It follows that $x - c = \delta - \epsilon \cong 0$, hence $x - c = 0$ (as $x - c$ is ordinary real), which would mean that $c \in Q$, a contradiction.

There are various ways to introduce new numbers. One way is done by means of infinite sequences of real numbers. In particular, the elements of $*R$ will be generated by means of infinite sequences of reals and it will be necessary to consider all such sequences. (Recall that the elements of R can be generated by means of rather special infinite sequence of rationals, i.e, the Cauchy sequences). More generally, given any set X the elements of its * transform $*X$ will be generated by means of infinite sequence of elements of X, quotionted by the equivalence class generated by the chosen ultra filter and again all such sequences must be taken into account. For example $\{1, 2, 3, ...\}$ generates a hyper large element of $*N$, and $\{3/2, 5/4, 9/8, ...\}$ generates a finite element of $*Q$, and an infinitesimal, generated by $\{1/2, 1/4, 1/8, ...\}$. Different sequences may generate the same elements of $*X$. In fact, given any $x \in *X$ there are many (uncountably many) different sequences which form an equivalence class under the ultra filter which represents the element $x \in *X$. As a consequence, changing finitely many terms of a generating sequence has no effect on the element generated.

2.2.4 The Purpose of Non-Standard Analysis

Starting from N, the sets Z, Q and R have been introduced in classical analysis (mathematics) in order to enrich analysis with more tools and to refine existing tools. The introduction of negative numbers, of fractions, and of irrational numbers is felt as a strong necessity, and without it mathematics would only be a small portion of what it actually is. The introduction of $*N, *Z, *Q$ and $*R$, however was not meant at all to enrich mathematical analysis (at least not when it all started), but only to simplifying it. In fact, definitions and theorems of classical analysis generally are greatly simplified in the context of non-standard analysis. Non-standard analysis has also been applied later in a more traditional way, namely to introduce new mathematical notions and models. Examples can be found in probability theory, asymptotic analysis, mathematical physics, economics etc.

As an example of a simpler definition, consider *continuity.* A function f from R to R is continuous at $c \in R$ if statement (i) holds:

$$\forall \epsilon \in R,\ \epsilon > 0 : \exists \delta \in R, \delta > 0 : \forall x \in R,$$
$$|x - c| < \delta :\ |f(x) - f(c)| < \epsilon \text{ ....(i)}$$

Now to f there corresponds a unique function $*f$, called the * transform of f, that is a function from $*R$ to $*R$, such that $*f(x) = f(x)$ if $x \in R$ and (i) is true iff (ii), which is the * transform of (i) is true:

$$\forall \epsilon \in *R,\ \epsilon > 0 : \exists \delta \in *R,\ \delta > 0 : \forall x \in *R,$$
$$|x - c| < \delta : |*f(x) - *f(c)| < \epsilon \text{ ....(ii)}$$

Moreover (i) is equivalent to much simpler statement (iii)

$$\forall \delta \in *R,\ \delta \cong 0 : *f(c + \delta) - *f(c) \cong 0 \text{....................(iii)}$$

An illustration of a simpler proof is that of the *Intermediate Value Theorem:*

If $f : R \to R$ is continuous in the closed interval $[a, b]$, $a < b$, a and b both finite, and $f(a) < 0,\ f(b) > 0$, then $f(c) = 0$ for some $c \in [a, b]$.

A non-standard proof of this theorem proceeds as follows:

Let, $m \in *N$ be hyper large. Divide $[a, b]$ into m equal subintervals, each of length $\delta = (b-a)/m$. Then $\delta \approx 0$. Let, n be the smallest element of $*N$ such that $*f(a+n\delta) > 0$, then $*f(a+(n-1)\delta) \leq 0$.

Let $c = st(a+n\delta)$, then, by continuity,

$*f(a+n\delta) - *f(c) = \epsilon_1$ and $*f(c) - *f(a+(n-1)\delta) = \epsilon_2$ for certain infinitesimals ϵ_1 and ϵ_2. Hence $-\epsilon_1 < f(c) = *f(c) \leq \epsilon_2$. But $f(c) \in R$, so $f(c) = 0$.

Terrence Tao, one of the most brilliant contemporary mathematicians, has been advocating strongly the use of nonstandard analysis as *soft* analysis rather than using only the classical *hard* analysis in partial differential equations and various other fields of applications in his blog page 'What's New'.

2.3 Non-Archimedean Ultra metric Theory

2.3.1 Absolute value on a Field

Definition 1. An absolute value on K is a function $|.| : K \to R_+$ that satisfies the following conditions :

1) $|x| = 0$ iff $x = 0$,

2) $|xy| = |x||y|$ for all $x, y \in K$,

3) $|x+y| \leq |x| + |y|$ for all $x, y \in K$.

We shall say an absolute value of K is non-archimedean if it satisfies the additional condition :

4) $|x+y| \leq max(|x|, |y|)$ for all $x, y \in K$.

Otherwise, the absolute value is archimedean.

Example- If we take,

$$|x| = \begin{cases} 1, & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases} \tag{2.2}$$

for any field K, then it is trivially a non-archimedean absolute value.

Theorem 1. *Let, $A \subset K$ be the image of Z in K. An absolute value $|.|$ on K is non-archimedean iff $|a| \leq 1$ for all $a \in A$. In particular, an absolute value on Q is non-archimedean iff $|n| \leq 1$ for any $n \in Z$.*

Proof. We have $|\pm 1| = 1$; hence, if $|.|$ is non-archimedean, we get that $|a \pm 1| \leq max\{|a|, 1\}$. By induction $|a| \leq 1$ for every $a \in A$.

For converse part, suppose that $|a| \leq 1$ for all $a \in A$. We want to prove that for any two elements $x, y \in K$, we have $|x + y| \leq max\{|x|, |y|\}$. If $y = 0$, this is obvious. If not, we can divide through by $|y|$, and we see that this is equivalent to the inequality $|x/y + 1| \leq max\{|x/y|, 1\}$.

This means that we need only to prove the inequality for the case when the second summand is 1, and the general fact will then follow. In other words, we want to prove that for any $x \in K$ we have $|x + 1| \leq max\{|x|, 1\}$.

Now let m be any positive integer. Then we have

$$\begin{aligned}
& |x+1|^m \\
= \; & |\sum_k ({}^mC_k x^k)| \\
\leq \; & |{}^mC_k||x^k| \\
\leq \; & \sum_k |x^k| \quad [\text{since } |({}^mC_k)| \leq 1] \\
= \; & \sum_k |x|^k \\
\leq \; & (m+1)\text{max}\{1, |x|^m\}
\end{aligned}$$

(for the last step, notice that the largest value of $|x|^k$ for $k = 0, 1, 2, ...m$ is equal to $|x|^m$ if $|x| > 1$ and equal to 1 otherwise).

Taking m-th root on both sides gives, $|x + 1| \leq (m + 1)^{1/m} max(1, |x|)$.

Now this inequality holds for every positive integer m, and we know that $\lim_{m\to\infty}(m+1)^{1/m} = 1$.

Therefore, if we let $m \to \infty$ we get $|x+1| \leq max\{|x|, 1\}$ which is what we wanted to prove.

■

Archimedean property

Given $x, y \in K, x \neq 0$, there exists a positive integer n such that $|nx| > |y|$.

The archimedean property is equivalent to the assertion that $\sup\{|n| : n \in Z\} = +\infty$.

Corollary 1. *An absolute value* $|.|$ *is non-archimedean iff* $\sup\{|n| : n \in Z\} = 1$.

2.3.2 Topology

The essential point of an absolute value is that it provides us with a notion of "size". In other words, once we have an absolute value, we can use it to measure distances between numbers, that is, to put a metric on our field. Having the metric, we can define open and closed sets, and in general investigate the (metric) topology of our field.

Definition 2. Let, K be a field and $|.|$ an absolute value on K. We define the distance $d(x, y)$ between two elements $x, y \in K$ by $d(x, y) = |x - y|$.

The function $d(x, y)$ is called the metric induced by the absolute value and the metric $d(x, y)$ has the following properties:

1) for any $x, y \in K$, $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$

2) for any $x, y \in K$, $d(x, y) = d(y, x)$

3) for any $x, y, z \in K$, $d(x, z) \leq d(x, y) + d(y, z)$

The last inequality is called triangle inequality.

Lemma 1. *Let* $|.|$ *be an absolute value on a field K, and define a metric by* $d(x, y) = |x - y|$. *Then* $|.|$ *is non-archimedean iff for any* $x, y, z \in K$, *we have* $d(x, y) \leq max\{d(x, z), d(z, y)\}$.

Proof. To get one way, apply the non-archimedean property to the equation $(x-y) = (x-z)+(z-y)$.

For the converse, take $y = -y$ and $z = 0$ in the inequality satisfied by $d(.,.)$. ∎

This inequality is known as the "Ultra metric inequality" and a metric for which it is true is called an "Ultra metric". A space with an ultra metric is called an "Ultra metric Space".

Proposition 1. *Let K be a field and let $|.|$ be a non-archimedean absolute value on K. If $x, y \in K$ and $x \neq y$, then $|x+y| = max\{|x|, |y|\}$.*

Proof. Exchanging x and y if necessary, we may suppose $|x| > |y|$. Then $|x+y| \leq |x| = max\{|x|, |y|\}$.

On the other hand $x = (x+y)-y$, so that $|x| \leq max\{|x+y|, |y|\} = |x+y|$.

So, $|x| = |x+y| \Rightarrow |x+y| = max\{|x|, |y|\}$. ∎

Corollary 2. *In an ultra metric space, all triangles are isocels.*

Definition 3. Let, K be a field with an absolute value $|.|$. Let $a \in K$ be an element and let $r \in R_+$ be a positive real number. The open ball of radius r and center a is the set $B(a,r) = \{x \in K : d(x,a) < r\} = \{x \in k : |x-a| < r\}$

The closed ball of radius r and centre a is the set $\bar{B}(a,r) = \{x \in k : d(x,a) \leq r\} = \{x \in k : |x-a| \leq r\}$.

Proposition 2. *Let K be a field with a non-archimedean absolute value.*

1) If $b \in B(a,r)$, then $B(a,r) = B(b,r)$; in other words, every point that is contained in an open ball is a centre of that ball.

2) If $b \in \bar{B}(a,r)$ then $\bar{B}(a,r) = \bar{B}(b,r)$; in other words, every point that is contained in a closed ball is a centre of that ball.

3) The set $B(a,r)[\bar{B}(a,r)]$ is both open and closed.

4) any two open balls (closed balls) are either disjoint or contained in one another.

Proof. 1) By definition $b \in B(a, r)$ iff $|b - a| < r$. Now, taking any x for which $|x-a| < r$, the non-archimedean property says that $|x-b| \leq max\{|x-a|, |b-a|\} < r$. So, $x \in B(b, r)$. That is $B(a, r) \subset B(b, r)$. Switching a and b, we get the opposite inclusion, so that the two balls are equal.

2) Similar as (1).

3) The open ball $B(a, r)$ is always an open set in any metric space. What we need to show is that in non-archimedean case, it is also closed. So, take an x in the boundary of $B(a, r)$; this means that any open ball centred in x must contain points those are in $B(a, r)$. Choose a number $s \leq r$. Now, since x is a boundary point, $B(a, r) \cap B(x, s) \neq \phi$; so that there exists an element $y \in B(a, r) \cap B(x, s)$.

This means that $|y - a| < r, |y - x| < s \leq r$

Now $|x - a| \leq max\{|x - y|, |y - a|\} < max\{s, r\} \leq r$, so $x \in B(a, r)$. This shows that boundary point of $B(a, r)$ belongs to $B(a, r)$, which means that $B(a, r)$ is closed.

Converse part is similar.

4) Let, $B(a, r)$ and $B(b, s)$ are two open balls such that $B(a, r) \cap B(b, s) \neq \phi$.

We can assume that $r \leq s$. If the intersection is not empty, there exists a $c \in B(a, r) \cap B(b, s)$. Then from (1) we have, $B(a, r) = B(c, r)$ and $B(b, s) = B(c, s)$. Hence $B(a, r) = B(c, r) \subset B(c, s) = B(b, s)$. ■

Definition 4. Let, K be a field with an absolute value $|.|$. We say a set $S \subset K$ is clopen if it is both an open and a closed set.

Thus it is clear that a non-archimedean ball does not have well defined centres, because every point of the ball can be called its centre. Also every open (closed) ball is clopen.

A Cauchy sequence in an ultra metric space X is a sequence $\{x_n\}$ such that for any $\epsilon > 0$, $\exists n_0 \in N$ such that for all $m, n \geq n_0$, $d(x_m, x_n) < \epsilon$. This implies $d(x_{n+1}, x_n) < \epsilon$ (by strong triangle inequality) for all $n \geq n_0$.

A space X is complete if every Cauchy sequence converges to a limit in X.

2.4 p-adic Number and Analysis

2.4.1 Introduction

In mathematics and chiefly in number theory, the p-adic number system [42] for any prime number p, extends the ordinary arithmetic of the rational numbers in a way different from the extension of the rational number system to the real and complex number systems. The extension is achieved by an alternative interpretation of the concept of absolute value. The p-adic numbers were motivated primarily by an attempt to bring the ideas and techniques of power series methods into number theory. Their influence now extends far beyond this. For example, the field of p-adic analysis essentially provides an alternative form of calculus.

More formally, for a given prime p, the field Q_p of p-adic numbers is a completion of rational numbers. The field Q_p is also given a topology derived from a metric, which is itself derived from an alternative valuation on the rational numbers. This metric space is complete in the sense that every Cauchy sequence converges to a point in Q_p. This is what allows the development of calculus on Q_p, and it is the interaction of this analytic and algebraic structure which gives p-adic number systems their power and utility.

2.4.2 p-adic expansions

If p is a fixed prime number, then any positive integer can be written in a base p expansion in the form $\sum_{i=1}^{n} a_i p^i$ where the a_i are integers in $\{0, 1, ...p-1\}$. For example, the binary expansion of 27 is $1.2^4 + 1.2^3 + 0.2^2 + 1.2^1 + 1.2^0$, often written in the short hand notation $(11011)_2$ The familiar approach to extending this description to the larger domain of the rationals (and, ultimately to the reals) is to use sums of the form: $\sum_{i=-m}^{n} a_i p^i$.

A definite meaning is given to these sums based on Cauchy sequences, using the absolute value as metric. Thus for example, 1/3 can be expressed in base 5 as the limit of the sequence $(0.13131313....)_5$. In this formulation, the integers are precisely those numbers for which $a_i = 0$ for all $i < 0$.

As an alternative, if we extend the base p expansions by allowing infinite sums of the form $\sum_{i=k}^{\infty} a_i p^i$ where k is some (not necessarily positive) integer, we obtain the p-adic expansions defining the field Q_p of p-adic numbers. Those p-adic numbers for which $a_i = 0$ for all $i < 0$ are called the p-adic integers. The p-adic integers form a subring of Q_p, denoted by Z_p.

2.4.3 The field of p-adic numbers Q_p

The basic example of a norm on the field Q of rational numbers is the absolute value $|.|$. The induced metric $d(x, y) = |x-y|$ is the ordinary Euclidean distance on the real line and the field of real numbers R is the completion of Q with respect to this norm. Now the question arises: Is the Euclidean distance between rational numbers really the most "natural" one? Is there any other way to describe the "closeness" between rationals? The new ways of measuring distance between rational numbers come from the following "arithmetical" construction.

Definition 1. Let, $p \in N$ be any prime number. Define a map $|.|_p$ on Q as follows:

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p x}, & \text{if} x \neq 0 \\ 0, & \text{if} x = 0 \end{cases} \tag{2.3}$$

where $\mathrm{ord}_p x =$ the highest power of p which divides x, if $x \in Z$ and $\mathrm{ord}_p x = \mathrm{ord}_p a - \mathrm{ord}_p b$, if $x = a/b$, $a, b \in Z$, $b \neq 0$ is the p-adic order or p-adic valuation of x.

Proposition 3. *$|.|_p$ is a non-archimedean norm on Q.*

Property (1) of non-archimedean norm follows from definition of $|.|_p$.

Again $\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$.

Therefore, $|xy|_p = p^{-\mathrm{ord}_p(xy)} = p^{-[\mathrm{ord}_p(x) + \mathrm{ord}_p(y)]} = p^{-\mathrm{ord}_p(x)} . p^{-\mathrm{ord}_p(y)} = |x|_p . |y|_p$.

So, property (2)of non-archimedean norm is satisfied.

Let us verify property (3). If $x = 0$, $y = 0$, (3) is trivial, so assume $x, y \neq 0$. Let, $x = a/b, y = c/d$.

Then we have

$x + y = \frac{ad+bc}{bd}$ and

$$\begin{aligned} & \operatorname{ord}_p(x+y) \\ = \; & \operatorname{ord}_p(ad+bc) - \operatorname{ord}_p(bd) \\ \geq \; & min(\operatorname{ord}_p(ad), \operatorname{ord}_p(bc)) - \operatorname{ord}_p(b) - \operatorname{ord}_p(d) \\ = \; & min(\operatorname{ord}_p a - \operatorname{ord}_p b, \operatorname{ord}_p c - \operatorname{ord}_p d) \\ = \; & min(\operatorname{ord}_p x, \operatorname{ord}_p y) \end{aligned}$$

Therefore $|x + y|_p = p^{-\operatorname{ord}_p(x+y)} \leq max(p^{-\operatorname{ord}_p x}, p^{-\operatorname{ord}_p y}) = max(|x|_p, |y|_p)$.

So, the strong ultra metric triangle inequality is satisfied by this norm. So, $|.|_p$ is non-archimedean.

Note that the usual absolute value is denoted by $|.|_\infty$ and is associated to the real numbers.

Definition 2. Two absolute values $|.|_1$ and $|.|_2$ on a field K are called equivalent if they define the same topology on K, that is, if every set that is open with respect to one is also open with respect to the other.

Lemma 2. *Let $|.|_1$ and $|.|_2$ be absolute values on a field K. The following statements are equivalent.*

1) $|.|_1$ and $|.|_2$ are equivalent absolute values;

2) for any $x \in K$ we have $|x|_1 < 1$ iff $|x|_2 < 1$;

3) there exists a positive real number α such that every $x \in K$ we have $|x|_1 = |x|_2^\alpha$.

Theorem 2. *(Ostrowski:) Every non-trivial absolute value on Q is equivalent to one of the absolute values $|.|_p$, where p is a prime number or $p = \infty$.*

Proposition 4. *For any $x \in Q - \{0\}$, we have $\prod_{p \leq \infty} |x|_p = 1$.*

Proof. It is easy to see that we only need to prove the formula when x is a positive integer, and that the general case will then follow. So, let x be a positive integer, which we can factor as $x = p_1^{\alpha_1}.p_2^{\alpha_2}.....p_k^{\alpha_k}$.

Then we have $|x|_q$ =1 if $q \neq p_i$

$|x|_{p_i} = p_i^{-\alpha_i}$ for i=1,2,....k

$|x|_\infty = p_1^{\alpha_1}.p_2^{\alpha_2}......p_k^{\alpha_k}$. The result then follows.

■

The field Q is not complete with respect to the usual absolute value $|.|_\infty$ and the set of real numbers R is the extension of Q which is complete field with respect to $|.|_\infty$.

Proposition 5. *The field Q of rational numbers is not complete with respect to $|.|_p$ for any prime p.*

Let, p be a fixed prime. We define Q_p to be the completion of Q with respect to the p-adic absolute value $|.|_p$. Therefore $(Q_p, |.|_p)$is a complete normed field and this is called the field of p-adic numbers. The elements of Q_p are the equivalent classes of Cauchy sequences in Q with respect to the extension of the p-adic norm. So, Q can be identified with the subfield of Q_p consisting of equivalence classes of constant Cauchy sequences.

For some $a \in Q_p$, let $\{a_n\}$ be a Cauchy sequence of rational numbers representing a. Then by definition $|a|_p = \lim_{n\to\infty} |a_n|_p$.

Definition 3. A p-adic number $a \in Q_p$ is said to be a p-adic integer if its canonical expansion contains only non-negative powers of p.

The set of p-adic integers is denoted by Z_p, so $Z_p = \{\sum_{i=0}^{\infty} a_i p^i\}$

It is easy to see that $Z_p = \{a \in Q_p | |a|_p \leq 1\}$.

So, we can summarize the results as follows:

- there is an absolute value $|.| = |.|_p$ on Q_p and Q_p is complete with respect to this absolute value;

- there is an inclusion $Q \to Q_p$ whose image is dense in Q_p, and the restriction of the absolute value $|.|_p$ to (the image of) Q coincides with the p-adic absolute value;

- the set of values of Q and Q_p under $|.|_p$ is the same; specially, the two sets $\{x \in R_+ : x = |\lambda|_p \text{ for some } \lambda \in Q\}$ and $\{x \in R_+ : x = |\lambda|_p \text{ for some } \lambda \in Q_p\}$ are both equal to the set $\{p^n : n \in Z\} \cup \{0\}$ of powers of p, together with 0.

Proposition 6. *For each $x \in Q_p$, $x \neq 0$ there exists an integer $n \in Z$ such that $|x|_p = p^{-n}$.*

The topology of Q_p

1) The open balls in Q_p are both open and closed.

2) If $b \in B(a, r)$, then $B(b, r) = B(a, r)$. in other words, every point of a ball is its centre.

3) Two balls in Q_p have a non-empty intersection iff one is contained in other.

4) The set of all balls in Q_p is countable.

5) The space Q_p is locally compact.

6) The space Q_p is totally disconnected.

Proof. (1), (2), (3) follows from the fact that Q_p is an ultra metric space.

