Chapter 2

BACKGROUND INFORMATION

2.1 Fractals

2.1.1 Introduction

The word 'fractal' is coined by Mandelbort from a Latin word *fractus* that denotes "a stone's shape after it was hit hard". It describes objects that are too irregular to fit into a traditional smooth (Euclidean) geometric setting. A fractal is usually considered to be an object of inquiry having finer structures. An important characteristic of fractal objects is the occurrence of some sort of a self-similarity. Generally, it can be expressed as a union of subsets, each of which is a reduced copy of the full set. Therefore the structure of the original set can be realized at smaller scales. Usually, the topological dimension of an object (i.e. a set)(in any dynamical problem) is a non-negative integer. But the dimension of a fractal set is a fractional (more correctly, real) number strictly exceeding the topological dimension. One of the major motivation in the study of fractals is the possibility of describing complex natural phenomena by only a finite set of parameters. But the methods of ordinary calculus is inapplicable in fractals as they are generally not smooth and are made of many fragmented geometric shapes. Therefore a proper development and application of an appropriate analytical framework for fractal sets is of considerable interest in the contemporary literature [3].

The main characteristic features of a fractal set $F \subset R^n$ can be stated as

1. It has fine structures on arbitrarily small scales.

2. F is too irregular to be described in the traditional geometrical setting.

3. It has fractional (fractal) dimension that exceeds its topological dimension.

4. The fractal set F has some sort of exact or approximate self-similarity.

5. The set F may be generated recursively following a simple (finite) set of rule.

Definition 1. *(self-similarity) The mapping $S_1, \ldots S_k : R^n \to R^n$ are called similarity transformations when $|S_i(x) - S_i(y)| = c_i\,|x - y|$ $(x, y \in R^n)$ and $0 < c_i < 1$ (c_i is called the scaling ratio of S_i). Each S_i transforms subsets of R^n into geometrically similar sets. A set that is invariant under such a collection of similarities is called a self-similar set.*

Middle third Cantor set, Sierpinski gasket and von Koch curve are all examples of self-similar sets.

2.1.2 Fractional Dimension

Conventionally, the dimension of a object is usually a non-negative integer that specify the number of coordinates that are necessary to describe the object (i.e. the elements of the set concerned) precisely. But fractal sets can not generally be described by simply by a finite set of coordinates. Thus, we have to look for a different definition of dimension that does not depend on the coordinates. Here, we shall consider two important dimensions namely Hausdorff dimension and Box dimension. We first briefly discuss some desirable properties which a definition of dimension is expected to satisfy [3]. Let $E, F \subset R^n$, then

1. Monotonicity: If $E \subset F$ then $\dim E \leq \dim F$.
2. Stability: $\dim(E \cup F) = \max(\dim E, \dim F)$.
3. Countable stability: $\dim\ (U_{i=1}^{\infty} F_i) = \sup\limits_{1 \leq i \leq \infty} \dim F_i$.

From (2) and (3) it is ensured that if we combine a set with other set having lower dimensions, the dimension will be same.

4. Geometric invariance: $\dim f(F) = \dim F$ provided f is a translation, rotation, similarity or affinity transformation.

5. Lipschitz invariance: Let $f : F \to R^n$ is a bi-lipschitz transformation, i,e. if $\exists$ c_1 and c_2 s.t. $c_1 \mid x - y \mid \leq \mid f(x) - f(y) \mid \leq c_2 \mid x - y \mid$, $x, y \in F$, and $0 < c_1 \leq c_2 < \infty$, then $\dim f(F) = \dim F$.

6. Countable sets: If F is finite or countable then $\dim F = 0$.

7. Open sets: If F is a open subset of R^n then $\dim F = n$.

8. Smooth manifold: If F is a n dimensional smooth manifold then $\dim F = n$.

Definitions of dimension generally satisfy monotonicity and stability, but some definitions fail to exhibit countable stability and may even ascribe a set of positive dimension to a countable set.

2.1.3 Hausdorff measure

Let U be a nonempty subset of n dimensional Euclidean space R^n. Diameter of U is defined as $\mid U \mid = \sup\{\mid x - y \mid : x, y \in U\}$. If $\{U_i\}$ be countable (or finite) collection of sets of diameter at most δ that cover F, i.e. $F \subset \cup_{i=1}^{\infty} U_i$, with $0 < \mid U_i \mid \leq \delta$, for each i, then we say that $\{U_i\}$is a $\delta-$cover of F. Suppose that F is a subset of $R^n$and we define

$$H_\delta^s(F) = \inf\left\{\sum_{i=1}^{\infty} \mid U_i \mid^s : \{U_i\} \text{ is a } \delta - \text{cover of } F\right\} \tag{2.1}$$

where s is a non-negative number and infimum is taken with all possible $\delta-$covers.

