CHAPTER 2

SOME RESULTS ON THE GROWTH PROPERTIES OF WRONSKIANS

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2.1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. In the sequel we use the following two notations:

$\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ *for* $k = 1, 2, 3, \cdots$ *and* $\log^{[0]} x = x$

and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ *for* $k = 1, 2, 3, \cdots$ and $\exp^{[0]} x = x$.

We recall the following definitions:

Definition 2.1.1 *The order ρ_f and lower order λ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If f is entire, one can easily verify that,

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

The results of this chapter have been published in **Archivum Mathematicum(BRNO)**, see **[10]**.

Definition 2.1.2 *The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f is defined as follows:*

$$\overline{\rho}_f = \limsup_{r\to\infty}\frac{\log^{[2]} T(r,f)}{\log r} \text{ and } \overline{\lambda}_f = \liminf_{r\to\infty}\frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire, then

$$\overline{\rho}_f = \limsup_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log r} \text{ and } \overline{\lambda}_f = \liminf_{r\to\infty}\frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 2.1.3 *The type σ_f of a meromorphic function f is defined as :*

$$\sigma_f = \limsup_{r\to\infty}\frac{T(r,f)}{r^{\rho_f}},\ 0<\rho_f<\infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho_f}},\ 0<\rho_f<\infty.$$

Definition 2.1.4 *A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f of finite lower order λ_f if*

(*i*) *$\lambda_f(r)$ is non negative and continuous for $r \geq r_0$, say,*

(*ii*) *$\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda_f'(r+0)$ and $\lambda_f'(r-0)$ exists,*

(*iii*) $\lim_{r\to\infty}\lambda_f(r) = \lambda_f$,

(*iv*) $\lim_{r\to\infty} r\lambda_f'(r)\log r = 0$ *and*

(*v*) $\liminf_{r\to\infty}\frac{T(r,f)}{r^{\lambda_f(r)}} = 1$.

Definition 2.1.5 *Let a be a complex number, finite or infinite. The Nevanlinna deficiency and Valiron deficiency of 'a' with respect to a meromorphic function f is defined as*

$$\delta(a;f) = 1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty}\frac{m(r,a;f)}{T(r,f)} \text{ and}$$
$$\Delta(a;f) = 1-\liminf_{r\to\infty}\frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}.$$

From the second fundamental theorem it follows that the set of values of $a \in C \cup \{\infty\}$ *for which* $\delta(a; f) > 0$ *is countable and* $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ *(cf.[17, p.43]) . If in particular* $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$ *, we say that* f *has the maximum deficiency sum.*

Definition 2.1.6 *A meromorphic function* $a = a(z)$ *is called small with respect to* f *if* $T(r, a) = S(r, f)$.

Definition 2.1.7 *Let* $a_1, a_2, \cdots, a_k$ *be linearly independent meromorphic functions and small with respect to* f. *We denote by* $L(f) = W(a_1, a_2, \cdots, a_k, f)$ *the wronskian determinant of* $a_1, a_2, \cdots, a_k, f$ *i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{(k)} & a_2^{(k)} & \cdots & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Since the natural extension of a derivative is a differential polynomial, in this chapter we prove our results for a special type of linear differential polynomials *viz.*, the wronskians. In the chapter we prove some new results depending on the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors which improve some earlier theorems.

2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.2.1 ([1]) *If* f *is meromorphic and* g *is entire then for all sufficiently large values of* r

$$T(f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 2.2.2 ([2]) *Let* f *be meromorphic and* g *be entire and suppose that* $0 < \mu < \rho_g \leq \infty$. *Then for a sequence of values of* r *tending to infinity,*

$$T(r, f \circ g) \geq T(\exp(r^{\mu}), f).$$

Lemma 2.2.3 ([25]) *Let f be a transcendental meromorphic function having maximum deficiency sum. Then*