4) Write the centre of the ball $B(a, p^{-s})$ in its canonical form $a = \sum_{n=-m}^{\infty} a_n p^n$ and let $a_0 = \sum_{n=-m}^{s} a_n p^n$.

Clearly a_0 is a rational number and $|a - a_0|_p < p^{-s}$

i.e. $a_0 \in B(a, p^{-s})$. Therefore, $B(a_0, p^{-s}) = B(a, p^{-s})$.

Here both the centres and radii come from countable sets. Therefore, the product set of all pairs (a_0, s) is also countable and so the set of all balls in Q_p is countable.

To prove (5) we recall that

a subset K in a metric space is called sequentially compact if every infinite sequence of points in K contains a subsequence converging to a point in K.

Further, every infinite sequence of p-adic integers has a convergent subsequence. Therefore Z_p is sequentially compact. Therefore Z_p is compact, and so is any ball in Q_p. So, Q_p is locally compact.

6) For any $a \in Q_p$ and each $n \in N$, the set $U_n(a) = \{x \in Q_p \| x - a|_p \leq p^{-n}\}$=$\{x \in Q_p \| x - a|_p < p^{-(n+1)}\}$ is an open and closed neighborhood of a. Suppose $a \in A$ so that $A \neq \{a\}$. Then there is an $n \in N$ such that $U_n(a) \cap A \neq A$. Therefore, $A = (U_n(a) \cap A) \cup (Q_P - U_n(a) \cap A)$, where both $U_n(a)$ and its complement $Q_p - U_n(a)$ are open and non-empty; this implies A is not connected and hence the result follows. ■

Chapter 3

NON-ARCHIMEDEAN EXTENSION OF REAL NUMBER SYSTEM

3.1 Introduction

The main objective of the present chapter is to study the formulation of a *scale invariant analysis* that aims at developing a coherent framework for analysis on the real line R as well as on Cantor like fractal subsets of R [13, 14]. The formulation of a scale invariant analysis was motivated by an effort in justifying the construction of the so called non-smooth (i.e. higher derivative discontinuous) solutions [15] of the simplest scale invariant Cauchy problem

$$x\frac{dX}{dx} = X, X(1) = 1 \tag{3.1}$$

in a rigorous manner. It is clear that the framework of classical analysis, because of Picard's uniqueness theorem, can not rigorously accommodate such solutions, except possibly only in an *approximate* sense. To bypass the obstacle, it becomes imperative to look for a non-archimedean extension of the classical setting, thus allowing for existence of non-trivial *infinitesimals* (and hence, by inversion, *infinities*). Robinson's original model of non-standard analysis appeared to be unsatisfactory, because (1) infinitesimals here are infinitesimals even in "values", since the value of an infinitesimal is the usual Euclidean value and (2) these are new extraneous elements in R. Although, the non-standard $*R$ is of course non-archimedean, but still an infinitesimal behaves more in a "real number like" manner; that is to say, in essence, it fails to have an identity, except for its *infinitesimal* Euclidean value. Such non-standard infinitesimals are known to generate proofs of harder theorems of mathematical analysis in a more intuitively appealing manner. Further, any new theorem proved in the non-standard approach is expected to have a classical proof, though, may be, using lengthier arguments. Justifying a higher derivative discontinuous solution of (3.1), therefore, appeared to be difficult even in the conventional non-standard analysis. To counter this problem, we develop a novel non-archimedean extension $\mathbf{R}$ of R by completing the rational number field Q under a novel *ultra metric* which treats arbitrarily small and large rational (and real) numbers [13] *separately.* The ultra metric reduces to the usual Euclidean value for finite real numbers, but, nevertheless, leads to a *new definition of scale invariant infinitesimals* in the present context.

Another motivation (as already mentioned) for this Chapter is to formulate "motion" (variation of a quantity) in a smooth differentiable sense on a zero or a positive measure Cantor like fractal sets which arise copiously in complex system studies.

3.2 Non-Archimedean model

Infinitesimals: Let $*R$ be a non-standard extension (c.f. Sec. 2.2) of the real number set R. Let $\mathbf{0}$ denote the set of infinitesimals in $*R$. Then an element of $*R$, denoted as $\mathbf{x}$, is written as $\mathbf{x}=x+\tau$, $x \in R$ and $\tau \in \mathbf{0}$. The set $\mathbf{0}$ and hence $*R$ is linearly ordered that matches with the ordering of R. The set $\mathbf{0}$ is thus of cardinality c, the continuum. The non zero elements of $\mathbf{0}$ are new numbers added to R which are constructed from the ring $\mathbf{S}$ of sequences of real numbers via a choice of an *ultra filter* to remove the zero divisors of $\mathbf{S}$. A non-standard infinitesimal is realized as an equivalence class of sequences under the ultra filter and may be considered extraneous to R. The magnitude of an element $\mathbf{x}$ of $*R$ is evaluated using the usual Euclidean absolute value $|\mathbf{x}|_e$.

We now give a *new* construction relating infinitesimals to arbitrarily small elements of R in a more *intrinsic* manner. The words "arbitrarily small elements" are made precise in a limiting sense in relation to a *scale.* The infinitesimals so defined are called *relative infinitesimals* [16, 17].

Definition 1. *Given an arbitrarily small positive real variable $x \to 0^+$, there exists a rational number $\delta > 0$ and a set I_{in}^+ of positive reals $\tilde{x}(x) = \tilde{x}(x, \lambda)$ satisfying $0 < \tilde{x}(x) < \delta < x$ and the inversion rule*

$$\frac{\tilde{x}(x)}{\delta} = \lambda(\delta)\frac{\delta}{x}, \tag{3.2}$$

where $0 < \lambda(\delta) << 1$, is a real constant, so that $\tilde{x}$ also satisfies the scale invariant equation

$$x\frac{d\tilde{x}}{dx} = -\tilde{x} \tag{3.3}$$

The elements $\tilde{x}(x)$ so defined are called relative infinitesimals relative to the scale δ. A necessary condition for relative infinitesimals is that $0 < \tilde{x}_1 < \tilde{x}_2 < \delta$ means $0 < \tilde{x}_1 + \tilde{x}_2 < \delta$. A relative infinitesimal $\tilde{x}$ is negative if $-\tilde{x}$ is a positive relative infinitesimals. Further, the associated scale invariant infinitesimal corresponding to the relative infinitesimal $\tilde{x}$ is defined by $\tilde{X} = \lim\limits_{\delta \to 0} \frac{\tilde{x}}{\delta}$.

Now, because of linear ordering of $\mathbf{0}^+$, the set of positive infinitesimals of $*R$, that is inherited from R, and the fact that the cardinality of $\mathbf{0}^+$ equals that of R, there is a one-one correspondence between $\mathbf{0}^+$ and $(0, \delta) \subset R$, which we can write as $\tau(\tilde{x}) = \tau_0(\tilde{x}/\delta)$ for an infinitesimal $\tau_0 \in \mathbf{0}^+$ and a relative infinitesimal $0 < \tilde{x} < \delta$, $\delta \to 0^+$. This may be interpreted as by saying that for each arbitrarily small $\delta > 0$, there exists in the non-standard $*R$ an infinitesimal $\tau_0 \in \mathbf{0}^+$ so that the dimensionless equality of the form $\tau/\tau_0 = \tilde{x}/\delta$ holds good independent of the scale δ. We, henceforth identify $\mathbf{0}^+$ with the set of relative infinitesimals I_{in}^+ in $I_\delta^+ = (0, \delta) \subset R$ so that $I_{\text{in}}^+ \subset I_\delta$. We use symbols $\mathbf{0}$ and I_{in} interchangeably henceforth to refer the set of (relative) infinitesimals. We remark that in this framework, a positive *real* variable x is defined relative to the scale δ by the condition $x > \delta$.

Infinitesimals , so modeled, will be assigned with a new absolute value. The real number set R equipped with this absolute value (denoted henceforth by $\mathbf{R}$) will be shown to support naturally the generalized class of solutions of eq(3.1).

Definition 2. *A relative infinitesimal $\tilde{x} \in I_{\text{in}} \subset I_\delta = (-\delta, \delta)$ $(\neq 0)$ is assigned with a new absolute value, $v(\tilde{x}) := \|\tilde{x}\| = \lim\limits_{\delta \to 0} \log_{\delta^{-1}} \tilde{x}_1^{-1}$, $\tilde{x}_1 = |\tilde{x}|_e/\delta$. We also set $\|0\| = 0$.*

Remark 1. We observe that there exists a nontrivial class of infinitesimals (viz., those satisfying $|\tilde{x}|_e \leq \delta.\delta^{\delta}$) for which the value $|\tilde{x}|$ assigned to an infinitesimal $\tilde{x}$ is a real number, i.e., $|\tilde{x}| \geq \delta$. One of our aims here is to point out nontrivial influence of these infinitesimals in real analysis. This is to be contrasted with the conventional approach. The Euclidean value of an infinitesimal in Robinson's non-standard analysis is numerically an infinitesimal. Further, the limit $\delta \to 0^+$ is, of course, considered in the above definitions in the Euclidean metric.

We also notice that the inversion in Definition 1 is nontrivial in the sense that in the absence of it, the scale δ can be chosen arbitrarily close to an infinitesimal $\tilde{x}$ (say), so that letting $\delta \to \tilde{x}$, which, in turn, $\to 0^+$, one obtains $|\tilde{x}| = 0$. Thus, dropping the inversion rule, we reproduce the ordinary real number system R with zero being the only infinitesimal.

Clearly, the above absolute value is well defined and also scale invariant. For, even as $\delta \to 0^+$ the relative ratio $\eta = x/\delta$ might be a constant (or approaches zero at a slower rate) in (0,1), so that Definition 2 can yield non-trivial values.

Remark 2. An infinitesimal $\tilde{x} \in I_\delta$ has a countable number of different realizations, each for a specific choice of the scale δ, having valuation $|\tilde{x}|_\delta$. Indeed, given a decreasing sequence of (*primary*) scales δ_n so that $\delta_n \to 0$ as $n \to \infty$, the limit in the Definition-2 can instead be evaluated over a sequence of *secondary* smaller scales of the form δ_n^m, $m \to \infty$ for each fixed n.

This observation allows one to extend that definition slightly which is now restated as

Definition 3. *i) Scale free (invariant) infinitesimals* $\tilde{X}_\delta = \tilde{x}_n/\delta^n$ *satisfying* $0 < \tilde{x}_n < \delta^n$ *(and* $\tilde{x}_n/\delta^n = (\tilde{x}/\delta)^n$ *as* $n \to \infty$*) are called (positive) scale-free δ-infinitesimals. By inversion, elements of* $|\tilde{X}_\delta^{-1}|_e > 1$ *are scale free δ-infinities. (ii) A relative (δ) infinitesimal* $\tilde{X}_\delta (\neq 0) \in I_\delta$ *is assigned with a new (δ dependent) absolute value* $v(\tilde{x}) = |\tilde{x}|_\delta = \lim_{n\to\infty} \log_{\delta^{-n}} (\tilde{x}_n/\delta^n)^{-1}$. *(In this scale free notation, all the finite real numbers* *are mapped to 1. We denote this set of δ infinitesimals and infinities by R_δ.)*

Clearly, the above definition yields $|\tilde{x}|_\delta = \log_{\delta^{-1}}(\delta/\tilde{x})$ for each fixed δ, which will have important applications in the following. The Euclidean absolute value, however, is uniquely defined $|x|_e = x,\ x > 0$. We also notice that for a (non-zero) real number $x \in R$, the only meaningful scale is $|x|$. It therefore makes sense to identify all finite real numbers to 1 in R_δ.

Proposition 7. *$|.|_\delta$ defines a nonarchimedean semi-norm on* $\mathbf{O}$.

Remark 3. To simplify notations, $|.|_\delta$, is written often as $|.|$. The $\delta-$ infinitesimals are also denoted simply by $\tilde{x}$. By a semi-norm we mean that $|.|$ satisfies three properties (i) $|\tilde{x}| > 0,\ \tilde{x} \neq 0$, (ii) $|-\tilde{x}| = |\tilde{x}|$ and (iii) $|\tilde{x}_1 + \tilde{x}_2| \leq \max\{|\tilde{x}_1|, |\tilde{x}_2|\}$. Property (iii) is called the *strong ultra metric triangle* inequality. Note that this definition of semi-norm on a set differs from the semi-norm on a vector space. However, this suffices our purpose here.

Proof. The first two are obvious from the definition. For the third, let $0 < \tilde{x}_2 < \tilde{x}_1$ in $\mathbf{O}$. Then there exists $\delta > 0$ so that $0 < \eta_2 < \eta_1 < 1$ where $\eta_i = \tilde{x}_i/\delta^n \neq 1$ and $|\tilde{x}_i| = \log_{\delta^{-n}} \eta_i^{-1},\ n \to \infty$. Clearly, $|\tilde{x}_2| > |\tilde{x}_1|$. Moreover, $0 < \eta_2 < \eta_1 < \eta_1 + \eta_2 < 1$. By Definition 2, we thus have $|\tilde{x}_1 + \tilde{x}_2| = \log_{\delta^{-n}}(\eta_1 + \eta_2)^{-1} \leq \log_{\delta^{-n}} \eta_2^{-1} \leq |\tilde{x}_2|$. Moreover, $|\tilde{x}_1 - \tilde{x}_2| = |\tilde{x}_1 + (-\tilde{x}_2)| \leq \max\{|\tilde{x}_1|, |\tilde{x}_2|\} = |\tilde{x}_2|$. ■

Now, to restore the product rule, viz., $|\tilde{x}_1\tilde{x}_2| = |\tilde{x}_1||\tilde{x}_2|$, we note that given $\tilde{x}$ and δ ($0 < \tilde{x} < \delta$), there exist $0 < \sigma(\delta) < 1$ and $a : \mathbf{O} \to R$ such that

$$\frac{\tilde{x}_n}{\delta^n} = (\delta^n)^{\sigma^{a(\tilde{x})}} \tag{3.4}$$

so that, in the limit $n \to \infty$, we have

$$v(\tilde{x}) = |\tilde{x}| = \sigma^{a(\tilde{x})}$$

For definiteness, we choose $\sigma(\delta) = \delta$ (this is justified later). The function $a(\tilde{x})$ is a (discretely valued) valuation satisfying (i) $a(\tilde{x}_1\tilde{x}_2) = a(\tilde{x}_1) + a(\tilde{x}_2)$ and (ii) $a(\tilde{x}_1 + \tilde{x}_2) \geq \min\{(a(\tilde{x}_1), a(\tilde{x}_2)\}$. As a result, $|\tilde{x}_1\tilde{x}_2| = |\tilde{x}_1||\tilde{x}_2|$ and hence we have deduced.

Proposition 8. $|.|$ *defines a nonarchimedean absolute value on* $\mathbf{0}$.

Remark 4. The above definition of valuation (3.4) can be extended further to include an extra piece in the exponent, viz.,

$$\tilde{x}_n/\delta^n = (\delta^n)^{(v(\tilde{x})+\xi(\tilde{x},\delta))} \tag{3.5}$$

where ξ (>0) vanishes with δ in such a manner that $(\delta^n)^\xi = 1$ in the limit. This observation offers an alternative definition of a scale free (δ) infinitesimal, viz., $\tilde{X} = \lim \frac{\tilde{x}_n}{(\delta^n)^{(1+|\tilde{x}|_\delta)}} = O(\delta^{n\xi})$, as $n \to \infty$, which will be useful in the following.

We now recall the general topological structure of a non-archimedean space.

Definition 4. *The set* $B_r(a) = \{x|\ |x-a| = v(x-a) < r\}$ *is called an open ball in* $\mathbf{0}$. *The set* $\bar{B}_r(a) = \{x|\ |x-a| \leq r\}$ *is called closed ball in* $\mathbf{0}$.

Lemma 3. *1) Every open ball is closed and vice versa(clopen ball).*

2) Every point $b \in B_r(a)$ *is a centre of* $B_r(a)$.

3) Any two balls in $\mathbf{0}$ *are either disjoint or one is contained in another.*

4) $\mathbf{0}$ *is the union of at most a countable family of clopen balls.*

5) The set $\mathbf{0}$ *equipped with the absolute value* $|.|$ *is totally disconnected.*

The proof of these assertions follows directly from the ultra metric property (See Chapter 2, Sec. 2.3) and the fact that $\mathbf{0}$ is an open set. Because of the property (4) the set $\mathbf{0}^+$ can be covered by at most a countable family of clopen balls viz., $\mathbf{0}^+ = \cup B(t_i)$ where t_i is a bounded sequence in $\mathbf{0}$, on each of which the absolute value $|.|$ can have a constant value. With this choice of absolute value $|.|$ is discretely valued.

Remark 5. To emphasize, the definition of relative infinitesimals takes note of relative position of $\tilde{x}$ with respect to δ, which could then be extended as a geometrical progression to a sequence $\tilde{x}_n$ satisfying $0 < \tilde{x}_n < \delta^n$ so that $\tilde{x}_n/\delta^n = (\tilde{x}/\delta)^n$ for the evaluation of $|\tilde{x}|_\delta$. Further, we use the symbol X to denote a scale free infinitesimal and the sequence $\tilde{x}_n$ of arbitrarily small real numbers are called the real valued realizations of the infinitesimal X. δ infinitesimals carry traces of residual influence of the scale, as reflected in the corresponding absolute values $|\tilde{x}|_\delta = \log_{\delta^{-1}} \delta/|\tilde{x}|_e$, where as a genuinely scale free one should be independent of any scale. We notice that the above absolute value awards the real number system R a novel structure, viz., for an arbitrarily small scale δ, numbers x and $\tilde{x}$ satisfying $x > \delta$ and $0 < \tilde{x} < \delta$ now are represented as

$$x = \delta.\delta^{-|\tilde{x}_0|} \text{ and } \tilde{x} = \lambda.\delta.\delta^{|\tilde{x}_0|}$$

for a λ ($0 < \lambda << 1$), so that the inversion rule is satisfied. Actually, $\tilde{x}$ belongs to an open set I^+_{in} (say) of $(0, \delta)$, the size of which is determined by λ. Here, $\tilde{x}_0$ is a special reference point in I^+_{in}, for instance, $\tilde{x}_0 = x^{-1} \in I^+_{\text{in}}$. It is also often useful to rewrite the inversion rule as an exponentiation: $\tilde{x}/\delta = (\delta/x)^\mu$ so that $\mu \log(x/\delta) = \log \lambda^{-1} + \log(x/\delta)$, for a given x and δ. It also follows that although λ is a constant, the exponent μ is actually a function both of the real variable x and the scale δ. For $x \to \delta$, and $\delta \to 0^+$, we have $\mu \to \infty$ and $|\tilde{x}_0| \to 0$ in such a manner that $|\tilde{x}|$ may have a finite value. For a δ infinitesimal, on the other hand, μ may tend to 1^+ as $\delta \to 0^+$. Indeed, in that case, we have, for a given arbitrarily small x and δ, a sequence $\tilde{x}_n$ such that $\tilde{x}_n = \delta^n \delta^{\mu|\tilde{x}_0|_\delta} = \delta^n \delta^{n(\tilde{\mu}|\tilde{x}_0|_\delta)}$ where $\tilde{\mu} = \mu/n \to 1$ for a sufficiently large n. Notice that such a sequence always exists. In the limit $\delta \to 0^+$, a δ infinitesimal should go over to a scale free infinitesimal. Letting $x_1 = x/\delta = \delta^{-|\tilde{x}_0|} \approx 1 + \eta$, so that $\eta \approx |\tilde{x}_0| \log \delta^{-1}$, we get $\tilde{x}_1 = \tilde{x}/\delta = \delta^{\mu|\tilde{x}_0|} \approx 1 - \mu\eta$, $\mu = O(1)(> 0)$. Moreover the rate of approach of a real variable x, which equals 1 in the ordinary analysis, gets slower in the presence of scale free infinitesimals. In fact x approaches 0 now as $x^{1-\alpha}$, $\alpha = |\tilde{x}|$, rather than simply as x.

Remark 6. The scale δ might correspond to the accuracy level in a computational problem. In this context, 0 in R is identified with the interval $\bar{I}_\delta$ (the closure of I_δ) and thus is raised to $\mathbf{0}$. A computation is therefore interpreted as an activity over an extended field $\tilde{R}$. By letting $\delta^n \to 0^+$ as $n \to \infty$, we consider an infinite precision computation, which is achieved progressively by increasing the accuracy level, when real numbers are represented as δ-adically, for instance, the binary or decimal representation correspond to $\delta = 1/2$ or $\delta = 1/10$ respectively. Consequently, one arrives at a class of (scale free) infinitesimals $X = \tilde{x}/\delta^n \in (-1, 1)$, $n \to \infty$, which seem to remain available even in the ordinary analysis. To avoid any conflict with the standard real analysis results (for instance, the Lebesgue measure of Cantor sets in R), the scale free infinitesimals may be assumed to live in a zero measure Cantor set. As a consequence, the topological dimension of R_δ is zero. To re-emphasize, a scale free infinitesimal X is an element of a Cantor set C_δ (with the scale factor δ) in I_δ, while the sequence of realizations $\tilde{x}_n$ corresponds to its nth iteration realization. In such a realization, $\tilde{x}_n$ (say) is an element of a closed (undeleted) subset F_{in} of I_δ, each element of which is mapped to the finite real number x by the inversion rule. The Cantor set structure of the scale free infinitesimals is consistent with the ultrametricity of $\mathbf{0}$.