As δ decreases, the class of permissible covers of F is reduced. Therefore, the infimum increases and so approaches a limit as $\delta \to 0$. We write

$$H^s(F) = \lim_{\delta \to o} H_\delta^s(F). \tag{2.2}$$

This limit exits for any subset F of R^n, though the limiting value can be 0 or ∞. We call $H^s(F)$, the $s-$dimensional Hausdorff measure of F. If $\{F_i\}$ is any countable collection of disjoint Borel sets, then

$$H^s\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} H^s(F_i). \tag{2.3}$$

Hausdorff measures generalize the familiar ideas of length, area, volume, etc. It may be shown that, for a subset of R^n, n-dimensional Hausdorff measure, to within a constant multiple, equals n-dimensional Lebesgue measure, i.e. the usual n-dimensional volume. More precisely if F is a borel subset of R^n, then

$$H^n(F) = c_n vol^n(F) \tag{2.4}$$

where the constant $c_n = 2^n \left(\frac{n}{2}\right)!/\pi^{\frac{n}{2}}$ is the reciprocal of the volume of an n-dimensional ball of diameter 1. Similarly, for 'nice' lower dimension subsets of R^n, we have $H^0(F)$ is the number of points of F; $H^1(F)$ gives the length of a smooth curve F; $H^2(F) = \left(\frac{4}{\pi}\right) \times area(F)$ if F is a smooth surface; $H^3(F) = \left(\frac{6}{\pi}\right) \times vol(F)$; and $H^m(F) = c_m vol^m(F)$ if F is a smooth m-dimensional sub-manifold of R^n.

The scaling properties of length, area and volume are well known. On magnification by a factor λ, the length of a curve is multiplied by λ, the area of a plane region is multiplied by λ^2, and the volume of a 3-dimensional object is multiplied by λ^3. As might be anticipated, s-dimensional Hausdorff measure scales with a factor λ^s. Such scaling properties are fundamental to the theory of fractals.

Lemma 1. *[3]* Scaling Property: *If $F \subset R^n$, and $\lambda > 0$, then $H^s(\lambda F) = \lambda^s H^s(F)$, where $\lambda F = \{\lambda x : x \in F\}$, i.e. the set F scaled by a factor λ.*

2.1.4 Hausdorff dimension

As $H^s_\delta(F)$ is non-increasing with s so that $H^s(F)$ is also non-increasing with s. Now, if $t > s$, and $\{U_i\}$ is $\delta-$cover of F, we have

$$\sum | U_i |^t \leq \delta^{t-s} \sum | U_i |^s \tag{2.5}$$

and so taking infimum for each fixed s,

$$H^t_\delta(F) \leq \delta^{t-s} H^s_\delta(F). \tag{2.6}$$

Letting $\delta \to 0$, we see that if $H^s(F) < \infty$, then $H^t(F) = 0$ for $t > s$. Thus it shows that there is a critical value of s at which $H^s(F)$ jumps from ∞ to 0. This critical value is called the Hausdorff dimension of F,

$$H^s(F) = \begin{cases} \infty & if\ s < \dim_H F \\ 0 & if\ s > \dim_H F. \end{cases} \tag{2.7}$$

The critical value $s_0 : 0 < s_0 < 1$ at which $H^s(F)$ jumps from ∞ to 0 is the Hausdorff dimension of F. It can be shown that for a totally disconnected uncountable set F, $0 < H^s(F) < \infty \Longleftrightarrow 0 < s_0 < 1$.

2.1.5 Box Dimension

Box dimension is one of the most widely used dimension. It is easy to use both analytically and numerically. Let F be any nonempty bounded subset of R^n and let $N_\delta(F)$ be the smallest number of sets of diameter *at most* δ which can cover F. Then the lower and upper box dimensions of F are respectively defined as

$$\underline{\dim}_b(F) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

$$\overline{\dim}_b(F) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Box dimension of F is defined as $\dim_b(F) = \underline{\dim}_b(F) = \overline{\dim}_b(F)$, whenever two limits are equal.

Box dimension is very simple but it gives in some cases inadmissible results. For example, Box dimension of countable sets may have dimension one. Consider the set of rational numbers of [0,1]. Let us cover this set by a partition with interval δ. Then $N(\delta) = \frac{1}{\delta}$. And hence the Box dimension of the set is 1. But the set of rational number is a countable set. The union of countable zero dimensional singleton sets has dimension zero. Inspite of this paradoxical result this definition is widely used mainly for its simplicity and geometrical appeal.

In case of Box dimension we essentially cover the set with boxes of fixed sizes where as in the Hausdorff dimension we allow all the sizes smaller than δ. This is the crucial difference between two definitions. One can give simple arguments to show that the Hausdorff dimension of the set of rational number is indeed zero. Set of rational number is countable and hence we can label each rational number by a positive integer K. Now cover the $K\ th$ rational number by an interval of length $\frac{\delta}{2^k}$. Then the sum becomes $\sum \frac{\delta^s}{2^{ks}}$. It is also bounded by $K\delta^s$. Now as $\delta \to 0$, for $s > 0$, the limit becomes zero. Hence Hausdorff dimension of the set becomes zero.

