Chapter 5

APPLICATION TO A CANTOR SET

5.1 *Introduction*

In this Chapter, we present a few new results on a Cantor set [19] exposing the precise nature of variability of a nontrivial valuation and hence of a Cantor function. This also constitutes an application of the scale invariant analysis on a Cantor subset of R. In Ref.[16, 17], the formulation of the scale invariant analysis on a Cantor set $C \subset [0,1]$ using the concepts of relative infinitesimals and the associated ultra metric norm are considered in detail (but not included in the present thesis). Here, we discuss in particular how an ordinary limiting variation $R \ni x \to 0$ is extended to a sub linear variation $x \log x^{-1} \to 0$ when $x \in C \subset [0,1]$. Finally, we derive the differential measure on a Cantor set C.

5.2 *Cantor Set: New Results*

It follows from the results presented in Chapters 3 and Chapter 4 that, because of scale invariant influence of relative infinitesimals, a nonzero real variable $x(> 0$ say,) in R and approaching 0 now gets a pair of *deformed* structures living in an associated *deformed real number system* $\mathcal{R} \supset R$ of the form $\mathcal{R} \ni X_{\pm}(x) = x \times x^{\mp v(\tilde{x}(x))}$, thus mimicking nontrivial effects of dynamic infinitesimals over the structure of the real number system R. Above ansatz works even for a finite $x\ (> 0)\ \in R$ when we express the above deformed representation in the form

$$X = x \times x^{\mp v(\tilde{x}(x'))} \tag{5.1}$$

for a set of infinitesimals $\tilde{x}$ living in $\mathbf{0}$. Here, x' denotes a positive real variable approaching 0^+ since an infinitesimal in the present formalism is defined relative to a limiting real variable. Clearly, $X = x \in R$ when $v = 0$ and hence $R \subset \mathcal{R}$. The corresponding scale invariant components $\tilde{X}_\pm := X_\pm/x$ now live, by definition, in a Cantor set (since by Theorem 3, Chapter 3, $v(\tilde{x})$ behaves as a Cantor function) and hence undergo changes by inversions of the form $X_+ = X_-^{-1}$ (c.f. Chapter 3).

We further recall that a given Cantor subset C of $I = [a, b] \subset R$ enjoys a ultra metric structure which is equivalent to the subspace topology inherited from the usual topology of R. In Ref.[14], it is, however, shown that the ultra metric valuation $v(\tilde{x})$ defined by Definition 2 of Chapter 3 is both metrically and topologically inequivalent to the usual ultra metric that a Cantor set carries naturally [13]. For a point x_0 in the Cantor set C, the representation Eq(5.1) now gives rise to a scale invariant ultra metric extension $\tilde{X}_\pm = X_\pm/x_0 = x_0^{\mp v(\tilde{x})}$ where the transition between two infinitesimally close scale invariant neighbors is mediated by more general inversions of the form $\tilde{X} \to \tilde{X}^{-h}$ for a real h, which *determines the jump size.* Notice that $\tilde{X}$ (and equivalently, $v(\tilde{x})$) is a locally constant Cantor function and solves $\frac{d\tilde{X}}{dx} = 0$ everywhere in I. The ordinary discontinuity of a Cantor function at an $x_0 \in C$ is removed, since in the present ultra metric extension, the point x_0 in C is replaced by an *inverted Cantor set* which is the closure of *gaps* of an infinitesimal Cantor set C_i that is assumed to be the residence Cantor set for the relevant infinitesimals $\tilde{x}$ living in the extended neighborhood $\mathbf{0}$ of 0. The gaps of C_i constitute a disjoint family of connected clopen intervals (represented in a scale invariant manner) over each of which scale invariant equation (3.1) of Chapter 3 are well defined [13]. Consequently, the valuation $v(\tilde{x})$, redefined slightly in the modified form

$$(x/x_0)^{\tilde{v}(x)} = x_0^{v(\tilde{x})} \tag{5.2}$$

(that is, $\tilde{v}(x)/v(\tilde{x}) = \log x_0/\log(x/x_0)$, exposing the relative variation of $\tilde{v}$ over v), x assuming values from the gaps in the neighborhood of x_0, is realized as a smooth function defined recursively in a scale invariant way by the equation