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)}=1+k-k\delta(\infty;f).$$

Lemma 2.2.4 *If f be a transcendental meromorphic function with the maximum deficiency sum, then the order and lower order of $L(f)$ are same as those of f also the type of $L(f)$ is $\{1+k-k\delta(\infty;f)\}$ times that of f when f is of finite positive order.*

Proof. By Lemma 2.2.3, we have

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)}=1+k-k\delta(\infty;f)=A\ \ (say)$$

$$i.e.,\ \limsup_{r\to\infty}\frac{T(r,L(f))}{T(r,f)}=A\ .$$

So for given $\varepsilon\,(0<\varepsilon<1)$ we get for all large values of r that

$$\frac{T(r,L(f))}{T(r,f)}\leq A+\varepsilon$$

$$i.e.,\ \log T(r,L(f))\leq\log(A+\varepsilon)+\log T(r,f)$$

$$i.e.,\ \frac{\log T(r,L(f))}{\log T(r,f)}\leq 1+\frac{\log(A+\varepsilon)}{\log T(r,f)}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log T(r,L(f))}{\log T(r,f)}\leq 1.$$

Again for given $\varepsilon\,(0<\varepsilon<1)$ we get for a sequence of values of r tending to infinity that

$$\frac{T(r,L(f))}{T(r,f)}\geq A+\varepsilon$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log T(r,L(f))}{\log T(r,f)}\geq 1\ .$$

Hence

$$\limsup_{r\to\infty}\frac{\log T(r,L(f))}{\log T(r,f)}=1\ .$$

Similarly

$$\liminf_{r\to\infty} \frac{\log T(r, L(f))}{\log T(r, f)} = 1 \ .$$

Thus it follows from above that

$$\lim_{r\to\infty} \frac{\log T(r, L(f))}{\log T(r, f)} = 1 \ .$$

So

$$\begin{aligned}\rho_{L(f)} &= \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log T(r, L(f))}{\log T(r, f)} \\ &= \rho_f .1 = \rho_f \ .\end{aligned}$$

Also

$$\begin{aligned}\lambda_{L(f)} &= \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log T(r, L(f))}{\log T(r, f)} \\ &= \lambda_f .1 = \lambda_f \ .\end{aligned}$$

Again

$$\begin{aligned}\sigma_{L(f)} &= \limsup_{r\to\infty} \frac{T(r, L(f))}{r^{\rho_{L(f)}}} \\ &= \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho_f}} . \lim_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} \\ &= \sigma_f \{1 + k - k\delta(\infty; f)\} .\end{aligned}$$

This proves the lemma. ■

Lemma 2.2.5 *Let f be a transcendental meromorphic function with the maximum deficiency sum. Then the hyper order and (hyper lower order) of $L(f)$ and f are equal.*

Proof. By Lemma 2.2.3, we have

$$\lim_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta(\infty; f) = A \ \ (say)$$

$$i.e., \ \limsup_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} = A \ .$$

Hence for given $\varepsilon\,(0<\varepsilon<1)$ we get for all large values of r that

$$\begin{aligned}\frac{T(r,L(f))}{T(r,f)} &\leq A+\varepsilon\\ i.e.,\ \log T(r,L(f)) &\leq \log(A+\varepsilon)+\log T(r,f)\\ &= \log T(r,f)\left\{1+\frac{\log(A+\varepsilon)}{\log T(r,f)}\right\}\\ i.e.,\ \log^{[2]} T(r,L(f)) &\leq \log^{[2]} T(r,f)+\log\{1+o(1)\}\\ i.e.,\ \frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)} &\leq 1+\frac{\log\{1+o(1)\}}{\log^{[2]} T(r,f)}\\ i.e.,\ \limsup_{r\to\infty}\frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)} &= 1.\end{aligned}$$

Again for given $\varepsilon\,(0<\varepsilon<1)$ we get for a sequence of values of r tending to infinity that

$$\begin{aligned}\frac{T(r,L(f))}{T(r,f)} &\geq A+\varepsilon\\ i.e.,\ \limsup_{r\to\infty}\frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)} &\geq 1\ .\end{aligned}$$