2.1.6 Examples of Fractals

We give here a few simple recursive constructions leading to a few interesting examples of fractal sets which arise significantly in various applications in recent literature [3]. The middle third Cantor set is one of the most well known and easily constructed fractals. It is constructed from a unit interval by a sequence of deletion operations of removing middle third portion of certain relative length at each scale. Stated in another way, the classical middle third Cantor set, for example, consists of all the points between 0 and 1 that can be represented using only 0's and 2's in ternary representation. It is self-similar, because at every scale, it is equal to two copies of itself, if each copy is shrunk by a factor of $\frac{1}{3}$ and translated. Another familiar fractal, though not considered in this work, is Von Koch curve. To construct Von Koch curve one have to begin again with the unit interval and remove then middle third of it by replacing the other two sides of the equilateral triangle based on the removed segments. Repeating the process the sequence of polygonal curves approaches a limiting curve, called the Von koch curve. Similarly, Sierpinski gasket is obtained by repeatedly removing of an equilateral triangle by the three trainless of half the hight. The highly intricate structure of the Julia set is constructed from the quadratic function $f(z) = z^2 + c$, for a suitable constant c. Although, the set is not strictly self-similar like as Cantor set or Von koch curve, it is quasi self-similar. These are a few examples of sets that are commonly known as fractals.

2.1.7 Occurrence of Fractals

There are abundance of natural objects and processes that have a fractal like structure. Fractal structures provide different ways of modeling biological systems [4]. The usage of fractals are common, for instance, in root system analysis, in the study of variation of shapes of dental crown pattern, and also in the analysis of cancer cells images. Human lungs, breathing patterns of mammals, branching of trees and so on, also have self-similar fractal like spatial and/or temporal structures. Fractal models are also used to understand the shape of neurons, growth of bacterial colonies, forest tree distributions, population distribution in metros and large cities. Geometry of fractals also appear in cloud boundaries, topographical surfaces, coastlines, turbulence in fluids, and in daily fluctuations in stock markets and other related fields of applied sciences [1]. However, most of these natural and biological/ financial objects are not actual fractals. Their fractal features disappear if they are viewed at sufficiently small scales. Only over a certain level of scales they appear and/or behave like fractals.

2.2 Ultrametric Space

An ultrametric space is a special kind of metric space in which the triangle inequality is replaced by a stronger inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ [20, 21, 22, 23]. The associated metric is also called non-Archimedean metric. To a beginner, properties of an ultrametric space may seem rather unusual but they, nevertheless, appear naturally in many applications. Many self-similar sets such as Cantor sets are bilipschitz equivalent to ultrametric spaces. Thus, ultrametric space is very relevant in the study of Cantor sets as well as other self similar sets.

Definition 2. *An ultrametric (or nonarchimedean metric) on a set X is a mapping $d : X \times X \to R$ with the following properties.*

(i) For $a, b \in X$, $d(a, b) \geqslant 0$ and $d(a, b) = 0$ if and only if $a = b$.

(ii) For $a, b \in X$, $d(a, b) = d(b, a)$.

(iii) For $a, b, c \in X$, $d(a, c) \leq \max\{d(a, b), d(b, c)\}$ (strong triangle inequality).

Note that if $d(a, b) \neq d(b, c)$, then $d(a, b) = \max\{d(a, b), d(b, c)\}$.

Definition 3. *Let K be a field. A nonarchimedean absolute value (norm) on K is a mapping $|.| : K \to R$ such that for any $a, b \in K$, we have*

(i) $|a| \geq 0$.

(ii) $|a| = 0$ *if and only if* $a = 0$.

(iii) $|ab| = |a|\,|b|$.

(iv) $|a + b| \leq \max(|a|, |b|)$.

It follows from the definition 3(iv) that $|n.1| \leq 1$, for any $n \in Z$. Also, if $|a| \neq |b|$ for some $a, b \in K$, then the triangle inequality becomes an equality $|a + b| = \max(|a|, |b|)$.

The set K becomes a ultrametric space via the metric induced by the non-archimedean norm $d(a, b) = |a - b|$. The subsets $\overline{D}(a, r) = \{b \in K : |b - a| \leq r\}$ for any $a \in K$ and any real number $r > 0$ are called closed balls or simply balls in K, Likewise the open balls $D(a, r) = \{b \in K : |b - a| < r\}$ form the neighbourhood of a in the metric space K.

A Cauchy sequence in X is a sequence $\{x_n\}_{n=1}^{\infty}$ such that for any $\epsilon > 0$, there exists $N \in \mathbf{N}$, ($\mathbf{N}$ being the set of Natural numbers), such that for all $m, n \geq N$ $(m, n \in \mathbf{N})$, $d(x_m, x_n) < \epsilon$ Note that by the strong triangle inequality, this is equivalent to $d(x_{n+1}, x_n) < \epsilon$ for all $n \in \mathbf{N}$. Further, a sequence $\{x_n\}$ converges to a non-zero limit x iff the sequence is eventually constant in the ultrametric, i.e. $|x_n| = |x|$ for $n > N$. As a consequence, nonarchimedean analysis turns out to be, in many situations, simpler than the traditional analysis on an archimedean field. X is complete if every Cauchy sequence converges to a limit (necessarily unique because of (i)).

