SOME PROBLEMS ON THE THEORY OF ENTIRE AND MEROMORPHIC FUNCTIONS

BY

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THESIS
SUBMITTED TO THE UNIVERSITY OF NORTH BENGAL
FOR
Ph.D. DEGREE IN SCIENCE (MATHEMATICS)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH BENGAL
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2011

Acknowledgement

It is a great pleasure to express my sincere gratitude and indebtedness to Dr. Sanjib Kumar Datta, Assistant Professor, Department of Mathematics, University of Kalyani and formerly Assistant Professor, Department of Mathematics, University of North Bengal under whose supervision the entire thesis has been carried out. I owe a lot to him for offering me valuable suggestions for corrections and improvement of the thesis and above all for sufficiently sharing his valuable time with me from his busy schedule.

I also wish to thank all the faculty members of the Department of Mathematics, University of North Bengal and also all the faculty members of the Department of Mathematics, University of Kalyani for their mental support. Also I am thankful to the infrastructural support of the Department of Mathematics. University of North Bengal as well as that of the Department of Mathematics, University of Kalyani to commence my research work.

I would like to express my sincere thanks to Mr. Jayanta Kumar Mallick for his excellent typing skill which transformed the hand written manuscript into a pleasant looking thesis.

I am also grateful to my fellow research scholars for their co-operation in different ways.

I must also thank to the authors of various papers and books which have been consulted during the preparation of the thesis. I will remain ever grateful to those including my family members who helped me in giving my research work a good finish. The author is also responsible for any possible errors and short comings, whatever it may be in the thesis despite the best attempt to make it immaculate.

Dated: April 29, 2011
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Darjeeling.

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INTRODUCTION

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1

INTRODUCTION

The theory of the distribution of values of entire and meromorphic functions was first developed by *R.* Nevanlinna in the year 1926. It was one of the most outstanding achievements in function theory in this century. Nevanlinna's fundamental theory, the famous First Fundamental theorem, the Second Fundamental theorem, as well as the related formulae of deficiencies are the foundation of the whole thesis. Before starting the discussion on Nevanlinna theory we state the following definitions.

Definition 1.1 A single valued function of one complex variable which has no singularities other than poles in the open complex plane (i.e., excluding the point at infinity) is called a meromorphic function.

Definition 1.2 A meromorphic function which has an essential singularity at the point at infinity is known as a transcendental meromorphic function.

Definition 1.3 A single valued function which is analytic in the open complex plane is defined as an entire function.

Definition 1.4 An entire function which has an essential singularity at the point at infinity is known as a transcendental entire function.

Let f be a meromorphic function in the finite complex plane $\mathbb{C}$. Also let $n(r, a\ ; f) \equiv n(r, a)$ which is a non-negative integer for each r, denote the number of a-points of f in $|z| \leq r$, counted with proper multiplicities, for a complex number a, finite or infinite. Obviously $n(r, \infty) \equiv n(r, f)$ represents the number of poles of f in $|z| \leq r$ counted with proper multiplicities. The function $N(r, a\ ; f) \equiv N(r, a)$ is defined as follows:

$$N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a) \log r \text{ and } N(r, \infty\ ; f) \equiv N(r, f).$$

Next let us define

$$\begin{aligned} \log^+ x &= \log x \text{ if } x \geq 1 \\ &= 0 \text{ if } 0 \leq x < 1. \end{aligned}$$

The following properties are then obvious

$$\begin{aligned} &(i) \log^+ x \geq 0 \text{ if } x \geq 0, \\ &(ii) \log^+ x \geq \log x \text{ if } x > 0, \\ &(iii) \log^+ x \geq \log^+ y \text{ if } x > y, \\ &(iv) \log x = \log^+ x - \log^+ \frac{1}{x} \text{ if } x > 0. \end{aligned}$$

The quantity $m(r, f)$ is defined as follows:

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta.$$

The term $m(r, f)$ is called the proximity function of f and is a sort of averaged magnitude of $\log |f(z)|$ on the arcs of $|z| = r$ where $|f(z)|$ is large. Now let us write

$$T(r, f) = m(r, f) + N(r, f).$$

The function $T(r, f)$ is called the Nevanlinna's characteristic function of f {p.4, [34]}. It plays an important role in the theory of meromorphic functions. Since for any positive integer p and complex number a_ν,

$$\log^+ |\Pi_{\nu=1}^p a_\nu| \leq \sum_{\nu=1}^p \log^+ |a_\nu|$$

$$\text{and } \log^+ \left| \sum_{\nu=1}^p a_\nu \right| \leq \log^+ (p \max_{\nu=1,2,\ldots p} |a_\nu|) \leq \sum_{\nu=1}^p \log^+ |a_\nu| + \log p,$$

it is easy to show that {p.5, [34] } for p meromorphic functions $f_1, f_2, \ldots f_p$

$$m(r, \Pi_{\nu=1}^p f_\nu) \leq \sum_{\nu=1}^p m(r, f_\nu)$$

$$\text{and } m(r, \sum_{\nu=1}^p f_\nu) \leq \sum_{\nu=1}^p m(r, f_\nu) + \log p.$$

Also one can easily verify that

$$N(r, \Pi_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} N(r, f_\nu)$$

$$\text{and } N(r, \sum_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} N(r, f_\nu).$$

$$\text{So } T(r, \sum_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} (T(r, f_\nu) + \log p$$

$$\text{and } T(r, \Pi_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} T(r, f_\nu).$$

Now we state the Poisson-Jensen formula {p.1, [34] } in the form of the following theorem:

Theorem 1.1 Suppose that f is meromorphic in $|z| \leq R(0 < R < \infty)$. Also let $a_\mu(\mu = 1, 2, ...M)$ and $b_\nu(\nu = 1, 2, ...N)$ denote the zeros and poles of f respectively in $|z| < R$. Then if $z = re^{i\theta}(0 < r < R)$ and if $f(re^{i\theta}) \neq 0, \infty$ we have

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\mathrm{Re}^{i\phi})\right| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$

$$+\sum_{\mu=1}^{M} \log\left|\frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z - b_\nu)}{R^2 \bar{b}_\nu z}\right|.$$

The theorem holds good also when f has zeros and poles on $|z| = R$. When $z = 0$, we obtain Jensen's formula

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\mathrm{Re}^{i\phi})\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R},$$

provided that $f(0) \neq 0, \infty$.

If f has a zero of order λ or a pole of order $-\lambda$ at $z = 0$ such that $f = C_\lambda Z^\lambda + ...$then Jensen's formula takes the form

$$\log|C_\lambda| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\mathrm{Re}^{i\phi})\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R} - \lambda\log R.$$

The complicated modification is one of the minor irritations of the theory. Generally we shall assume that our function behave in such a way that the

terms in the Jensen's formula do not become infinite in our use of that formula knowing that exceptional cases can be treated.
When f has no a-points (i.e., the roots of the equation $f = a$) at $z = 0$, then it follows from Riemann-Stieltjes integral that

$$\sum_{0<|a_\nu|<r} \log \frac{r}{|a_\nu|} = \int_0^r \frac{n(t,a)}{t} dt, \text{ where } a_\nu\text{'s are the } a\text{-points of } f \text{ in } |z| \leq r.$$

Again since $N(r, 0\ ; f) = N(r, \frac{1}{f})$, from Jensen's formula we get that

$$\log |f(0)| = m(R, f) - m(R, \frac{1}{f}) + N(R, f) - N(R, \frac{1}{f})$$

$$\text{i.e., } T(R, f) = T(R, \frac{1}{f}) + \log |f(0)|.$$

For any finite complex number a let us denote by $m(r, a)$ the function $m(r, \frac{1}{f-a})$ and $m(r, \infty) = m(r, f)$.
Now we express Nevanlinna's First Fundamental theorem in the following form:
Theorem 1.2 {p.6, [34] }. Let f be a meromorphic function in $|z| < \infty$ and a be any complex number, finite or infinite, then

$$m(r, a) + N(r, a) = T(r, f) + O(1).$$

This result shows the remarkable symmetry exhibited by a meromorphic function in its behavior relative to different complex number a, finite or infinite. The sum $m(r, a) + N(r, a)$ for different values of a maintains a total, given by the quantity $T(r, f)$ which is invariant upto a bounded additive term involving r.
One part of this invariant sum, the quantity $N(r, a\ ; f)$ hints how densely the roots of the equation $f = a$ are distributed in the average in the disc $|z| < r$. The large the number of a-points the faster this counting function for a-points grows with r.
The first term $m(r, a)$ which is defined to be the mean value of

$$\log^+ |1/f - a| \ \ (\text{or } \log^+ |f| \text{ if } a = \infty)$$

on the circle $|z| = r$, receives a remarkable contribution only from those arcs on the circle where the functional values differ very little from the given value a. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle $|z| = r$ of the functional value f from the value a.

If the a-points of a meromorphic function are relatively scarce for a certain 'a', this fact finds expression analytically in the relatively slow growth of the function $N(r, a)$ as $r \to \infty$; in the extreme case where a is a Picard's exceptional value of the function (so that $f \neq a$ in $|z| < \infty$), $N(r, a)$ is identically zero. But this fact on a-points finds a compensation.

