Chapter 4

RELATIVE L-ORDER AND RELATED COMPARATIVE GROWTH PROPERTIES OF ENTIRE FUNCTIONS ON THE BASIS OF THEIR MINIMUM MODULUS

CHAPTER

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4.1 Introduction, Definitions and Notations.

Let f and g be two entire functions and

$$F(r) = max\{|f(z)| : z = r\}, G(r) = max\{|g(z)| : |z| = r\}.$$

If f is non-constant then $F(r)$ is strictly increasing and continuous and its inverse

$$F^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$$

exists and is such that

$$\lim_{s \rightarrow \infty} F^{-1}(s) = \infty.$$

The results of this chapter have been published in **International Mathematical Forum**, see [**21**].

Bernal[3] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

The definition coincides with the classical one[65] if

$$g(z) = \exp z.$$

Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

Somasundaram and Thamizharasi [63] introduced the notions of L-order, L-lower order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.

$$L(ar) \sim L(r) \text{ as } r \to \infty$$

for every constant 'a'. Their definitions are as follows:

Definition 4.1.1 *[63] The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

Definition 4.1.2 *[63] The L-type σ_f^L of an entire function f with L-order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$

Similarly one can define the L-hyper order and L-hyper lower order of entire functions. So with the help of the above notion one can easily define the relative L-order and relative L-lower order of entire functions.

Definition 4.1.3 *The relative L-order $\rho_g^L(f)$ and the relative L-lower order $\lambda_g^L(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]} \text{ and } \lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]}.$$

Definition 4.1.4 *The relative L-hyper order $\overline{\rho}_g^L(f)$ and the relative L-hyper lower order $\overline{\lambda}_g^L(f)$ of an entire function f with respect to another entire function g are defined as*

$$\overline{\rho}_g^L(f) = \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log[rL(r)]} \text{ and } \overline{\lambda}_g^L(f) = \liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log[rL(r)]}.$$

In this chapter we establish some results on the growth properties of entire functions on the basis of relative L-order and relative L-lower order where $L \equiv L(r)$ is a slowly changing function. In the sequel we use the following notations:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1,2,3, \ldots\ldots \text{and } \log^{[0]} x = x.$$

The more generalised concept of L-order and L-type of entire and meromorphic functions are respectively L^*-order and L^*-type. Their definitions are as follows:

Definition 4.1.5 *The L^*-order, L^*-lower order and L^*-type of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}$$
$$\text{and } \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}, \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}$$
$$\text{and } \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{Lr}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty.$$

Definition 4.1.6 *The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[re^{L(r)}]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[re^{L(r)}]}.$$

Definition 4.1.7 *The relative L^*-hyper order $\overline{\rho}_g^{L^*}(f)$ and the relative L^*-hyper lower order $\overline{\lambda}_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as*

$$\overline{\rho}_g^{L^*}(f) = \limsup_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log[re^{L(r)}]} \text{ and } \overline{\lambda}_g^{L^*}(f) = \liminf_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log[re^{L(r)}]}.$$

In order to develop our results we shall need various kinds of measures and densities for sets of points on the positive axis. Let E be such a set and let $E[a,b]$ denote the part of E for which $a < r < b$. The linear and logarithmic measures of E are defined to be

$$m(E) = \int_E dr \text{ and } lm(E) = \int_{E(1,\infty)} \frac{dr}{r} \text{ respectively.}$$

These may be finite or infinite. We also define the lower and upper densities of E by

$$\begin{aligned} dens\ E(upper) &= \limsup_{r\to\infty}\frac{m(E(0,r))}{r} \\ \text{and } dens\ E(lower) &= \liminf_{r\to\infty}\frac{m(E(0,r))}{r} \end{aligned}$$

and also the upper and lower logarithmic densities of E by

$$\begin{aligned} \log\ densE(upper) &= \limsup_{r\to\infty}\frac{\lim(E(1,r))}{\log r} \\ \text{and } \log\ densE(lower) &= \liminf_{r\to\infty}\frac{\lim(E(1,r))}{\log r}. \end{aligned}$$

$$\text{Also let } f(r) = m(r,f) = \inf_{|z|=r}|f(z)|$$

which is known as the minimum modulus of an entire function f. In this chapter we also estimate some growth properties of composite entire functions in terms of their minimum modulus. In fact all the definitions in the chapter can also be stated in terms of minimum modulus on a set of logarithmic density 1.