Definition 4. *The field K is called nonarchimedean if it is equipped with* *a nonarchimedean absolute value such that corresponding metric space K is complete (that is, every Cauchy sequence in K converges).*

In the following, a few salient features of ultrametric topology is enlisted. The strong triangle inequality leads to several geometrical as well as topologically interesting consequences. For example,

Proposition 1. *Let K be a complete non- archimedean field. Then*

i) Given $a,\ b \in K$ and $s \geq r > 0$ such that $a \in D(b,s)$, we have $D(a,r) \subset D(b,s)$, and $D(a,s) = D(b,s)$.

ii) Given $a,\ b \in K$ and $s \geq r > 0$ such that $a \in \overline{D}(b,s)$, we have $\overline{D}(a,r) \subset \overline{D}(b,s)$, and $\overline{D}(a,s) = \overline{D}(b,s)$.

iii) If $D_1, D_2 \subseteq C_k$ are two balls such that $D_1 \cap D_2 \neq \phi$ then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

iv) All balls in K are both open and closed topologically. Such balls are called clopen balls.

v) K is totally disconnected as topological space; that is, only nonempty connected subsets are singletons.

Proof. i) Given $x \in D(a,r)$, because $|x-a| < r \leq s$ and $|a-b| < s$, we have $|x-b| \leq \max\{|x-a|, |a-b|\} < s$, and hence $x \in D(b,s)$. The reverse inclusion in the case $r = s$ is similar.

ii) Given $x \in \overline{D}(a,r)$, because $|x-a| < r \leq s$ and $|a-b| < s$, we have $|x-b| \leq \max\{|x-a|, |a-b|\} < s$, and hence $x \in \overline{D}(b,s)$. The reverse inclusion in the case $r = s$ is similar.

iii) Pick $c \in D_1 \cap D_2$. From (i) and (ii) we can say that c is a centre of each ball. That is each D_i can be written as either $D(c, r_i)$ or

$\overline{D}(c, r_i)$. After possibly exchanging D_1 and D_2, either $r_1 > r_2$ or else $r_1 = r_2$ with either D_1 closed or D_2 open (or both) . Then $D_1 \supseteq D_2$.

iv) To show that an open ball $D(a, r)$ is topologically closed, pick any $x \in C_k \setminus D(a, r)$. If the two balls $D(x, r)$ and $D(a, r)$ intersects, then one contains the other by part (iii) and hence they coincide by part (i). That contradicts our assumption that $x \notin D(a, r)$, and therefore $D(a, r) \subseteq C_k \setminus D(a, r)$, as desired.

To show that a closed ball $\overline{D}(a, r)$ is open, pick any $x \in \overline{D}(a, r)$. Since the balls $\overline{D}(x, r)$ and $\overline{D}(a, r)$ intersects at x, one contains the other by part (iii), and hence coincide by part (ii). Thus, $D(x, a) \subseteq \overline{D}(x, r) = \overline{D}(a, r)$, as desired.

v) Suppose $X \subseteq K$ is set containing two distinct points a, b. Let $r = |a - b| > 0$. Then by part (iv), $X \cap D(a, r) \ni a$ and $X \setminus D(a, r) \ni b$ are both nonempty open subsets of X, and hence X is disconnected. Thus, the only connected subsets of K are the empty set and singletons.

■

Thus, it is clear that a non-archimedean ball does not have well defined centres; indeed every point of the ball can be called its centre. Further, the clopen balls form the basis for the induced topology on a ultrametric space.

Example 1. *Let n be a natural number. Then it has a unique representation as a product of powers of distinct primes. Let p be a prime from this product of primes. Denote by $ord_p n$, exponent of p in this representation and put $|n|_p = p^{-ord_p n}$, called the p-adic norm on the set of natural numbers . Indeed, (i) and (ii) of Definition obviously hold for the defined* *norm. Moreover, the stronger inequality (iii) also holds. This definition of norm can easily be extended over the set of rational Q. Let* $Q \ni r = \frac{n}{m}$, $(n, m) = 1$, $m \neq 0$. *Then* $|r|_p = p^{-ord_p r}$, $ord_p r = ord_p n - ord_p m$. *The completion of the p-adic norm over the field of rational Q leads to the local field* Q_p *for each prime p.*

The set of real numbers R with usual metric is Archimedean. That is, it satisfies the Archimedean axioms that can be geometrically stated as follows. Let us consider a segment of real line of length s and another smaller segment of length l. Then there exists a natural number n such that $n.l > s$. That is to say, if we append a short segment of line to itself sufficient number of times we get a longer segment. Let us now give a similar geometric argument to clarify the significance of the corresponding non-archimedean property: In the ring of p-adic integers Z_p, defined by $|x|_p \leq 1$, $x \in Z_p$, appending a segment to itself one could make the resulting segment shorter than the original one. Let $p = 2$ and let L be some 'segment of length' $\frac{1}{2}$, say, $L = 2$. Then doubling the segment we, obtain a 'segment' $2 \cdot L = 4$, and for which we have $|4|_2 = \frac{1}{2^2}$. Thus the 'doubled segment' becomes twice as short as the original in a 2-adic space, in contradistinction with our usual Archimedean commonsense. The origin of this peculiarity is again hidden in the strong triangle inequality that plays a crucial role in a non-Archimedean space.

We say K is discretely valued if its value group $|K| = \{|x| : x \in K\}$ is a discrete subgroup of R; that means it must be isomorphic to Z. Some examples of complete, discretely valued, ultrametric fields are

(a) the field of formal power series $K\,(G)$ over the field K.

(b) the local fields Q_p of p-adic numbers.

(c) any finite extension of either of these.