The function deviates in the mean slightly from the value a in question, the corresponding proximity function $m(r, a)$ will be relatively large, so that the sum $m(r, a) + N(r, a)$ reaches the magnitude $T(r, f)$, characteristic function of f.

For an entire function f, $N(r, f) = 0$ and so $T(r, f) = m(r, f)$, i.e., in the case of an entire function, the Nevanlinna's characteristic function and the proximity function are same.

Let us consider that f be an entire function, i.e., a function of a complex variable regular in the whole finite complex plane $\mathbb{C}$. By Taylor's theorem such a function has an everywhere convergent power series expansion as

$$f = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n + \ldots \tag{1.1}$$

which forms a natural generalization of the polynomials.

The degree of a polynomial which is equal to its number of zeros estimates the rate of growth of the polynomial as the independent variable moves without bound. So the more zeros, the greater is the growth.

An analogous property that relate the set of zeros and the growth of a function can be developed for arbitrary entire functions.

Establishing relations between the distribution of the zeros of an entire function and its asymptotic behavior as z tending to infinity enriched most of the classical results of the theory of entire functions. The classical investigations of Borel, Hadamard and Lindelof are of this kind.

To characterise the growth of an entire function and the distribution of its zeros a special growth scale called maximum modulus function of f on $|z| = r$ is introduced as $M(r) = M(r, f) = \max\limits_{|z|=r} |f(z)|$.

It plays an important role in the theory of entire functions. Since by Li-

ouville's theorem a bounded entire function is constant, it follows that for non-constant f the maximum modulus function $M(r)$ is unbounded.
The following theorem is due to Cauchy.
Theorem 1.3 {Theorem 1, p. 5, [66] }. The maximum of the modulus of a function f, which is regular in a closed connected region D, bounded by one or more curves C, is attained on the boundary.
This theorem implies that when f is an entire function, $M(r)$ is a non-decreasing function of r for all values of r.Using the uniform continuity of f in any closed region and the above theorem, i.e., the value $M(r)$ is attained by f on $|z| = r$, it follows that $M(r)$ is a continuous function of r. Also $M(r)$ is differentiable in adjacent intervals {Theorem 10, p. 27, [66] }. In view of Hadamard's theorem {Theorem 9, p. 20, [66]} we know that $\log M(r)$ is a continuous, convex and ultimately increasing function of $\log r$.
For an entire function f the study of the comparative growth properties of $T(r, f)$ and $\log M(r, f)$ is a popular problem among the researchers. Now we express a fundamental inequality relating $T(r, f)$ and $\log M(r, f)$.
Theorem 1.4 {p. 18, [34] }. If f is regular for $|z| \leq R$ then

$$T(r, f) \leq \log^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f), 0 \leq r < R.$$

In case of a transcendental entire function f, $M(r)$ grows faster than any positive power of r.Thus in order to estimate the growth of transcendental entire functions we choose a comparison function $e^{r^k}, k > 0$ that grows more rapidly than any positive power of r.
More precisely f is said to be a function of finite order if there exists a positive constant k such that $\log M(r) < r^k$ for all sufficiently large values of $r (r > r_0(k)$; say).The infimum of such k's is called the order of f. If no such k exists, f is said to be of infinite order.
For example the order of the function e^z is 1 i.e., finite but that of e^{e^z} is infinite.
Let ρ be the order of f. It can be easily shown that the order ρ of f has the following alternative definition

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}.$$

The lower order λ of f is defined as follows

$$\lambda = \liminf_{r \to \infty} \frac{\log \log M(r)}{\log r}.$$

Clearly $\lambda \leq \rho$.
If in particular for a function f, $\lambda = \rho$, then f is said to be of regular growth. For example a polynomial or the functions $e^z, \cos z$ etc. are of regular growth. With known order $\rho (0 < \rho < \infty)$ the growth of an entire function can be characterised more precisely by the type of the function. The number τ given by

$$\tau = \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho}, 0 < \rho < \infty$$

is called the type of f.
Between two functions of same order one can be characterised to be of greater growth if its type is greater. The quantities ρ, λ and τ are extensively used to the study of growth properties of f. At this stage we note the following definition.
Definition 1.5 {p.16, [34]}. Let S be a real and non-negative function increasing for $0 < r_0 \leq r < \infty$. The order k and the lower order λ of the function $S(r)$ are defined as

$$k = \limsup_{r \to \infty} \frac{\log S(r)}{\log r}$$
$$\text{and } \lambda = \liminf_{r \to \infty} \frac{\log S(r)}{\log r}.$$

Moreover if

$$0 < k < \infty, \text{we set } C = \limsup_{r \to \infty} \frac{S(r)}{r^k}$$

and distinguish the following possibilities:

(a) $S(r)$ has maximal type if $C = +\infty$;

(b) $S(r)$ has mean type if $0 < C < +\infty$;

(c) $S(r)$ has minimal type if $C = 0$;

(d) $S(r)$ has convergence class if $\int_{r_0}^{\infty} \frac{S(t)}{t^{k+1}} dt$ converges

Now we state the following theorem.

Theorem 1.5 {p.18, [34]}. If f is an entire function then the order k of the function $S_1(r) = \log^+ M(r, f)$ and $S_2(r) = T(r, f)$ is the same. Further if $0 < k < \infty$, $S_1(r)$ and $S_2(r)$ belong to the same classes (a), (b), (c) and (d). Also we note that $S_1(r)$ and $S_2(r)$ have the same lower order.

A function f meromorphic in the plane is said to have order ρ, lower order λ and maximal, minimal, mean type or convergence class if the function $T(r, f)$ has this property. For entire functions these coincide by the above theorem with the corresponding definition in terms of $M(r, f)$ which is classical. The type of a meromorphic function f is defined by

$$\tau = \limsup_{r \to \infty} \frac{T(r, f)}{r^\rho}, 0 < \rho < \infty.$$

We know that the order and the lower order of an entire function f and its derivative are equal. The same result holds for a meromorphic function also. After revealing the important symmetry property of a meromorphic function f, which is expressed in the first fundamental theorem through the invariance of the sum $m(r, a) + N(r, a)$, it is natural to attempt for a more careful investigation of the relative strength of two terms in the sum, of the proximity component $m(r, a)$ and of the counting component $N(r, a)$. Individual results have been obtained in this direction {p.234, [34]}.

1, Picard's theorem shows that the counting function for a non-constant meromorphic function in the finite complex plane vanish for almost two values of a.

2. For a meromorphic function of finite non-integral order there is almost one Picard's exceptional value.

3. That the counting function $N(r, a)$ is in general i.e., for the great majority of the values of a, large in comparison with the proximity function.

We now state Nevanlinna's Second Fundamental theorem.

Theorem 1.6 {p.31, [34]}. Suppose that f is a non-constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, ...a_q (q \geq 2)$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\gamma| \geq \delta$ for $1 \leq \mu < \gamma \leq q$. Then

$$m(r, \infty) + \sum_{\gamma=1}^{q} m(r, a_\gamma) \leq 2T(r, f) - N_1(r) + S(r),$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N(r, \frac{1}{f'}) + 2N(r, f) - N(r, f')$$

$$\text{and } S(r) = m(r, \frac{f'}{f}) + m\{r, \sum_{\gamma=1}^{q} \frac{f'}{(f - a_\gamma)}\} + q \log^+ \frac{3q}{\delta} + \log 2 + \log \frac{1}{|f'(0)|},$$

with modifications if $f(0) = 0$ or ∞ and $f'(0) = 0$.
The quantity $S(r)$ will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.
The following theorem gives an estimation of $S(r)$.
Theorem 1.7{p.34, [34]}. Let f be a meromorphic function and not constant in $|z| < R_0 \leq \infty$ and that $S(r) \equiv S(r, f)$ is defined as in the above theorem. Then we have
(i) If $R_0 = +\infty$, $S(r, f) = O\{\log T(r, f)\} + O(\log r)$ as $r \to \infty$ through all values if f has finite order and as $r \to \infty$ outside a set E of finite linear measure otherwise
(ii) If $0 < R_0 < \infty, S(r, f) = O\{\log^+ T(r, f) + \log 1/(R_0 - r)\}$ as $r \to R_0$ outside a set E such that $\int_E \frac{dr}{R_0 - r} < +\infty$.
Further there is a point r outside E for which $\rho < r < \rho'$ provided that $0 < R - \rho' < e^{-2}(R - \rho)$.
Consequently we get the following theorem.
Theorem 1.8{p.41, [34]}. Let f be meromorphic and non-constant in $|z| \leq R_0$. Then

$$\frac{S(r, f)}{T(r, f)} \to 0 \text{ as } r \to R_0 \text{ with the following provisions:} \qquad (*)$$

$\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to R_0$ with the following provisions:
(a)(*) holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.
(b)(*) If f has infinite order in the plane, (*) still holds as $r \to \infty$ outside a certain exceptional set E_0 of finite length.
Here E_0 depends only on f.
(c) If $R_0 < +\infty$ and $\limsup\limits_{r \to \infty} \frac{T(r,f)}{\log\{\frac{1}{(R_0 - r)}\}} = +\infty$,
then (*) holds as $r \to R_0$ through a suitable sequence r_n, which depends on f only. This theorem points out why $S(r)$ plays the role of an unimportant

error term.
Let f be meromorphic and not constant in the plane. We shall denote by $S(r,f)$ any quantity $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set r of finite linear measure. Also we shall denote by a, a_0, a_1 etc. functions meromorphic in the plane and satisfying $T(r,a) = S(r,f)$ as $r \to \infty$. Now we introduce Milloux's theorem which is important in studying the properties of the derivatives of meromorphic functions.

