Chapter 5

RELATIVE (p,q)TH ORDER AND RELATED GROWTH ESTIMATES OF ENTIRE FUNCTIONS ON THE BASIS OF THEIR MINIMUM MODULUS

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5.1 Introduction, Definitions and Notations.

Let f and g be two entire functions and

$$F(r) = \max\{\,|f(z)| : |z| = r\}, G(r) = \max\{\,|g(z)| : |z| = r\}.$$

If f is non-constant then $F(r)$ is strictly increasing and continuous and its inverse

$$F^{-1} : (\,|f(0)|\,, \infty) \rightarrow (0, \infty)$$

exists and is such that

$$\lim_{s \to \infty} F^{-1}(s) = \infty.$$

Bernal [3] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf\ \{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\}$$

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [**22**].

$$=\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log r}.$$

The definition coincides with the classical one [65] if

$$g(z)=\exp z$$

Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f)=\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log r}.$$

Juneja, Kapoor and Bajpai [38] defined the (p,q)th order and (p,q)th lower order of an entire function f respectively as follows:

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p+1]}M(r,\ f)}{\log^{[q+1]}r}\text{ and }\lambda_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p+1]}M(r,\ f)}{\log^{[q+1]}r}.$$

where p,q are positive integers and $p>q$. So with the help of the above notion one can easily define the relative (p,q)th order and relative (p,q)th lower order of entire functions.

Definition 5.1.1 *The relative (p,q)th order $\rho_g^f(p,q)$ and the relative (p,q)th lower order $\lambda_g^f(p,q)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}G^{-1}F(r)}{\log^{[q]}r}\textit{ and }\lambda_g^f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}G^{-1}F(r)}{\log^{[q]}r},$$

where p,q are positive integers and $p>q$.In the chapter we establish some results on the comparative growth properties of entire functions on the basis of relative (p,q)th order and relative (p,q)th lower order where p,q are positive integers with $p>q$. In the sequel we use the following notations:

$$\log^{[k]}x=\log\left(\log^{[k-1]}x\right)\textit{ for }k=1,2,3,\ldots\ldots\textit{and}\log^{[0]}x=x.$$

In order to develop our results we shall need various kinds of measures and densities for sets of points on the positive axis. Let E be such a set and let $E[a,b]$ denote the part of E for which $a<r<b$. The linear and logarithmic measures of E are defined to be

$$m(E)=\int_E dr\textit{ and }lm(E)=\int_{E(1,\infty)}\frac{dr}{r}\textit{ respectively.}$$

These may be finite or infinite. We also define the lower and upper densities of E by

$$\textit{dens } E(\textit{upper}) = \limsup_{r\to\infty} \frac{m(E(0,r))}{r}$$

$$\textit{and dens } E(\textit{lower}) = \liminf_{r\to\infty} \frac{m(E(0,r))}{r}$$

and also the upper and lower logarithmic densities of E by

$$\log \ \textit{dens}E(\textit{upper}) = \limsup_{r\to\infty} \frac{\lim (E(1,r))}{\log r}$$

$$\textit{and } \log \ \textit{dens}E(\textit{lower}) = \liminf_{r\to\infty} \frac{\lim (E(1,r))}{\log r}.$$

$$\text{Also let } \ f(r) = m(r, f) = \inf_{|z|=r} |f(z)|$$

which is known as the minimum modulus of an entire function f. In this chapter we also estimate some comparative growth properties of composite entire functions in terms of their minimum modulus. In fact all the definitions in this chapter can also be stated in terms of minimum modulus on a set of logarithmic density 1.

5.2 Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 5.2.1 *{[2], [36]}. Let $f(z)$ be an entire function such that*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq c < \frac{1}{4e}.$$

If $0 < 4ec < \delta < 1$ then outside a set of upper logarithmic density at most δ,

$$\frac{m(r,f)}{M(r,f)} > k(\delta, c) = \frac{1-2.2\tau}{1+2.2\tau} \ \textit{where } \tau = \exp\{-\delta/(4ec)\}.$$

If in particular $c = 0$ then

$$\frac{m(r,f)}{M(r,f)} \to 1 \ \textit{as } r\to\infty$$

on a set of logarithmic density 1.

5.3 Theorems.

In this section we present the main results of this chapter.

In the following theorems we see the application of relative (p,q)th order and relative (p,q)th lower order in the growth properties of entire functions where p,q are positive integers with $p>q$.