On the other hand, infinitesimals in nonstandard analysis are considered as examples of non-discretely valued ultrametric field. Here, any infinitesimal number can be represented as a certain equivalence classes of sequence of real numbers.

Before closing this subsection, let us prove an important result revealing the generic structure of a discretely valued field K for which the multiplicative value group $|K^\times|$ is a discrete subset of $R_+^\times$.

Proposition 2. *[18] The subgroup $|K^\times| \subseteq R_+^\times$ either is dense or is discrete; in the latter case there is a real number $0 < r < 1$ such that $|K^\times| = r^Z$.*

Proof. Let us assume that the multiplicative group $|K^\times|$ is not dense in $R_+^\times$. Then the additive group $\log|K^\times|$ is not dense in R. Set $\rho = \sup(\log|K^\times| \cap (-\infty, 0))$. We claim that ρ actually is the maximum of this set. Otherwise there is a sequence $\rho_1 < \rho_2 < \cdots$ in $\log|K^\times|$ which converges to ρ. But then $(\rho_i - \rho_{i+1})$ is a sequence in $(\log|K^\times| \cap (-\infty, 0))$ converging to zero which implies that $\rho = 0$. In that case we find for any $\epsilon > 0$ a $\sigma \in \log|K^\times|$ such that $-\epsilon < \sigma < 0$. Consider now an arbitrary $\tau \in R$ and choose an integer $m \in Z$ such that $m\sigma \leq \tau < (m-1)\,\sigma$. It follows that $0 \leq \tau - m\sigma < -\sigma < \epsilon$ and hence that $\log|K^\times|$ is dense in R which is a contradiction. This establishes the existence of this maximum and consequently also the existence of $r = \max(|K^\times| \cap (0,1))$. Given any $s \in |K^\times|$ there is an $m \in Z$ such that $r^{m+1} < s \leq r^m$. We then have $r < s/r^m \leq 1$ which by the maximality of r, implies that $s = r^m$. This shows that $|K^\times| = r^Z$. ■

2.3 Nonstandard Analysis

Robinson showed that proper extension $^*R$ of the field of real numbers R could be constructed, which contains infinitely small and infinitely large numbers [35]. The theory, first evolved by using free ultrafilters and equivalence classes of sequences of reals, was later formalized by Nelson as an axiomatic extension of Zermelo set theory. We do not intend to give here a detailed account of the field which is now developed as a major branch of Mathematical analysis. We shall just recall the results which we think to be relevant for our subsequent presentation [36].

Let us briefly recall the ultrapower construction of Robinsion. Though less direct than the axiomatic approach, it allows one to get a more intuitive contact with the origin of the new structure. Indeed the new infinite and infinitesimal numbers are formulated as equivalence classes of sequences of real numbers, in a way quite similar to the construction of R from rationals.

Let N be the set of natural numbers. A free ultrafilter u on N is defined as follows:

u is a non empty set of subsets of N [$p(N) \supset u \supset \phi$], such that:

(1) $\phi \in u$.

(2) $A \in u$ and $B \in u \Longrightarrow A \cap B \in u$.

(3) $A \in u$ and $B \in p(N)$ and $B \supset A \Longrightarrow B \in u$.

(4) $B \in p(N) \Longrightarrow$ either $B \in u$ or $\{j \in N : j \notin B\} \in u$, but not both.

(5) $B \in p(N)$ and B is finite $\Longrightarrow B \notin u$.

Then the set $^*R$ is defined as the set of the equivalence classes of all sequences of real numbers modulo the equivalence relation:

$a \equiv b$, provided $\{j : a_j = b_j\} \in u$,, a and b being the two sequences $\{a_j\}$ and $\{b_j\}$.

Similarly, a given relation is said to hold between elements of $^*R$ if it holds termwise for a set of indices which belongs to the ultrafilter. For example:

$a < b \Longleftrightarrow \{j : a_j < b_j\} \in u.$

R is isomorphic to a subset of $^*R$, since one can identify any real $r \in R$ with the class of sequences $\{r, r, \ldots\}$. It is the axiom of maximality (4) which ensures $^*R$ to be an order field. In particular, thanks to this axiom, a sequences which takes its value in a finite set of numbers is equivalent to one of these numbers, depending on the particular ultrafilter u. This allows one to solve the problem of zero divisors: indeed the fact that $(0, 1, 0, 1, \ldots).(1, 0, 1, 0, \ldots) = (0, 0, 0, \ldots)$ does not imply that there are zero divisors, since axiom (4) ensures that one of the sequences is equal to 0 and other to 1.

That $^*R$ contains new elements with respect to R become evident when one considers the sequence $\{\omega_j = j\} = \{1, 2, 3, \ldots n, \ldots\}$. The equivalence class of this sequence, ω, is larger than any $r \in R$, $\{j : \omega_j > r\} \in u$, so that what ever $r \in R$, $\omega > r$. It is straightforward that the inverse of ω is infinitesimal.