Theorem 1.9{p.55, [34]}. Let l be a positive integer and $\psi = \sum_{\gamma=0}^{l} a_\gamma f^{(\gamma)}$. Then $m(r, \frac{\psi}{f}) = S(r,f)$ and $T(r,\psi) \le (l+1)T(r,f) + S(r,f)$.
Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of $f = a$ by roots of the equation $\psi = b$, where ψ is given as in the above theorem. In this connection we state the following theorem.

Theorem 1.10{p.57, [34]}. Let f be meromorphic and non-constant in the plane and $\psi = \sum_{\gamma=0}^{l} a_\gamma f^{(\gamma)}$, where l is a positive integer. If ψ is non-constant then

$$T(r,f) < \overline{N}(r,f) + N(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{\psi-1}) - N_0(r,\frac{1}{\psi'}) + S(r,f)$$

where in $N_0(r,\frac{1}{\psi'})$ only zeros of ψ' not corresponding to the repeated roots of $\psi = 1$ are to be considered.
Now we set

$$\delta(a) = \delta(a;f) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)},$$
$$\Theta(a) = \Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)},$$

where $\overline{N}(r,a\,;f) \equiv \overline{N}(r,a)$ is the counting function for distinct a-points,

$$\theta(a) = \theta(a;f) = \liminf_{r\to\infty} \frac{N(r,a) - \overline{N}(r,a)}{T(r,f)}.$$

Evidently, given $\varepsilon(>0)$, we have for sufficiently large values of r,

$$N(r,a) - \overline{N}(r,a) > \{\theta(a) - \varepsilon\}T(r,f),$$
$$N(r,a) < \{1 - \delta(a) + \varepsilon\}T(r,f)$$

and hence

$$\overline{N}(r,a) < \{1-\delta(a)-\theta(a)+2\varepsilon\}T(r,f)$$

so that

$$\Theta(a) \geq \delta(a)+\theta(a).$$

The quantity $\delta(a)$ is known as the deficiency of the value 'a' and $\theta(a)$ is called the index of multiplicity. Evidently $\delta(a)$ is positive only if there are relatively few roots of the equation $f=a$, while $\theta(a)$ is positive if there are relatively many multiple roots.

Let us now state a fundamental theorem called Nevanlinna's theorem on deficient values.

Theorem 1.11 {p.43, [34]}. Let f be a non-constant meromorphic function defined on the plane. Then the set of values a for which $\Theta(a) > 0$ is countable and we have, on summing over all such values a

$$\sum_a \{\delta(a)+\theta(a)\} \leq \sum_a \Theta(a) \leq 2.$$

The magnitude of the deficiency $\delta(a)$ lies in the closed unit interval $[0,1]$ and it gives us a very accurate measure for the relative density of the points where the function f assumes the value 'a' in question. The larger the deficiency is, the more rare are latter points. The deficiency reaches its maximum value 1 when the latter have been very sparsely distributed, as for example, in the extreme case where the value 'a' is a Picard exceptional value i.e., a complex number which is not assumed by the function. We shall call every value of vanishing deficiency $\delta(a)$, a normal value in contrast to the deficient values for which $\delta(a)$ is positive.

We know from Picard's theorem that a meromophic function can have at most two Picard exceptional values. This theorem follows easily from Nevanlinna's theorem on deficient value because as we have stated before that for a Picard exceptional value a, $\delta(a)=1$.

The quantity

$$\triangle(a\,;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a)}{T(r,f)}$$

gives another measure of deficiency and is called the Valiron deficiency.

Clearly

$$0 \leq \delta(a\,;f) \leq \triangle(a\,;f) \leq 1.$$

An entire function $f(z)$ has a Taylor's expansion about any point a in the complex plane of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Since this series is absolutely convergent everywhere in the plane, the terms $|a_n|$ must approach 0. Consequently, there exists for each a an index $n_0 = n(a)$ for which $|a_n|$ is a maximal coefficient. B. Lepson [41]raised the problem of characterising entire functions for which $n(a)$ is bounded. The latter are called functions of bounded index.

Definition 1.6. An entire function f is said to be of bounded index if and only if there exists an integer N, such that for all z

$$\max\left(|f|,\ \left|f^{(1)}\right|),\ \frac{\left|f^{(2)}\right|}{2!}, \ldots, \frac{\left|f^{(N)}\right|}{N!} \right) \geq \frac{\left|f^{(j)}\right|}{j!},$$

$$\text{where } j = 0, 1, 2, 3 \ldots \text{and } f^{(0)} \text{denotes } f. \tag{1.2}$$

We shall say that f is of index N, if N is the smallest integer for which (1.2) holds.

An entire function which is not of bounded index is said to be of unbounded index.

A function of bounded index satisfies

$$\sum_{i=0}^{N} \frac{\left|f^{(i)}\right|}{i!} \geq \frac{\left|f^{(j)}\right|}{j!}, j = 0, 1, 2, 3, \ldots\ldots$$

Furthermore, if (1.3) holds then

$$\max\left(|f|,\ \left|f^{(1)}\right|,\ \frac{\left|f^{(2)}\right|}{2!}, \ldots\ldots, \frac{f^{(N)}}{N!} \right) \geq \frac{1}{(N+1)} \cdot \frac{\left|f^{(j)}\right|}{j!}; j = 0, 1, 2, 3, \ldots\ldots$$

These facts suggests

Definition 1.7. An entire function $f(z)$ is said to be of non-uniform bounded index if and only if there exist integers N_j, such that

$$\sum_{i=0}^{N} \frac{\left|f^{(i)}(z)\right|}{i!} \geq c. \frac{\left|f^{(j)}(z)\right|}{j!} \text{ for } |z| > N_j,$$

where $j = 0, 1, 2, 3, \ldots\ldots$ and c is any fixed constant.

For bounded regions Lepson [41]proved that

Theorem 1.12. If $f(z) \neq 0$ is an entire function and R is any bounded set, then there is an integer N such that for any z in R and any non-negative integer n

$$\frac{|f^{(n)}(z)|}{n!} \leq \max\left(|f(z)|, \left|f^{(1)}(z)\right|, \frac{|f^{(2)}(z)|}{2!}, \ldots\ldots, \frac{|f^{(n)}(z)|}{N!}\right).$$

Example 1.1. The function $\exp z$ is obviously of bounded index.

Example 1.2. Let k be a positive integer. The entire function $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(kn)!}$ is of non-uniform bounded index which can be proved by the following theorem of F. Gross [31].

Theorem 1.13. Let t be a positive integer. If $g(z) = f(z^{\frac{1}{t}})$ is entire and $f(z)$ is of bounded index (say of index k) then $g(z)$ is of non-uniform bounded index. We now discuss two special type of orders of entire functions.

During the past decades, several authors {cf. [54], [57], [58] and [61]} made close investigations on the properties of entire Dirichlet series related to Ritt order. Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1.4}$$

where $0 < \lambda_n < \lambda_{n+1} (n \geq 1)$, $\lambda_n \to \infty$ as $n \to \infty$ and a_n's are complex constants.

If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1.4) then in this clearly $\sigma_c = \sigma_a = \infty$.

$$\text{Let } F(\sigma) = \underset{-\infty<t<\infty}{lub} |f(\sigma + it)|. \tag{1.5}$$

Then the Ritt order [59] of $f(s)$ denoted by $\rho(f)$ is given by

$$\rho(f) = \limsup_{\sigma\to\infty} \frac{\log\log F(\sigma)}{\sigma} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma}.$$

In other words

$$\rho(f) = \inf\{\mu > 0 : \log F(\sigma) < \exp(\sigma\mu) \text{ for all } \sigma > R(\mu)\}.$$

Similarly the lower Ritt order of $f(s)$ denoted by $\lambda(f)$ may be defined.