Hence

$$\limsup_{r\to\infty}\frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)}=1\ .$$

Similarly

$$\liminf_{r\to\infty}\frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)}=1\ .$$

Thus it follows from above that

$$\lim_{r\to\infty}\frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)}=1\ .$$

So

$$\begin{aligned}\overline{\rho}_{L(f)} &= \limsup_{r\to\infty}\frac{\log^{[2]} T(r,f)}{\log r}.\lim_{r\to\infty}\frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)}\\ &= \overline{\rho}_f.1=\overline{\rho}_f\ .\end{aligned}$$

Also

$$\overline{\lambda}_{L(f)} = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} . \lim_{r\to\infty} \frac{\log^{[2]} T(r,L(f))}{\log^{[2]} T(r,f)}$$
$$= \overline{\lambda}_f .1 = \overline{\lambda}_f \; .$$

This proves the lemma. ■

Lemma 2.2.6 *For a meromorphic function f of finite lower order, lower proximate order exists.*

The lemma can be proved in the line of Theorem 1[21] and so the proof is omitted.

Lemma 2.2.7 *Let f be a meromorphic function of finite lower order λ_f. Then for $\delta\,(>0)$ the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is ultimately an increasing function of r.*

Proof. Since

$$\frac{d}{dr} r^{\lambda_f+\delta-\lambda_f(r)} = \left\{\lambda_f + \delta - \lambda_f(r) - r\lambda_f'(r)\log r\right\} r^{\lambda_f+\delta-\lambda_f(r)-1} > 0$$

for all sufficiently large values of r, the lemma follows. ■

Lemma 2.2.8 ([20]) *Let f be an entire function of finite lower order. If there exists entire functions $a_i\,(i = 1, 2, \cdots, n; n \leq \infty)$ satisfying $T(r, a_i) = o\{T(r,f)\}$ and $\sum_{i=1}^{n} \delta(a_i; f) = 1$, then $\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = \frac{1}{\pi}$.*

2.3 Theorems.

In this section we present the main results of the chapter.

Theorem 2.3.1 *Let f be a meromorphic function and g be a transcendental entire function satisfying*

(i) λ_f, λ_g *are both finite and*

(ii) $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$, *then*

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r, L(g))} \leq \frac{3.\rho_f.2^{\lambda_g}}{1 + k - k\delta(\infty; g)}.$$

Proof. If $\rho_f = \infty$, the theorem is obvious. So we suppose that $\rho_f < \infty$. Since $T(r,g) \leq \log^+ M(r,g)$, in view of Lemma 2.2.1, we get for all sufficiently large values of r,

$$T(f \circ g) \leq \{1 + o(1)\} T(M(r,g), f).$$

$$i.e., \ \log T(r, f \circ g) \leq \log\{1 + o(1)\} + \log T(M(r,g), f)$$
$$\leq o(1) + (\rho_f + \varepsilon) \log M(r,g)$$

$$i.e., \ \liminf_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r,g)} \leq (\rho_f + \varepsilon) \liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,g)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r,g)} \leq \rho_f . \liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,g)}. \tag{2.3.1}$$

As $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1$, so for given $\varepsilon\,(0 < \varepsilon < 1)$ we get for a sequence of values of r tending to infinity,

$$T(r,g) \leq (1+\varepsilon)\, r^{\lambda_g(r)} \tag{2.3.2}$$

and for all sufficiently large values of r,

$$T(r,g) \geq (1-\varepsilon)\, r^{\lambda_g(r)}. \tag{2.3.3}$$

Since $\log M(r,g) \leq 3T(2r,g)$, we have by (2.3.2),for a sequence of values of r tending to infinity,

$$\log M(r,g) \leq 3T(2r,g) \leq 3(1+\varepsilon)(2r)^{\lambda_g(2r)}. \tag{2.3.4}$$