4.2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 4.2.1 *{[2], [36]}. Let $f(z)$ be an entire function such that*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \le c < \frac{1}{4e}.$$

If $0 < 4ec < \delta < 1$ then outside a set of upper logarithmic density at most δ,

$$\frac{m(r,f)}{M(r,f)} > k(\delta, c) = \frac{1-2.2\tau}{1+2.2\tau} \text{ where } \tau = \exp\{-\delta/(4ec)\}.$$

If in particular $c = 0$ then

$$\frac{m(r,f)}{M(r,f)} \to 1 \text{ as } r \to \infty$$

on a set of logarithmic density 1.

4.3 Theorems.

In this section we present the main results of the chapter. In the following theorems we see the application of relative L-order and relative L-lower order in the growth properties of entire functions.

Theorem 4.3.1 *Let f, g and h be three entire functions such that*

$$0 < \lambda_g^L(f) \le \rho_g^L(f) < \infty \text{ and } 0 < \lambda_g^L(h) \le \rho_g^L(h) < \infty. \text{ Then}$$

$$\frac{\lambda_g^L(f)}{\rho_g^L(h)} \le \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \le \frac{\lambda_g^L(f)}{\lambda_g^L(h)} \le \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \le \frac{\rho_g^L(f)}{\lambda_g^L(h)}.$$

Proof. From the definition of relative L-order and relative L-lower order we have for arbitrary positive ε and for all large values of r,

$$\log G^{-1}F(r) \ge (\lambda_g^L(f) - \varepsilon)\log[rL(r)] \tag{4.3.1}$$

$$\text{and } \log G^{-1}H(r) \le (\rho_g^L(h) + \varepsilon)\log[rL(r)]. \tag{4.3.2}$$

Now from (4.3.1) and (4.3.2) it follows for all large values of r,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)-\varepsilon}{\rho_g^L(h)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)}{\rho_g^L(h)}. \tag{4.3.3}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \leq (\lambda_g^L(f)+\varepsilon)\log[rL(r)] \tag{4.3.4}$$

and for all large values of r,

$$\log G^{-1}H(r) \geq (\lambda_g^L(h)-\varepsilon)\log r. \tag{4.3.5}$$

So combining (4.3.4) and (4.3.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^L(f)+\varepsilon}{\lambda_g^L(h)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^L(f)}{\lambda_g^L(h)}. \tag{4.3.6}$$

Also for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r) \leq (\lambda_g^L(h)+\varepsilon)\log[rL(r)]. \tag{4.3.7}$$

Now from (4.3.1) and (4.3.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)-\varepsilon}{\lambda_g^L(h)+\varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)}{\lambda_g^L(h)}. \tag{4.3.8}$$

Also for all large values of r,

$$\log G^{-1}F(r) \leq (\rho_g^L(f) + \varepsilon)\log[rL(r)]. \tag{4.3.9}$$

So from (4.3.5) and (4.3.9) it follows for all large values of r,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f) + \varepsilon}{\lambda_g^L(h) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f)}{\lambda_g^L(h)}. \tag{4.3.10}$$

Thus the theorem follows from (4.3.3), (4.3.6), (4.3.8) and (4.3.10). ■

Remark 4.3.1 *Under the same conditions stated in Theorem 4.3.1, the conclusion of the theorem can also be drawn by using Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1.*

Theorem 4.3.2 *Let f, g, h be three entire functions with*

$$0 < \lambda_g^L(f) \leq \rho_g^L(f) < \infty \text{ and } 0 < \rho_g^L(h) < \infty. \text{ Then}$$

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f)}{\rho_g^L(h)} \leq \limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.$$

Proof. From the definition of relative L-order we get for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r) \geq (\rho_g^L(h) - \varepsilon)\log[rL(r)]. \tag{4.3.11}$$

Now from (4.3.9) and (4.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f) + \varepsilon}{\rho_g^L(h) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f)}{\rho_g^L(h)}. \tag{4.3.12}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \geq (\rho_g^L(f) - \varepsilon)\log[rL(r)]. \tag{4.3.13}$$

So combining (4.3.2) and (4.3.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^L(f) - \varepsilon}{\rho_g^L(h) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^L(f)}{\rho_g^L(h)}. \tag{4.3.14}$$

Thus the theorem follows from (4.3.12) and (4.3.14). ∎

Remark 4.3.2 *Under the same conditions stated in Theorem 4.3.2, the conclusion of the theorem can also be deduced in view of Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1.The following theorem is a natural consequence of Theorem 4.3.1 and Theorem 4.3.2.*

Theorem 4.3.3 *Let f, g and h be three entire functions with*

$$0 < \lambda_g^L(f) \leq \rho_g^L(f) < \infty \text{ and } 0 < \lambda_g^L(h) \leq \rho_g^L(h) < \infty. \text{ Then}$$

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} &\leq \min\{\frac{\lambda_g^L(f)}{\lambda_g^L(h)}, \frac{\rho_g^L(f)}{\rho_g^L(h)}\} \\ &\leq \max\{\frac{\lambda_g^L(f)}{\lambda_g^L(h)}, \frac{\rho_g^L(f)}{\rho_g^L(h)}\} \leq \limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.\end{aligned}$$

The proof is omitted.