Remark 7. **Relationship between $|X|$ and $|X|_\delta$:**

We have already noticed that for an arbitrarily small $\delta \to 0^+$, we have the asymptotic representations $x = \delta\delta^{-|\tilde{x}_0|}$ and $\tilde{x} = \lambda\delta\delta^{|\tilde{x}_0|}$. Accordingly, $|\tilde{x}|$ may have a finite value even as $\tilde{x}$ vanishes as $\delta \to 0$. For a δ infinitesimal $\tilde{X}$, on the other hand the analogous representations are $x = \delta\delta^{-|\tilde{x}_0|_\delta}$ and $\tilde{x}_n = \delta^n(\delta^n)^{\tilde{\mu}(\delta)|\tilde{x}_0|_\delta}$, where $n \to \infty$ but δ is kept fixed, and $\tilde{\mu}$ now depends on δ. It follows that $|\tilde{x}|_\delta = \lim_{n\to\infty} \log_{\delta^{-n}}(\delta/\tilde{x})^n = \log_{\delta^{-1}}(\delta/\tilde{x}) = \tilde{\mu}(\delta)|\tilde{x}_0|_\delta$. Recalling that $\tilde{\mu} = 1 + \sigma_\delta(x)$, where $\sigma_\delta \to 0$ with δ, we therefore write, $|\tilde{x}|_\delta = (1 + \sigma_\delta(x))|\tilde{x}_0|_\delta := \tilde{x}_1$. From the remarks following the proof of Proposition 7 it also follows that $|\tilde{x}_0|_\delta = \delta^\alpha \leq 1$ for $\alpha \geq 0$.

Example 1. Let us recall that a p-adic integer is given by $X_r = p^r(1 + \sum_0^\infty a_i p^i)$, $r > 0$, where a_i assumes values from 0,1,2,....(p-1) with p-adic norm $|X_r|_p = p^{-r}$. As an example of δ infinitesimals we now consider p-infinitesimals, which are related to the p-adic integers in $Z_p \subset Q_p$. Let $\delta = 1/p$, p being a prime. Then there exist a class of p infinitesimals X_p (actually an equivalence class of such infinitesimals) which are ordered according to the primes. Let, $x = p^{-(1-1/p^r)}$, for some positive integer r, be a given value of a real variable x relative to the scale $1/p$. Then we have a class of p-infinitesimals $0 < X_{rp} < 1/p$ given by $X_{rp} = p^{-n\mu_p(x)(1+1/p^r)}$ where $0 < r \leq n$ and $\mu_p = (1 + \sigma_p(x)), \sigma_p(x)$ being a small positive variable and goes to zero faster than $1/p$. Then we have $|X_{rp}| = p^{-r}(1 + \sigma_p)$. When $\sigma_p = 0$, one obtains a p-adic integer, realized as a p-infinitesimal, because in that case we have $|X_{rp}| = p^{-r}$. The sequence of partial sums $S_m = p^r(1 + \sum_0^n a_i p^i)$, which is divergent in the usual metric of Q and is an infinitely large element in the conventional non-standard models of Q, is realized in the present model as a p-infinitesimal X_{rp}.

Lemma 4. *A closed ball in* $\mathbf{0}$ *is both complete and compact.*

Proof. The proof follows from the following observations. Given $\epsilon > 0$, consider a closed interval $[a, b] \subset \mathbf{0}$ (in the usual topology) such that $0 < a < b < \epsilon$. The valuation v realizes this closed interval as an ultra metric (sub) space U of $\mathbf{0}$ which is an union of at most of a countable family of disjoint clopen balls (by Lemma 3).

Now we consider completeness. A sequence $\{x_n\} \subset U$ is Cauchy $\Leftrightarrow v(x_m - x_n) \to 0 \Leftrightarrow v(x_{n+1} - x_n) \to 0 \Rightarrow \exists N > 0$ such that $v(x_{n+1}) = v(x_n)$ for all $n \geq N$. Now since for a non-zero infinitesimal x_n, the associated valuation is non-zero, it follows that $x_n \to x_N \in U$ in the ultra metric in the sense that $v(x_n) = v(x_N)$ as $n \to \infty$. Compactness is a consequence of the fact that any sequence in U has a convergent subsequence. Indeed, a sequence $\{x_n\}$ in U can not be divergent in the given ultra metric since $0 \leq v(x_n) \leq 1$. ■

Next, we extend this nonarchimedean structure of $\mathbf{0}$ on the whole of R, which is already assumed as an *intrinsically* nonstandard extension $\tilde{R}$.

Definition 5. *Let* $I_\delta(r) = r + I_\delta(0)$ $I_\delta(0) = (-\delta, \delta)$, $\delta > 0$ *for a real number* $r \in R$. *For a finite* $r \in R$, *i.e., when* $r \notin I_\delta(0)$, *we have* $||r|| = |r|_e = r$. *For an* $r \in I_\delta(0)$, *on the other hand, we have* $||r|| = |r| = \lim_{\delta \to 0} \log_{\delta^{-1}}(\delta/r) = v(r)$, *while, for an arbitrarily* *large r ($\rightarrow \infty$), i.e., when $|r|_e > N,\ N > 0$, we define $||r|| = |r^{-1}|$ which is evaluated with the scale $\delta \leq 1/N$.*

Proposition 9. *$||.||$ is a nonarchimedean absolute value on R. It is discretely valued over the set of infinitely small and large numbers.*

Notation 1. *The ultra metric space $\{R, ||.||\}$ is denoted as $\mathbf{R}$.*

Proof. For an infinitely small or large r, the proof follows from Proposition 8. For a finite (non-zero) value of $r \in R$, we have, on the other hand, $r = s + \tau(t),\ s = r - t, \tau(t) = t,\ t \in I_\delta^+(0)$, so that $||r|| = \max\{||s||, ||\tau(t)||\} = s = |r|_e$, by letting $t \rightarrow 0$. Discreteness on the set of infinitesimals and infinities follows from the discreteness of $|.|$. ■

Corollary 3. *$\mathbf{R}$ is a locally compact complete (ultra-)metric space.*

The proof follows from Lemma 4 and Proposition 9.

Proposition 10. *The topology induced by $||.||$ on R is equivalent to the usual topology. Moreover, the embedding $i : R \rightarrow \mathbf{R}$ is continuous.*

Proof: It is easy to verify that an open set of R in usual topology is open in $||.||$ and conversely since $||.||$ reduces to the usual absolute value on R and the continuity also follows from Definition 5.

Definition 6. *A (δ) infinitesimal $\tilde{X}$ is an (non-archimedean) integer if $|\tilde{X}| < 1$. It is a unit if $|\tilde{X}| = 1$.*

Lemma 5. *A scale free (δ) unit $\tilde{X}_u$ on $\mathbf{R}$ has the form $\tilde{X}_u = 1 + \tilde{X}$ where $|\tilde{X}| < 1$.*

Proof. A scale free (δ) unit is defined by $||X_u|| = 1$. According to the valuation equation (3.5), we have

$$\tilde{x}_{un}/\delta^n = (\delta^n)^{1+\xi(\tau_u,\delta)} \tag{3.6}$$

since $v(X_u) = 1$ ($\tilde{x}_{un}$ are various realizations of X_u). We also assume $\xi > 0$. Thus $\tilde{x}_{un}/\delta^{2n} = O(\delta^{n\xi}) \to 1$, as $n \to \infty$ and subsequently $\delta \to 0$. Thus writing $\tilde{X}_n = \xi \log \delta^n$ (Remark 4), we have the lemma, since, as $n \to \infty$, $X_u = \lim \tilde{x}_{un}/\delta^{2n} = \lim e^{\tilde{X}_n} = 1 + \tilde{X}_0$, when we have $||\tilde{X}_n(1 + o(\tilde{X}_n)|| = ||\tilde{X}_0||$, (because of the ultrametricity of $||.||$) for a $\tilde{X}_0 = \tilde{X}_n(1 + o(\tilde{X}_n))$. ■

Lemma 6. *Let X_i be two δ_i infinitesimals, $i = 1, 2$ such that $\delta_1 > \delta_2$. Then there is a canonical decomposition $X_1 = \tilde{X}_1(1 + X_2)$ where $|X_1| = |\tilde{X}_1| < 1$.*

Proof. Recall that the positive δ-infinitesimals live in $(0, 1) \subset R_\delta$, which is covered by clopen balls $B(\tilde{X}_{1j})$, $j = 1, 2, \ldots$. Let $X_1 \in B(\tilde{X}_{1j})$ for some j. Suppose X_{1j} is defined by (3.4). For a general infinitesimal X_1 we have, on the other hand, the extended definition given by (3.5), viz., $X_1 = \tilde{x}_{1n}/\delta_1^n = (\delta_1^n)^{(|X_{1j}|_{\delta_1} + \xi(X_1, \delta_2))}$, where $\xi > 0$ goes to zero faster than $|X_{1j}| = v(X_{1j})$ such that $(\delta_1^n)^\xi = 1$ as $\delta_1^n \to 0$. Now writing $\xi = (\tilde{\xi}) \log \delta_2 / \log \delta_1$ and using Lemma 5, we obtain $\tilde{x}_{1n}/\delta_1^n = \tilde{x}_{1jn}/\delta_1^n \times \tilde{x}_{un}/\delta_2^{2n}$ and so taking limit $n \to \infty$ the desired result follows. ■

Let us recall that $dX/dx = 0$ means X=constant, on R. However, in a non-archimedean space **R**, X can be a *locally* constant function, which we call here a *slowly varying* function. In a non-archimedean extension of R, λ (in Definition 1) may be a slowly varying function. Thus, the nonsmooth solutions of R (Section 3.1) are realized as smooth in the non-archimedean space **R**. We recall that differentiability in a non-archimedean space is defined in the usual sense by simply replacing the usual Euclidean metric by the ultra metric $||x - y||$, $x, y \in \mathbf{R}$.

Definition 7. *Let $f : \mathbf{R} \to \mathbf{R}$ be a mapping from $\mathbf{R}$ to itself. Then f is differentiable at $x_0 \in \mathbf{R}$ if $\exists\ l \in \mathbf{R}$ such that given $\varepsilon > 0, \exists\ \eta > 0$ so that*

$$\left| \frac{\| f(x) - f(x_0) \|}{\| x - x_0 \|} - \| l \| \right| < \varepsilon \tag{3.7}$$

when $0 < \| x - x_0 \| < \eta$, and we continue to write the standard notation $f'(x_0) = \frac{df(x_0)}{dx} = l$.

Remark 8. The above definition is in conformity with the more conventional definition, viz.,

$$\| \frac{f(x)-f(x_0)}{x-x_0} - l \| < \varepsilon \tag{3.8}$$

since $\left| \frac{\|f(x)-f(x_0)\|}{\|x-x_0\|} - \| l \| \right|_e < \| \frac{f(x)-f(x_0)}{x-x_0} - l \| < \varepsilon$. As long as $|x-x_0|_e \to 0^+$, but $|x-x_0| \geq O(\delta)$, the above definition reduces to the ordinary differentiability. But when $|x-x_0|_e \to \delta$ the above gets extended to the logarithmic derivative $x\frac{d\log f(x_0)}{dx} = l$, when we make use of the nonarchimedean absolute value $|.|$.

So far in the above discussion the scale δ is unspecified. In the following, we introduce the nonarchimedean absolute value (Definition 10) on the field of rational numbers Q, construct its Cauchy completion and finally, because of the Ostrowski theorem, relate it to the local p-adic fields. We get a *minimal* nonarchimedean extension of R (and which is a subset of the above $\mathbf{R}$) for each given scale δ, thus sufficing our purpose of relating the (secondary) scales δ with the inverse primes viz., $\delta = 1/p$. Consequently, we have a countable number of distinct field extensions $\mathbf{R}_p$ of R depending on the scale at which the origin 0 of R is probed.

However, before proceeding further, let us collect a few more general properties of the valuation $v(\tilde{x})$ in the following subsection.

3.3 Non-archimedean valuation: A few more properties

The set of infinitesimals $\mathbf{0}$ reduces to the singleton $\{0\}$ when $\delta \to 0$ classically. However, the corresponding asymptotic expressions for the scale free (invariant) infinitesimals are non-trivial, in the sense that the associated valuations (Definition 3) can be shown to exist as finite real numbers. Below we give a definite construction indicating the exact sense how relative infinitesimals and associated values could arise in a limiting problem.

Fix a value $\delta = \delta_0$ and let $C_{\delta_0} \subset [0, \delta_0] = I^+_{\delta_0}$ be a Cantor set defined by an IFS of the form $f_1(x) = \lambda x$, $f_2(x) = \lambda x - (\lambda/\delta_0 - 1)\delta_0$ where $\lambda = \beta\delta_0$, $0 < \beta < 1$

and $\alpha + 2\beta = 1$. Thus at the first iteration an open interval O_{11} of size $\alpha\delta_0$ is removed from the interval $I^+_{\delta_0}$, at the second iteration two open intervals O_{21} and O_{22} each of size $\alpha\delta_0(\beta)$ are removed and so on, so that a family of gaps O_{ij} of size $\alpha\delta_0(\beta)^{i-1}$, $j = 1, 2, ...2^{i-1}$ are removed in subsequent iterations from each of the closed subintervals I_{ij}, $j = 1, 2, ...2^i$ of $I^+_{\delta_0}$. Consequently, $C_{\delta_0} = I^+_{\delta_0} - \cup_i O_{ij} = \cap_i \cup_j I_{ij}$. Notice that the total length removed is $\sum \alpha\delta_0(2\beta)^{i-1} = \delta_0$, so that the linear Lebesgue measure $m(C_{\delta_0}) = 0$.

Next, consider $\tilde{I}_N = [0, \beta^N]$ and let $N = n + r$ and $N \to \infty$ as $n \to \infty$ for a fixed $r \geq 0$. Choose the scale $\delta = \alpha\beta^n\delta_0$ and define $\tilde{x}_r \in [0, \alpha\beta^N\delta_0]$, a relative infinitesimal (relative to the scale δ) provided it also satisfies the inversion rule $\tilde{x}/\delta = \lambda\delta/x$, for a real constant $\lambda(\delta)$ $(0 << \lambda \leq 1)$. For each choice of x and δ, we have a unique $\tilde{x}$ for a given $\lambda \in (0, 1)$. Consequently, by varying λ in an open subinterval of (0,1), we get an open interval of relative infinitesimals in the interval $(0, \delta)$, all of which are related to x by the inversion formula. In the limit $\delta \to 0$, relative infinitesimals $\tilde{x}_r$, of course, vanish identically. However, the corresponding scale invariant infinitesimals $\tilde{X}_r = \tilde{x}_r/\delta$, $\delta \to 0$ are, nevertheless, nontrivial and are weighted with new scale invariant absolute values (norms) (Definition 3).

The set of infinitesimals are uncountable, and as already shown the above norm satisfies the stronger triangle inequality $v(x + y) \leq max\{v(x), v(y)\}$. Accordingly, the zero set $\mathbf{0}=\{0, \pm\delta\tilde{X}_r |\ \tilde{X}_r \in (0, \beta^r), r = 0, 1, 2, ..., \delta \to 0^+\}$ may be said to acquire dynamically the structure of a Cantor like ultra metric space, for each $\beta \in (0, 1/2)$. The set $\mathbf{0}$ indeed is realized as a set of nested circles $S_r = \{\tilde{x}_r |\ v(\tilde{x}_r) = \alpha_r\}$ in the ultra metric norm, when we order, with out any loss of generality, $\alpha_0 > \alpha_1 >$ ..... The ordinary 0 of R is replaced by this set of scale free infinitesimals $0 \to \bar{\mathbf{0}} = \mathbf{0}/\sim = \{0, \cup S_r\}$; $\bar{\mathbf{0}}$ being the equivalence class under the equivalence relation $\sim$, where $x \sim y$ means $v(x) = v(y)$.

Remark 9. The concept of infinitesimals and the associated absolute value considered here become significant only in a limiting problem (process), which is reflected in the explicit presence of "$\lim\limits_{\delta\to 0}$" in the relevant definitions. For the continuous real valued function $f(x) = x$, the statement $\lim\limits_{x\to 0} x = 0$, means that the statement $x \to 0$ essentially means that $x = 0$ (i.e. x not only tends to 0 but, in fact, assumes the value 0 "exactly"). This may be considered to be a passive evaluation of limit. The present approach is active (dynamic), in the sense that it offers not only a more refined *intermediate* stages in the evaluation of the limit, but also provides a clue how one may induce new (nonlinear) structures in the limiting (asymptotic) process. The inversion rule is one such non linear structure which may act non trivially as one investigates more carefully the motion of a real variable x (and hence of the associated scale $\delta < x$) as it goes to 0 more and more accurately. Notice that at any "instant", elements defined by $0 < \tilde{x} < \delta < x$ in a limiting process are well defined; relative infinitesimals are meaningful only in that dynamic sense (classically, these are all zero, as x itself is zero). Scale invariant infinitesimals $\tilde{X}$, however, may or may not be zero classically. $\tilde{X} = \mu(\neq 0)$, a constant, for instance, is non zero even when x and δ go to zero. On the other hand, $\tilde{X} = \delta^{\alpha}$, $0 < \alpha < 1$, of course, vanish classically, but as shown below, are non trivial in the present formalism. As a consequence, relative infinitesimals may be said to exit even as real numbers in this dynamic sense. The accompanying metric $|.|$, however is an ultra metric.

Remark 10. A genuine (nontrivial) scale free infinitesimal $\tilde{X}$ can not be a constant. Let, $\tilde{x} = \mu\delta$, $0 < \mu < 1, \mu$ being a constant. Then $v(\tilde{x}_0) = \lim\limits_{\delta\to 0} \log_{\delta^{-1}} \mu^{-1} = 0$, so that $\tilde{x}_0$ is essentially the trivial infinitesimal 0. More precisely, such a relative infinitesimal belongs to the equivalence class of 0.

Example 2. *Let,* $x_n = \epsilon^{n(1-l)}$, $0 < l < 1$, $0 < \epsilon < 1$. *Then scale invariant infinitesimals are* $\tilde{X}_{n\lambda} = \lambda\epsilon^{nl}$, $0 < \lambda < 1$, *when* $\delta = \epsilon^n$, *for a sufficiently large* n, *is chosen as a scale. Analogously, for a continuous variable* x *approaching* 0^+, *say, and considered as a scale, a class of relative infinitesimals are represented as* $\tilde{x} = x^{1+l}(1 + o(x))$, $0 < l < 1$, *so that the corresponding scale invariant infinitesimals* *are defined by the asymptotic formula* $\tilde{X} = \lambda x^l + o(x^m)$, $m > l$. *The corresponding scale invariant absolute value has the non-trivial value* $|\tilde{x}| = \lim_{x\to 0} \log_{x^{-1}} \frac{x}{\tilde{x}} = l$. *Notice that a scale invariant infinitesimals goes to zero at a smaller (ultra metric) rate* $l : \tilde{X} = \lambda x^l \Rightarrow d\log\tilde{X}/d\log x = l$

Remark 11. The scale free infinitesimals of the form $\tilde{X}_m \approx \delta^{\alpha_m} + o(\delta^{\beta}), \beta > \alpha$ goes to 0 at a slower rate compared to the linear motion of the scale δ. The associated non trivial absolute value $v(\tilde{x}_m) = \alpha_m$ essentially quantifies this decelerated motion.

Theorem 3. *The norm* v *has the following properties*

1) v *is an ultra metric, and hence* $\mathbf{0}$ *equipped with* v *is an ultra metric space (non-archimedean space).*

2) v *is a locally constant Cantor function. Conversely, given a Cantor function* $\phi(\tilde{x})$, *there exists a class of scale invariant infinitesimals determined by* $\phi(\tilde{x})$, *those live on the extended ultra metric neighborhood* $\mathbf{0}$ *of 0.*

Remark 12. The part 1 of the theorem is already proved in proposition 7 and Proposition 8. We present here a slightly improved concise proof of the same.

Proof. 1)(a) v is well defined. Indeed, the open set $\mathbf{0}$ is written as a countable union of disjoint open intervals I_{δ_i} of relative infinitesimals, i.e. $\mathbf{0}=\cup I_{\delta_i}$. Let $v(\tilde{x}_i) = \alpha_i$, a constant for all $\tilde{x}_i(=\lambda\delta\delta^{\alpha_i}) \in \bar{I}_{\delta_i}$, the closure of I_{δ_i}. Thus v exists and well defined.

(b) Let $0 < \tilde{x}_2 < \tilde{x}_1 < \tilde{x}_1 + \tilde{x}_2 < \delta$ be two relative infinitesimals. We have, $0 < \tilde{X}_2 < \tilde{X}_1 < \tilde{X}_1 + \tilde{X}_2 < 1$ and $v(\tilde{x}_2) > v(\tilde{x}_1) > v(\tilde{x}_1 + \tilde{x}_2)$, thus proving the strong triangle inequality $v(\tilde{x}_1 + \tilde{x}_2) \leq max\{v(\tilde{x}_1), v(\tilde{x}_2)\}$. Next, given $0 < \tilde{x} < \delta$, there exists a constant $0 < \sigma(\delta) < 1$ and $a : \mathbf{0} \to \mathrm{R}$, such that $\tilde{X} = \lambda\delta^{v(\tilde{x})}$ and $v(\tilde{x}) = \sigma^{a(\tilde{x})}$. Accordingly, $a(\tilde{x})$ is a discrete valuation satisfying (i) $a(\tilde{x}_1\tilde{x}_2) = a(\tilde{x}_1) + a(\tilde{x}_2)$, (ii) $a(\tilde{x}_1 + \tilde{x}_2) \geq min\{a(\tilde{x}_1), a(\tilde{x}_2)\}$. As a result, $v(\tilde{x}_1\tilde{x}_2) = v(\tilde{x}_1)v(\tilde{x}_2)$. Hence $\{\mathbf{0}, v\}$ is an ultra metric space.