$$\frac{d\tilde{v}(x)}{d\xi} = -\tilde{v}(x) \tag{5.3}$$

where $\xi = \log\log(x/x_0)$, $x \in C_i$. Recall that $\tilde{x}$ resides in the gaps of nontrivial neighborhood of 0 instead. As a consequence, $\tilde{v}$ may be written as $\tilde{v}(x) = (\log(x/x_0)^k)^{-1}$, where k may be allowed to assume values from a set of scale factors related to that of the Cantor set. This form is clearly consistent with (5.2). Assuming x is drawn from a specific gap of a given size, the same, written more effectively as $\tilde{v}(x)/v(\tilde{x}) = (\log_{x_0}(x/x_0))^{-1}$, yields, in the limit of vanishingly small gaps (i.e., as $x \to x_0$ and vice versa), the limiting value $\tilde{v}_0(x)/v_0(\tilde{x}) = 1/s$, since $\lim \log_{x_0}(x/x_0) = s$ *equals the finite Hausdorff dimension of the Cantor set* C_i. Let us first note that if one replaces the Cantor set by a segment of a line of the form $(0, \delta)$, then $x/x_0 = 1$, in that limit ($\delta \to 0$) gives $s = 0$, which is consistent with the fact that the line segment reduces to a point, viz. 0 in the said limit. In the general case, $x/x_0 \propto N$, the number of clopen balls that covers the fattened gap of the form $(x, x_0) \subset (0, \delta)$ (size of balls are determined by the gap). Letting $x_0 \to 0$ (following the relevant scale factors $\beta^n \to 0$), the above limit therefore mimics the box dimension, which also equals the Hausdorff dimension of the Cantor set concerned. The topological in equivalence of the present ultra metric arises from the possible dichotomy in the choice of C_i.

5.3 *Limit on a Cantor set*

Let us now show that when the ordinary 0 of R is replaced by an infinitesimal Cantor set C_i, the ordinary limit $\epsilon \to 0$ in R is altered. This follows because of scale invariant dynamic infinitesimals with valuations given by $\log \tilde{x}/\epsilon \approx v(\tilde{x}) \log \epsilon^{-1} \approx \epsilon \log \epsilon^{-1}$, when the relative infinitesimal $\tilde{x}$ is considered to lie on a fattened (connected) gap, so that the ultra metric valuation may be assumed to coincide with the usual (Euclidean) value viz. $v(\tilde{x}) \approx \epsilon$. However, assuming $\tilde{\epsilon}$ $(= \beta^n,\ n \to \infty)$ to be an infinitesimal scale of the Cantor set concerned, we also have $\log \tilde{x}/\tilde{\epsilon} \approx \tilde{\epsilon}^s \log \tilde{\epsilon}^{-1}$, since the valuation is identified with the associated Cantor function $\phi(\tilde{x}) \approx \tilde{\epsilon}^s \approx \epsilon$, s being, as usual, the corresponding Hausdorff dimension. Reverting back to the ordinary scale ϵ (and keeping in view the associated scale invariance), this scaling can be identified with $\epsilon^{\tilde{s}} \approx O(1)\epsilon \log \epsilon^{-1}$, for an $\tilde{s}$ given by $\tilde{s} \approx 1 - \frac{\log\log \epsilon^{-1}}{\log \epsilon^{-1}}$. *As a consequence, in the presence of an ultra metric space, in the neighborhood of 0 the ordinary limit $\epsilon \to 0$ is replaced by the sub linear limit*

$$\epsilon^{\tilde{s}} = \epsilon \log \epsilon^{-1} \to 0, \tag{5.4}$$

$0 < \tilde{s} < 1$, *as* $\epsilon \to 0$. A real variable x in R approaching to (or flowing out from) 0 will experience this scale invariant sub linear behavior in an incredibly small neighborhood of **0** in **R** and should have a deep significance in number theory and other areas [13, 19, 21]. One application is already presented in the new proof of the prime number theorem. If the variable x changes only over a Cantor set C and approaches 0 through points of C by hoppings (inversion induced jumps) then the above sub linear asymptotic behavior is obviously remain valid. As a consequence *the limit $\epsilon \to 0$ on C is interpreted as the sub linear limit $\epsilon \log \epsilon^{-1} \to 0^{+}$ on R.*

To justify further the above claim, let us suppose that the original Cantor set C and the infinitesimal Cantor set C_i have Hausdorff dimensions s and s' respectively. Any point $\mathbf{x}$ of the fattened set $\mathbf{C} = C + C_i$ is given as $\mathbf{x} = x + \tilde{x}$, $x \in C$, $\tilde{x} \in C_i$. It is well known that $\mathbf{C} = I$, for *almost every s'*, for a given s [64]. Accordingly, it follows that *given a Lebesgue measure zero Cantor set C, the above smooth differentiable structure is a.s (almost surely) realized on $\mathbf{C}$, which is nothing but I, though in an appropriate (scale free) logarithmic variable.*

We note that similar behavior is also reported recently in the context of diffusion in an ultra metric Cantor set in a non commutative space [49]. Finally, the ultra metric induced by the valuation $v(\tilde{x})$ coincides with the natural ultra metric only when the scaling properties of C_i coincide with that of C. In this chapter we adhere to the latter possibility.