Remark 4.3.3 *Under the same conditions stated in Theorem 4.3.3, the conclusion of the theorem can also be drawn in view of Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1. In the line of Theorem 4.3.1, Theorem 4.3.2 and Theorem 4.3.3 we may now prove similar results for relative hyper order and relative hyper lower order.*

Theorem 4.3.4 *Let f, g and h be three entire functions such that*

$$0 < \overline{\lambda}_g^L(f) \leq \overline{\rho}_g^L(f) < \infty \text{ and } 0 < \overline{\lambda}_g^L(h) \leq \overline{\rho}_g^L(h) < \infty. \text{ Then}$$

$$\begin{aligned}\frac{\overline{\lambda}_g^L(f)}{\overline{\rho}_g^L(h)} &\leq \liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\lambda}_g^L(f)}{\overline{\lambda}_g^L(h)} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\rho}_g^L(f)}{\overline{\lambda}_g^L(h)}.\end{aligned}$$

Theorem 4.3.5 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^L(f) \leq \overline{\rho}_g^L(f) < \infty \text{ and } 0 < \overline{\rho}_g^L(h) < \infty. \text{ Then}$$

$$\liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\rho}_g^L(f)}{\overline{\rho}_g^L(h)} \leq \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}.$$

The following theorem is a natural consequence of Theorem 4.3.4 and Theorem 4.3.5.

Theorem 4.3.6 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^L(f) \leq \overline{\rho}_g^L(f) < \infty \text{ and } 0 < \overline{\lambda}_g^L(h) < \infty. \text{ Then}$$

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} &\leq \min\left\{\frac{\overline{\lambda}_g^L(f)}{\overline{\lambda}_g^L(h)}, \frac{\overline{\rho}_g^L(f)}{\overline{\rho}_g^L(h)}\right\} \\ &\leq \max\left\{\frac{\overline{\lambda}_g^L(f)}{\overline{\lambda}_g^L(h)}, \frac{\overline{\rho}_g^L(f)}{\overline{\rho}_g^L(h)}\right\} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}.\end{aligned}$$

Remark 4.3.4 *Under the same conditions respectively stated in Theorem 4.3.4, Theorem 4.3.5 and Theorem 4.3.6 the conclusions of the theorems can also be drawn with the help of Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$*

instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1. In the following theorems we see some comparative growth properties of entire functions on the basis of relative L^*-order and relative L^*-lower order where $L \equiv L(r)$ is a slowly changing function.

Theorem 4.3.7 *Let f, g and h be three entire functions such that $0 < \lambda_g^{L^*}(f) \leq \rho_g^{L^*}(f) < \infty$ and $0 < \lambda_g^{L^*}(h) \leq \rho_g^{L^*}(h) < \infty$. Then*

$$\frac{\lambda_g^{L^*}(f)}{\rho_g^{L^*}(h)} \leq \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)}$$
$$\leq \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f)}{\lambda_g^{L^*}(h)}.$$

Proof. From the definition of relative L^*-order and relative L^*-lower order we have for arbitrary positive ε and for all large values of r,

$$\log G^{-1}F(r) \geq (\lambda_g^{L^*}(f) - \varepsilon) \log[re^{L(r)}] \tag{4.3.15}$$

$$\text{and } \log G^{-1}H(r) \leq (\rho_g^{L^*}(h) + \varepsilon) \log[re^{L(r)}].$$

Now from (4.3.15) and (4.3.16) it follows for all large values of r,

$$\frac{G^{-1}F(r)}{G^{-1}H(r)} \geq \frac{\lambda_g^{L^*}(f) - \varepsilon}{\rho_g^{L^*}(h) + \varepsilon}. \tag{4.3.16}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{G^{-1}F(r)}{G^{-1}H(r)} \geq \frac{\lambda_g^{L^*}(f)}{\rho_g^{L^*}(h)}. \tag{4.3.17}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \leq (\lambda_g^{L^*}(f) + \varepsilon) \log[re^{L(r)}] \tag{4.3.18}$$

and for all large values of r,

$$\log G^{-1}H(r) \geq (\lambda_g^{L^*}(h) - \varepsilon) \log[re^{L(r)}]. \tag{4.3.19}$$