2) Let $\bar{\mathbf{0}} = (\cup\bar{I}_{\delta i})\bigcup(\cup J_k)$, the closure of $\mathbf{0}$. The open intervals J_k are gaps between two consecutive closed intervals $\bar{I}_{\delta i}$. J_k's actually contain new points those arise as

the limit points of sequences of the end points of the open intervals $I_{\delta i}$. Clearly, $\bar{\mathbf{0}}$ is connected in usual topology. However, in the ultra metric topology, both $I_{\delta i}$ and J_k are clopen sets and $\bar{\mathbf{0}}$ is totally disconnected. Since, it is bounded and also is perfect, $\mathbf{0}$ is equivalent to an ultra metric Cantor set.

Now, the local constancy of v in the ultra metric $\bar{\mathbf{0}}$ follows from the definition:

$$\frac{dv(\tilde{x})}{dx} = \lim_{\delta\to 0^+}\frac{d}{dx}\left(\frac{\log x}{\log \delta}+1\right) = 0 \tag{3.9}$$

The vanishing derivative above arises from a logarithmic divergence arising from the nontrivial finer scales. This is unlike the ordinary analysis, when one interprets $\bar{\mathbf{0}}$ as a connected subset of R, thereby forcing v to vanish uniquely, so as to recover the usual structure of R. The above vanishing derivative can be interpreted non trivially as a LCF [43] when $x \in R$ is supposed to belong to a Cantor subset of R.

Eq(3.9) also reveals the *reparameterization invariance* of a locally constant valuation $v(x)$. As a consequence, v may be a function of any reparametrized monotonic variable $\tilde{x} = \tilde{x}(x)$ with $\tilde{x}'(x) > 0$, instead being simply a function of the original real variable x.

Now to construct a general class of locally constant functions in the ultra metric space, let us proceed as in 1a) above, with the supposition that the constants α_i's are arranged in ascending order. Thus, $v(\tilde{x}_i) = \alpha_i$, $\alpha_i \leq \alpha_j \Leftrightarrow i \leq j$ for all $\tilde{x}_i \in I_i$(we drop the suffix δ for simplicity). Clearly, Definition 5 holds over for all I_i. On the other hand, for an $\tilde{x} \in J_k$, where J_k separates two consecutive I_i and I_{i+1}, say, so that $\tilde{x}_i < \tilde{x} < \tilde{x}_{i+1}$, where $\tilde{x}_i$ is the right end point of I_i and $\tilde{x}_{i+1}$ is the left end point of I_{i+1}, we have $v(\tilde{x}_{i+1})-v(\tilde{x}_i) = (\alpha_{i+1}-\alpha_i)$. Because of ultrametricity, one can always choose $\alpha_i = \beta_{ij_i}\sigma(i)^s$, for $\beta_{ij_i} >0$ ascending and $\sigma(i) \to 0$ as $i \to \infty$ and $j_i = 0, 1, 2, ...k(i)$ for some i dependent constant $k(i)$. Consequently, $v(\tilde{x}_{i+1})-v(\tilde{x}_i) = (\beta_{(i+1)j_{i+1}}-\beta_{ij_i})\sigma(i)^s$. It follows that the sequence $v(\tilde{x}_{i+1})$ is decreasing and $v(\tilde{x}_i)$ is increasing. Thus, $v(\tilde{x}) := \lim v(\tilde{x}_i)$ as $i \to \infty$. Hence, $v : \mathbf{0} \to \mathbf{I}^+$ is indeed a Cantor function.

Conversely, given a Cantor function $\phi(x)$, $x \in I^+$, one can define a class of infinitesimals $\tilde{x} \approx \delta\delta^{\phi(\tilde{x}/\delta)}$ belonging to the extended set $\mathbf{0}$ for $\delta \to 0^+$. This completes the proof. ■

Definition 8. *Besides the usual Euclidean value, a real variable* $x \neq 0$*, but* $x \to 0^+$ *gets a deformed ultra metric value given by* $v(x) := \lim_{\delta\to 0^+} \log_{\delta^{-1}}(x/\delta)$.

Lemma 7. $v(x) = v(\tilde{x})$.

Because of inversion rule, $x/\delta = \lambda(\delta/\tilde{x})$, $0 < \lambda < 1$, and hence $v(x) = v(\tilde{x})$ since $\lim \log_{\delta^{-1}} \lambda^{-1}$=0. □

Lemma 8. *Let,*$0 < |x| < |x'|$ *be two arbitrarily small real variables and* δ *be a scale such that* $0 < \delta < |x - x'| < |x| < |x'|$. *Then* $v(x') = v(x)$.

From Definition 8, $v(x - x') < v(x) < v(x')$. But $x' = x + (x' - x)$. So, by ultra metric inequality, $v(x') \leq max\{v(x), v(x' - x)\} \leq v(x)$ and hence the result. □

Lemma 9. *Let* $0 < |x| < |x'|$ *be two arbitrarily small real variables and* δ *and* δ' *be two scales such that* $0 < \delta < |x| < \delta' < |x'|$. *The corresponding scale invariant infinitesimals are* $\tilde{X}$ *and* $\tilde{X}'$ *with associated valuations* $v(x)$ *and* $v(x')$. *Then* $v(x') = (\alpha/s)v(x)$, *where* $\alpha = \lim \log_{\tilde{X}} \tilde{X}'$, *determines the gap size between* $\tilde{X}$ *and* $\tilde{X}'$ *and* $s = \lim \log \delta' / \log \delta$ *is the Hausdorff dimension of the Cantor set of infinitesimals as* $x, x' \to 0$.

Proof. The proof follows from

$$\frac{v(x')}{v(x)} = \lim \frac{\log(x'/\delta')}{\log(x/\delta)} \times \lim \frac{\log \delta}{\log \delta'} \tag{3.10}$$

so that $\alpha = \lim \log_{x/\delta}(x'/\delta') = \lim \log_{\tilde{X}} \tilde{X}' \Rightarrow \tilde{X}' = X^{\alpha}(1 + O(\beta(x, x')))$, $\beta \to 0$ faster than the linear approach $x \to 0$. The exponent α gives a measure of the said gap size. ■

Corollary 4. *Let* $0 < \delta < \delta' < x$ *be two scales in association with an arbitrarily small real variable and* $\tilde{X} = (x/\delta)^{-1}$ *and* $\tilde{X}' = (x/\delta')^{-1}$ *be the corresponding scale* *invariant infinitesimals. Then* $v(x') = (\alpha/s)v(x)$*, where* $\alpha = \lim \log_{\tilde{X}} \tilde{X}'$*, determines the gap size between* $\tilde{X}$ *and* $\tilde{X}'$ *and* $s = \lim \log \delta / \log \delta'$ *is the Hausdorff dimension of the Cantor set of infinitesimals as* $x, x' \to 0$*. The exponent* α *gives a measure of the said gap size.*

Definition 9. *A scale invariant jump is defined by the pure inversion* $\tilde{X}' = X^{-\alpha}$ *with the scale invariant minimal jump size* $\alpha = 1$*. The (scale invariant) jump size* α *thus runs over the set of natural numbers N.*

Remark 13. Lemma 8 characterizes the equivalence classes of infinitesimals with identical valuations. Subsequent lemma (and corollary) tells that the valuation v changes only when an infinitesimal from one equivalence class switches over to another class.

Summing up the above observations, we now state a general representation of relative infinitesimals and corresponding valuation.

Lemma 10. *A relative infinitesimal* $\tilde{x}$ *relative to the scale* δ *has the asymptotic form*

$$\tilde{x} = \delta \times \delta^{l} \times \delta^{\phi(\tilde{x}/\delta)}(1 + o(1)) \tag{3.11}$$

with associated valuation $v(\tilde{x}) = l + \phi(\tilde{x}/\delta)$ *where* $l \geq 0$ *is a constant and* ϕ *is a nontrivial Cantor function.*

The locally constant $v = v_0 + v_1$ solves $dv/dx = 0$ and so the above ansatz is the more general solution, with the trivial ultra metric valuation $v_0 = l$ and the nontrivial valuation $v_1 = \phi$. The representation for $\tilde{x}$ now follows from definition. □

Remark 14. As a real variable x and the associated scale $\delta < x$ approach 0, the corresponding infinitesimals $0 < \tilde{x} < \delta$ may also live (in contrast to measure zero Cantor sets considered so far) in a positive measure Cantor set C_p. Such a possibility is already considered in [14] in relation to an interesting phenomenon of growth of measure. In such a case $v_0(\tilde{x}) = m(C_p) = l$, the Lebesgue measure of C_p. The nontrivial component v_1 then relates to the uncertainty (fatness) exponent of the positive measure 1-set. In this extended model, the valuation quantifies the presence of nontrivial motion in a limiting process: v_0 gives the uniform scale invariant motion when v_1 arises from the associated non-uniformity stemming out from measure zero Cantor sets. We, however, do not consider this aspect of the analysis any further in this thesis.

3.4 Completion of the Field of Rational Numbers

We have already shown that the real number system R gets extended over a locally compact, complete ultra metric space $\mathbf{R}$ under the nonarchimedean norm $||.||$. However, we haven't yet shown that the ultra metric space $\mathbf{R}$ is a field. In the present Section, we present yet another route extending R *minimally* as an ultra metric field completion $\mathbf{R}$ of the field of rationals Q (with a slight abuse of notation, we use same notation for both type of ultra metric extensions). Because of the Frobenius field extension theorem, this field extension must be of infinite dimensional. In the next chapter, a new elementary proof of the Prime Number Theorem will be presented. Applications to differential equations on the extended field will be studied in the subsequent latter chapter.

On the field of rationals Q, we introduce the definition of the valuation $||.|| : Q \to R_+$ as follows.

Definition 10. *Let $I_\delta(r) = r + I_\delta, I_\delta = (-\delta, \delta)$, $\delta > 0$ for rational numbers r and $\delta \in Q$. For a finite $r \in Q$, i.e., when $r \notin I_\delta(0)$, we have $||r|| = |r|_e = r$. For an $r \in I_\delta(0)$, on the other hand, we have $||r|| = |r| = \lim_{\delta\to 0} \log_{\delta^{-1}}(\delta/r) = v(r)$, while, for an arbitrarily large $r(\to\infty)$, i.e., when $|r|_e > N, N > 0$, we define $||r|| = |r^{-1}|$ with a scale $\delta \leq 1/N$.*

Proposition 11. $||.||$ *is a non-archimedean absolute value over* Q.

Proof is similar to that on R which is given in the previous Section. □

Now by the Ostrowski theorem [42], any non-trivial absolute value on Q must be equivalent to any of the p-adic absolute values $|.|_p$, $p > 1$ being a prime and $|.|_\infty = |.|_e$ is the usual Euclidean absolute value. From Definition 10, finite rationals of Q get the Euclidean value, while $|.|$, on the arbitrarily small and large values, must be related to the p-adic valuations. Consequently, the set of *primary scales* are represented uniquely by the inverse primes $\delta = p^{-1}$.

To construct the completion of Q under $||.||$ we first consider the ring $\mathcal{S}$ of all sequences of Q. The zero divisors in $\mathcal{S}$ are removed by the choice of an ultra filter, as in the usual non-standard models of R [52]. The quotient set of the Cauchy sequences C ($\subset$S), under the usual absolute value, modulo the maximal ideal $\mathcal{N}$ consisting of sequences converging to 0, gives rise to the ordinary real number set R=$\mathcal{C} - \mathcal{N}$. The elements of diverging sequences in $\mathcal{S}_{div} = \mathcal{S} - \mathcal{C}$ correspond to the infinitely large elements, when the inverse $\{a_n^{-1}\} \in \mathcal{S}_{div}$ of a divergent sequence $\{a_n\}$ leads to an infinitesimal in the conventional approaches of non-standard analysis. In our approach, this realization is, however, somewhat reversed.

Notice that the set of divergent sequences $\mathcal{S}_{div}$ is quite a large set. Now among the all possible divergent sequences, there exists a subset $\mathcal{S}_p$ of sequences which are nevertheless p (-adically) convergent ($\mathcal{S}_p \subset \mathcal{S}_{div}$, since the sequence $\{n\}$ is p-adically divergent for each p). For each fixed p, let us consider the Cauchy completion of p convergent sequences $\{a_n^p\}$ (say) (modulo the sequences p-adically converging to zero), viz., the local field Q_p. We identify, by definition, the p-adic integers $X \in Z_p \subset Q_p$ with $|X|_p \leq 1$ as the p infinitesimals. On the other hand, the elements $\tilde{X}$ of Q_p with $|\tilde{X}|_p > 1$ are identified with infinitely large numbers of *type p*. In other words, $\tilde{X}$ denotes the $p-$adic limit of an inverse sequence of the form $\{(a_n^p)^{-1}\}$, leading to the inversion symmetry $\tilde{X} = X^{-1}$ which is valid for a suitable p infinitesimal X. The absolute value $||.||$ when restricted to $\mathcal{S}_p$ thus relates an infinitesimal X, i.e. an element of $*R$ (a non-standard model of R), to a countable number of p-adic realizations $X_p \in Z_p$ with valuations $||X|| = \mu_p|X_p|_p$, p=2,3,....; $p \neq \infty, \mu_p$ being a constant for each p, as the neighborhood of 0 in R is probed deeper and deeper by letting $\delta = p^{-1} \to 0$ as $p \to \infty$. In the computational model (c.f Sec.3.2, Remark 6), this might be interpreted as (in equivalence classes of) higher precision models of a computation. Consequently, equipped with $||.||$, the set $\mathcal{S}_p$ decomposes into (a countably infinite Cartesian product of) local fields Q_p in a hierarchical sense as detailed in the Lemma 11. We note that any element X of $\mathcal{S}_p$ is an equivalence class of sequences of rational numbers under the chosen ultra filter. In each of such a class there exists a unique sequence $\{a_n^p\}$, say, converging to a p-adic integer or its inverse X_p. A scale free infinitesimal X then relates to X_p, and that X indeed is an infinitesimal tells that $||X|| = \mu_p|X_p|_p \leq 1$. We say that 0 of R is probed at the depth of the (secondary) scale $1/p$ when a scale free (δ)- infinitesimal X is related to a p-adic infinitesimal X_p. We, henceforth, denote infinitesimals, as usual, by $\mathbf{0}$, when p- infinitesimals are denoted as $\mathbf{0}_p$ ($\mathbf{0}$ is identified with $\mathbf{0_p}$) at the level of the (secondary) scale $\tilde{\delta} = 1/p$.

Example 3. *Let $a_{pn} = 1+\sum_1^n \alpha_i p^i$, where α_i assumes values from 0,1,.....(p-1). The sequence a_{pn} is divergent in R for each prime p. In the non-standard set $*R, \{a_{pn}\}$ denote a distinct infinitely large number for each p. p-adically, however, a_{pn} converges to the unity X_{pu} ($|X_{pu}|_p = 1$). The scale free unity $X_u \in \mathcal{S}_p$ now denotes the larger sequence $\{a_{pn}, \forall p\}$. At the level of secondary scale $\tilde{\delta} = 1/p$, unity X_u is realized as X_{pu}.*

Remark 15. By "hierarchical" we mean that as a scale free real variable $\tilde{x} = x/\delta^n, n \to \infty$, approaches 0 from the initial value 1 through the secondary scales $1/p$, $p \to \infty$ and $\delta = p^{-1}$, the ordinary real variable x ∈R would experience changes over various local fields Q_p successively by inversions.

Lemma 11. *Let $X_p \in Z_p$ and $X_q \in Z_q$, q being the immediate successor of the prime p. Then an infinitesimal $X \in \mathbf{0}$ when realized as a p- infinitesimal has the* *representation*

$$X = X_p(1 + X_q) \tag{3.12}$$

Further, a p−unit is given as $X_{pu} = 1 + X_p, |X_p| < 1$.

Let us fix the scale at $\delta = 1/p^n$. Then the proof follows from Lemma 5 and 6 . □

Corollary 5. *One also has the following adelic extension*

$$X = X_p \prod_{q>p}(1 + X_q) \tag{3.13}$$

where the product is over all the primes q greater than p.

Proof. This follows from the above Lemma when the valuation formula (3.5) is extended further

$$X/\delta^n = (\delta^n)^{(|X_i| + \sum_m \xi_m(X,\delta_m))} \tag{3.14}$$

where $|X| = |X_i|$, δ_m^{-1} $(m > 1)$ are primes greater than p, $\delta = 1/p$ and each of the indeterminate functions ξ_m satisfies conditions analogous to that in the formula in Lemma 6 . Further, ξ_q goes to zero faster than ξ_p if $q > p$. ■

Collecting together the above results, we have

Theorem 4. *The completion of Q under the absolute value* $||.||$ *yields a countable number of complete scale free models* $\mathbf{R}_p$ *of R, such that each element* $x \in \mathbf{R}_p$ *has the form* $x_p = x(1 + X_p \prod_{q>p}(1 + X_q))$, $x \in R$, $X_p \in Q_p$, *where* X_p *is given by the asymptotic expression* $X_p = (p^{-n})^{(1+|X_p|_p(1+\sigma(\eta)))}$ *and* $\eta = O(\delta)$ *is a real variable. Finally,* $\mathbf{R}_p$ *locally has the Cartesian product form* $\mathbf{R}_p = R \times Q_p \times \prod_{q>p} Q_q$.

The only missing element in the proof of the above is the completeness. Let us first fix a scale $\delta = 1/p^r, r > 0$. Let $\{a_n\}$ be a Cauchy sequence in $\mathbf{R}_p$. Then it is Cauchy either in p-adic metric or in the usual metric, finishing the proof. □

In the following chapter we discuss the nature of influences that the scale free non-archimedean extensions of R would have on the basic structure of R itself.

Chapter 4

APPLICATION TO NUMBER THEORY: PRIME NUMBER THEOREM

4.1 A Brief History

Analytic number theory is a branch of number theory that uses methods from mathematical analysis to solve problems about natural numbers [45, 47]. The modern study of analytic number theory may be said to have begun in the eighteenth century with Euler's proof of the divergence of the series of inverse primes $\sum \frac{1}{p} = \infty$ and later with Dirichlet's introduction of Dirichlet L-functions in the first half of the nineteenth century to give the first proof of Dirichlet's theorem on arithmetic progressions [45]. Another major interest in this subject is the Prime Number Theorem.

Here we shall focus mainly on the Prime Number Theorem (PNT).

Statement of Prime Number Theorem :

Let $\Pi(x)$ be the prime counting function that gives the number of primes less than or equal to x, for any positive real number x. For example, $\Pi(10) = 4$ because there are four prime numbers (2,3,5 and 7) less than or equal to 10. The PNT then states that the limit of the quotient of the two functions $\Pi(x)$ and $x/\ln(x)$ as x approaches infinity is 1, which is expressed by the formula $\lim_{x\to\infty} \frac{\Pi(x)}{x/\ln(x)} = 1$, known as the asymptotic law of distribution of prime numbers. Using asymptotic notation this result can be restated as $\Pi(x) \sim x/\ln(x)$.

This notation (and the theorem) does not say anything about the limit of difference of the two functions as x approaches infinity. Indeed, the behavior of this difference is very complicated and related to the Riemann hypothesis (RH) [47]. Instead, the theorem states that $x/\ln(x)$ approaches $\Pi(x)$ in the sense that the relative error of this approximation approaches 0 as x approaches infinity. According to the RH , the relative correction (error) should be given by $\Pi(x) \times (x/\ln(x)) = 1 + O(x^{-\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$. So far no proof of the PNT could retrieve and substantiate the RH correction term, although all the current experimental searches on primes are known to agree with the RH value [47].

The prime number theorem is equivalent to the statement that the nth prime number p_n is approximately equal to $n\ln(n)$, again with the relative error of this approximation approaching 0 as n approaches infinity.

Based on some deep results derived on 1859 by B. Riemann on the relationship of PNT and the complex zeros of the Riemann Zeta function, the first proof of the PNT was given independently by J Hadamard and de la Vallee Poussin on 1896 using methods of advanced theory of complex analysis. The first *elementary* proof of the PNT without using Complex analysis was obtained by A. Selberg and P. Erdos on 1949.

4.2 New Elementary Proof

4.2.1 Introduction

We present a new proof of the PNT [13]. We call it elementary because the proof does not require any advanced techniques from the analytic number theory and complex analysis. Although the level of presentation is truly elementary even in the standard of the elementary calculus, except for some basic properties of non-archimedean spaces [42, 48], some of the novel analytic structures that have been uncovered here seem to have significance not only in number theory but also in other areas of mathematics, for instance the non commutative geometry [49], infinite trees [50] and network [51] and emergence of nonlinear complex structures.

The proof of the PNT is derived on the scale invariant, non-archimedean model **R** of real number system R, involving non-trivial infinitesimals and infinities which have been introduced in the previous chapter. The model **R** is realized as a completion of the field of rational numbers Q under a new non-archimedean absolute value $||.||$, which treats arbitrarily small and large numbers separately from any finite number. The model constructed is distinct from the usual non standard models of R in two ways: (1) infinitesimals arise because of our nontrivial scale invariant treatments of small and large elements and so may be regarded members of R itself and (2) it is a completion of Q under the new absolute value. The so-called scale-invariant, infinitesimals are therefore modeled as p-adic integers X_i with $|X_i|_p < 1$, $|.|_p$ being the p-adic absolute value and is given by the adelic formula $X = X_p \prod_{q>p}(1 + Xq)$. By inversion, infinities are identified with a general p-adic number X with $|X|_p > 1$. The infinitesimals considered here are said to be *active* as the definition involves an asymptotic limit of the form $x \to 0^+$, thereby letting an infinitesimal *directed* i.e. having a direction. We show that as a consequence the value of a scale invariant infinitesimal X would undergo infinitely slow variations over p-adic local fields Q_p as a scale free real variable x^{-1}, called the *internal time variable*, approaches ∞ through the sequence of primes p. We show that these p-adic infinitesimals leaving in **R** conspire, via non trivial absolute values, to have an influence over the structure of the ordinary real number system R thereby extending it into an associated infinite dimensional Euclidean space $\mathcal{R}$, so that a finite real number r gets an infinitely small correction term given by $r_{cor} = r + \epsilon(x)||X||$, where $\epsilon(x^{-1}) = \log x^{-1}/x^{-1}$ is the inverse of the asymptotic PNT formula of the prime counting function $\Pi(x^{-1}) = \sum_{p<x^{-1}} 1$. In the ordinary analysis, there is no room for such an ϵ thus making the value of r exact, viz., $r_{cor} = r$.