Hence the set $^*R$ of hyper-real numbers, as it is also called sometime in the literature, is a totally ordered and non-Archimedean field, of which the set R of standard numbers is a subset. $^*R$ contains infinite elements, i.e. numbers A such that $\forall n \in N$, $|A| > n$ (where N refers to the integers). It also contains infinitesimal elements, i.e. numbers ϵ such that $\forall n \neq 0 \in N$, $|\epsilon| < \frac{1}{n}$. A finite element C is also defined formally as: $\exists\, n \in N$, $|C| < n$. Now all hyper-integers ${}^*N$ (of which N and the set of infinite hyper-integers ${}^*N_\infty$ are subsets), hyper-rationales ${}^*Q$, positive or negative numbers, odd or even hyperintegers, etc. may be defined systematically.

An important result is that any finite number a can be spilt up in a unique way as the sum of a standard real number $r \in R$ and an infinitesimal number $\epsilon \in J : a = r + \epsilon$. In other words the set of finite hyper-reals consists of a set of new real numbers (a) clustered infinitesimally closely around each ordinary real r. The set of these additional numbers $\{a\}$ is called monad of r. More generally, one may demonstrate that any hyper-real number A may be decomposed in a unique way as $A = N + r + \epsilon$, where $N \in {}^*N$, $r \in R \cap [0, 1)$ and $\epsilon \in J$.

The real r is said to be the 'standard part' of finite hyper-real a, this function being denoted by $r = st(a)$. This new operation, " take the standard part of " play a crucial role in the theory, since it allows one to solve the contradictions which prevented previous attempts, such as Leibniz's, to be developed rigorously. Indeed, apart from the strict equality "=", one introduces an equivalence relation, "$\approx$", meaning "infinitely close to" defined by $a \approx b \Longleftrightarrow st(a - b) = 0$. Hence the two numbers of the same monad are infinitely close to one another, but not strictly equal. Similarly the derivative of a function will be written in the form $\frac{df}{dx} = st\left\{\left[f\left(x+\epsilon\right) - f\left(x\right)\right]/\epsilon\right\}$, provided the expression is finite and independent of ϵ.

2.4 Cantor set

2.4.1 Introduction

A Cantor set is a compact, perfect, totally disconnected, metrisable topological space. In this thesis we consider a Cantor set C that is realized as a (proper) subset of the real line. It is of measure zero if the Lebesgue measure of the set is zero. Otherwise this has a positive measure. Such a set is also said to be a fat Cantor set. Topological dimension of a Cantor set is also zero. Although, both the linear Lebesgue measure and the usual sense of dimension are trivial, a Cantor has the cardinality of the continuum c. To reveal the intricate geometric structure of such a set, nonlinear Hausdorff measure and Hausdorff dimension are generally considered to be most useful. A set C is said to be an s- set if the corresponding Hausdorff measure has a finite non-zero value viz; $0 < H^s(C) < \infty$ [3]. The real number s then denotes the Hausdorff dimension of the Cantor set.

As already noted in introduction, a Cantor set is an example of a self-similar fractal set that arises in various fields of applications. The chaotic attractors of a number of one dimensional maps; such as the logistic maps, turn out to be topologically equivalent to Cantor sets. Cantor set also arises in electrical communications [1], in biological systems [2], and diffusion processes [10]. Recently there have been a lot of interest in developing a framework of analysis on a Cantor like fractal sets [12, 13]. Because of the disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approaches based on the fractional derivatives [14, 15] and the measure theoretic harmonic analysis [16] have already been considered at length in the literature. Parvate and Gangal [19], for instance, considered the so called staircase functions, having a Cantor function like properties, in their formulation of the analysis. Their approach is based mainly on developing a formalism for replacing the linear Lebesgue measure (variable) viz., $x \in C \subset [0,1]$ by a nonlinear Hausdorff measure theoretic variable, viz., the integral staircase function $S_C^s(x) \approx x^s$ when $x(\approx 0) \in C$ and s is the Hausdorff dimension of C. However, a simpler intuitively appealing approach is still considered to be welcome.

2.4.2 Basic Definitions

A Cantor set C is defined as a countable intersection of finite unions of closed (and bounded) subsets of R. For definiteness, let $C \subset I = [0,1]$. Then, by definition, $C = \bigcap_1^\infty F_n = \bigcap_{n=1}^\infty \bigcup_{m=1}^{p^n} F_{nm}$ where $F_{nm} \subset I$ are closed with $F_{00} = I$. Equivalently, C is also defined as $C = I - \bigcup_{i=1}^\infty O_i$ where O_i are open intervals which are deleted recursively from I. Consequently, a Cantor set is often defined as the limit set of an iterated function system (IFS) $f = \{f_i \mid f_i : I \to I,\ i = 1, 2....,p\}$ so that $C = f\ (C)$. For definiteness, we consider binary Cantor sets in which each application of the IFS removes an open interval from a closed subinterval splitting it into two disjoint closed subintervals of the form

$$F = F_0 \cup O \cup F_1. \tag{2.8}$$

The deleted interval O is called the gap and the two closed components are the bridges. As an example let us consider a middle α Cantor set C_α