So combining (4.3.18) and (4.3.19) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^{L^*}(f) + \varepsilon}{\lambda_g^{L^*}(h) - \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\leq\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)}. \tag{4.3.20}$$

Also for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r)\leq(\lambda_g^{L^*}(h)+\varepsilon)\log[re^{L(r)}]. \tag{4.3.21}$$

Now from (4.3.15) and (4.3.21) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\geq\frac{\lambda_g^{L^*}(f)-\varepsilon}{\lambda_g^{L^*}(h)+\varepsilon}.$$

Choosing $\varepsilon\to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\geq\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)}. \tag{4.3.22}$$

Also for all large values of r,

$$\log G^{-1}F(r)\leq(\rho_g^{L^*}(f)+\varepsilon)\log[re^{L(r)}]. \tag{4.3.23}$$

So from (4.3.19) and (4.3.23) it follows for all large values of r,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\leq\frac{\rho_g^{L^*}(f)+\varepsilon}{\lambda_g^{L^*}(h)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\leq\frac{\rho_g^{L^*}(f)}{\lambda_g^{L^*}(h)}. \tag{4.3.24}$$

Thus the theorem follows from (4.3.17), (4.3.20), (4.3.22) and (4.3.24). ■

Theorem 4.3.8 *Let f, g and h be three entire functions with*

$0<\lambda_g^{L^*}(f)\leq\rho_g^{L^*}(f)<\infty$ *and* $0<\rho_g^{L^*}(h)<\infty$. *Then*

$$\liminf \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)} \leq \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.$$

Proof. From the definition of relative L^*-order we get for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r) \geq (\rho_g^{L^*}(h) - \varepsilon)\log[re^{L(r)}]. \tag{4.3.25}$$

Now from (4.3.9) and (4.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f) + \varepsilon}{\rho_g^{L^*}(h) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}. \tag{4.3.26}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \geq (\rho_g^{L^*}(f) - \varepsilon)\log[re^{L(r)}]. \tag{4.3.27}$$

So combining (4.3.16) and (4.3.27) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^{L^*}(f) - \varepsilon}{\rho_g^{L^*}(h) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}. \tag{4.3.28}$$

Thus the theorem follows from (4.3.26) and (4.3.28). The following theorem is a natural consequence of Theorem 4.3.7 and Theorem 4.3.8. ■

Theorem 4.3.9 *Let f, g and h be three entire functions with*

$$0 < \lambda_g^{L^*}(f) \leq \rho_g^{L^*}(f) < \infty \text{ and } 0 < \lambda_g^{L^*}(h) \leq \rho_g^{L^*}(h) < \infty. \text{ Then}$$

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \le \min\{\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)}, \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}\}$$
$$\le \max\{\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)}, \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}\}$$
$$\le \limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.$$

Remark 4.3.5 *Under the same conditions respectively stated in Theorem 4.3.7, Theorem 4.3.8 and Theorem 4.3.9 the conclusions of the theorems can also be deduced by using Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1. We may prove similar results for relative L^*-hyper order and relative L^*-hyper lower order.*

Theorem 4.3.10 *Let f, g and h be three entire functions such that*

$$0 < \overline{\lambda}_g^{L^*}(f) \le \overline{\rho}_g^{L^*}(f) < \infty \text{ and } 0 < \overline{\lambda}_g^{L^*}(h) \le \overline{\rho}_g^{L^*}(h) < \infty. \text{ Then}$$

$$\frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)} \le \liminf_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \le \frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}$$
$$\le \limsup_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \le \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}.$$

Theorem 4.3.11 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^{L^*}(f) \le \overline{\rho}_g^{L^*}(f) < \infty \text{ and } 0 < \overline{\rho}_g^{L^*}(h) < \infty. \text{ Then}$$

$$\liminf_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \le \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)} \le \limsup_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}.$$

The following theorem is a natural consequence of Theorem 4.3.10 and Theorem 4.3.11.

Theorem 4.3.12 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^{L^*}(f) \leq \overline{\rho}_g^{L^*}(f) < \infty \text{ and } 0 < \overline{\lambda}_g^{L^*}(h) < \infty. \text{ Then}$$

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} &\leq \min\{\frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}, \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)}\} \\ &\leq \max\{\frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}, \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)}\} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}.\end{aligned}$$

Remark 4.3.6 *Under the same hypothesis respectively stated in Theorem 4.3.10, Theorem 4.3.11 and Theorem 4.3.12 the conclusions of the theorems can also be drawn by using Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1.*

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