The proof of the PNT in the present formulation is accomplished by proving that the value $||X||$ of a scale free infinitesimal actually corresponds to the prime counting function $\Pi(x^{-1})$ as the internal time x^{-1} approaches infinity through larger and larger scales denoted by primes p. To this end we consider an equivalent (infinite dimensional) extension $\mathcal{R}$ of R, in the usual metric topology, however, with a caveat that increments of a variable are mediated by a combination of *linear translations* *and inversions.* We show that there exist two types of inversions,, viz. the *global or growing mode* leading to an asymptotic finite order variation in the value of a dynamic variable of $\mathcal{R}$ following the asymptotic growth formula of the prime counting function. On the other hand, the *localized inversion mode* is shown to lead to an asymptotic (golden ratio) scaling to a directed (dynamic) infinitesimal and the relative correction to the PNT.

4.2.2 Dynamical Properties

To recapitulate, we note that the set R, in the presence of non-trivial scales, proliferates into the above p-adically induced extensions $\mathbf{R}_p$. We now investigate the converse question, "How do these field extensions influence the standard asymptotic behaviors in R?" Let us recall that the standard asymptotic behavior of an ordinary real variable x as it approaches 0 is that x *vanishes linearly* as $x \to 0$ (at the uniform rate 1). In the presence of non trivial scales the situation is altered significantly. The point 0 is now identified with the set $I_\delta = [-\delta, \delta], \delta = 1/p^r$, for some $r > 0$ inhabiting infinitesimals as in the Definition 1 of previous chapter. Corresponding to these infinitesimals there exists scale free (p)- infinitesimals $X_p = \lim \tilde{x}_n/\delta^n, n \to \infty$ with absolute values of the form $||X_p|| = |X_p| = |X_{rp}|_p(1 + \sigma(\eta))$ when we choose $\delta = p^{-r}$. Here, η is defined by the real variable x=$\delta(1 + \eta)$ approaching 0^+, viz., δ from the right. In the ordinary real analysis, δ =0 and the limiting value of x, viz., 0 is attained exactly. For a non zero δ, this exact value is attained by the rescaled variable $x_1 = x/\delta$, although the value attained is now 1 instead, i.e., $x_1 = x/\delta = 1$ (which also means equivalently $\log x_1 = 0$). We are thus still in the framework of the real analysis (and the computational models based on this analysis). The presence of infinitesimals of the form $\tilde{x}$ (in the conventional sense) does not appear to induce any new structure beyond those already existing in the system. The definition of scale free infinitesimals and the associated nonarchimedean absolute values now provide us with a new input.

Definition 11. *The scale free (p)- infinitesimals, in association with absolute values $|.|$, are called valued (scale-free) infinitesimals. These infinitesimals are also said to be "active" (directed), when infinitesimals of conventional non-standard models are inactive (or passive, non-directed).*

Remark 16. Because of scale free infinitesimals, the exact equality of R in the ordinary analysis is replaced by an approximate equality, for instance, the equality $x = 1$ is now reinterpreted as $x = O(1)$. In an associated non-archimedean realization $\mathbf{R_p}$ the exact equality is again realized, albeit, in the ultra metric absolute value viz., $||x|| = 1$. Further, the rescalings defined by the inversion rule (so that a variable x is replaced by the rescaled variable x/δ) accommodates also a *residual rescalings* (because of the nontrivial factor of the form $\delta^{\pm|X_p|}$ in the representations $x = \delta \times \delta^{-|X_p|}$ and $\tilde{x} = \lambda\delta \times \delta^{|X_p|}$ (c.f. Remark 7) when the absolute value of a valued infinitesimal X_p at the scale $\delta = 1/p^{-n}$ is given by $|X_p| = |X_{np}|_p(1 + \sigma(x))$, $|X_{np}|_p = p^{-n}$).

Influence of infinitesimals on R: We consider a class of infinitesimals $\tilde{\tilde{x}}$ as defined by $0 < \tilde{\tilde{x}} < \epsilon < \tilde{x} < \delta < x$ (ϵ is determined shortly). We show that valued infinitesimals from $(0, \epsilon]$ would affect the *ordinary value* of x non-trivially. Infinitesimals in (ϵ, δ) are (relatively) passive. As stated above, in ordinary analysis, the limit $x \to 0^+$ in the presence of a scale δ is *evaluated exactly* viz., $x_1 = x/\delta = 1$ i.e., $\log x_1 = 0 := O(\delta)$. Infinitesimals, in conventional scenario, are *passive* in the sense that their values remain always infinitesimally small in any linear process (or dynamical problem). In the present case, however, the numerically small (in the ordinary Euclidean sense) infinitesimals lying closer to 0 relative to δ, are *dominantly valued*, so as to induce a *nontrivial influence* over a finite real variable (number), because of the definitions of valued infinitesimals.

To state formally, we reinterpret *the concerned effect* of the valuation (in the sense of an absolute value), in the context of ordinary analysis, as one admitting an extension of the ordinary (positive) real line from (δ, ∞) to the larger set $(\epsilon, \delta] \cup (\delta, \infty)$ so that ordinary zero is now identified as $[0, \epsilon]$, where the value of ϵ is defined by

$\epsilon = \tilde{\tilde{x}}_1^{-1} \delta \log \delta^{-1}$ for an O(1) rescaled (renormalized) variable $\tilde{\tilde{x}}_1 = |\tilde{\tilde{x}}| = \delta^{-\alpha} > 1$ for $\alpha > 0$ (c.f. Remark 7).

Indeed, from Definitions 1,2 and Remarks 5 & 6 , a variable x approaching 0^+ is replaced by the dressed representation $x = \delta\delta^{-|\tilde{\tilde{x}}|}(1 + o(1))$ for an infinitesimal $\tilde{\tilde{x}} = \lambda\delta\delta^{|\tilde{\tilde{x}}|}(1 + o(1))$, so that the classical limit $x/\delta \approx 1$ is replaced by $x \approx \delta + |\tilde{\tilde{x}}|\delta \log \delta^{-1}$, as $\delta \to 0^+$. The presence of the extra logarithmic term now facilitates the above mentioned extension. Infact, we notice from Remark 7 that the absolute value of an infinitesimal of the form $\tilde{\tilde{x}}$ can exceed 1 for a negative valuation i.e. $|\tilde{\tilde{x}}| = \delta^{-\alpha} > 1$ for $\alpha > 0$ which is attained by a sufficiently small infinitesimal $\tilde{\tilde{x}} << \delta$ relative to the scale δ, so that the size of the linear neighborhood $[0, \delta]$ gets extended to the level $[\delta, \tilde{\epsilon}]$ for an $\tilde{\epsilon} = |\tilde{\tilde{x}}|\delta \log \delta^{-1}$. Defining $\epsilon = \tilde{\tilde{x}}_1^{-1} \delta \log \delta^{-1} < \delta$ proves the desired assertion when α is sufficiently large.

As a consequence, we shall now have $\log x_1 = O(\epsilon)$ improving the classical value $\log x_1 = O(\delta)$. In the language of a *computational model, the accuracy of the model is therefore increased to the level denoted by* ϵ. So, from Remark 7 we can write

$$x\tilde{\tilde{x}}_1 = x(|X_r|_p(1 + \sigma_p(\eta)) + o(1)) = O(\delta^2)$$

since both x and $|\tilde{\tilde{x}}| \sim O(\delta)$. Thus, fixing $r = n$, so that $|X_n|_p = p^{-n} = \delta$, we get the first correction

$$(1 + \eta)(1 + \sigma_p(\eta)) = O(1) \tag{4.1}$$

to the ordinary (classical) value of $x_1 = 1 + \eta$ from (p) infinitesimals, even for a fixed value of η.

We note that $\tilde{x}_{1p} := 1 + \sigma_p(\eta)$ (c.f. notation in Remark 7) and σ_p >0 must be of higher degree in the real variable η for a $\tilde{x}_p \in \mathbf{R}_p$. Eq.(4.1) is *interpreted as one encoding the influence of the (first order) (p) infinitesimals.* Taking into account successively the higher order (p) infinitesimals, and iterating the above steps on each rescaled variables $\tilde{x}_{1p} = O(1)$, one obtains $\log(1 + \sigma_p(\eta)) = O(\tilde{\epsilon})$, where $\tilde{\epsilon} = \tilde{x}_2 \times \tilde{\delta} \log \tilde{\delta}^{-1}$, $\tilde{\delta} = q^{-n}$, q being the immediate successor to the prime p, and so on, so that we get finally an extended version of the equality $x_1 = O(1) \in R$ when the effective influence of infinitesimals $\tilde{x}$ living in $\mathbf{R}$ on R is encoded as

$$\mathcal{X}(\eta) := (1+\eta)\prod_{q\geq p}(1+\sigma_q(\eta))(=O(1)) \tag{4.2}$$

The variable $x = (1+\eta) \in R$ is thus replaced by the modified variable $\mathcal{X} \in \mathcal{R}$ (where $\mathcal{R}$ is the infinite dimensional Euclidean (Archimedean) extension of R) and hence, in this extended framework, a solution of

$$x d\tilde{x}/dx = -\tilde{x} \tag{4.3}$$

is written, for a $x > 1$, as $0 < \tilde{x}(x) = (\mathcal{X}(\eta))^{-1} < 1$, which belongs to the class of nonsmooth solutions of Eq. (3.1) [15, 18]. Here, $\sigma_p' s$ take care of the residual rescalings of Ref.[15], and thereby introduce small scale variations in the value of η.

Remark 17. It is important to note that the above derivation is performed purely in the framework of the ordinary analysis, except for the fact that we make use of the special representations of x and $\tilde{x}$ as induced from the definitions of valued infinitesimals. Consequently, (i) the transitions between real and infinitesimals are interpreted as being facilitated by inversions (for instance, either as $x_- \mapsto x_-^{-1} = \tilde{x}_+$, or as $x_+ \mapsto \tilde{x}_- = x_+^{-1}$, as the case may be) as opposed to linear shifting operations only, and (ii) the non-archimedean p-adic absolute value $|X_r|_p$ generates a scale factor in the smaller scale logarithmic variables. In fact, this correspondence could be made more precise.

Proposition 12. 1. **[43, pages 14,15]** *Let $X_p \in Q_p$, so that $X_p = p^r(1+\sum_1^\infty a_i p^i)$, where a_i assumes values from $1, 2, \dots (p-1)$ and $r \in Z$. Then there exists a one to one continuous mapping $\phi : Q_p \to R_+$ given by $\phi(X_p) = p^{-r}(1+\sum a_i p^{-2i})$.*

2. **[44, pages 63-65]** *The set of p-adic integers Z_p is homeomorphic to a Cantor set C_p under the homeomorphism $\psi : Z_p \to C_p$ defined by*

$$\psi(X_p) = (2p-1)^{-r}(1+\textstyle\sum_1^\infty \frac{2a_i}{(2p-1)^{i+1}})$$

where $r > 0$.

We denote $R_p = \phi(Q_p)$. It then follows that any bounded subset of R_p is a zero (Lebesgue) measure Cantor set C_p in R. Accordingly, the treatments of ordinary analysis can be extended in a scale free manner (though remaining in the framework of the usual topology of R) over a *more general metric space* $\mathcal{R}$ accommodating the above new structure. In view of Theorem 4 of previous chapter, $\mathcal{R}$ is locally a Cartesian product, viz.,$\mathcal{R}$=$R \times \prod_p R_p$.

The product space is interpreted as an hierarchical sense. An ordinary real variable x is extended over $\mathcal{R}$ as $\mathcal{X}$=$x \prod x_P$ where $x_p = 1 + \epsilon_p X_p$, $X_p \in R_P$ and $\epsilon_P \approx \delta p^{-1} \log(p\delta^{-1})$ denotes the enhanced level of accuracy because of valued infinitesimals $\tilde{x} \in \mathbf{R}$ at the scale $\tilde{\delta} = \delta p^{-1}, \delta \to 0^+$. Noting that $\epsilon_q = p/q \times \epsilon_p \to 0$, as $\delta \to 0^+$, for $q > p$, one may re-express $\mathcal{X}$ as $\mathcal{X}$=$x(1 + \epsilon X)$, where $\epsilon \approx \delta \log \delta^{-1}$ when the scale is identified with $\delta = 2^{-(n-1)}$ so that $p = 2$, and the scale free infinitesimal X now resides and varies in $\prod R_p$ in an orderly (hierarchical) manner as detailed below, as $x \to 0^+ \equiv O(\delta)$.

Let us first recall that the concept of relative infinitesimals is introduced originally in the context of a Cantor set [16, 17]. Because of the existence of relative infinitesimals, each element of a Cantor set, denoted here as C_p, for each class of (p) infinitesimals, is effectively identified with a closed and bounded interval of R_p at every level of the scale δ, as $\delta = 1/p \to 0$. In the presence of infinitesimals a Cantor set is thus realized as a compact subset of R_p except for the fact that the *motion* on C_p is now *visualized as a combination of linear shift (along a compact and connected line segment) together with an inversion in the vicinity of a Cantor point.* As a consequence, the generalized, inversion mediated metric space $\mathcal{R}$ is denoted locally as $\mathcal{R}$=$R \times \prod R_p$. A *generalized motion on* $\mathcal{R}$ *now is represented as follows.*

Definition 12. *The set* $\mathcal{R}$ *is interpreted as having several branches* R *and* R_p*,* p *being a prime. The branches* R_p *accommodating scale free Euclidean (Archimedean) infinitesimals and infinites are thought to be knotted at (the scale free number) 1. A real variable* $x \in R$ *approaching* 0 $(\equiv O(\delta))$ *from 1(say) is replaced, because of the* *scale invariance, by the scale free variable* $x_1 = x/\delta$. *For simplicity of notation we continue to denote* x_1 *by* x *and call it a scale free variable. So, as the scale free* $x \to 1^+$, *the unique linear motion is replaced by two* **inversion mediated modes***:* *(i) Local or vertical mode:* x_+ *is replaced by* $x_+ \mapsto x_+^{-1} = \tilde{x}_-$ *which takes note of the* **localized** *effects of (Euclidean) infinities on an (Euclidean) infinitesimal and (ii) global or horizontal (growing) mode: a scale free (Euclidean) infinitesimal* $0 < \tilde{x} \in R_p$ *grows infinitely slowly following the linear law until it shifts to a O(1) variable living possibly in another branch* R_q *by inversion* $\tilde{x}_- \mapsto \tilde{x}_-^{-1} = x_+ = 1 + \tilde{\tilde{x}}$ *where* $0 \leq \tilde{\tilde{x}} \in R_q$.

Remark 18. We disregard henceforth distinguishing Euclidean (Archimedean) and Non-Archimedean infinitesimals and infinities explicitly in notations. We indicate the difference whenever there is a room for confusion. The basic characteristic of an Euclidean infinitesimal is the explicit presence of a logarithmic factor.

Remark 19. The unidirectional motion of a real variable $x \in R$ approaching 0^+ thus bifurcates into two possible modes: a variable x approaching 0 ($\equiv O(\delta)$) from above will experience, as it were, a bounce at $x \approx \delta$ and so would get replaced by the inverted rescaled infinitesimal variable $\tilde{x} = \delta/x$ living in a scale free branch R_p (say). A fraction of the asymptotic limit $x \to 0^+ \equiv O(\delta)$ of R (viz., $x \mapsto x_+ = x/\delta = 1 + \eta, \eta \to 0^+$) is therefore replaced by a growing mode of the rescaled variable $\tilde{x}_-^{-1} = x_{p^+} = 1 + \eta_p, x_{p^+}, \eta_p \in R_P$ and $\eta_p \geq 0$ initially but subsequently growing to $\to 1^-$ in the branch R_p. Besides this growing mode, another fraction of the decreasing (decaying) mode of the flow in R, viz., $x \mapsto x_+ = x/\delta = 1+\eta, \eta \to 0^+$, is also available as a localized mode in another branch R_q (say) in the form $x_+^{-1} = (1 + \eta_q)^{-1} = \tilde{x}_{q^-}$, where again $\eta_q \approx 0$ initially, but grows subsequently to $\eta_q \to 1^-$ slowly. As a consequence, the limiting value 1 of the rescaled variable in R, now, gets a dynamic (multiplicative) partitioning of the form $\tilde{x}_{q^-} x_{p^+} \approx 1 \Rightarrow (1 - \mu(\eta_q)\eta_q)(1 + \eta_p) = O(1), 0 \leq \eta_p \in R_p,\ 0 \leq \eta_q \in R_q$, which equivalently can also be written more conveniently as $(1 - x^{\eta_q})(1 + \eta_p) = O(1)$. The local and global modes in $\mathcal{R}$ therefore induce, as it were, a competition between the effects generated by infinitesimals and infinities living in $\mathbf{R}$, leading to this dynamic partitioning of the unity. The localized factor $(1 - x^{\tilde{X}})$ arising from active infinitesimals $\tilde{X}$ living in a branch of $\mathcal{R}$ will lead to an asymptotic scaling of any scale free (locally constant) variable $\tilde{\mathcal{X}} = \mathcal{X}/x$ in $\mathcal{R}$ (see Sec. Scaling(4.2.4)). Accordingly, $d\tilde{\mathcal{X}}/dx = 0$ and hence $\tilde{X} \in \prod R_p$ satisfies the scale free equation

$$\log x \frac{d\tilde{X}}{d\log x} = -\tilde{X} \tag{4.4}$$

Consequently, asymptotic limits either of the forms $x \to 0^+$ or $x \to \infty$, in R would ultimately behave as a *directed* (monotonically increasing) variable in $\mathcal{R}$. Moreover, as $x^{-1} \to \infty$, the ordinary linear motion of $x \to 0$ will undergo small scale mutations, because of zigzag motion of the inverted variables $\tilde{x}_p(\sim O(1))$, living successively in the rescaled branches R_p and mapping recursively the smaller and smaller neighborhoods of 0 to the smaller and smaller neighborhoods of 1. As a consequence, extended real numbers $\mathcal{X}$ of $\mathcal{R}$ are *directed*, since each of the (p) infinitesimals are, by definition, directed.

Definition 13. Intrinsic (Internal) Time: *A continuous monotonically increasing variable $\tilde{x}$ living in the product space $\prod R_p$, from the initial value 1, is called an internal evolutionary time. The rate of variations of $\tilde{x}$ is infinitely small because of the presence of scale factors of the form $\delta p^{-1}, \delta \to 0^+$.*

Any variable $X \in \prod R_p$ is called dynamic since it undergoes spontaneous changes (mutations) relative to the (scale free) internal time $\tilde{x}$.

With this dynamic interpretation of $\mathcal{R}$, it now follows that the new solution constructed in Eq.(4.2) is indeed smooth in $\mathcal{R}$ (as it is evident from the derivation). However, because of the presence of the irreducible O(1) correction factors this solution can not be accommodated in the ordinary analysis (i.e., even in the context of $\mathcal{R}$) in an exact sense. In the non-archimedean extensions $\mathbf{R}_p$, such a solution is not only admissible and smooth but also exact, in the sense of absolute values, viz., $||X|| = 1$, since $||x|| = ||\tilde{x}_i|| = 1$ for each i, thus retrieving the ordinary equality $|x_i|_e = 1$ in the ultra metric sense.

We restate the above deductions as the following Lemma.

Lemma 12. *The ordinary analysis on R is extended over $\mathcal{R}$ with new structures as detailed above (Definition 12, Remark 19). In this extended formalism accommodating dynamic infinitesimals, $0 \in R^+$ (the set of positive reals) is identified with $[0, \epsilon], \epsilon = x_1^{-1}.\delta \log \delta^{-1}$, where $x_1 = x/\delta$ and $x \to \delta^+$. As a consequence, a constant in R becomes a variable over infinitesimals of $\mathcal{R}$.*

Proof. We have already seen in the above that $0 \in R^+$ is extended over $[0, \epsilon]$ in $\mathcal{R}$. We justify it further by showing that an equation of the form

$$\frac{d\phi}{dx} = 0 \tag{4.5}$$

for finite real values of x is transformed into

$$\frac{d \log \phi}{d \log x} = O(1) \tag{4.6}$$

for a relative infinitesimal $\tilde{x}$ satisfying $x/\delta = \lambda\delta/\tilde{x} = \delta^{-||\tilde{x}||}, 0 < \tilde{x} < \delta \leq x, x \to 0^+, \lambda > 0$ and $||\tilde{x}|| = \tilde{x}_1 (= |\tilde{x}|_p(1+\sigma(\eta)))$, when one interprets 0 in relation to the scale δ as $O(\frac{\delta^2}{x} \log \delta^{-1})$. Indeed, we first notice that Eq.(4.5) means $d\phi = 0 = O(\delta), dx \neq 0$ for an ordinary real variable. However, as $x \to \delta$, that is, as $dx = \eta \to 0 = O(\epsilon)$ (i.e. η is an infinitesimal in the present sense lying in $[0, \epsilon]$), the ordinary variable x gets replaced by the above extended variable, so that $d \log_{\delta^{-1}}(x/\delta) = d\tilde{x}_1 = O(\epsilon)$. As a consequence, in the infinitesimal neighborhood of 0, Eq. (4.5) is transformed into an equation of the form Eq. (4.6); since in that neighborhood, x and ϕ, are both represented as $x = \delta.\delta^{-\tilde{x}_1}$ and $\phi = \phi_0 \delta^{k\tilde{x}_1}$ for a real constant k, whence we get Eq.(4.6).