Theorem 5.3.1 *Let f,g and h be three entire functions such that*

$$0<\lambda_g^f(p,q)\le\rho_g^f(p,q)<\infty$$
$$\textit{and } 0<\lambda_g^h(m,q)\le\rho_g^h(m,q)<\infty$$

where p,q,m are positive integers with $p>q$ and $m>q$. Then

$$\frac{\lambda_g^f(p,q)}{\rho_g^h(m,q)}\le\liminf_{r\to\infty}\frac{\log^{[p]}G^{-1}F(r)}{\log^{[m]}G^{-1}H(r)}\le\frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}$$
$$\le\limsup_{r\to\infty}\frac{\log^{[p]}G^{-1}F(r)}{\log^{[m]}G^{-1}H(r)}\le\frac{\rho_g^f(p,q)}{\lambda_g^h(m,q)}.$$

Proof. From the definition of relative (p,q)th order and relative (p,q)th lower order we have for arbitrary positive ε and for all large values of r,

$$\log^{[p]}G^{-1}F(r)\ge(\lambda_g^f(p,q)-\varepsilon)\log^{[q]}r \tag{5.3.1}$$

and

$$\log^{[m]}G^{-1}H(r)\le(\rho_g^h(m,q)+\varepsilon)\log^{[q]}r. \tag{5.3.2}$$

Now from (5.3.1) and (5.3.2) it follows for all large values of r,

$$\frac{\log^{[p]}G^{-1}F(r)}{\log^{[m]}G^{-1}H(r)}\ge\frac{\lambda_g^f(p,q)-\varepsilon}{\rho_g^h(m,q)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}G^{-1}F(r)}{\log^{[m]}G^{-1}H(r)}\ge\frac{\lambda_g^f(p,q)}{\rho_g^h(m,q)}. \tag{5.3.3}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]}G^{-1}F(r)\le(\lambda_g^f(p,q)+\varepsilon)\log^{[q]}r \tag{5.3.4}$$

and for all large values of r,

$$\log^{[m]} G^{-1}H(r) \geq (\lambda_g^h(m,q) - \varepsilon)\log^{[q]} r. \tag{5.3.5}$$

So combining (5.3.4) and (5.3.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\lambda_g^f(p,q) + \varepsilon}{\lambda_g^h(m,q) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}. \tag{5.3.6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \leq (\lambda_g^h(m,q) + \varepsilon)\log^{[q]} r. \tag{5.3.7}$$

Now from (5.3.1) and (5.3.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\lambda_g^f(p,q) - \varepsilon}{\lambda_g^h(m,q) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}. \tag{5.3.8}$$

Also for all large values of r,

$$\log^{[p]} G^{-1}F(r) \leq (\rho_g^f(p,q) + \varepsilon)\log^{[q]} r. \tag{5.3.9}$$

So from (5.3.5) and (5.3.9) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q) + \varepsilon}{\lambda_g^h(m,q) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q)}{\lambda_g^h(m,q)}. \tag{5.3.10}$$

Thus the theorem follows from (5.3.3), (5.3.6), (5.3.8) and (5.3.10). ■

Remark 5.3.1 *Under the same conditions stated in Theorem 5.3.1, the conclusion of the theorem can also be drawn with the help of Lemma 5.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ respectively on a set of logarithmic density 1.*

Theorem 5.3.2 *Let f, g, h be three entire functions with*

$$0 < \lambda_g^f(p,q) \leq \rho_g^f(p,q) < \infty \text{ and } 0 < \rho_g^h(m,q) < \infty$$

where p, q, m are positive integers with $p > q$ and $m > q$. Then

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)}.$$

Proof. From the definition of relative (p, q)th order we get for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \geq (\rho_g^h(m,q) - \varepsilon) \log^{[q]} r. \tag{5.3.11}$$

Now from (5.3.9) and (5.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q) + \varepsilon}{\rho_g^h(m,q) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}. \tag{5.3.12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \geq (\rho_g^f(p,q) - \varepsilon) \log^{[q]} r. \tag{5.3.13}$$

So combining (5.3.2) and (5.3.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\rho_g^f(p,q) - \varepsilon}{\rho_g^h(m,q) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}. \tag{5.3.14}$$

Thus the theorem follows from (5.3.12) and (5.3.14). ■

Remark 5.3.2 *Under the same hypothesis stated in Theorem 5.3.2, the conclusion of the theorem can also be deduced in view of Lemma 5.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ respectively on a set of logarithmic density 1.The following theorem is a natural consequence of Theorem 5.3.1 and Theorem 5.3.2.*

Theorem 5.3.3 *Let f, g, h be three entire functions with*

$$0 < \lambda_g^f(p,q) \leq \rho_g^f(p,q) < \infty \text{ and } 0 < \lambda_g^h(m,q) \leq \rho_g^h(m,q) < \infty$$

where p, q, m are positive integers with $p > q$ and $m > q$. Then

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} &\leq \min\{\frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}, \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}\}\\ &\leq \max\{\frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}, \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}\}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)}.\end{aligned}$$

The proof is omitted.

Remark 5.3.3 *Under the same conditions stated in Theorem 5.3.3, the conclusion of the theorem can also be drawn with the help of Lemma 5.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ respectively on a set of logarithmic density 1.*

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