Combining (2.3.3) and (2.3.4) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} . \frac{(2r)^{\lambda_g(2r)}}{r^{\lambda_g(r)}}.$$

Now for any $\delta\,(> 0)$, for a sequence of values of r tending to infinity,

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} . \frac{(2r)^{\lambda_g+\delta}}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}} . \frac{1}{r^{\lambda_g(r)}}$$

$$i.e., \frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)}.2^{\lambda_g+\delta} \tag{2.3.5}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately an increasing function of r by Lemma 2.2.7. Since $\varepsilon\,(>0)$ and $\delta\,(>0)$ are arbitrary, it follows from (2.3.5) that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \leq 3.2^{\lambda_g}. \tag{2.3.6}$$

Thus from (2.3.1)and (2.3.6) we obtain that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)} \leq 3.\rho_f.2^{\lambda_g}. \tag{2.3.7}$$

Now in view of Lemma 2.2.3 and (2.3.7) we get

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,L(g))} &= \liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{T(r,L(g))}\\ &\leq \frac{3.\rho_f.2^{\lambda_g}}{1+k-k\delta(\infty;g)}.\end{aligned}$$

This proves the theorem. ∎

Theorem 2.3.2 *Let f be meromorphic and g be transcendental entire such that $\rho_f<\infty, \lambda_g<\infty$ and $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[2]} T(r,f\circ g)}{\log T(r,L(g))} \leq 1.$$

Proof. Since $T(r,g)\leq \log^{+} M(r,g)$, in view of Lemma 2.2.1, we get for all sufficiently large values of r,

$$\begin{aligned}\log T(r,f\circ g) &\leq \log\{1+o(1)\}+\log T(M(r,g),f)\\ &\leq o(1)+(\rho_f+\varepsilon)\log M(r,g)\end{aligned}$$

$$i.e., \ \log^{[2]} T(r,f\circ g) \leq \log^{[2]} M(r,g)+O(1). \tag{2.3.8}$$

It is well known that for any entire function g, $\log M(r,g) \leq 3T(2r,g)$ $\{cf.[17]\}$. Then for $0<\varepsilon<1$ and $\delta\,(>0)$, for a sequence of values of r tending to infinity it follows from (2.3.5) that

$$\log^{[2]} M(r,g) \leq \log T(r,g)+O(1). \tag{2.3.9}$$

Now combining (2.3.8) and (2.3.9) we obtain for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, f \circ g) \leq \log T(r, g) + O(1)$$

$$i.e., \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, g)} \leq 1. \tag{2.3.10}$$

As by Lemma 2.2.3, $\lim\limits_{r \to \infty} \frac{\log T(r,g)}{\log T(r,L(g))}$ exists and is equal to 1, from (2.3.10) we obtain that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, L(g))} = \liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, g)} . \lim_{r \to \infty} \frac{\log T(r, g)}{\log T(r, L(g))}$$
$$\leq 1.1 = 1.$$

Thus the theorem is established. ■

Remark 2.3.1 *The inequality sign in Theorem* 2.3.2 *is best possible in the sense that '$\leq$' cannot be replaced by '$<$' only which is evident from the following example.*

Example 2.3.1 *Let* $f = g = \exp z$ *and* $L(g) = \begin{vmatrix} a_1 & g \\ a_1' & g' \end{vmatrix}$.
Taking $a_1 = 1$ *we see that* $L(g) = \exp z$.
Then $\rho_f = 1,\ \lambda_g = 1$ *and* $\sum\limits_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$.
Also $T(r, f \circ g) \sim \frac{e^r}{(2\pi^3 r)^{\frac{1}{2}}}$ *and* $T(r, L(g)) = \frac{r}{\pi}$.
Hence

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, L(g))} = 1 .$$

Remark 2.3.2 *The condition* $\rho_f < \infty$ *is essential in Theorem* 2.3.2 *as we see in the following example.*