■

Before proving the PNT in the present dynamic extension of R, we need two more ingredients: viz., the origin of the prime counting function and the asymptotic scaling of active infinitesimals.

4.2.3 Prime Counting Function

The prime counting function arises in connection with the growing mode of a dynamic variable in $\mathcal{R}$, when we *assume* that a scale free variable x varies over all possible prime-adic branches R_p in an *orderly manner following the order of the primes.*

Recall that the usual $\epsilon - \delta$ definition of limit (in R) does not characterize explicitly the actual motion of a real variable x (in fact, it is taken in granted that x varies in uniform rate 1) approaching a fixed number, 0, say. In the present formalism infinitesimals are defined by *intermediate* asymptotic scaling formulas (which would ordinarily correspond simply to zero), and so may be considered to carry *an evolutionary arrow.* A limit of the form $x \to 0$ in $\mathcal{R}$ would be interpreted in the context of a dynamical problem, so that x^{-1} may be identified with the (physical) time. More precisely, when the ordinary R component of $\mathcal{R}$=R$\times \prod R_p$ is free of any arrow, the *non-trivial components* R_p do *carry an evolutionary arrow.* A problem involving the asymptotic limit $x \to 0^+$ (equivalently $x \to \infty$) in ordinary analysis is raised in the present context over $\mathcal{R}$ as the asymptotic limit of the extended variable $\mathcal{X}$=$x(1+\epsilon X)$ where X is an O(1) growing dynamic variable which lives hierarchically in the sets R_p, $p = 2, 3, 5, \ldots$ as explained in Definition 12 and Remark 19 (a growing dynamic infinitesimal η is represented now as $\eta = \epsilon X$).

Thus *the ordinary limit of* $\mathcal{X}$ *as* $x \to 0^+$, *that is,* $\mathcal{X} = 0$, *is interpreted in the present context as* $\log(\mathcal{X}/x) = \log(1 + \epsilon X) = O(\epsilon\Pi(x^{-1}))$ *as* $x \to 0^+$ *hierarchically through scales* δp^{-1}.

In fact, as pointed out above (in Remark 19), as $x \to \delta^+$, it changes over to various branches R_p by assuming the guise of several variables $\tilde{x}_i's$ (all of which are different R_p valued realizations of X) having the forms $\tilde{x}_i = 1 + \eta_i, x_0 \equiv x$ and i runs over the primes. Consequently, as $\tilde{x}_2 = (\lambda_1)x_1^{-1} \in R_2$, $x_1 = x/\delta$ approaches $1/2^-$, we get the next level variable $\tilde{x}_3 = (2\lambda_2)\tilde{x}_2^{-1} \in R_3$ and so on and so forth, adding one unit to the prime counting function Π at every change of the prime-adic scale. Indeed, infinitesimal η_i grows linearly (and spontaneously) to O(1) whence it undergoes inversion mediated transition of the form $\eta_{i-} \mapsto \eta_{i-}^{-1} = 1 + \eta_j$, where j being the next prime. As a consequence, we have

Theorem 5. *The ordinary (linear) limiting behavior of a real variable* $x \to 0^+$ *in the real number set* R *is raised, in the inversion mediated set* $\mathcal{R}$, *to the asymptotic limit of the extended variable* $\mathcal{X}=x(1+\epsilon X)$, *where the O(1) dynamic variable* X *is realized in relation to every secondary (prime-adic) scale* $1/p$ *as a variable of the form* $x_p \geq 1$. *As a consequence, the asymptotic limit of* $\mathcal{X} = X/x$ *as* X $(\equiv x_p) \to 1/p, p \to \infty$ *is given by* $\log(\mathcal{X}/x) = O(\epsilon\Pi(x^{-1}))$, *for a locally constant infinitesimal* $\epsilon = O(\delta \log \delta^{-1}) = O(x \log x^{-1})$, *when the real variable* $x^{-1} \to \infty$

In the next section we consider the scaling of $\mathcal{X}(x) \in \mathcal{R}$ as $x \to 0$ in R.

4.2.4 Scaling

Let us begin by recalling that the main characteristic of both the inverted motions is the inherent directed sense. That is to say, although $x_{1+} = 1 + \eta, \eta \downarrow 0^+$ in R, in either of the inverted motions, we have however, $x_{1+} = 1 + \tilde{\eta}, \tilde{\eta} \approx 0$, initially, but $\tilde{\eta} \uparrow 1^-$, slowly, when $x_{1+} \in R_p$. As shown in the above sections, the growing mode induces the global evolutionary sense leading to the prime counting function. Here we study the local motion leading to the asymptotic scaling for a small scale variable $\tilde{\mathcal{X}} \in \mathcal{R}$.

Because of the valued infinitesimals in **R** that contribute non-trivially to the ordinary value of an arbitrarily small $x \in R$, the scaling behavior of the corresponding extended variable $\mathcal{X} \in \mathcal{R}$ is also nontrivial. As explained above, an ordinary, arbitrarily small $x \in R$ is extended in $\mathcal{R}$ as $\mathcal{X}/x = (1 - O(x^{\tilde{X}(x^{-1})}))\phi(x^{-1})$, for a class of (Euclidean) infinitesimals $\tilde{X}(x^{-1})$. Our aim here is to estimate $\lim \tilde{X}$ as $x \to 0$. Notice that the (-) sign in the first factor makes it a true dynamic infinitesimal living in R_p. The second factor ϕ corresponds to the growing mode of a dynamic infinitesimal and is considered in Theorem 5.

We recall that the above limit may have a constant (non zero)(ultra metric) value. Indeed, as x approaches 0, following δ, the ordinary variable x gets extended to the rescaled variable $\tilde{\mathcal{X}}_- = \mathcal{X}_-/x = 1 - O(x^{\tilde{X}})$, which now approaches 0^+, via a combination of inversions and translations. Indeed, as $x \to 0$ in R, $\tilde{\mathcal{X}}_-$ in $\mathcal{R}$ is realized as a locally constant function satisfying $d\tilde{\mathcal{X}}_-/dx = 0$ so that $\tilde{X} \in \prod R_p$ now satisfies the Eq.(4.4), i.e.

$$\log x \frac{d\tilde{X}}{d\log x} = -\tilde{X} \tag{4.7}$$

and changes from one copy of R_p to another near the scale $1/p$ by inversions via a sequence of distinct realizations $\tilde{x}_i$, i being a prime. To see in detail, let $\tilde{\eta}_- = x^{X(x^{-1})}$. As $x \to 0 \equiv O(\delta)$ linearly and the motion should have terminated at δ in R, now, instead is picked up by the rescaled variable which shifts by inversion to $\tilde{\eta}_{2-} = x^{2\tilde{x}_2}, \tilde{x}_2 \approx 0$. The limiting motion is now transmitted over to the next generation variable $\tilde{x}_2 \in R_2$, which grows to $1/2^-$ linearly, until the motion is again transferred to the next level by inversion viz., $2\tilde{x}_2 = 1/(1+3\tilde{x}_3)$, where $\tilde{x}_3(\approx 0) \in R_3$. Recall that this (and the following) local inversions essentially inject into an infinitesimal *higher order influences from infinities.* The new rescaled variable $\tilde{x}_3$ now grows to $1/3^-$ and transmits its motion to $\tilde{x}_5 \in R_5$ near $\tilde{x}_5 \to 1/5^-$ by inversion, and so on successively over all the higher prime-adic scales. The exponent in $\tilde{\eta}_-$ now asymptotically assumes the form of the *golden ratio continued fraction*, i.e., $\tilde{\eta}_{\infty-} = x^{\frac{1}{1+\frac{1}{1+\dots}}}$, so that the exponent has the value $\nu = \frac{1}{1+\frac{1}{1+\dots}} = \frac{\sqrt{5}-1}{2}$ and therefore $\tilde{\eta}_{\infty-} = x^{\nu}$. As a consequence, the asymptotic small scale variations (mutations) in the dynamic infinitesimal follow a generic golden ratio scaling exponent [20].

Combining this local asymptotic scaling together with the global asymptotic of Theorem 5, one finally arrives at the asymptotic law.

Theorem 6. *The generic asymptotic behavior of a dynamic variable $\mathcal{X} \in \mathcal{R}$, extending the ordinary real variable x, is given by*

$$\log \mathcal{X}/x = \epsilon O(\Pi(x^{-1}))(1 - O(x^{\nu})) \tag{4.8}$$

as $x^{-1} \to \infty$.

The above asymptotic formula is the main result of this Chapter. Over any finite (time) x scale, the right hand side effectively reduces to zero, recovering the standard variable $\mathcal{X} = x$. However, in any dynamic process which persists over many (infinitely large) (time) scales, the correction factor may become significant leading to a finite observable correction to the evolving quantity $\mathcal{X} = xe^{O(1)}$ which may arise from the annihilation (cancellation) of the infinitesimal (locally constant variable) ϵ by the growing mode of the prime counting function. The proof of the PNT now follows as a corollary to the Theorem 6.

4.2.5 Prime Number Theorem

The locally constant infinitesimal $\epsilon(x^{-1}) = O(x \log x^{-1})$ clearly corresponds to the inverse of the PNT asymptotic formula for the prime counting function $\Pi(x^{-1})$. The O(1) correction to any dynamic variable $\mathcal{X} \in \mathcal{R}$ is realized for a sufficiently large value of x^{-1} provided

$$\epsilon(x^{-1})\Pi(x^{-1}) = (1 + O(x^{\nu})), x^{-1} \to \infty \tag{4.9}$$

with the relative correction (error) $\Pi(x^{-1})\epsilon(x^{-1}) - 1 = O(x^{\nu})$, which clearly respects the Riemann's hypothesis since $x^{\nu} \leq Mx^{(1/2-\sigma)}$ for a suitable M >0 and for any $\sigma > 0$, $x \to 0^{+}$. $\square$

This completes the derivation of the PNT on a deformed real number system $\mathcal{R}$ accommodating scale invariant infinitesimals and the inversion induced nonlinear jump modes for infinitesimal increments. We close this section with another application of the scale invariant formalism to the prime counting function $\Pi(x)$. We recall that $\Pi(x)$ has by definition the structure of an irregular step function (i.e. a devil's straicase function).

Proposition 13. *The prime counting function* $\Pi(x)$ *is a locally constant function on* $\mathcal{R}$.

To prove this, let us first consider the step function

$$f(x) = \begin{cases} a, & 0 < x < p, \\ b, & x > p \end{cases} \tag{4.10}$$

with a finite discontinuity at $x = p$ in the usual sense. In the present scale invariant formalism with inversion mode for increments, we now show that f solves $x\frac{df}{dx} = 0$ every where, that is, even at $x = p$. As x increases toward p from the left linearly, the graph of f is a straight line parallel to the x-axis. In the left neighborhood of p, $x = p - \eta = px_-$, $x_- = 1 - \eta/p$. Analogously, in the right neighborhood, we have $x = p + \tilde{\eta} = px_+$, $x_+ = 1 + \tilde{\eta}/p$, so that $x_+ = x_-^{-1}$. Let us assume that η and $\tilde{\eta}$ are sufficiently small, so that the point set $\{p\}$ is identified with the closed interval $I_p = [1 - \eta/p, 1 + \tilde{\eta}/p]$, and so defines the accuracy level of a given computational problem. The interval I_p corresponds to an infinitesimally small neighborhood of p. At the level of this infinitesimal scale, the function f is interpolated by the scale invariant formula

$$\tilde{f}(x) = \begin{cases} a, & 0 < x < px_-, \\ a + (b-a)\phi_p(x), & x \in pI_p, \\ b, & px_+ < x. \end{cases} \tag{4.11}$$

where $x = (p - \eta) + (\tilde{\eta} + \eta)\tilde{x}$, $0 \le \tilde{x} \le 1$. Clearly, $\tilde{f}(x) = f(x)$, in the limit η, $\tilde{\eta} \to 0$. Moreover, $x\frac{d\tilde{f}}{dx} = 0$ everywhere, including $x = p$, since the locally constant Cantor function $\phi_p(x)$ on I_p does. It follows, therefore, that as x approaches to p from left and arrives at a point of the form $x = px_-$, it switches smoothly to $x = px_+$ at the right of p by inversion $x_-^{-1} = x_+$. The associated value of the function f i.e. a, however, changes over to b by a cascade of smaller scale self similar smooth jumps as represented by the Cantor function $\phi_p(x)$. The cost of this smoothness, however, is the arbitrariness in the formalism that is introduced via the arbitrariness of the choice of the Cantor function.

The prime counting function $\Pi(x)$ is a step function in the neighborhood of every prime. Hence the result. $\square$

Chapter 5

APPLICATION TO A CANTOR SET

5.1 Introduction

In this Chapter, we present a few new results on a Cantor set [19] exposing the precise nature of variability of a nontrivial valuation and hence of a Cantor function. This also constitutes an application of the scale invariant analysis on a Cantor subset of R. In Ref.[16, 17], the formulation of the scale invariant analysis on a Cantor set $C \subset [0,1]$ using the concepts of relative infinitesimals and the associated ultra metric norm are considered in detail (but not included in the present thesis). Here, we discuss in particular how an ordinary limiting variation $R \ni x \to 0$ is extended to a sub linear variation $x \log x^{-1} \to 0$ when $x \in C \subset [0,1]$. Finally, we derive the differential measure on a Cantor set C.

5.2 Cantor Set: New Results

It follows from the results presented in Chapters 3 and Chapter 4 that, because of scale invariant influence of relative infinitesimals, a nonzero real variable $x(> 0$ say,) in R and approaching 0 now gets a pair of *deformed* structures living in an associated *deformed real number system* $\mathcal{R} \supset R$ of the form $\mathcal{R} \ni X_{\pm}(x) = x \times x^{\mp v(\tilde{x}(x))}$, thus mimicking nontrivial effects of dynamic infinitesimals over the structure of the real number system R. Above ansatz works even for a finite x (> 0) $\in R$ when we express the above deformed representation in the form

$$X = x \times x^{\mp v(\tilde{x}(x'))} \tag{5.1}$$

for a set of infinitesimals $\tilde{x}$ living in $\mathbf{0}$. Here, x' denotes a positive real variable approaching 0^+ since an infinitesimal in the present formalism is defined relative to a limiting real variable. Clearly, $X = x \in R$ when $v = 0$ and hence $R \subset \mathcal{R}$. The corresponding scale invariant components $\tilde{X}_\pm := X_\pm/x$ now live, by definition, in a Cantor set (since by Theorem 3, Chapter 3, $v(\tilde{x})$ behaves as a Cantor function) and hence undergo changes by inversions of the form $X_+ = X_-^{-1}$ (c.f. Chapter 3).

We further recall that a given Cantor subset C of $I = [a, b] \subset R$ enjoys a ultra metric structure which is equivalent to the subspace topology inherited from the usual topology of R. In Ref.[14], it is, however, shown that the ultra metric valuation $v(\tilde{x})$ defined by Definition 2 of Chapter 3 is both metrically and topologically inequivalent to the usual ultra metric that a Cantor set carries naturally [13]. For a point x_0 in the Cantor set C, the representation Eq(5.1) now gives rise to a scale invariant ultra metric extension $\tilde{X}_\pm = X_\pm/x_0 = x_0^{\mp v(\tilde{x})}$ where the transition between two infinitesimally close scale invariant neighbors is mediated by more general inversions of the form $\tilde{X} \to \tilde{X}^{-h}$ for a real h, which *determines the jump size.* Notice that $\tilde{X}$ (and equivalently, $v(\tilde{x})$) is a locally constant Cantor function and solves $\frac{d\tilde{X}}{dx} = 0$ everywhere in I. The ordinary discontinuity of a Cantor function at an $x_0 \in C$ is removed, since in the present ultra metric extension, the point x_0 in C is replaced by an *inverted Cantor set* which is the closure of *gaps* of an infinitesimal Cantor set C_i that is assumed to be the residence Cantor set for the relevant infinitesimals $\tilde{x}$ living in the extended neighborhood $\mathbf{0}$ of 0. The gaps of C_i constitute a disjoint family of connected clopen intervals (represented in a scale invariant manner) over each of which scale invariant equation (3.1) of Chapter 3 are well defined [13]. Consequently, the valuation $v(\tilde{x})$, redefined slightly in the modified form

$$(x/x_0)^{\tilde{v}(x)} = x_0^{v(\tilde{x})} \tag{5.2}$$

(that is, $\tilde{v}(x)/v(\tilde{x}) = \log x_0 / \log(x/x_0)$, exposing the relative variation of $\tilde{v}$ over v), x assuming values from the gaps in the neighborhood of x_0, is realized as a smooth function defined recursively in a scale invariant way by the equation

$$\frac{d\tilde{v}(x)}{d\xi} = -\tilde{v}(x) \tag{5.3}$$

where $\xi = \log\log(x/x_0)$, $x \in C_i$. Recall that $\tilde{x}$ resides in the gaps of nontrivial neighborhood of 0 instead. As a consequence, $\tilde{v}$ may be written as $\tilde{v}(x) = (\log(x/x_0)^k)^{-1}$, where k may be allowed to assume values from a set of scale factors related to that of the Cantor set. This form is clearly consistent with (5.2). Assuming x is drawn from a specific gap of a given size, the same, written more effectively as $\tilde{v}(x)/v(\tilde{x}) = (\log_{x_0}(x/x_0))^{-1}$, yields, in the limit of vanishingly small gaps (i.e., as $x \to x_0$ and vice versa), the limiting value $\tilde{v}_0(x)/v_0(\tilde{x}) = 1/s$, since $\lim \log_{x_0}(x/x_0) = s$ *equals the finite Hausdorff dimension of the Cantor set* C_i. Let us first note that if one replaces the Cantor set by a segment of a line of the form $(0, \delta)$, then $x/x_0 = 1$, in that limit ($\delta \to 0$) gives $s = 0$, which is consistent with the fact that the line segment reduces to a point, viz. 0 in the said limit. In the general case, $x/x_0 \propto N$, the number of clopen balls that covers the fattened gap of the form $(x, x_0) \subset (0, \delta)$ (size of balls are determined by the gap). Letting $x_0 \to 0$ (following the relevant scale factors $\beta^n \to 0$), the above limit therefore mimics the box dimension, which also equals the Hausdorff dimension of the Cantor set concerned. The topological in equivalence of the present ultra metric arises from the possible dichotomy in the choice of C_i.

5.3 *Limit on a Cantor set*

Let us now show that when the ordinary 0 of R is replaced by an infinitesimal Cantor set C_i, the ordinary limit $\epsilon \to 0$ in R is altered. This follows because of scale invariant dynamic infinitesimals with valuations given by $\log \tilde{x}/\epsilon \approx v(\tilde{x}) \log \epsilon^{-1} \approx \epsilon \log \epsilon^{-1}$, when the relative infinitesimal $\tilde{x}$ is considered to lie on a fattened (connected) gap, so that the ultra metric valuation may be assumed to coincide with the usual (Euclidean) value viz. $v(\tilde{x}) \approx \epsilon$. However, assuming $\tilde{\epsilon}$ $(= \beta^n,\ n \to \infty)$ to be an infinitesimal scale of the Cantor set concerned, we also have $\log \tilde{x}/\tilde{\epsilon} \approx \tilde{\epsilon}^s \log \tilde{\epsilon}^{-1}$, since the valuation is identified with the associated Cantor function $\phi(\tilde{x}) \approx \tilde{\epsilon}^s \approx \epsilon$, s being, as usual, the corresponding Hausdorff dimension. Reverting back to the ordinary scale ϵ (and keeping in view the associated scale invariance), this scaling can be identified with $\epsilon^{\tilde{s}} \approx O(1)\epsilon \log \epsilon^{-1}$, for an $\tilde{s}$ given by $\tilde{s} \approx 1 - \frac{\log\log\epsilon^{-1}}{\log\epsilon^{-1}}$. *As a consequence, in the presence of an ultra metric space, in the neighborhood of 0 the ordinary limit $\epsilon \to 0$ is replaced by the sub linear limit*

$$\epsilon^{\tilde{s}} = \epsilon \log \epsilon^{-1} \to 0, \tag{5.4}$$

$0 < \tilde{s} < 1$, *as* $\epsilon \to 0$. A real variable x in R approaching to (or flowing out from) 0 will experience this scale invariant sub linear behavior in an incredibly small neighborhood of **0** in **R** and should have a deep significance in number theory and other areas [13, 19, 21]. One application is already presented in the new proof of the prime number theorem. If the variable x changes only over a Cantor set C and approaches 0 through points of C by hoppings (inversion induced jumps) then the above sub linear asymptotic behavior is obviously remain valid. As a consequence *the limit $\epsilon \to 0$ on C is interpreted as the sub linear limit $\epsilon \log \epsilon^{-1} \to 0^+$ on R.*

To justify further the above claim, let us suppose that the original Cantor set C and the infinitesimal Cantor set C_i have Hausdorff dimensions s and s' respectively. Any point x of the fattened set $\mathbf{C} = C + C_i$ is given as $\mathrm{x} = x + \tilde{x}$, $x \in C$, $\tilde{x} \in C_i$. It is well known that $\mathbf{C} = I$, for *almost every* s', for a given s [64]. Accordingly, it follows that *given a Lebesgue measure zero Cantor set C, the above smooth differentiable structure is a.s (almost surely) realized on* $\mathbf{C}$, *which is nothing but I, though in an appropriate (scale free) logarithmic variable.*

We note that similar behavior is also reported recently in the context of diffusion in an ultra metric Cantor set in a non commutative space [49]. Finally, the ultra metric induced by the valuation $v(\tilde{x})$ coincides with the natural ultra metric only when the scaling properties of C_i coincide with that of C. In this chapter we adhere to the latter possibility.