Let us denote by C^n and R^n the complex and real $n-$space respectively. We

indicate the point $(z_1, \ldots z_n)$, $(m_1, \ldots m_n)$ of C^n or I^n by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. The modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + \ldots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^m will denote $z_1^{m_1} \ldots z_n^{m_n}$ and $||m|| = m_1 + \ldots + m_n$.
Let $D \subset C^n$ be an arbitrary bounded complete n-circular domain with center at the origin of coordinates. Then for the analytic function f and $R > 0$, $M_{f,D}(R) = \sup_{z \epsilon D_R} |f(z)|$, where a point $z \epsilon D_R$ if and only if $\frac{z}{R} \epsilon D$.
Definition 1.8 {[30], p. 339} The Gol'dberg order (briefly G-order) ρ_D^f of f with respect to the domain D is defined as

$$\rho_D(f) \equiv \rho_D^f = \limsup_{R \to \infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

The lower Gol'dberg order λ_D^f of f with respect to the domain D is defined as

$$\lambda_D(f) \equiv \lambda_D^f = \liminf_{R \to \infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

We say that f is of regular growth if $\rho_D^f = \lambda_D^f$.
Further theories of entire and meromorphic functions can be found in [4] and [13] and those of analytic functions are available in [15] and [30]. Extensive works on growth properties of entire and meromorphic functions related to relative order, relative lower order, relative sharing of values, relative Ritt order, relative Gol'dberg order etc. have been done in [6], [7], [8], [9], [10], [11], [45], [49], [51], [52], [53], [54], [55], [56] and [60]. Also growth properties based on differential monomials, differential polynomials and wronskians, all generated by entire or meromorphic functions have been studied in[5], [47] and [50]. One can find the theories of Dirichlet series and their related problems in [33] and [61]. Further growth properties related to convex or entire functions as well as the potential theory in several complex variables can be found in [39] and [40]. For results related to deficiencies of meromorphic functions one can see [44] and [69].
Apart from **Chapter 1** the thesis consists of ten other chapters.

- **In Chapter 2** we study the growth properties of composite entire and meromorphic functions using L-order and L^*-order improving some earlier results where $L \equiv L(r)$ is a slowly changing function. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [19].

- **In Chapter 3** we consider the growth properties of composite entire and meromorphic functions using (p, q)th order which improve some earlier results, where p, q are positive integers and $p > q$. Some results in the form of remarks based upon minimum modulus of entire functions have also been stated in this chapter. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences**, see [20].

- **In Chapter 4** we discuss the growth properties of entire functions on the basis of relative L-order where $L \equiv L(r)$ is a slowly changing function. Also some growth properties in terms of minimum modulus of entire functions have been discussed in this chapter. The results of this chapter have been published in **International Mathematical Forum**, see [21].

- **In Chapter 5** we study the comparative growth properties of entire functions on the basis of relative $L - (p, q)$th order where p, q are positive integers and $p > q$. A few results in the form of remarks on the basis of minimum modulus of entire functions have also been stated in this chapter. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [22].

- **In Chapter 6** we study the ${}_{(t)}L^* - (p, q)$th order of entire and meromorphic functions based on sharing of values of them. The results of this chapter have been published in **International Mathematical Forum**, see [23].

- **In Chapter 7** we generalise the results of Datta and Mondal [18]. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences,** see [24].

- **In Chapter 8** we study regular L relative growth and regular L^*relative growth of an entire function with respect to another entire function and show that under certain conditions they are equal where $L \equiv L(r)$ is a slowly changing function. The results of this chapter have been published in **International Mathematical Forum**, see [25].

- **In Chapter 9** we establish some results on the comparative growth properties related to generalised L^*-Ritt order of entire Dirichlet series. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [26].

- **In Chapter 10** we study the comparative growth properties of composite entire functions of two complex variables. We also introduce the notion of zero order of entire functions of two complex variables and discuss some related results. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences**, see [27].

- **In Chapter 11** we discuss about the comparative growth properties related to the Gol'dberg order of composition of two entire function of several complex variables. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [28].

From Chapter 2 onwards when we write Theorem a.b.c (Corollary a.b.c. etc.) where a,b and c are positive integers, we mean the c-th Theorem (c-th Corollary etc.) of the b-th Section in the a-th chapter. Also by equation number (a.b) we mean the b-th equation in the a-th chapter for positive integers a and b. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

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Chapter 2

ESTIMATION OF GROWTH OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS OF ORDER ZERO ON THE BASIS OF SLOWLY CHANGING FUNCTIONS

CHAPTER

2

ESTIMATION OF GROWTH OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS OF ORDER ZERO ON THE BASIS OF SLOWLY CHANGING FUNCTIONS

2.1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. In the sequel we use the following notation:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, \ldots \text{and } \log^{[0]} x = x.$$

We recall the following definitions:

Definition 2.1.1 *The order ρ_f and lower order λ_f of a meromorphic func-*

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [**19**].

tion f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$
$$\text{and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$
$$\text{and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2.1.2 *The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f are defined as follows:*

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}$$
$$\text{and } \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$
$$\text{and } \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 2.1.3 *[46] Let f be meromorphic function of order zero. Then ρ_f^*, λ_f^* and $\overline{\rho}_f^*, \overline{\lambda}_f^*$ are defined as follows:*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r},$$
$$\lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$
$$\text{and } \overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r},$$
$$\overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}$$

If f is entire, then clearly

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r},$$

$$\lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$

$$\text{and } \overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}$$

$$\overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

Definition 2.1.4 *The type σ_f of a meromorphic function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

If f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

Somasundaram and Thamizharasi [63] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Their definitions are as follows:

Definition 2.1.5 *[63] The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}$$

$$\text{and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}$$

$$\text{and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

Definition 2.1.6 *[63]. The L-type σ_f^L of an entire function f with L-order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$

For meromorphic f, the L-type σ_f^L becomes

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r,f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$

Similarly one can define the L-hyper order and L-hyper lower order of entire and meromorphic f.The more generalised concept of L-order and L-type of entire and meromorphic functions are L^*-order and L^*-type. Their definitions are as follows:

Definition 2.1.7 *The L^*-order, L^*-lower order and L^*-type of a meromorphic function f are defined by*

$$\begin{aligned}
\rho_f^{L^*} &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]},\\
\lambda_f^{L^*} &= \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}\\
\text{and } \sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{T(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty.
\end{aligned}$$

When f is entire, one can easily verify that

$$\begin{aligned}
\rho_f^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]},\\
\lambda_f^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}\\
\text{and } \sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty.
\end{aligned}$$

In this chapter we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of L-order and L^*-order improving some previous results.

2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.2.1 *[14] If f and g are two entire functions, then for all sufficiently large values of r*

$$M(\frac{1}{8}M(\frac{r}{2},g)-|g(0)|,f)\leq M(r,fog)\leq M(M(r,g),f).$$

Lemma 2.2.2 *[62] Let f be entire and g be a transcendental entire function of finite lower order. Then for any $\delta>0$,*

$$M(r^{1+\delta},fog)\geq M(M(r,g),f)(r\geq r_0).$$

Lemma 2.2.3 *[29] Let f be a meromorphic function and g be transcendental entire. If $\lambda_{fog}<\infty$ then $\lambda_f=0$.In the line of Lemma 2.2.3 we may state the following lemma without proof.*

Lemma 2.2.4 *Let f be a meromorphic function and g be transcendental entire. If $\lambda^L_{fog}<\infty$ then $\lambda^L_f=0$.*

Lemma 2.2.5 *[31] Let f be meromorphic and g be transcendental entire. If $\rho_{fog}<\infty$ then $\rho_f=0$.*

Lemma 2.2.6 *Let f be meromorphic and g be transcendental entire. If $\rho^L_{fog}<\infty$ then $\rho^L_f=0$.*

Lemma 2.2.7 *Let f be a meromorphic function and g be transcendental entire. If $\lambda^{L^*}_{fog}<\infty$ then $\lambda^{L^*}_f=0$.*

Lemma 2.2.8 *Let f be meromorphic and g be transcendental entire. If $\rho^{L^*}_{fog}<\infty$ then $\rho^{L^*}_f=0$.*

Lemma 2.2.9 *Let f be entire and g be transcendental entire with $\lambda^L_g<\infty$.Also let $\rho^L_{fog}=0$.Then ${}^*\rho_f\ {}^*\lambda^L_g\leq{}^*\ \rho^L_{fog}\leq{}^*\rho_f\ {}^*\rho^L_g$.*

Proof. By Lemma 2.2.2,

$$\begin{aligned}{}^{*}\rho_{fog}^{L} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[rL(r)]} \\ &\geq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[rL(r)]} \\ &= \rho_f^{*}\,{}^{*}\lambda_g^{L}.\end{aligned}$$

Again by Lemma 2.2.1,

$$\begin{aligned}{}^{*}\rho_{fog}^{L} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]}[rL(r)]} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[rL(r)]} \\ &= \rho_f^{*}\,{}^{*}\rho_g^{L}.\end{aligned}$$

From the above two inequalities we get that $\rho_f^{*}\,{}^{*}\lambda_g^{L} \leq {}^{*}\rho_{fog}^{L} \leq \rho_f^{*}\,{}^{*}\rho_g^{L}$. ■

Remark 2.2.1 *The second part of Lemma 2.2.9 is also valid under the same conditions for meromorphic f and entire g.*

Lemma 2.2.10 *Let f and g be two entire functions such that $\rho_f^L = 0$ and $\lambda_g^L < \infty$. Also let g be transcendental entire. Then*

$$\lambda_f^{*}\rho_g^{L} \leq \rho_{fog}^{L} \leq \rho_f^{*}\rho_g^{L}.$$

Proof. In view of Lemma 2.2.1, we get that

$$\begin{aligned}\rho_{fog}^{L} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[rL(r)]} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[rL(r)]} \\ &= \rho_f^{*}\rho_g^{L}.\end{aligned}$$