To understand more clearly the above smooth scale invariant structure let us consider the classical middle third Cantor set $C_{1/3}$ with scale factors $\tilde{\epsilon} = 3^{-n}$. A point $x_0 = 3^{-n}\sum a_i 3^{-i}$, $a_i \in \{0,2\}$ of $C_{1/3}$ is raised to the scale free $\dot{\mathrm{x}}$ which is a variable living in a family of fattened gaps, attached and structured hierarchically at the point x_0 (or equivalently, by scale invariance, at 1), over each of which scale free equations of the form equation (5.3) are valid. The infinitesimals are elements of the gaps "closest" to 0, viz. the open intervals $3^{-n-m}(1,2)$, in the limit $n \to \infty$, for a fixed $0 < m < n$, which are assigned nontrivial values analogous to the Cantor function $v(\tilde{x}) = i3^{-ms}$, $i = 1, 2, \ldots 2^m - 1$ [16]. Over each of the finite size gaps, on the other hand, the valuation $v(x)$ is awarded as $v(x) = 3^{-sn}$. Both these valuations are not only continuous but also smooth since the corresponding Cantor function is realized as a smooth function via the logarithmic ansatz for a substitution of the form $3^n\Delta x_n = 3^n(x - x_n) \to n \log \frac{\mathrm{x}}{x_n}$, as $n \to \infty$, thereby removing the derivative discontinuity at the points of scale changes (c.f. [16]), so that $\frac{dv(x)}{dx} = 0$, every where on the Cantor set concerned. Notice that gaps scale as $\epsilon = 2^{-n}$ (recall the binary representation for points on a connected segment of the real line) when a closed interval containing points like $x_0 \in C_{1/3}$ scales as 3^{-n}, so that the Hausdorff dimension is $s = \log_3 2$. By equation (5.2), the variability of the valuation $\tilde{v}(x)$ in the limit of vanishing gap sizes is obtained as $\tilde{v}_0(x) \propto 3^{-sn}s^{-1}, n \to \infty$.

5.4 Differential increments

Next, we determine the incremental *measure*, denoted $d_j\tilde{X}$, of smooth self similar jump processes of (gap) "size" (in the sense of a weight) ϵ (2^{-n}, for $C_{1/3}$, say) in the neighborhood of the scale invariant 1. To this end, let us first recall that pure translations follow a linear law: $y = Tx = x + h$. The *instantaneous pure jumps* (of unit length close to the scale invariant 1), on the other hand, follow a *hyperbolic law*: $\tilde{X} \to Y = \tilde{X}^{-1} \Rightarrow \log Y + \log X = 0$, which tells, in turn, that the corresponding translational increment, even in the log scale, is indeed zero. This actually is the case for the valuation defined in terms of the locally constant Cantor function. The (manifestly scale invariant) multiplicative valuation $\tilde{v}(x) = \log_{x^{-1}}(X/x)$, however, gives the correct linear measure for a single jump relative to the point x (for the above hyperbolic type jump, $v(x) = 1$ relative to x itself as the scale). The corresponding *multiplicative increment* is denoted as $\delta_j \tilde{X} = (x/x_0)^{\tilde{v}(x)}$. More importantly, this valuation is realized as a smooth measure and may be considered to contribute an independent component in the ordinary measure of R. Further, the total self similar jump mediated increments over a spectrum of gaps of various sizes of the forms $\epsilon_n = 2^{-n}\epsilon_0$ in the neighborhood of a (middle third) Cantor point x_0 (say) is now obtained as

$$\Delta_j \tilde{X} = (x/x_0)^{s^{-1} 2^{-m} \sum_n 2^{-n}} \tag{5.5}$$

which in the limit $x \to x_0$, that is, $m \to \infty$ yields the *jump differential* $d_j \tilde{X} = (d\tilde{x})^{s^{-1}}$, where $\tilde{x} = \lim(x/x_0)^{2^{-m}}$, is a deformed variable close to 1. Such a variable ($\neq 1$ exactly) exists because of a nontrivial g.l.b. of gap sizes (another manifestation of the sub linear asymptotic). Incidentally, we note that the essential singularity in $s = 0$ tells that in the absence of inversion mediated jumps, the whole structure of gaps collapses to a point (singleton set, devoid of any nontrivial infinitesimals). The divergence in the jump measure then reflects the ordinary non differentiable structure of the Cantor set. On the other hand, on any connected segment of R, $s = 1$, and the jump measure reduces to the ordinary linear measure dx. To summarize, *the significantly new insight that emerges from the above analysis is that an infinitesimal scale invariant increment on an ultra metric space must have the form* $\tilde{X} = 1 + \epsilon^{1/s}$, $\epsilon \to 0^+$ *on a connected segment close to 0.* Recalling $\tilde{\epsilon} = \epsilon^{1/s}$, the above infinitesimal jump increment $\tilde{X} = 1 \pm \tilde{\epsilon}$ reduces to the usual increment on a Cantor set in the usual metric, but at the cost of the smooth structure.