Example 2.3.2 *Let* $f = \exp^{[2]} z,\ g = \exp z$ *and* $L(g) = \begin{vmatrix} a_1 & g \\ a_1' & g' \end{vmatrix}$.
Taking $a_1 = 1$ *we see that* $L(g) = \exp z$.
Then $\rho_f = \infty,\ \lambda_g = 1$ *and* $\sum\limits_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$.
Since $T(r, f) \leq \log^+ M(r, f)$, *we have*

$$T(r, f \circ g) = T\left(r, \exp^{[3]} z\right) \leq \log M\left(r, \exp^{[3]} z\right) = \exp^{[2]} r .$$

Now in view of the inequality

$$T(r,f) \le \log M(r,f) \le 3T(2r,f) \quad (cf.\ p.\ 18, [17])$$

we have

$$\begin{aligned} 3T(2r, f\circ g) &\ge \log M(r, f\circ g) \\ i.e.,\ T(r, f\circ g) &\ge \frac{1}{3}\log M\left(\frac{r}{2}, f\circ g\right) \\ i.e.,\ T(r, f\circ g) &\ge \frac{1}{3}\log M\left(\frac{r}{2}, \exp^{[3]} z\right) \\ i.e.,\ T(r, f\circ g) &\ge \frac{1}{3}\log\left(\exp^{[3]}\frac{r}{2}\right) = \frac{1}{3}\exp^{[2]}\frac{r}{2} \\ i.e.,\ \log^{[2]} T(r, f\circ g) &\ge \frac{r}{2} + O(1)\ . \end{aligned}$$

Also

$$\log T(r, L(g)) = \log r + O(1)\ .$$

Hence

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log T(r, L(g))} = \infty\ ,$$

which is contrary to Theorem 2.3.2.

Theorem 2.3.3 *Let f and g be two transcendental entire functions such that $\lambda_f > 0$, $\rho_g < \lambda_f \le \rho_f < \infty$, and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2 = \sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g)$. Also there exist entire functions b_i $(i = 1, 2, \cdots, n; n \le \infty)$ with*

(i) $T(r, b_i) = o\{T(r,g)\}$ as $r \to \infty$ for $i = 1, 2, \cdots, n$ and

(ii) $\sum_{i=1}^{n} \delta(b_i; g) = 1$. Then

$$\lim_{r\to\infty} \frac{\{\log T(r, f\circ g)\}^2}{T(r, L(f))\, T(r, L(g))} = 0.$$

Proof. In view of the inequality $T(r,g) \leq \log^{+} M(r,g)$ and Lemma 2.2.1 we obtain for all sufficiently large values of r,

$$\begin{aligned} T(f \circ g) &\leq \{1 + o(1)\} T(M(r,g), f) \\ i.e., \ \log T(r, f \circ g) &\leq \log\{1 + o(1)\} + \log T(M(r,g), f) \\ &\leq o(1) + (\rho_f + \varepsilon) \log M(r,g) \\ &\leq o(1) + (\rho_f + \varepsilon) r^{\rho_g + \varepsilon} . \end{aligned} \quad (2.3.11)$$

Again in view of Lemma 2.2.4, we get for all sufficiently large values of r,

$$\begin{aligned} \log T(r, L(f)) &> \left(\lambda_{L(f)} - \varepsilon\right) \log r \\ i.e., \ \log T(r, L(f)) &> (\lambda_f - \varepsilon) \log r \\ i.e., \ T(r, L(f)) &> r^{\lambda_f - \varepsilon} . \end{aligned} \quad (2.3.12)$$

Now combining (2.3.11) and (2.3.12) it follows for all sufficiently large values of r,

$$\frac{\log T(r, f \circ g)}{T(r, L(f))} \leq \frac{o(1) + (\rho_f + \varepsilon) r^{\rho_g + \varepsilon}}{r^{\lambda_f - \varepsilon}} . \quad (2.3.13)$$