Also from Lemma 2.2.2 it follows that

$$\begin{aligned}
\rho^L_{fog} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log[rL(r)]^{1+\delta}} \\
&\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[rL(r)]^{1+\delta}} \\
&= \lambda^*_f \rho^L_g .
\end{aligned}$$

Now combining the above two inequalities we obtain that

$$\lambda^*_f \rho^L_g \leq \rho^L_{fog} \leq \rho^*_f \rho^L_g .$$

■

Remark 2.2.2 *Under the conditions of Lemma 2.2.10,*

$$\rho^*_f \lambda^L_g \leq \rho^L_{fog} \leq \rho^*_f \rho^L_g .$$

Lemma 2.2.11 *If f be an entire function and g be transcendental entire with*

$$\lambda^L_{fog} = 0, \lambda^L_g < \infty.$$

$$\textit{Then } {}^*\lambda^L_{fog} \geq {}^*\lambda^{L*}_f \lambda^L_g .$$

Proof. By Lemma 2.2.2,

$$\begin{aligned}
{}^*\lambda^L_{fog} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[rL(r)]^{1+\delta}} \\
&\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[rL(r)]} \\
&= \lambda^*_f \, {}^*\lambda^L_g .
\end{aligned}$$

This proves the lemma. ■

Lemma 2.2.12 *Let f be entire and g be transcendental entire with $\lambda^{L^*}_g < \infty$. Also let $\rho^{L^*}_{fog} = 0$. Then*

$${}^*\rho^*_f \lambda^{L^*}_g \leq^* \rho^{L^*}_{fog} \leq^* \rho^*_f \rho^{L^*}_g .$$

Proof. By Lemma 2.2.2,

$$
\begin{aligned}
{}^{*}\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[re^{L(r)}]} \\
&\geq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]} \\
&= \rho_f^{*}\, {}^{*}\lambda_g^{L^*}.
\end{aligned}
$$

Again by Lemma 2.2.1,

$$
\begin{aligned}
{}^{*}\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]}[re^{L(r)}]} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]} \\
&= \rho_f^{*}\, {}^{*}\rho_g^{L^*}.
\end{aligned}
$$

From the above two inequalities we get that

$$
\rho_f^{*}\, {}^{*}\lambda_g^{L^*} \leq^{*} \rho_{fog}^{L^*} \leq \rho_f^{*}\, {}^{*}\rho_g^{L^*}.
$$

This proves the lemma. ■

Remark 2.2.3 *The second part of Lemma 2.2.12 is also valid under the same conditions for meromorphic f and entire g.*

Lemma 2.2.13 *Let f and g be two entire functions such that $\rho_f^{L^*} = 0$ and $\lambda_g^{L^*} < \infty$. Also let g be transcendental entire. Then*

$$
\lambda_f^{*}\rho_g^{L^*} \leq \rho_{fog}^{L^*} \leq \rho_f^{*}\rho_g^{L^*}
$$

Proof. In view of Lemma 2.2.1, we get that

$$
\begin{aligned}
\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log[re^{L(r)}]} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]} \\
&= \rho_f^{*}\rho_g^{L^*}.
\end{aligned}
$$

Again from Lemma 2.2.2 it follows that

$$\begin{aligned}
\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log[re^{L(r)}]^{1+\delta}} \\
&\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} \cdot \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log[re^{L(r)}]^{1+\delta}} \\
&= \lambda_f^* \rho_g^{L^*}.
\end{aligned}$$

Now combining the above two inequalities we obtain that

$$\lambda_f^* \rho_g^{L^*} \leq \rho_{fog}^{L^*} \leq \rho_f^* \rho_g^{L^*}.$$

Thus the lemma is established. ■

Remark 2.2.4 *Under the conditions of Lemma 2.2.13,*

$$\rho_f^* \lambda_g^{L^*} \leq \rho_{fog}^{L^*} \leq \rho_f^* \rho_g^{L^*}.$$

Lemma 2.2.14 *If f is an entire function and g be transcendental entire with*

$$\lambda_{fog}^{L^*} = 0, \lambda_g^{L^*} < \infty.$$

Then

$$^*\lambda_{fog}^{L^*} \geq {}^*\lambda_f^{L^*}\, {}^*\lambda_g^{L^*}.$$

Proof. By Lemma 2.2.2,

$$\begin{aligned}
^*\lambda_{fog}^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r^{1+\delta}, fog)}{\log^{[2]}[re^{L(r)}]^{1+\delta}} \\
&\geq \liminf_{r\to\infty} \frac{\log^{[2]} M(M(r,g),f)}{\log^{[2]} M(r,g)} \cdot \liminf_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log^{[2]}[re^{L(r)}]} \\
&= \lambda_f^*\, {}^*\lambda_g^{L^*}.
\end{aligned}$$

This proves the lemma. ■

2.3 Theorems.

In this section we present the main results of this chapter.

Theorem 2.3.1 *Let f be meromorphic and g be transcendental entire such that $\lambda_{fog}^{L} > 0$. Then for every positive number A,*

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f^{(k)})} \leq \frac{{}^*\lambda_{fog}^{L}}{A^*\lambda_f^L}, \text{ where } k = 0, 1, 2, 3, \ldots$$

Proof. Case (a). If $\lambda_{fog}^{L} = \infty$, the theorem is obvious.

Case (b). If $\lambda_{fog}^{L} < \infty$, then by Lemma 2.2.4, $\lambda_f^L = 0$ and the theorem follows. ■

Theorem 2.3.2 *Let f be meromorphic and g be transcendental entire such that $\rho_{fog}^{L} = 0$. Also, let $0 < {}^*\lambda_{fog}^{L} \leq {}^*\rho_{fog}^{L} < \infty$ and $0 <^* \lambda_f^L \leq^* \rho_f^L < \infty$. Then for any positive number A,*

$$\frac{{}^*\lambda_{fog}^{L}}{A^*\rho_f^L} \leq \liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\lambda_{fog}^{L}}{A^*\lambda_f^L}$$
$$\leq \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho_{fog}^{L}}{A^*\lambda_f^L}.$$

Proof. Since $\rho_{fog}^{L} = 0 < \infty$ by Lemma 2.2.6, $\rho_f^L = 0$. Now from the definition of ${}^*\rho_f^L$ and ${}^*\lambda_f^L$ we have for arbitrary positive ε and for all large values of r,

$$\log T(r, fog) \geq ({}^*\lambda_{fog}^{L} - \varepsilon) \log^{[2]}[rL(r)] \tag{2.3.1}$$

$$\text{and } \log T(r^A, f) \leq A({}^*\rho_f^L + \varepsilon) \log^{[2]}[rL(r)]. \tag{2.3.2}$$

Now from (2.3.1) and (2.3.2) it follows for all large values of r,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\lambda_{fog}^{L} - \varepsilon}{A({}^*\rho_f^L + \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\lambda_{fog}^{L}}{A^*\rho_f^L}. \tag{2.3.3}$$

Again for a sequence of values of r tending to infinity,

$$\log T(r, fog) \leq ({}^*\lambda_{fog}^{L} + \varepsilon) \log^{[2]}[rL(r)] \tag{2.3.4}$$

and for all large values of r,

$$\log T(r^A, f) \geq A({}^*\lambda_f^L - \varepsilon)\log^{[2]}[rL(r)]. \tag{2.3.5}$$

Combining (2.3.4) and (2.3.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\lambda_{fog}^L + \varepsilon}{A({}^*\lambda_f^L - \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\lambda_{fog}^L}{A^*\lambda_f^L}. \tag{2.3.6}$$

Also, for a sequence of values of r tending to infinity,

$$\log T(r^A, f) \leq A({}^*\lambda_f^L + \varepsilon)\log^{[2]}[rL(r)]. \tag{2.3.7}$$

Now from (2.3.1) and (2.3.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\lambda_{fog}^L - \varepsilon}{A({}^*\lambda_f^L + \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary, we get that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\lambda_{fog}^L}{A^*\lambda_f^L}. \tag{2.3.8}$$

Also for all large values of r,

$$\log T(r, fog) \leq ({}^*\rho_{fog}^L + \varepsilon)\log^{[2]}[rL(r)]. \tag{2.3.9}$$

From (2.3.5) and (2.3.9) it follows for all large values of r,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho_{fog}^L + \varepsilon}{A({}^*\lambda_f^L - \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho_{fog}^L}{A^*\lambda_f^L}. \tag{2.3.10}$$

Thus the theorem follows from (2.3.3), (2.3.6), (2.3.8) and (2.3.10). ■

Theorem 2.3.3 *Let f be entire and g be transcendental entire satisfying the following conditions*

$$(i)\rho^L_{fog} = 0 \text{ and } \lambda^L_g < \infty$$
$$(ii) 0 <^* \lambda^L_{fog} \leq^* \rho^L_{fog} < \infty$$
$$\text{and } (iii) 0 <^* \lambda^L_f \leq^* \rho^L_f < \infty.$$