To understand more clearly the above smooth scale invariant structure let us consider the classical middle third Cantor set $C_{1/3}$ with scale factors $\tilde{\epsilon} = 3^{-n}$. A point $x_0 = 3^{-n}\sum a_i 3^{-i}$, $a_i \in \{0,2\}$ of $C_{1/3}$ is raised to the scale free $\dot{x}$ which is a variable living in a family of fattened gaps, attached and structured hierarchically at the point x_0 (or equivalently, by scale invariance, at 1), over each of which scale free equations of the form equation (5.3) are valid. The infinitesimals are elements of the gaps "closest" to 0, viz. the open intervals $3^{-n-m}(1,2)$, in the limit $n \to \infty$, for a fixed $0 < m < n$, which are assigned nontrivial values analogous to the Cantor function $v(\tilde{x}) = i3^{-ms}$, $i = 1, 2, \ldots 2^m - 1$ [16]. Over each of the finite size gaps, on the other hand, the valuation $v(x)$ is awarded as $v(x) = 3^{-sn}$. Both these valuations are not only continuous but also smooth since the corresponding Cantor function is realized as a smooth function via the logarithmic ansatz for a substitution of the form $3^n \Delta x_n = 3^n(x - x_n) \to n \log \frac{\mathrm{x}}{x_n}$, as $n \to \infty$, thereby removing the derivative discontinuity at the points of scale changes (c.f. [16]), so that $\frac{dv(x)}{dx} = 0$, every where on the Cantor set concerned. Notice that gaps scale as $\epsilon = 2^{-n}$ (recall the binary representation for points on a connected segment of the real line) when a closed interval containing points like $x_0 \in C_{1/3}$ scales as 3^{-n}, so that the Hausdorff dimension is $s = \log_3 2$. By equation (5.2), the variability of the valuation $\tilde{v}(x)$ in the limit of vanishing gap sizes is obtained as $\tilde{v}_0(x) \propto 3^{-sn}s^{-1}, n \to \infty$.

5.4 *Differential increments*

Next, we determine the incremental *measure*, denoted $d_j\tilde{X}$, of smooth self similar jump processes of (gap) "size" (in the sense of a weight) ϵ (2^{-n}, for $C_{1/3}$, say) in the neighborhood of the scale invariant 1. To this end, let us first recall that pure translations follow a linear law: $y = Tx = x + h$. The *instantaneous pure jumps* (of unit length close to the scale invariant 1), on the other hand, follow a *hyperbolic law*: $\tilde{X} \to Y = \tilde{X}^{-1} \Rightarrow \log Y + \log X = 0$, which tells, in turn, that the corresponding translational increment, even in the log scale, is indeed zero. This actually is the case for the valuation defined in terms of the locally constant Cantor function. The (manifestly scale invariant) multiplicative valuation $\tilde{v}(x) = \log_{x^{-1}}(X/x)$, however, gives the correct linear measure for a single jump relative to the point x (for the above hyperbolic type jump, $v(x) = 1$ relative to x itself as the scale). The corresponding *multiplicative increment* is denoted as $\delta_j \tilde{X} = (x/x_0)^{\tilde{v}(x)}$. More importantly, this valuation is realized as a smooth measure and may be considered to contribute an independent component in the ordinary measure of R. Further, the total self similar jump mediated increments over a spectrum of gaps of various sizes of the forms $\epsilon_n = 2^{-n}\epsilon_0$ in the neighborhood of a (middle third) Cantor point x_0 (say) is now obtained as

$$\Delta_j \tilde{X} = (x/x_0)^{s^{-1} 2^{-m} \sum_n 2^{-n}} \tag{5.5}$$

which in the limit $x \to x_0$, that is, $m \to \infty$ yields the *jump differential* $d_j \tilde{X} = (d\tilde{x})^{s^{-1}}$, where $\tilde{x} = \lim (x/x_0)^{2^{-m}}$, is a deformed variable close to 1. Such a variable ($\neq 1$ exactly) exists because of a nontrivial g.l.b. of gap sizes (another manifestation of the sub linear asymptotic). Incidentally, we note that the essential singularity in $s = 0$ tells that in the absence of inversion mediated jumps, the whole structure of gaps collapses to a point (singleton set, devoid of any nontrivial infinitesimals). The divergence in the jump measure then reflects the ordinary non differentiable structure of the Cantor set. On the other hand, on any connected segment of R, $s = 1$, and the jump measure reduces to the ordinary linear measure dx. To summarize, *the significantly new insight that emerges from the above analysis is that an infinitesimal scale invariant increment on an ultra metric space must have the form* $\tilde{X} = 1 + \epsilon^{1/s}$, $\epsilon \to 0^+$ *on a connected segment close to 0.* Recalling $\tilde{\epsilon} = \epsilon^{1/s}$, the above infinitesimal jump increment $\tilde{X} = 1 \pm \tilde{\epsilon}$ reduces to the usual increment on a Cantor set in the usual metric, but at the cost of the smooth structure.

Chapter 6

APPLICATIONS TO DIFFERENTIAL EQUATIONS

6.1 Ordinary Differential Equations

We begin applications of the present nonlinear analytic formalism to differential equations starting from the very elementary level. The real number system R is realized as an extended deformed real line $\mathcal{R}$ which reduces to the *connected* real line R only at the O(1) scale. At infinitesimally (infinitely) small (large) scales, $\mathcal{R}$ degenerates into a set having the structure of a positive measure *totally disconnected* Cantor set. In the presence of scale invariant infinitesimals, the usual point like structure of an element of the Cantor set is extended to an infinitesimal connected line segment (at the level of a well defined infinitesimal scale) on which a self similar replica of the original O(1) differential equation on R is written in a scale invariant manner using appropriate logarithmic variables. Two self similar replica equations on the disconnected branches are matched smoothly at the point of disconnection by inversion induced smooth jumps. We recall that such a smooth matching is not admissible in the ordinary analysis since a point of disconnection is interpreted as a singular point in the conventional treatment. The present nonlinear framework accommodating smooth jump as a nonlinear incremental mode is formulated mainly to bypass the difficulties in formulating differential equations on a disconnected setting involving discontinuous coefficients and/or data. Notice that the scale free equation (3.1) (Chapter 3) already plays a key role in the formulation of the analysis.

6.1.1 *First order Equation*

Let us consider the simplest differential equation

$$\frac{dx}{dt} = 1 \tag{6.1}$$

This may be assumed to represent the uniform motion (of the centre of mass) of a rigid ball. Accordingly, the ball will roll at uniform rate 1 along the x-axis when the position at any instance may, for instance, be given by $x(t) = t$. This is what follows according to Newton's first law of motion. The ball will continue to roll for ever along the rectilinear path unless impressed by an externally applied force.

Now suppose the above uniform motion of the rigid ball continues to hold good even as $t \to \infty$. Let $t = \frac{1}{\epsilon}\tau$, $\epsilon > 0$. Then for $\epsilon \to 0^+$, $t \to \infty$ for a finite O(1) non-zero τ. In the new rescaled variable τ, Eq(6.1) assumes the form of an singularly perturbed problem

$$\epsilon\frac{dx}{d\tau} = 1 \tag{6.2}$$

For an arbitrarily small but fixed $\epsilon \neq 0$, both the above equations are identical and yield the same solution $x(t) = t = \frac{1}{\epsilon}\tau$ (and also satisfies the initial condition $x(0) = 0$). It is also clear that the parameter ϵ can go arbitrarily close to 0, but can not exactly vanish. In other words, the singularity at ∞ for Eq(6.1) is realized as a singularity at 0 for Eq(6.2). Consequently, this singular problem is further reducible to the scale free equation

$$\tau\frac{dx}{d\tau} = x \tag{6.3}$$

when τ here is a small scale variable and tends to 0 satisfying : $0 < \epsilon < \tau$, $\tau \to 0$. This follows once one replaces ϵ in the left hand side by τ/x, since $x = t$ for any finite value of t.

Because of scale invariance, the above equation (6.3) is assumed to be valid on the deleted set (neighborhood) $I = (-1, 1)\backslash\{0\}$. Since, $\tau = 0$ is *unattainable*, and the size of the hole of I has no *positive lower bound*, it might be imagined to have the shape of a totally disconnected Cantor set having a countable number of disjoint gaps (open intervals) of arbitrarily small lengths. Indeed, exploiting scale invariance, Eq(6.3) may be rewritten as

$$\tau_1 \frac{dx}{d\tau_1} = x \tag{6.4}$$

where $\tau_1 = \tau/\epsilon$ and $0 < \epsilon < \tau$ so that as $\tau \to 0^+$, $\epsilon \to 0^+$ in such a manner that $\tau_1 = \epsilon^{-v(\tilde{\tau}(\tau))}$ goes to zero at a much slower rate. Here, v denotes an ultra metric valuation defined over the class of infinitesimals $\tilde{\tau}(\tau)$ those are assumed to reside in $(0, \epsilon)$ (c.f. Chapter 3). We remark here once more that the scale ϵ may be identified with an accuracy level in the sense that beyond which elements of $(0, \epsilon)$, are practically invisible (undetectable) relative to the real variable (defined by) $\tau > \epsilon$. Let $T := \log_{\epsilon^{-1}} \tau_1 = v(\tilde{\tau})$. We call T as a *dressed (deformed) value* for the original linear variable (asymptotic time) τ relative to the scale ϵ. Exactly in a similar manner we also write $X := \log_{\epsilon^{-1}} x_1$ where $x_1 = x/\epsilon$.

We now have an important observation. The limit $\epsilon \to 0^+$ realizes a *non classical* extension of the ordinary *linear* neighborhood of 0 of the form $(-\epsilon, \epsilon)$ into a *topologically inequivalent* ultra metric neighborhood $\tilde{U}$ so that the singleton set $\{0\}$ of R is extended into a positive measure Cantor like fractal set $\tilde{C} \subset \mathcal{R}$ (more precisely, the gaps of $\tilde{C}$) accommodating infinitesimals $\tilde{\tau}$. The valuation $v(\tilde{\tau})$ now replaces the ordinary linear measure τ for the closed interval $[0, \tau]$ by the *nonlinear, but, nevertheless, smooth* measure $dT = dv(\tilde{\tau})$ defined instead over an infinitesimal neighborhood (in the form of open gaps) of $\tilde{C} \subset (-\epsilon, \epsilon)$, as $\epsilon \to 0^+$. Clearly, the original *initial value (consistency constraint)* $\dot{x}(0) = 1$ for (6.1) gets *extended smoothly* over to a differential equation over $\tilde{C}$ in the form

$$\frac{dX}{dT} = 1 \tag{6.5}$$

using the deformed variables X and T, thereby replicating the original ODE (6.1) on the deleted neighborhood $(-1, 1)\backslash\{0\}$ onto an infinitesimally small (deleted) neighborhood $\tilde{C}$ of 0. Accordingly, the original $t - x$ plane in the neighborhood of 0 is extended over to a $T - X$ plane (realized as a subset of the product space $\tilde{C} \times \tilde{C}$), which is nothing but the $t - x$ plane in the log-log scale, though however, in the asymptotic limit $\epsilon \to 0$.

To summarize, the extension of the singleton set of the form $\{0\}$ of R over to a deleted set of the form $(-1,1)\backslash\{0\}$ is realized explicitly in the context of the linear equation (6.1). Plugging in all the steps together we can also write down a generalized class of solutions of this equation in the form

$$X(t) = t(1+\epsilon \times (\epsilon t \uparrow)^{-X(v(\tilde{\tau}))}) = t(1+\epsilon \times \tau^{-s(\tau)}) \tag{6.6}$$

where symbols are already introduced above. We remark that t in the right hand sides of the new solution is an undirected real variable, but $t \uparrow$ within the bracketed expression in the r.h.s (first equality) is an asymptotically increasing variable. For arbitrarily small $\epsilon > 0$ and $t \sim O(1)$, the new extended solution reduces to the standard solution $x(t) = t$. However, as t grows to the level of $t \sim O(\epsilon^{-1})$ so that $\tau \sim O(1)$, the second terms in the two equalities can become significant, defining not only the above mentioned extension of the size of the ordinary neighborhoods of a point, but should also have important applications in various nonlinear complex physical, biological and other problems.

It also follows that the ordinary non-directed (classical/Newtonian) time t is extended to the deformed time $T(t) = t(1+\epsilon \times (\epsilon t \uparrow)^{-T(v(\tilde{\tau}))}) = t(1+t^{-1} \times \tau^{-s(\tau)})$ with an O(1) directed multiplicative component (c.f. Definition 13, Chapter 4) for a $\tau = \epsilon t \sim O(1)$.

6.1.2 Harmonic Oscillation

Consider an orbit of the Harmonic oscillator

$$\ddot{x} + x = 0 \tag{6.7}$$

In $\mathcal{R}$ this equation is written as $X'' + X = 0$ where $'$ denotes derivation with the deformed time $T = t(1+t^{-1} \times (\tau)^{-s(t)})$, so that $dT \approx dt$, both for $t \to 0$ or ∞. For t finite, $s(t) \approx 0$, but $\tau^{-s} = O(1)$ when $t \to \infty$. Suppose the corresponding deformed orbit X is given by $X = xe^{\tilde{\phi}(\eta, x(\eta))}$, $\eta = \log T/t$. One verifies that nonlinear late time fluctuation $\phi(t) = \tilde{\phi}(\eta, x(\eta))$ satisfies the driven nonlinear equation

$$\ddot{\phi} + (2\dot{x}/x + \dot{\phi})\dot{\phi} = -(\ddot{x} + x)/x. \tag{6.8}$$

when the explicit dependence of ϕ on x is disregarded, for simplicity (i.e., $|\phi_x| << |\phi_t|$ etc). As a consequence, in a world admitting nontrivial small scale structures in the real number system, the classically sinusoidal orbits of an harmonic oscillator are expected to undergo a highly *nonlinear late time evolution* governed by (6.8) exposing an emergent interaction of slowly evolving nonlinear waves with (linear) sinusoidal wave. For finite values of t, ϕ and its derivatives are negligible and we recover the pure harmonic oscillation. However, as $t \to \infty$, small scale fluctuation in t is magnified, so that the scale invariant logarithmic variable $\eta \approx \tau^{-s(t)} \approx (t_0/t)^{s(t)} \sim O(1)$, for a sufficiently large scale $t_0 = 1/\epsilon$ for the ordinary time. As a consequence, the original harmonic oscillation calibrated in ordinary time t now is transferred to the new O(1) fluctuating motion in η in which derivatives of ϕ ie, $\frac{d\phi}{d\eta}$ etc are non negligible, and the original linear oscillation would experience nonlinear perturbations.

Rewriting Eq(6.8) as

$$\ddot{x} + 2\dot{\phi}\dot{x} + (1 + \dot{\phi}^2)x = -\ddot{\phi}x \tag{6.9}$$

We note that for a *given* deformation factor $\phi(t)$ original simple harmonic oscillation gets deformed into a driven Lienard type system (under the above simplifying assumption disregarding x dependence of ϕ) because of the late time influence of the nonlinear internal time. For instance, choosing a deformation factor of the form $\dot{\phi} = \frac{\epsilon}{2}(\frac{1}{2}x^2 - 1),\ 0 < \epsilon << 1$ so that $\dot{\phi}^2$ term may be dropped, a late time nonlinear oscillation may be designed as the *intrinsically generated* Van der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \tag{6.10}$$

The closed circular phase paths of the harmonic oscillations would therefore be broken into families of phase paths spiraling towards the unique limit cycle of the Van der Pol equation Eq(6.10) in an asymptotically late time when the deformed nonlinear component of time t is chosen suitably as above. For a small nonlinearity parameter $0 < \epsilon << 1$, the unique limit cycle of the Van der Pol limit cycle has the exact amplitude (i.e. the value of $x > 0$ when $\dot{x} = 0$) $a(\epsilon) = 2$. According to the present scenario, as t approaches $1/\epsilon$, all the harmonic oscillator orbits, excepting the one with amplitude 2 would open up and spiral toward the unique stable cycle (either inwardly or outwardly) in the asymptotically generated nonlinear system.

The most general amplitude variation is expected to be described by a nonlinear partial differential equation replacing Eq(6.8). Further details of this linear to nonlinear transition and nonlinear amplitude variations will be considered elsewhere [67].

6.1.3 *Van der Pol Equation*

As pointed out already, the above derivation has a simple interpretation. A simple harmonic oscillator left undisturbed for an indefinite time would experience late time nonlinear oscillation because of the influence of hidden nonlinear structures which become activated at an asymptotic time. We gave a derivation of the Van der Pol oscillator like variations when the late time variation is modeled as a special amplitude variation for the original harmonic oscillator. Here we take the reversed problem. We begin with the Van der Pol oscillator equation

$$x'' + \epsilon(x^2 - 1)x' + x = 0 \tag{6.11}$$

and assume that explicit nonlinearity in the equation induces a transformation from the linear time variable t to a nonlinear time $\tau = \phi(\epsilon t)$ with the condition that $\phi(t)$ is monotonic increasing and $\dot{\phi}(0) = 1$ so that $d\phi(t) \approx dt$ for $t \approx 0$. This, in turn, implies that time derivatives (i.e. x' and x'') in the above equation are in fact with respect to τ rather than t. As a consequence, re-expressing this equation in the ordinary time t we have

$$\frac{1}{\dot{\phi}^2}\ddot{x} - \frac{\ddot{\phi}}{\dot{\phi}^3}\dot{x} + \frac{1}{\dot{\phi}}\epsilon(x^2 - 1)\dot{x} + x = 0 \tag{6.12}$$

so that fixing ϕ by $\dot{\phi}(t)^{-1} = 1 - \epsilon \int^t (x^2(\tau) - 1)\, dt$, we reproduce the harmonic oscillator equation for *any* $\epsilon > 0$ when $0 < t << 1$, and we set $\dot{\phi}^2 \approx 1$ (derivatives $\dot{\phi}$ etc are evaluated with the rescaled variable t). With this choice of ϕ the two middle terms in the above equation cancel each other leading to the harmonic oscillator equation for a time scale arbitrarily close to 0.

The advantage of this observation is the following. We now have a new non-perturbative method for computing solutions and estimating amplitude, frequency etc of the closed cycle of the Van der Pol equation. To briefly outline the actual procedure for computing the amplitude for the closed cycle of the Van der Pol oscillator, one begins with the harmonic oscillator solution $x(t) = a \sin t$ for $t \approx 0$. For a finite time t, we replace t by the nonlinear time $\tau = \phi(t) := t + q(\epsilon t)$, so that a solution of the nonlinear oscillator now is written as $x(\tau) = a(q) \sin(t + q(\epsilon t))$ where the functions $\phi(t)$ and the associated $q(\epsilon t)$ are introduced as above. Consequently, $q(0) = 0$ and $q(1) = 1$ as $t \sim 1/\epsilon$. Writing $x(t + q) = \sum x_n q^n$ and $a(q) = \sum a_n q^n$, and noting that the definition of q already involves x, one can now develop an algorithm for calculating iteratively approximate solutions x_n and the corresponding approximate amplitudes of the limit cycle. In general, the calculated phase trajectory need not represent a closed cycle, but as $t \to 1/\epsilon$, all the trajectories of the Van der Pol system would be attracted toward the unique limit cycle, so the computed solution $x(\tau)$, in the limit $q \to 1^-$ should correspond to the orbit of the limit cycle.

Details of the explicit computations will be taken up elsewhere. We close this Section dealing with the problem of transition of a linear system into a nonlinear mode and vice versa in the deformed real number system $\mathcal{R}$ with the following remark:

In the present scenario, neighborhoods of $t = 0$ and $t = \infty$ are disconnected sets. As a consequence, a linear harmonic oscillator in the O(1) linear time t in the connected segment of the form $(\epsilon, \epsilon^{-1})$, will be transported into a nonlinear Van der Pol type nonlinear oscillator in an asymptotically distant connected line segment in the region $t > \epsilon^{-1}$. On the other hand, a Van der Pol system in ordinary O(1) time variable t will be reduced into a linear harmonic oscillator in an infinitesimally small connected segment in $0 < t < \epsilon$. Although realized in a different setting, such a transition from a linear system to a nonlinear system (and vice versa) was also proposed in the homotopy analysis method of Liao [2] which seems to offer a much improved method for computing solution and relevant parameters in a nonlinear differential system. We shall explore the interesting problem of a homotopy interpretation of the above mentioned transition based on the deformed nonlinear time in future. The linear oscillator in the disconnected O(1) component $(\epsilon, \epsilon^{-1})$ and a nonlinear oscillator in a disjoint component when $t > \epsilon^{-1}$ as $\epsilon \to 0^+$ can indeed be visualized as a homotopy transformation [2] on the (topological) space of differential operators.

6.2 Diffusion to Anomalous Diffusion

6.2.1 Introduction

As an application of the present formalism to the linear partial differential equations, we consider the linear one dimensional diffusion (heat) equation

$$\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2}, \ -\infty < x < \infty, \ t > 0 \tag{6.13}$$

where we choose, for simplicity, the diffusion constant to be unity [19]. This parabolic equation is known to admit self similar fundamental solution $W(x,t)$ which represents the Gaussian probability density function for a diffusing particle in Brownian motion and is characterized by the linear growth of the mean square deviation $\langle \Delta x^2(t) \rangle = t$.