Since $\rho_g < \lambda_f$, we can choose $\varepsilon (> 0)$ in such a way that

$$\rho_g + \varepsilon < \lambda_f - \varepsilon. \quad (2.3.14)$$

So in view of (2.3.13) and (2.3.14) it follows that

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} = 0 . \quad (2.3.15)$$

Again from Lemma 2.2.4 and Lemma 2.2.8 we get for all sufficiently large values of r,

$$\begin{aligned} \frac{\log T(r, f \circ g)}{T(r, L(g))} &\leq \frac{o(1) + (\rho_f + \varepsilon) \log M(r,g)}{T(r, L(g))} \\ i.e., \ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} &\leq (\rho_f + \varepsilon) \limsup_{r \to \infty} \frac{\log M(r,g)}{T(r, L(g))} \\ i.e., \ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} &\leq (\rho_f + \varepsilon) \limsup_{r \to \infty} \frac{\log M(r,g)}{T(r,g)} . \lim_{r \to \infty} \frac{T(r,g)}{T(r, L(g))} \\ i.e., \ \limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} &\leq (\rho_f + \varepsilon) . \pi . \frac{1}{1 + k - k\delta(\infty; g)} . \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,L(g))}\leq\rho_f.\pi.\frac{1}{1+k-k\delta(\infty;g)}\,. \tag{2.3.16}$$

Combining (2.3.15) and (2.3.16) we obtain that

$$\begin{aligned}\limsup_{r\to\infty}\frac{\{\log T(r,f\circ g)\}^2}{T(r,L(f))\,T(r,L(g))} &= \lim_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,L(f))}.\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,L(g))}\\ &\leq 0.\frac{\pi\rho_f}{1+k-k\delta(\infty;g)}=0\end{aligned}$$

$$i.e.,\ \lim_{r\to\infty}\frac{\{\log T(r,f\circ g)\}^2}{T(r,L(f))\,T(r,L(g))}=0\,.$$

This proves the theorem. ■

Theorem 2.3.4 *If f and g be two entire functions with f to be transcendental satisfying the following conditions:*
(i) $\lambda_f>0$, *(ii)* $\overline{\rho}_f<\infty$, *(iii)* $0<\lambda_g\leq\rho_g$ *and (iv)* $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$,
then

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f\circ g)}{\log^{[2]}T(r,L(f))}\geq\max\left\{\frac{\lambda_g}{\overline{\lambda}_f},\frac{\rho_g}{\overline{\rho}_f}\right\}.$$

Proof. We know that for $r>0$ $(cf.[28])$ and for all sufficiently large values of r,

$$T(r,f\circ g)\geq\frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1)\,,f\right\}. \tag{2.3.17}$$

Since λ_f and λ_g are the lower orders of f and g respectively then for given $\varepsilon\,(>0)$ and for large values of r we obtain that
$\log M(r,f)>r^{\lambda_f-\varepsilon}$ and $\log M(r,g)>r^{\lambda_g-\varepsilon}$ where $0<\varepsilon<\min\{\lambda_f,\lambda_g\}$. So

from (2.3.17) we have for all sufficiently large values of r,

$$T(r, f\circ g) \geq \frac{1}{3}\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + o(1)\right\}^{\lambda_f - \varepsilon}$$

$$i.e.,\ T(r, f\circ g) \geq \frac{1}{3}\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right)\right\}^{\lambda_f - \varepsilon}$$

$$i.e.,\ \log T(r, f\circ g) \geq O(1) + (\lambda_f - \varepsilon)\log M\left(\frac{r}{4}, g\right)$$

$$i.e.,\ \log T(r, f\circ g) \geq O(1) + (\lambda_f - \varepsilon)\left(\frac{r}{4}\right)^{\lambda_g - \varepsilon}$$

$$i.e.,\ \log^{[2]} T(r, f\circ g) \geq O(1) + (\lambda_g - \varepsilon)\log r. \tag{2.3.18}$$