Then

$$\frac{{}^*\lambda^{L*}_f \lambda^L_g}{A^* \rho^L_f} \leq \frac{{}^*\lambda^L_{fog}}{A^* \rho^L_f} \leq \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \leq \frac{{}^*\lambda^L_{fog}}{A^* \lambda^L_f}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r^A, f)} \leq \frac{\rho^L_{fog}}{A^* \lambda^L_f} \leq \frac{{}^*\rho^{L*}_f \rho^L_g}{A^* \lambda^L_f}.$$

Proof. In view of Lemma 2.2.11 and the second part of Lemma 2.2.9, Theorem 2.3.3 follows from Theorem 2.3.2. ∎

Theorem 2.3.4 *Let f be meromorphic and g be entire such that $\rho^L_{fog} = 0$. Also let $0 < {}^*\lambda^L_{fog} \leq {}^*\rho^L_{fog} < \infty$ and $0 < {}^*\rho^L_f < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho^L_{fog}}{A^* \rho^L_f} \leq \limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)}.$$

Proof. In view of Lemma 2.2.6, $\rho^L_{fog} = 0$ implies that $\rho^L_f = 0$.From the definition of L-order we get for a sequence of values of r tending to infinity,

$$\log T(r^A, f) \geq A({}^*\rho^L_f - \varepsilon) \log^{[2]}[rL(r)]. \tag{2.3.11}$$

Now from (2.3.9) and (2.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho^L_{fog} + \varepsilon}{A({}^*\rho^L_f - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain,

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \leq \frac{{}^*\rho^L_{fog}}{A^* \rho^L_f}. \tag{2.3.12}$$

Again for a sequence of values of r tending to infinity,

$$\log T(r, fog) \geq ({}^*\rho^L_{fog} - \varepsilon)\log^{[2]}[rL(r)]. \tag{2.3.13}$$

So combining (2.3.2) and (2.3.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\rho^L_{fog} - \varepsilon}{A({}^*\rho^L_f + \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, f)} \geq \frac{{}^*\rho^L_{fog}}{A{}^*\rho^L_f}. \tag{2.3.14}$$

Thus the theorem follows from (2.3.12) and (2.3.14). ■

Theorem 2.3.5 *If f be an entire function and g be a transcendental entire function satisfying*

$$(i)\rho^L_{fog} = 0 \text{ and } \lambda^L_g < \infty,$$
$$(ii)0 <^* \lambda^L_{fog} \leq^* \rho^L_{fog} < \infty$$
$$\text{and } (iii)0 <^* \rho^L_f < \infty, \text{ then}$$

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \geq^* \lambda^L_g$$
$$\text{and } \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \leq^* \rho^L_g.$$

Proof. In view of Lemma 2.2.9 we obtain from Theorem 2.3.4 for $A = 1$,

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \geq \frac{{}^*\rho^{L*}_f\lambda^L_g}{{}^*\rho^L_f} =^* \lambda^L_g$$
$$\text{and } \liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \leq \frac{{}^*\rho^{L*}_f\rho^L_g}{{}^*\rho^L_f} =^* \rho^L_g.$$

Thus the theorem follows from the above two inequalities. The following theorem is a natural consequence of Theorem 2.3.2 and Theorem 2.3.4. ■

Theorem 2.3.6 *Let f be meromorphic and g be entire such that $\rho^L_{fog} = 0$. Also let*

$$0 < {}^*\lambda^L_{fog} \leq {}^*\rho^L_{fog} < \infty$$
$$\text{and } 0 <^* \lambda^L_f \leq^* \rho^L_f < \infty.$$

Then for any positive number A,

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log T(r,fog)}{\log T(r^A,f)} &\leq \min\{\frac{{}^*\lambda^L_{fog}}{A^*\lambda^L_f}, \frac{{}^*\rho^L_{fog}}{A^*\rho^L_f}\}\\ &\leq \max\{\frac{{}^*\lambda^L_{fog}}{A^*\lambda^L_f}, \frac{{}^*\rho^L_{fog}}{A^*\rho^L_f}\}\\ &\leq \limsup_{r\to\infty} \frac{\log T(r,fog)}{\log T(r^A,f)}.\end{aligned}$$

The proof is omitted.

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Chapter 3

ON THE GROWTH OF COMPOSITION OF ENTIRE FUNCTIONS WITH RESPECT TO MINIMUM MODULUS

CHAPTER

3

ON THE GROWTH OF COMPOSITION OF ENTIRE FUNCTIONS WITH RESPECT TO MINIMUM MODULUS

3.1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. In the sequel we use the following notation:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, \ldots \text{and } \log^{[0]} x = x.$$

The following definition is well known.

Definition 3.1.1 *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences**, see [**20**].

Juneja, Kapoor and Bajpai [38] defined the (p,q)th order and (p,q)th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}.$$

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r},$$

where p,q are positive integers and $p > q$.In this chapter we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of (p,q)th order and minimum modulus of integral (entire) functions improving some previous results where p,q are positive integers and $p > q$.

3.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.2.1 *[14] If f and g are two entire functions, then for all sufficiently large values of r*

$$M(\frac{1}{8}M(\frac{r}{2},g) - |g(0)|, f) \leq M(r, fog) \leq M(M(r,g),f).$$

Lemma 3.2.2 *[62] Let f be entire and g be a transcendental entire function of finite lower order. Then for any $\delta > 0$,*

$$M(r^{1+\delta}, fog) \geq M(M(r,g),f) \; (r \geq r_0).$$

Lemma 3.2.3 *{[2], [36]}. Let $f(z)$ be an entire function such that*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \leq c < \frac{1}{4e}.$$

If $0 < 4ec < \delta < 1$ then outside a set of upper logarithmic density at most δ,

$$\frac{m(r,f)}{M(r,f)} > k(\delta,c) = \frac{1-2.2\tau}{1+2.2\tau} \text{ where } \tau = \exp\{-\delta/(4ec)\}.$$

If in particular $c = 0$ *then*

$$\frac{m(r,f)}{M(r,f)} \to 1 \text{ as } r \to \infty$$

on a set of logarithmic density 1, where

$$m(r,f) = \inf_{|z|=r} |f(z)|\,, \text{ the minimum modulus of } f.$$

3.3 Theorems.

In this section we present the main results of the chapter.

Theorem 3.3.1 *Let* f *be entire and* g *be transcendental entire with* $\lambda_g < \infty$.

Then

$$\rho_f(p,m)\ \lambda_g(m,q) \ \leq \rho_{fog}(p,q) \ \leq \rho_f(p,m)\rho_g(m,q),$$

where p, q, m, n are positive integers such that $p > m > q$.
Proof. In view of Lemma 3.2.2

$$\begin{aligned}\rho_{fog}(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p]} M(r^{1+\delta}, fog)}{\log^{[q]} r^{1+\delta}} \\ &\geq \limsup_{r\to\infty} \frac{\log^{[p]} M(M(r,g),f)}{\log^{[m]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]} r} \\ &= \rho_f(p,m).\lambda_g(m,q).\end{aligned}$$

Again by Lemma 3.2.1,

$$\begin{aligned}\rho_{fog}(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p]} M(r, fog)}{\log^{[q]} r} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[p]} M(M(r,g),f)}{\log^{[m]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]} r} \\ &= \rho_f(p,m)\rho_g(m,q).\end{aligned}$$

From the above two inequalities we get that

$$\rho_f(p,m)\lambda_g(m,q) \leq \rho_{fog}(p,q) \leq \rho_f(p,m)\rho_g(m,q).$$

This proves the theorem. ■

Corollary 3.3.1 *Under the same conditions of Theorem 3.3.1,*

$$\rho_{fog}(p,q) \geq \lambda_f(p,m)\rho_g(m,q).$$

Proof. By Lemma 3.2.2,

$$\begin{aligned}
\rho_{fog}(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p]} M(r^{1+\delta}, fog)}{\log^{[q]} r^{1+\delta}} \\
&\geq \liminf_{r\to\infty} \frac{\log^{[p]} M(M(r,g),f)}{\log^{[m]} M(r,g)}. \limsup_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]} r} \\
&= \lambda_f(p,m).\rho_g(m,q).
\end{aligned}$$

Thus the corollary is established. ■

Theorem 3.3.2 *If f is an entire function and g be transcendental entire with $\lambda_g(m,q) < \infty$. Then*

$$\lambda_{fog}(p,q) \geq \lambda_f(p,m)\lambda_g(m,q),$$

where p,q,m,n are positive integers such that $p > q > m$.