which arises as the limit set under the IFS

$$f_i(x) = \beta x + i(1-\beta), \quad i = 0, 1 \tag{2.9}$$

where the scale factor β is defined by $\alpha + 2\beta = 1$. Each iteration of the IFS removes an open interval (i.e. a gap) of length proportional to α from a closed subinterval of I, leaving out two bridges of size proportional to β each. The IFS (2.8) satisfies the *open set condition* (OSC) if there exists a non-empty bounded open set S such that $\underset{i}{\cup} f_i\ (S) \subseteq S$. It follows accordingly that $\beta \in (0, \frac{1}{2})$. Since the total length of the disjoint open intervals viz., $\sum_{i=1}^{\infty} \mid O_i \mid = \sum_{i=1}^{\infty} \alpha(2\beta)^{i-1} = 1$, the middle α Cantor set is of measure zero with the Hausdorff dimension $s = \frac{\log 2}{\log \frac{1}{\beta}}$. For latter reference, let us recall that a point $x \in C_\alpha$ has the unique infinite word representation

$$x = (1-\beta)\sum_{0}^{\infty} x_i \beta^i = x_0 x_1 x_2 \cdots, \ x_i \in \{0, 1\}.$$

More generally, when q open intervals each of size α are deleted leaving out p equal closed intervals of size β so that $q\alpha + p\beta = 1$, then the OSC gives $\beta \in (0, \frac{1}{p})$. The length of the deleted open intervals add up to 1 viz., $\Sigma(q\alpha)(p\beta)^{n-1} = 1$. The corresponding measure zero set $C_{\alpha,p}$ has the Hausdorff dimension $\frac{\log p}{\log \frac{1}{\beta}}$.

Returning to the discussion of the binary Cantor set we recall that the set C_α is also a homogeneous and uniform Cantor set. It is homogeneous since the scale factors in each component of the IFS are same. The set is uniform because each deleted open interval also is of constant proportion α of the length of the previous (defining) closed interval.

A positive 1-set $\tilde{C}$, on the other hand, is obtained if the deletion process removes open intervals of variable sizes.

Example 2. *Let at each step we remove α_n portion of the length of each component of the previous closed set F_{n-1} so that $F_{n-1} = F_{n0} \cup O_n \cup F_{n1}$ and $\mid O_n \mid = \alpha_n \mid F_{n-1} \mid, \mid F_{n0} \mid = \mid F_{n1} \mid = \frac{1}{2}(1-\alpha_n) \mid F_{n-1} \mid$. By induction, each of 2^n components of F_n has length $\mid F_{ni} \mid = \frac{1}{2^n}\prod_0^n (1-\alpha_j)$, $i = 1, 2, \cdots 2^n$. Consequently, $m\ (\tilde{C}) = \lim_{n\to\infty} \mid F_{n-1} \mid = \prod_0^\infty (1-\alpha_i) > 0$ when $\Sigma\ \alpha_n < \infty$.*

Example 3. *Suppose at the nth step an open interval of length $\frac{\delta}{3^n}$, $(0 < \delta < 1)$ is removed from each of the 2^n components of F_n. The length of each component of F_n is $\frac{1}{2^n}\ (1 - \frac{\delta}{3} - \cdots \frac{2^{n-1}\delta}{3^n})$. The sum of the lengths of all the open intervals removed is $\Sigma\ \frac{2^n\delta}{3^{n+1}} = \delta$, so that $m\ (\tilde{C}) = 1 - \delta$.*

In both the above examples, $0 < H^s(\tilde{C}) = m(\tilde{C}) = l < \infty$ when $s = 1$. Even as the Cantor set $\tilde{C}$ is nowhere dense in I, it has a non-zero measure because of the fact that the defining iteration process now removes the open interval at a slower rate in comparison to the middle α set C_α. In fact, the relative difference of the lengths of the deleted open middle α and α_n intervals $O_{\alpha n}$ and $\tilde{O}_{\alpha n}$ respectively is given by

$$\begin{aligned} \Big|\mid O_{\alpha n} \mid - \mid \tilde{O}_{\alpha n} \mid\Big| &= \left|\alpha\beta^n - \frac{1}{2^n}\alpha_n \prod_1^n (1-\alpha_i)\right| \\ &= \left|1 - \frac{\alpha_n}{\alpha}\prod_1^n\Big(\frac{1-\alpha_i}{1-\alpha}\Big)\right| |O_{\alpha n}| \\ &\geq |1-\gamma||O_{\alpha n}| \end{aligned}$$

where $\frac{\alpha_n}{\alpha}\prod\limits_1^n(\frac{1-\alpha_i}{1-\alpha}) \to \gamma$ for $n \to \infty$. Note that the above lower bound exists, otherwise $\tilde{C}$ would have been a set of measure zero. For the modified middle $\frac{1}{3}$rd set (Example 2) one has the exact equality

$$\left| \mid O_{\frac{1}{3}n} \mid - \mid \tilde{O}_\delta \mid \right| = (1-\delta) \mid O_{\frac{1}{3}n} \mid .$$

The emergence of the positive measure of $\tilde{C}$ can, therefore, be explained in a dynamical sense provided the said set $\tilde{C}$ is seen as being evolved from a given zero measure set because of a possible *principle* allowing for a deformation of scales in the deletion process. As is evident the relative scaling of the lengths of infinitesimal elements of the deleted open intervals $\tilde{O}_{\alpha n}$ over the corresponding α interval $O_{\alpha n}$, indeed captures the origin of the positive measure even in the totally disconnected perfect set $\tilde{C}$. In the following we offer a new ultrametric explanation of the growth of the positive measure of $\tilde{C}$ over C_α [chapter 7]. The class of ultrametrics that we consider is not only *scale invariant* (in the sense of a power law), but also *reparametrisation invariant* (that is, the invariance under a reparatrisation invariance of the form $X(t) \to \tilde{X}(t) = X(f(t))$ where the otherwise arbitrary function f satisfies the conditions $f' > 0$, along with the boundary condition $f(0) = 0$, $f(1) = 1$). As will be explained later, the scale variation can, therefore, be interpreted as a reflection of the underlying reparametrisation invariance of the valuation. Incidentally, we note that for a measure 1 set the functions defining an IFS may not have a closed form [45].