Again in view of Lemma 2.2.5, we get for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, L(f)) \leq \left(\overline{\lambda}_{L(f)} + \varepsilon\right)\log r$$

$$i.e.,\ \log^{[2]} T(r, L(f)) \leq \left(\overline{\lambda}_f + \varepsilon\right)\log r. \tag{2.3.19}$$

Combining (2.3.18)and (2.3.19), it follows for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \frac{O(1) + (\lambda_g - \varepsilon)\log r}{\left(\overline{\lambda}_f + \varepsilon\right)\log r}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \frac{\lambda_g}{\overline{\lambda}_f}. \tag{2.3.20}$$

Again from (2.3.17) we get for a sequence of values of r tending to infinity,

$$\log T(r, f\circ g) \geq O(1) + (\lambda_f - \varepsilon)\left(\frac{r}{4}\right)^{\rho_g - \varepsilon}$$

$$i.e.,\ \log^{[2]} T(r, f\circ g) \geq O(1) + (\rho_g - \varepsilon)\log r. \tag{2.3.21}$$

Also in view of Lemma 2.2.5, for all sufficiently large values of r, we have

$$\log^{[2]} T(r, L(f)) \leq \left(\overline{\rho}_{L(f)} + \varepsilon\right)\log r = \left(\overline{\rho}_f + \varepsilon\right)\log r. \tag{2.3.22}$$

Now from (2.3.21) and (2.3.22) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \frac{O(1) + (\rho_g - \varepsilon)\log r}{\left(\overline{\rho}_f + \varepsilon\right)\log r}.$$

As $\varepsilon\,(0<\varepsilon<\rho_g)$ is arbitrary, we obtain from above that

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f\circ g)}{\log^{[2]}T(r,L(f))}\geq\frac{\rho_g}{\overline{\rho}_f}. \tag{2.3.23}$$

Therefore from (2.3.20) and (2.3.23) we get that

$$\limsup_{r\to\infty}\frac{\log^{[2]}T(r,f\circ g)}{\log^{[2]}T(r,L(f))}\geq\max\left\{\frac{\lambda_g}{\overline{\lambda}_f},\frac{\rho_g}{\overline{\rho}_f}\right\}.$$

Thus the theorem is established. ■

Theorem 2.3.5 *Let f be transcendental meromorphic and g be entire such that (i) $0<\overline{\lambda}_f<\overline{\rho}_f$, (ii) $\rho_g<\infty$, (iii) $\rho_f<\infty$ and (iv) $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[2]}T(r,f\circ g)}{\log^{[2]}T(r,L(f))}\leq\min\left\{\frac{\lambda_g}{\overline{\lambda}_f},\frac{\rho_g}{\overline{\rho}_f}\right\}.$$

Proof. In view of Lemma 2.2.1 and the inequality $T(r,g)\leq\log^{+}M(r,g)$, we obtain for all sufficiently large values of r

$$\log T(r,f\circ g)\leq o(1)+(\rho_f+\varepsilon)\log M(r,g). \tag{2.3.24}$$

Also for a sequence of values of r tending to infinity,

$$\log M(r,g)\leq r^{\lambda_g+\varepsilon}. \tag{2.3.25}$$

Combining (2.3.24) and (2.3.25) it follows for a sequence of values of r tending to infinity,

$$\begin{aligned}\log T(r,f\circ g)&\leq o(1)+(\rho_f+\varepsilon)\,r^{\lambda_g+\varepsilon}\\ \text{i.e., }\log T(r,f\circ g)&\leq\{(\rho_f+\varepsilon)+o(1)\}\,r^{\lambda_g+\varepsilon}\\ \text{i.e., }\log^{[2]}T(r,f\circ g)&\leq O(1)+(\lambda_g+\varepsilon)\log r.\end{aligned} \tag{2.3.26}$$

Again in view of Lemma 2.2.5, we have for all sufficiently large values of r,

$$\log^{[2]}T(r,L(f))>\left(\overline{\lambda}_{L(f)}-\varepsilon\right)\log r=\left(\overline{\lambda}_f-\varepsilon\right)\log r. \tag{2.3.27}$$