Proof. By Lemma 3.2.2

$$\begin{aligned}
\lambda_{fog}(p,q) &= \liminf_{r\to\infty} \frac{\log^{[p]} M(r^{1+\delta}, fog)}{\log^{[q]} r^{1+\delta}} \\
&\geq \liminf_{r\to\infty} \frac{\log^{[p]} M(M(r,g),f)}{\log^{[m]} M(r,g)}. \liminf_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]} r} \\
&= \lambda_f(p,m)\lambda_g(m,q).
\end{aligned}$$

This proves the theorem. ■

Remark 3.3.1 *Under the same hypothesis respectively stated in Theorem 3.3.1, Corollary 3.3.1 and Theorem 3.3.2 the conclusions of the theorems can also be drawn by using Lemma 3.2.3 on a set of logarithmic density 1.*

Remark 3.3.2 *The second part of Theorem 3.3.1 is also valid under the same conditions for meromorphic f and entire g.*

Theorem 3.3.3 *Let f be meromorphic and g be entire such that*

$0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty$ *and* $0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty$. *Then*

$$\frac{\lambda_{fog}(p,q)}{\rho_g(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q)}{\lambda_g(m,q)}$$

where p, q, m are positive integers such that $p > q > m$ and $k = 0, 1, 2, \ldots\ldots$

Proof. From the definition of (p,q)th order and (p,q)th lower order we have for arbitrary positive ε and for all large values of r,

$$\log^{[p]} T(r, fog) \geq (\lambda_{fog}(p,q) - \varepsilon) \log^{[q]} r. \tag{3.3.1}$$

$$\text{and } \log^{[m]} T(r^A, g^{(k)}) \leq (\rho_g(m,q) + \varepsilon) \log^{[q]} r. \tag{3.3.2}$$

Now from (3.3.1) and (3.3.2) it follows for all large values of r,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}(p,q) - \varepsilon}{(\rho_g(m,q) + \varepsilon)}. \tag{3.3.3}$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}(p,q)}{\rho_g(m,q)}.$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} T(r, fog) \leq (\lambda_{fog}(p,q) + \varepsilon) \log^{[q]} r \tag{3.3.4}$$

and for all large values of r,

$$\log^{[m]} T(r^A, g^{(k)}) \geq (\lambda_g(m,q) - \varepsilon) \log^{[q]} r. \tag{3.3.5}$$

So combining (3.3.4) and (3.3.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}(p,q) + \varepsilon}{(\lambda_g(m,q) - \varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}. \tag{3.3.6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} T(r^A, g^{(k)}) \leq (\lambda_g(m,q)+\varepsilon)\log^{[q]} r. \tag{3.3.7}$$

Now from (3.3.1) and (3.3.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}(p,q)-\varepsilon}{(\lambda_g(m,q)+\varepsilon)}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}. \tag{3.3.8}$$

Also for all large values of r,

$$\log^{[p]} T(r, fog) \leq (\rho_{fog}(p,q)+\varepsilon)\log^{[q]} r. \tag{3.3.9}$$

So from (3.3.5) and (3.3.9) it follows for all large values of r,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q)+\varepsilon}{A(\lambda_g(m,q)-\varepsilon)}$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q)}{A\lambda_g(m,q)}. \tag{3.3.10}$$

Thus the theorem follows from (3.3.3), (3.3.6), (3.3.8) and (3.3.10). ■

Theorem 3.3.4 *Let f be meromorphic and g be entire such that*

$$0<\lambda_{fog}(p,q) \leq \rho_{fog}(p,q)<\infty \text{ and } 0<\rho_g(m,q)<\infty. \text{ Then}$$

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r, g^{(k)})} \leq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r, g^{(k)})}$$

where p, q, m are positive integers such that $p>q>m$ and $k=0,1,2,...$

Proof. From the definition of (p,q)th order we get for a sequence of values of r tending to infinity,

$$\log^{[m]} T(r^A, g^{(k)}) \geq (\rho_g(m,q) - \varepsilon) \log^{[q]} r. \tag{3.3.11}$$

Now from (3.3.9) and (3.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q) + \varepsilon}{A(\rho_g(m,q) - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q)}{A\rho_g(m,q)}. \tag{3.3.12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} T(r, fog) \geq (\rho_{fog}(p,q) - \varepsilon) \log^{[q]} r. \tag{3.3.13}$$

So combining (3.3.2) and (3.3.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\rho_{fog}(p,q) - \varepsilon}{A(\rho_g(m,q) + \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\rho_{fog}(p,q)}{A\rho_g(m,q)}. \tag{3.3.14}$$

Thus the theorem follows from (3.3.12) and (3.3.14). ■

In view of Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.3.3 we may state the following theorem without proof.

Theorem 3.3.5 *Let f be meromorphic and g be entire such that*

$$0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty \text{ and } 0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty.$$

Then for any positive number A,

$$\begin{aligned}\frac{\lambda_f(p,m)\lambda_g(m,q)}{A\rho_g(m,q)} \leq \frac{\lambda_{fog}(p,q)}{A\rho_g(m,q)} &\leq \liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \\ \leq \frac{\lambda_{fog}(p,q)}{A\lambda_g(m,q)} &\leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \\ \leq \frac{\rho_{fog}(p,q)}{A\lambda_g(m,q)} &\leq \frac{\rho_f(p,m)\rho_g(m,q)}{A\lambda_g(m,q)},\end{aligned}$$

where p, q, m *are positive integers such that* $p > q > m$ *and* $k = 0, 1, 2, \ldots$

The following theorem is a natural consequence of Theorem 3.3.3 and Theorem 3.3.4.

Theorem 3.3.6 *Let* f *be meromorphic and* g *be entire such that*

$$0 < \lambda_{fog}(p, q) \leq \rho_{fog}(p, q) < \infty \text{ and } 0 < \lambda_g(m, q) \leq \rho_g(m, q) < \infty.$$

Then for any positive number A,

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} &\leq \min\left\{\frac{\lambda_{fog}(p, q)}{A\lambda_g(m, q)}, \frac{\rho_{fog}(p, q)}{A\rho_g(m, q)}\right\} \\
&\leq \max\left\{\frac{\lambda_{fog}(p, q)}{A\lambda_g(m, q)}, \frac{\rho_{fog}(p, q)}{A\rho_g(m, q)}\right\} \\
&\leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})},
\end{aligned}$$

where p, q, m *are positive integers such that* $p > q > m$ *and* $k = 0, 1, 2, \ldots$ *The proof is omitted.*

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Chapter 4

RELATIVE L-ORDER AND RELATED COMPARATIVE GROWTH PROPERTIES OF ENTIRE FUNCTIONS ON THE BASIS OF THEIR MINIMUM MODULUS

CHAPTER

4

RELATIVE L-ORDER AND RELATED COMPARATIVE GROWTH PROPERTIES OF ENTIRE FUNCTIONS ON THE BASIS OF THEIR MINIMUM MODULUS

4.1 Introduction, Definitions and Notations.

Let f and g be two entire functions and

$$F(r) = max\{|f(z)| : z = r\}, G(r) = max\{|g(z)| : |z| = r\}.$$

If f is non-constant then $F(r)$ is strictly increasing and continuous and its inverse

$$F^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$$

exists and is such that

$$\lim_{s \rightarrow \infty} F^{-1}(s) = \infty.$$

The results of this chapter have been published in **International Mathematical Forum**, see [**21**].

Bernal[3] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^\mu) \text{ for all } r > r_0(\mu) > 0\} = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

The definition coincides with the classical one[65] if

$$g(z) = \exp z.$$

Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

Somasundaram and Thamizharasi [63] introduced the notions of L-order, L-lower order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.

$$L(ar) \sim L(r) \text{ as } r \to \infty$$

for every constant 'a'. Their definitions are as follows:

Definition 4.1.1 *[63] The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

Definition 4.1.2 *[63] The L-type σ_f^L of an entire function f with L-order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, 0 < \rho_f^L < \infty.$$

Similarly one can define the L-hyper order and L-hyper lower order of entire functions. So with the help of the above notion one can easily define the relative L-order and relative L-lower order of entire functions.

Definition 4.1.3 *The relative L-order $\rho_g^L(f)$ and the relative L-lower order $\lambda_g^L(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]} \text{ and } \lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]}.$$

Definition 4.1.4 *The relative L-hyper order $\overline{\rho}_g^L(f)$ and the relative L-hyper lower order $\overline{\lambda}_g^L(f)$ of an entire function f with respect to another entire function g are defined as*

$$\overline{\rho}_g^L(f) = \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log[rL(r)]} \text{ and } \overline{\lambda}_g^L(f) = \liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log[rL(r)]}.$$

In this chapter we establish some results on the growth properties of entire functions on the basis of relative L-order and relative L-lower order where $L \equiv L(r)$ is a slowly changing function. In the sequel we use the following notations:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, \ldots\ldots \text{and } \log^{[0]} x = x.$$

The more generalised concept of L-order and L-type of entire and meromorphic functions are respectively L^*-order and L^*-type. Their definitions are as follows:

Definition 4.1.5 *The L^*-order, L^*-lower order and L^*-type of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}$$
$$\text{and } \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}, \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}$$
$$\text{and } \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{Lr}]^{\rho_f^{L^*}}}, 0 < \rho_f^{L^*} < \infty.$$

Definition 4.1.6 *The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[re^{L(r)}]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[re^{L(r)}]}.$$

Definition 4.1.7 *The relative L^*-hyper order $\overline{\rho}_g^{L^*}(f)$ and the relative L^*-hyper lower order $\overline{\lambda}_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as*

$$\overline{\rho}_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log[re^{L(r)}]} \text{ and } \overline{\lambda}_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log[re^{L(r)}]}.$$

In order to develop our results we shall need various kinds of measures and densities for sets of points on the positive axis. Let E be such a set and let $E[a,b]$ denote the part of E for which $a < r < b$. The linear and logarithmic measures of E are defined to be

$$m(E) = \int_E dr \text{ and } lm(E) = \int_{E(1,\infty)} \frac{dr}{r} \text{ respectively.}$$

These may be finite or infinite. We also define the lower and upper densities of E by

$$dens\ E(upper) = \limsup_{r\to\infty} \frac{m(E(0,r))}{r}$$
$$\text{and } dens\ E(lower) = \liminf_{r\to\infty} \frac{m(E(0,r))}{r}$$

and also the upper and lower logarithmic densities of E by

$$\log\ densE(upper) = \limsup_{r\to\infty} \frac{\lim(E(1,r))}{\log r}$$
$$\text{and } \log\ densE(lower) = \liminf_{r\to\infty} \frac{\lim(E(1,r))}{\log r}.$$

Also let $f(r) = m(r,f) = \inf_{|z|=r} |f(z)|$

which is known as the minimum modulus of an entire function f. In this chapter we also estimate some growth properties of composite entire functions in terms of their minimum modulus. In fact all the definitions in the chapter can also be stated in terms of minimum modulus on a set of logarithmic density 1.