2.4.3 Thickness

The measure of thickness of a Cantor set has various applications in number theory and dynamical systems [39, 40]. To recall the definition, let $F_i = I - \bigcup_{l=0}^{i} O_l$ form a defining sequence of the Cantor set C. The 2^i components of F_i are the closed intervals F_{ij}, $j = 1, 2, \cdots 2^i$, which are the bridges. The deleted intervals O_i are the gaps of C. Let O_{i_k} denote the open deleted subinterval of a bridge F_{ij} of F_i which divides F_{ij} into two smaller bridges F_{ij}^L and F_{ij}^R of F_i. Let

$$\tau(F_i) = \inf_j \left\{ \frac{\mid F_{ij}^L \mid}{\mid O_{i_k} \mid}, \frac{\mid F_{ij}^R \mid}{\mid O_{i_k} \mid} \right\}.$$

The thickness $\tau(C)$ is defined by $\tau(C) = \sup_i \tau\ (F_i)$, where sup is evaluated over the defining sequences of C. For a set A containing an interval, $\tau(A) = \infty$, by definition.

For the middle α- Cantor set C_α, it follows that

$$\frac{\mid F_{ij}^L \mid}{\mid O_{i_k} \mid} = \frac{\mid F_{ij}^R \mid}{\mid O_{i_k} \mid} = \frac{\beta \mid F_{ij} \mid}{\alpha \mid F_{ij} \mid} = \frac{\beta}{\alpha}$$

so that $\tau(C) = \frac{\beta}{\alpha}$.

For a positive measure set $\tilde{C}$, on the other hand, one has $\frac{|F_{ij}^L|}{|O_{i_k}|} = \frac{|F_{ij}^R|}{|O_{i_k}|} = \frac{1-\alpha_n}{2\alpha_n}$, leading to $\tau(\tilde{C}) = \infty$, as expected.

2.4.4 Cantor Function

A Cantor function [7] is a nonconstant and non-decreasing continuous function $\phi : [0, 1] \to [0, 1]$ such that $\phi'(x) = 0$ a.e. with points of nondifferentiability x lying, for instance, in the Cantor set C_α .

To construct ϕ explicitly, let $\phi(0) = 0$, $\phi(1) = 1$. Assign $\phi(x)$ a constant value $\phi(x) = i2^{-n}$, $i = 1, 2, ..., 2^n - 1$ on each of the deleted open intervals (including the end points of the deleted interval) of C_α. Next, let $x \in C_\alpha$. Then, at the nth iteration, x belongs to the interior of exactly one of the 2^n remaining closed intervals each of length β^n. Let $[a_n, b_n]$ be one such intervals. Then $b_n - a_n = \beta^n$. Moreover, $\phi(b_n) - \phi(a_n) = 2^{-n}$. At the next iteration, assuming $x \in [a_{n+1}, b_{n+1}]$, ($a_n = a_{n+1}$), say, we have $\phi(a_n) \leq \phi(a_{n+1}) < \phi(b_{n+1}) \leq \phi(b_n)$. Define $\phi(x) = \lim_{n\to\infty} \phi(a_n) = \lim_{n\to\infty} \phi(b_n)$. Then $\phi : [0, 1] \to [0, 1]$ is a continuous, non-decreasing function. Also $\phi'(x) = 0$ for $x \in I \backslash C_\alpha$ when it is not differentiable at any $x \in C_\alpha$. (c.f., [24, 25]).

2.4.5 Ultrametric

The topology of a Cantor set is equivalent to an ultrametric topology. As already noted, a point $x \in C_\alpha$ has the unique infinite word representation

$$x = (1 - \beta)\sum_0^\infty x_i\beta^i = x_0x_1x_2 \cdots , \ x_i \in \{0, 1\}.$$

Let $L(x, y) = n$ such that $x_0 = y_0, \cdots, x_{n-1} = y_{n-1}$, $x_n \neq y_n$, $x, y \in C_\alpha$. The ultrametric $\tilde{d}_u$ is defined by $\tilde{d}_u(x, y) = p^{-L(x,y)}$ for any $p > 1$. This ultrametric is equivalent to the usual metric

$$C_1\tilde{d}_u(x, y) \leq d(x, y) \leq C_2\tilde{d}_u(x, y)$$

for two positive constants C_1 and C_2, where $d(x, y)$ is the usual metric.

The Cantor set thus consists of towers of closed balls (intervals) with countable intersection property. Further the fundamental neighbourhood system of any point consists of clopen balls.