Anomalous diffusion, on the other hand, is known to occur ubiquitously in diverse complex systems enjoying fine "structure with variations" [53] such as in disordered or fractal media, thus giving rise to fat tailed, broad (probability) distributions and/or long range spatio-temporal correlations [54, 55]. The hallmark of such a diffusion process is the occurrence of an anomalous law for the mean square displacement (deviation/fluctuation), viz., $\langle \Delta x^2(t) \rangle = t^\nu$, with $\nu \neq 1$. Sub-diffusive ($\nu < 1$) behavior is usually predominant in disordered systems, for instance, in spin glasses, amorphous semiconductors, lipid bilayers, living cells, transport in fractal sets and many others where a broad (fat tailed) distribution for local trapping times of the diffusive test particle gradually build up [54]. Super-diffusion ($\nu > 1$), on the other hand, may arise from long range correlations in velocity fields of turbulent flows, Levy flights and so on [55]. The emergence of nonlinear growth of the mean square fluctuation tells that the Gaussian central limit theorem may not be applicable in the underlying random walk processes (We note, incidentally, that complex systems with linear mean square fluctuations in Brownian like motion with nongaussian distribution are reported in literature recently [56]. A nonlinear growth, however, could be considered as a clear signature of the break down of the (finite variance) central limit theorem). An important problem is to look for a generic (universal) mechanism for the emergence of anomalous diffusion in such diverse phenomena.

In the present Section, we offer one such dynamical principle that might be at play at the heart of complex systems, besides more specific, system dependent mechanisms. We show that *the anomalous mean square fluctuations can arise naturally from the ordinary diffusion equation interpreted scale invariantly in the formalism endowing real numbers with a nonarchimedean multiplicative structure* (c.f. Chapter 3-5). In Chapter 5, it is shown that a variable t approaching 0 linearly in the ordinary analysis would enjoy a sub linear $t \log t^{-1}$ flow in the presence of this scale invariant structure. Diffusion on an ultra metric Cantor set is also generically sub diffusive with the above seemingly universal sub linear mean square deviation. The present study seems to offer a new interpretation of a possible emergence of complex patterns from an apparently simple system. This, in turn, appears also to suggest a realization of the philosophical principle, " Nature can produce complex structures even in simple situations, and can obey simple laws even in complex situation [53]."

There is already a vast body of studies on anomalous diffusion and its origin that are available in literature [54, 55]. Even with this back ground, the present investigation aims at offering a potentially new insight into the actual mechanism of the dynamics of an anomalous motion. As it is well known, there are actually two distinct types of motion observed in Nature: smooth, regular motion, like the Newtonian (two body) planetary motion, and random, highly irregular motion, as in the Brownian motion of a fine pollen particle in a liquid at rest [55]. A smooth motion, at least on a moderate time scale, is expected to be predictable and so are deterministic in nature, when Brownian type motion requires statistical (stochastic) methods. Of course, the deterministic chaos falls in between, and several authors, for instance, Ref. [57], discussed the problem of offering a dynamical interpretation for the Brownian like motion based on the deterministic Hamiltonian models in the phase space. At a more elementary level, on the other hand, the classical Brownian motion can, in fact, be considered to enjoy a bit of a deterministic flavor as the relevant Gaussian probability distribution (transition probability/ propagator) is known to follow the linear homogeneous diffusion equation (6.13). Accordingly, variations of macroscopic variables such as the number of diffusive particles, concentration and similar other quantities are governed effectively by a smooth deterministic law. Mathematically, Brownian motion is a process realized in a homogeneous smooth manifold where Taylor's expansion and other relevant analytic function theoretic resources are available. Moreover, the probability density is also a smooth function having finite mean and variance, so that the universality of the central limit theorem drives the force law to be smooth (for another explanation, see [57]).

On the other hand, if the underlying space of diffusion has a manifestly disordered, fractal structure, the above Gaussian, and/or the function theoretic smoothness is generally lost (see for instance [29, 34, 49, 54, 58, 59, 60, 61, 62, 63]). All these studies tried to offer precise mathematical justifications leading to anomalous mean square variations in such fractal sets. It follows also that the emergence of a *smooth* effective deterministic law (in the sense of differentiable functions in a Euclidean space) at the macroscopic scale of such a system is generally lost because of the inherent loss of smoothness in the random process. The most of the above approaches are based mainly on new developments in the areas of geometric measure theory, harmonic analysis, functional analysis, probability theory and so on and are technically more involved. We also note that a macroscopic law derived from some sort of an integral principle based on Lebesgue integration need not be smooth in the sense considered here. A "fractionally smooth" macroscopic law following a fractional differential equation, for example, is supposedly nonsmooth (in the ordinary sense) and reflects the presence of randomness.

As already detailed in Chapter 3-5, the scale invariant nonlinear analysis [16, 17] aims at integrating the framework of the standard analysis on, for instance, the Euclidean spaces and those on a fractal like space. Another motivation was to investigate *if a seemingly smooth deterministic evolution, even in the absence of any external influences, may lead naturally to a complex pattern over an asymptotically long (or short) time scale* (c.f. Sec.6.1.1-3 for applications of this principle in ODEs). It will become clear that the presence of *dynamically* generated ultra metric Cantor sets at infinitesimally small finer scales in the (deformed) real line would force a linear diffusive process to evolve *anomalously* over *infinitely long time scales*, that are available naturally in the scale invariant formalism. Our approach, in comparison to the above cited references, namely, [29, 34, 49, 54, 58, 59, 60, 61, 62, 63], is conceptually more simpler and appealing and may be considered to give yet another concrete example of the extension of the ordinary analysis on R over a positive measure Cantor set.

6.2.2 Diffusion

Coming back to the diffusion equation Eq(6.13), let us next recall the scale invariance and self similarity of the solutions of the same. Writing $W(x,t) = t^{-1/2}w(z),\ z = \frac{x}{2\sqrt{t}}$, the scaling function w satisfies the first order ordinary differential equation

$$\frac{dw}{du} = -w,\ u = z^2 \tag{6.14}$$

giving rise to the Gaussian propagator $W(x,t) = At^{-1/2}e^{-\frac{x^2}{4t}}$ by a direct integration. Now, in the present analysis, the real variables x and $t > 0$ must be assumed to live in the corresponding scale invariant deformed extensions $\mathcal{R}$ and $\mathcal{R}_+$ (set of nonnegative numbers), so that the scaling variable z gets extended to a deformed variable $\tilde{Z} = \frac{z}{z_0} = \frac{\tilde{X}}{\sqrt{\tilde{T}}} \in \mathbf{1}$ in the extended (deformed) neighborhood of a point $(x_0, t_0) \in R \times R_+$. In fact, we have deformed extension $(x_0, t_0) \to (X, T) = (x_0\tilde{X}, t_0\tilde{T})$ leading to the above scale invariant ratio for $\tilde{Z}$ The scale invariant variables $\tilde{X}$ and $\tilde{T}$ belong to two Cantor sets C_s and C_t respectively with scale invariant measures $d_j\tilde{X} = dx^{\alpha}$ and

$d_j\tilde{T} = dt^\beta$, where α and β are the respective inverse Hausdorff dimensions and x and t are two scale invariant variables near 1 of R (c.f., jump differential following Eq(5.5) of Chapter 5). As a consequence, Eq.(6.13), defined originally on R, now automatically gets extended to one in the new rescaling symmetric variable $\tilde{Z}$ living on a ultra metric Cantor set in the extended neighborhood of every point $(x,t) \in R \times R_+$ and hence supports nontrivial solutions analogous to Eq.(5.1) and Eq.(6.6). Thus, integrating Eq.(6.14) in that class of new solutions, we get

$$W_C(\tilde{X},\tilde{T}) = At^{-\beta/2}e^{-(\frac{x^{2\alpha}}{t^\beta})^{1+\nu}} \tag{6.15}$$

as a *stretched exponential,* fat tailed propagator for a diffusive process (random walker) on a Cantor set. Indeed, W_C satisfies the scale invariant diffusion equation

$$\frac{\partial W}{\partial \tilde{T}} = \frac{\partial^2 W}{\partial \tilde{X}^2} \tag{6.16}$$

which is defined close to every point $\tilde{X} \in C_s$ and at any instant $\tilde{T} \in C_t$. Although the derivatives are evaluated with jump differentials, these are equivalent to the usual partial derivatives, but in the deformed (scaling) variables x^α and t^β respectively (recall differential jump measure of Chapter 5). Further, the exponent ν in Eq.(6.15) is a valuation so that $\tilde{Z}^{\nu(\tilde{Z})}$ is a locally constant Cantor function that arises in connection with the residence Cantor set for the variable z (c.f. Eq.(5.1) of Chapter 5) [1]. Because of the sub linear asymptotic increments of the form Eq.(5.4) (of Chapter 5), this equation is also considered to be valid on a connected line segment close to $t = 0$, so that Eq.(6.16) is also valid for $\tilde{T} \approx 0$. Next, we note that the scale invariant factors of a real variable (viz. Eq.(5.1)) become significant only for an asymptotically large time. Consequently, the above scale invariant solution Eq.(6.15) of the diffusion equation Eq.(6.13) is expected to arise naturally in any diffusive process that persists over *many longer time scales* living in a set of the form $\mathbf{R}$ and hence in $\mathcal{R}$. Further, the anomalous mean square deviation is given generically as $\langle \Delta x^2(t) \rangle = t^{\beta/\alpha}$, where $t \to 0^+$ [54], for every scale invariant x near 1.

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[1]If $x \in C_s$ and $t \in C_t$, then z, in general, would belong to a Cantor set, when thickness of the original sets satisfy certain restrictions [66].

We conclude that a simple diffusion process if allowed to evolve over many longer and longer time scales as those available for natural processes will ultimately give away naturally to a *stretched exponential* nongaussian distribution of the form Eq.(6.15) leading to an anomalous mean square fluctuation; reflecting, in turn, the universally present scale invariant numerical fluctuations [65].

Diffusion on a Cantor set, on the other hand, when examined in the framework of the ordinary Newtonian time (i.e., when C_t reduces to the singleton $\{0\}$) is generally sub diffusive with exponent $\alpha^{-1} = s : 0 < s < 1$, s being the Hausdorff dimension of the diffusive medium (since $\beta = 1$) [54]. The mean square deviation has the generic form $\langle \Delta x^2(t) \rangle = t^s \approx t \log t^{-1}$, $t \to 0^+$, because of the sub linear asymptotic flow on a Cantor set (Chapter 5).

For a fractal time process (i.e. when time t itself varies over a Cantor set) [54] the sub diffusion occurs for $s < \tilde{s}$ and super diffusion for $s > \tilde{s}$, $\tilde{s}$ being the Hausdorff dimension for underlying Cantor like set for the fractal time. However, for $s = \tilde{s}$, the gaussian like linear mean square variation may be observed even for a fractal time process [56]. Analogous smoothening in the asymptotic scaling of the eigen value counting function was also noticed by Freiberg [60].

To conclude, the scale invariant formalism is shown to have interesting applications both for simple ordinary and partial differential equations. Besides offering new *non-classical* asymptotic late time behaviors for simple differential systems on the deformed real line $\mathcal{R}$, both simple and fractal time diffusion on a Cantor like fractal medium can also be treated easily as a smooth (sub or super-diffusive) process.

Chapter 7

CONCLUDING REMARKS

An approach to a scale-invariant non-linear analysis on R is presented. In a computational problem a number $x = 1$, for example, is represented up to a finite accuracy; i.e., up to a scale ϵ, say. Then the numbers in the interval $(1-\epsilon, 1+\epsilon)$ are computationally unobservable and identified, as a whole, as the number 1. The non-trivial construction presented in this thesis now tells that the points in that computationally inaccessible limiting interval may get aligned dynamically as non intersecting clopen balls of a Cantor set C endowed with the non-trivial ultra metric value, thereby extending the ordinary real number set R to an infinite dimensional, scale free, non-archimedian space **R** accommodating dynamically active scale invariant infinitesimals and infinities.

Because of scale invariant infinitesimals, a non zero real variable x (> 0) say, in R approaching 0 now gets a pair of deformed structures living in an associated deformed real number system $\mathcal{R} \supset R$ of the form $\mathcal{R} \ni \mathcal{X}_{\pm}(x) = x.x^{\mp v(\tilde{x}(x))}$, thus mimicking non-trivial effects of dynamic infinitesimals over the structure of real number system R.

In this thesis we have studied a few non-trivial influences of the dynamical infinitesimals in the asymptotic estimates of number theory, more specifically prime number theorem. We have also presented some applications of dynamical infinitesimals in some simple ordinary differential equations and also in diffusion equation leading to the emergence of anomalous mean square fluctuations when a diffusive system is allowed to execute motion over infinitely long time scales.

We aim to make more detailed applications of this nonlinear analysis to other well known differential equations as well as dynamical systems in future. Applications to analytic number theory and fractal sets will also be considered. The status of homotopy analysis method in the present formalism will also be explored [67].

BIBLIOGRAPHY

[1] G.Adomain, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers (1994).

[2] S.Liao, Homotopy Analysis Method in Nonlinear Differential Equations, Springer Berlin Heidelberg (2012).

[3] J.H.He, Homotopy Perturbation Method; a new nonlinear analytical technique, Applied Mathematics and Computation, Vol.135, (2003), 73-79.

[4] L.-Y.Chen, N.Goldenfeld and Y.Oono, Renormalization group and Singular Perturbations: Multiplee scales, boundary layers and reductiive perturbation theory, phys.Rev.E, 543(1), (1996), 376-394.

[5] Carl M Bender, Steven A.Orszag, Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory, Springer.

[6] D.W.Jordan, P.Smith, Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers (4th ed.), Oxford University Press (2007).

[7] Kevorkian,J; Cole,J.D., Multiple Scale and Singular Perturbation Methods, Springer, (1996).

[8] L.Perko, Differential Equations and Dynamical Systems, Springer-Verlag (2001).

[9] A.Katok, B.Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press (1995).

[10] F.Herbert, Geometric Measure Theory, Series die Grundlehren der mathematischen wissenschaften, Band 153, NewYork, Springer-Verlag, (1969).

[11] L.Cesari and R-Kannan, Functional Analysis and Nonlinear Differential Equations, Bull.Amer. Math.Soc, Volume 79, Number 6 (1973), 1216-1219.

[12] S.Abbas, M.Benchora, G.M.Guerekata, Topics in Fractional Differential Equations, Springer, (2012).

[13] D.P.Datta and A.Roy Choudhuri, Scale free analysis and prime number theorem, Fractals, 18, (2010), 171-184.

[14] D.P.Datta, S.Raut and A.Roy Choudhuri, Ultrametric Cantor sets and Growth of measure, P-adic Numbers, Ultrametric Analysis and Applications, Vol.3,No.1, (2011), 7-22.

[15] D.P.Datta and M.K.Bose, higher derivative discontinuous solutions to linear ordinary differential equations: a new route to complexity? Chaos, Solitons and Fractals, 22, (2004), 271-275.

[16] S.Raut and D.P.Datta, Analysis on a Fractal set, Fractals 17, (2009), 45-52.

[17] S.Raut and D.P.Datta, Nonarchimedean Scale invariance and Cantor sets, Fractals 18, (2010), 111-118.

[18] D.P.Datta and S.Raut, The arrow of time, complexity and the scale free analysis, Chaos, Soliton & Fractals, 28, (2006), 581-589.

[19] D.P.Datta, S.Raut and A.Ray Chaudhur, Diffusion in a Class of Fractal Sets, International Journal of Applied Mathematics & Statistics, Vol.30; Issue No.6 (2012), 34-50.

[20] D.P.Datta, A new class of scale-free solutions to linear ordinary differential equations and the universality of the golden mean $\frac{\sqrt{5}-1}{2} = 0.618033....$, Chaos Solitons Fractals 17, (2003) 621-630.

[21] A.Ray Chaudhuri, D.P.Datta, Rescaling Symmetry, Ultrametricity and Emergent Nonlinearity, Communicated (2013).

[22] E.Stein, T.S.Murphy, Harmonic Analysis: Real variable methods, orthogonally and oscillatory integral, Princeton University Press (1993).

[23] A.Connes, Non-commutative geometry, Academic Press, (1994).

[24] Papoulis, Athanasions and Pillai, S.Unnikrishna, Probability, Random Variables and Stochastic Processes, McGraw-Hill Science/Engineering/Math, (2001).

[25] M.Shesinger, J.Klafter and Y.M.Wong, Random Walks with Infinite Spatial and Temporal Moments, J.Statist.Phys. 27, (1982), 499-512.

[26] A.Robinson, Non-standard analysis, North-Holland Publishing Co., Amsterdam, 1966.

[27] H.O.Peitgen, H.J$\ddot{U}$rgens, and D.Saupe, Chaos and Fractals: New Frontiers of Science 2nd ed., N.Y., N.Y.: Springer Verlag, (2004), 65.

[28] Jose Ferreiros, Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics, Basel, Switzerland: Birkh$\ddot{a}$user Verlag, (1999), 162-165.

[29] J.Kigami, Analysis on Fractals, Cambridge University Press, (2000).

[30] R.Dalrymple, R.S.Strichartj and J.P.Vinson. Fractal differential equations on the Sierpenski gasket, J.Fourier Anal. Appl. 5 (1999), 205-287.

[31] Ian Stewart, Does God Play Dice?: The New Mathematics of Chaos.

[32] B.Mandelbrot, Fractal Geometry of Nature, Free man and company, (1982).

[33] B.J.West and W.Deering, Fractal physiology for physicists, Phys. Rept. 246, (1994), 1.

[34] M.T.Barlow, Diffusion on Fractals, Springer, (1998).

[35] R.Hilfer and L.Anton, Fractional master equations and fractal time random walk, Phys. Rev. E 51 (2), (1994), 848-851.

[36] K.M.Kolwankar and A.D.Gangal, Fractional differentiability, Chaos 6, (1996), 505-512.

[37] U.Freiberg and M.Zahle, Harmonic calculus on fractals, a measure geometric approach I, Potential Anal. 16, (2002), 265-277.

[38] Curt Schmieden and Detlef Laugwitz: Eine Erweiterung der Infinitesimalrechnung, Mathematische Zeitschrift 69 (1958), 1-39.

[39] Edward Nelson: Internal Set Theory: A New Approach to Nonstandard Analysis, Bulletin of the American Mathematical Society, Vol 83, Number 6, (1977).

[40] Vopenka, P.Mathematics in the Alternative Set Theory. Teubner, Leipzig, (1979).

[41] Hensel, Kurt, "*Über* eine new Begrundung der Theorie der algebraischen Zahlen". Jahresbericht der Deutschen Mathematiker-Vereinigung 6(3): (1897), 83-88.

[42] Gouvea, Q. Fernando P-adic numbers: An Introduction (2nd ed.). Springer, (2000).

[43] V.S.Viadimirow, I.V.Volovich, E.I.Zelenov, P-adic Analysis and Mathematical Physics, World Scientific Press.

[44] S.Katok, P-adic Analysis compared with Real, American Mathematical Society, (2007).

[45] Apostol, Tom. M (1976), Introduction to analytic number theory, Undergraduate Texts in mathematics, NewYork -Heidelberg: Springer-Verlag, (1976), 7

[46] L.J.Goldstein, A history of prime number theorem, Amer.Math.Monthly 80, (1973), 599-615.

[47] H.M.Edwards, Riemann's Zeta function, Dover Publications, New York, (2001).

[48] P.Schneider, Non-archimedean Functional Analysis, Springer-Berlin, (2002).

[49] J.Pearson and J.Bellissard, Non commutative Riemannian geometry and diffusion on ultrametric cantor sets, arxive: math/0802.13362V2 (2008).

[50] B.Bollobas, Modern Graph Theory, Springer- Berlin, (1998).

[51] R.Albert and A.L.Barabasi, Statistical mechanics of complex networks, Rev.Modern Phys.74 (2002), 47-101.

[52] P.Laubeinheimer, T.Schick and U.Stuhler, Completion of countable nonstandard models of Q, arxive: math/0604466, (2006).

[53] N.Goldenfeld and L.P.Kadanoff, Simple lessons from complexity, Science, 28, (1999), 87-89.

[54] R.Metzater and J Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Reports. 339,(2000), 1-77.

[55] A.A.Dubkov, B.Spagnolo, V.V.Uchaikin, Levi flight super diffusion: an introduction, Int.J.of Bifurcation and Chaos, 18, (2008), 2649-2672.

[56] B.Wang, S.M.Anthony, S.C.Bau and S.Granick, Anomalous yet Brownian, PNAS, 206, (2009),1 5140-15146.

[57] P.Allegrini, P.Grigolini, and B.J.West, Dynamical approach to Levy statistics, Phys.Rev.E., 54, (1996), 4760-4767.

[58] R.C.Blei, Analysis in Integer and Fractional Dimensions, Cambridge University Press, (2003).

[59] J.Schweizer and B.Frank, Calculus on linear Cantor sets, Archiv der Mathematik, Vol.79,(2002), 46-50.

[60] U.Freiberg, Spectral asymptotic of generalized measure geometric Laplacian on Cantor like sets, Forum.Math. 17, (2005), 87-104.

[61] Y.Bakhtin, Self similar Markov process on Cantor set, Arxive:0810.0326v2 (math.PR) (2008).

[62] S.Alveberio and W.Karowowski, A random walk on P-adics: the generator and spectrum, Stochastic Process Appl. 53, (1994), 1-22.

[63] S.Alveberio and W.Karowowski, Jump processes on the leaves of multi branching trees, J.Math.Phys. 49, (2008).

[64] B.Solomyak, On the measure of arithmetic sums of Cantor sets, Indagationes Math.N.S. 8, 133-141,(1997), 133-141,.

[65] I.Volovich, The number theory as the ultimate physical theory, P-adic Numbers, Ultrametric Analysis and Applications, 2,(2010), 77-87.

[66] S.Astels, Cantor sets and numbers with restricted partial sums, Transactions(AMS), 352, (2000), 133-170.

[67] A.Ray Chaudhuri, D.P.Datta, in preparation.