Now from (2.3.26) and (2.3.27) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{O(1) + (\lambda_g + \varepsilon) \log r}{(\overline{\lambda}_f - \varepsilon) \log r}.$$

As $\varepsilon (> 0)$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{\lambda_g}{\overline{\lambda}_f}. \tag{2.3.28}$$

In view of Lemma 2.2.1 we get for all sufficiently large values of r,

$$\log^{[2]} T(r, f \circ g) \leq O(1) + (\rho_g + \varepsilon) \log r. \tag{2.3.29}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, L(f)) > (\overline{\rho}_{L(f)} - \varepsilon) \log r = (\overline{\rho}_f - \varepsilon) \log r. \tag{2.3.30}$$

Combining (2.3.29) and (2.3.30) we have for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{O(1) + (\rho_g + \varepsilon) \log r}{(\overline{\rho}_f - \varepsilon) \log r}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{\rho_g}{\overline{\rho}_f}. \tag{2.3.31}$$

Now from (2.3.28) and (2.3.29) we get that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, L(f))} \leq \min \left\{ \frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f} \right\}.$$

This proves the theorem. ■

The following theorem is a natural consequence of Theorem 2.3.4 and Theorem 2.3.5.

Theorem 2.3.6 *Let f be a transcendental entire function and g be an entire function such that* $(i)\ 0 < \overline{\lambda}_f \leq \overline{\rho}_f < \infty, (ii)\ 0 < \lambda_f \leq \rho_f < \infty, (iii)\ 0 < \lambda_g \leq$

$\rho_g < \infty$ *and (iv)* $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, L(f))} \leq \min\left\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\right\}$$
$$\leq \max\left\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, L(f))}.$$

Theorem 2.3.7 *Let f be transcendental meromorphic and g be entire such that $0 < \lambda_f \leq \rho_f < \infty$ and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$. Then for $0 < \mu < \rho_g < \infty$,*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log^{[2]} T(\exp(r^{\mu}), L(f))} = \infty.$$

Proof. Let $0 < \mu' < \rho_g$. Then in view of Lemma 2.2.2 we get for a sequence of values of r tending to infinity,

$$\begin{aligned}
\log T(r, f \circ g) &\geq \log T\left(\exp\left(r^{\mu'}\right), f\right) \\
&\geq (\lambda_f - \varepsilon) \log\left(\exp\left(r^{\mu'}\right)\right) \\
&\geq (\lambda_f - \varepsilon) r^{\mu'} \\
\text{i.e.,}\ \log^{[2]} T(r, f \circ g) &\geq O(1) + \mu' \log r.
\end{aligned}$$

So for a sequence of values of r tending to infinity,

$$\begin{aligned}
\log^{[2]} T(\exp(r^{\rho_g}), f \circ g) &\geq O(1) + \mu' \log(\exp(r^{\rho_g})) \\
\text{i.e.,}\ \log^{[2]} T(\exp(r^{\rho_g}), f \circ g) &\geq O(1) + \mu' r^{\rho_g}. \qquad (2.3.32)
\end{aligned}$$

Again in view of Lemma 2.2.4, we have for all sufficiently large values of r,

$$\begin{aligned}
\log T(\exp(r^{\mu}), L(f)) &\leq \left(\rho_{L(f)} + \varepsilon\right) \log(\exp(r^{\mu})) \\
\text{i.e.,}\ \log T(\exp(r^{\mu}), L(f)) &\leq (\rho_f + \varepsilon) r^{\mu} \\
\text{i.e.,}\ \log^{[2]} T(\exp(r^{\mu}), L(f)) &\leq O(1) + \mu \log r. \qquad (2.3.33)
\end{aligned}$$

Now combining (2.3.32) and (2.3.33) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log^{[2]} T(\exp(r^{\mu}), L(f))} \geq \frac{O(1) + \mu' r^{\rho_g}}{O(1) + \mu \log r},$$

from which the theorem follows. ■