4.2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 4.2.1 *{[2], [36]}. Let $f(z)$ be an entire function such that*

$$\limsup_{r\to\infty}\frac{\log M(r,f)}{(\log r)^2}\le c<\frac{1}{4e}.$$

If $0<4ec<\delta<1$ then outside a set of upper logarithmic density at most δ,

$$\frac{m(r,f)}{M(r,f)}>k(\delta,c)=\frac{1-2.2\tau}{1+2.2\tau}\ \textit{where}\ \tau=\exp\{-\delta/(4ec)\}.$$

If in particular $c=0$ then

$$\frac{m(r,f)}{M(r,f)}\to 1\ \textit{as}\ r\to\infty$$

on a set of logarithmic density 1.

4.3 Theorems.

In this section we present the main results of the chapter. In the following theorems we see the application of relative L-order and relative L-lower order in the growth properties of entire functions.

Theorem 4.3.1 *Let f,g and h be three entire functions such that*

$$0<\lambda_g^L(f)\le\rho_g^L(f)<\infty\ \textit{and}\ 0<\lambda_g^L(h)\le\rho_g^L(h)<\infty.\ \textit{Then}$$

$$\frac{\lambda_g^L(f)}{\rho_g^L(h)}\le\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\le\frac{\lambda_g^L(f)}{\lambda_g^L(h)}\le\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\le\frac{\rho_g^L(f)}{\lambda_g^L(h)}.$$

Proof. From the definition of relative L-order and relative L-lower order we have for arbitrary positive ε and for all large values of r,

$$\log G^{-1}F(r)\ge(\lambda_g^L(f)-\varepsilon)\log[rL(r)] \tag{4.3.1}$$

$$\text{and}\ \log G^{-1}H(r)\le(\rho_g^L(h)+\varepsilon)\log[rL(r)]. \tag{4.3.2}$$

Now from (4.3.1) and (4.3.2) it follows for all large values of r,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)-\varepsilon}{\rho_g^L(h)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)}{\rho_g^L(h)}. \tag{4.3.3}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \leq (\lambda_g^L(f)+\varepsilon)\log[rL(r)] \tag{4.3.4}$$

and for all large values of r,

$$\log G^{-1}H(r) \geq (\lambda_g^L(h)-\varepsilon)\log r. \tag{4.3.5}$$

So combining (4.3.4) and (4.3.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^L(f)+\varepsilon}{\lambda_g^L(h)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^L(f)}{\lambda_g^L(h)}. \tag{4.3.6}$$

Also for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r) \leq (\lambda_g^L(h)+\varepsilon)\log[rL(r)]. \tag{4.3.7}$$

Now from (4.3.1) and (4.3.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)-\varepsilon}{\lambda_g^L(h)+\varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\lambda_g^L(f)}{\lambda_g^L(h)}. \tag{4.3.8}$$

Also for all large values of r,

$$\log G^{-1}F(r) \leq (\rho_g^L(f) + \varepsilon)\log[rL(r)]. \tag{4.3.9}$$

So from (4.3.5) and (4.3.9) it follows for all large values of r,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f) + \varepsilon}{\lambda_g^L(h) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f)}{\lambda_g^L(h)}. \tag{4.3.10}$$

Thus the theorem follows from (4.3.3), (4.3.6), (4.3.8) and (4.3.10). ■

Remark 4.3.1 *Under the same conditions stated in Theorem 4.3.1, the conclusion of the theorem can also be drawn by using Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1.*

Theorem 4.3.2 *Let f, g, h be three entire functions with*

$$0 < \lambda_g^L(f) \leq \rho_g^L(f) < \infty \text{ and } 0 < \rho_g^L(h) < \infty. \text{ Then}$$

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f)}{\rho_g^L(h)} \leq \limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.$$

Proof. From the definition of relative L-order we get for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r) \geq (\rho_g^L(h) - \varepsilon)\log[rL(r)]. \tag{4.3.11}$$

Now from (4.3.9) and (4.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f) + \varepsilon}{\rho_g^L(h) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^L(f)}{\rho_g^L(h)}. \tag{4.3.12}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \geq (\rho_g^L(f) - \varepsilon) \log[rL(r)]. \tag{4.3.13}$$

So combining (4.3.2) and (4.3.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^L(f) - \varepsilon}{\rho_g^L(h) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^L(f)}{\rho_g^L(h)}. \tag{4.3.14}$$

Thus the theorem follows from (4.3.12) and (4.3.14). ■

Remark 4.3.2 *Under the same conditions stated in Theorem 4.3.2, the conclusion of the theorem can also be deduced in view of Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1.The following theorem is a natural consequence of Theorem 4.3.1 and Theorem 4.3.2.*

Theorem 4.3.3 *Let f, g and h be three entire functions with*

$$0 < \lambda_g^L(f) \leq \rho_g^L(f) < \infty \text{ and } 0 < \lambda_g^L(h) \leq \rho_g^L(h) < \infty. \text{ Then}$$

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} &\leq \min\left\{\frac{\lambda_g^L(f)}{\lambda_g^L(h)}, \frac{\rho_g^L(f)}{\rho_g^L(h)}\right\} \\ &\leq \max\left\{\frac{\lambda_g^L(f)}{\lambda_g^L(h)}, \frac{\rho_g^L(f)}{\rho_g^L(h)}\right\} \leq \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.\end{aligned}$$

The proof is omitted.

Remark 4.3.3 *Under the same conditions stated in Theorem 4.3.3, the conclusion of the theorem can also be drawn in view of Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1. In the line of Theorem 4.3.1, Theorem 4.3.2 and Theorem 4.3.3 we may now prove similar results for relative hyper order and relative hyper lower order.*

Theorem 4.3.4 *Let f, g and h be three entire functions such that*

$$0 < \overline{\lambda}_g^L(f) \leq \overline{\rho}_g^L(f) < \infty \text{ and } 0 < \overline{\lambda}_g^L(h) \leq \overline{\rho}_g^L(h) < \infty. \text{ Then}$$

$$\begin{aligned} \frac{\overline{\lambda}_g^L(f)}{\overline{\rho}_g^L(h)} &\leq \liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\lambda}_g^L(f)}{\overline{\lambda}_g^L(h)} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\rho}_g^L(f)}{\overline{\lambda}_g^L(h)}. \end{aligned}$$

Theorem 4.3.5 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^L(f) \leq \overline{\rho}_g^L(f) < \infty \text{ and } 0 < \overline{\rho}_g^L(h) < \infty. \text{ Then}$$

$$\liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\rho}_g^L(f)}{\overline{\rho}_g^L(h)} \leq \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}.$$

The following theorem is a natural consequence of Theorem 4.3.4 and Theorem 4.3.5.

Theorem 4.3.6 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^L(f) \leq \overline{\rho}_g^L(f) < \infty \text{ and } 0 < \overline{\lambda}_g^L(h) < \infty. \text{ Then}$$

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} &\leq \min\left\{\frac{\overline{\lambda}_g^L(f)}{\overline{\lambda}_g^L(h)}, \frac{\overline{\rho}_g^L(f)}{\overline{\rho}_g^L(h)}\right\} \\ &\leq \max\left\{\frac{\overline{\lambda}_g^L(f)}{\overline{\lambda}_g^L(h)}, \frac{\overline{\rho}_g^L(f)}{\overline{\rho}_g^L(h)}\right\} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}. \end{aligned}$$

Remark 4.3.4 *Under the same conditions respectively stated in Theorem 4.3.4, Theorem 4.3.5 and Theorem 4.3.6 the conclusions of the theorems can also be drawn with the help of Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$*

instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1. In the following theorems we see some comparative growth properties of entire functions on the basis of relative L^*-order and relative L^*-lower order where $L \equiv L(r)$ is a slowly changing function.

Theorem 4.3.7 *Let f, g and h be three entire functions such that $0 < \lambda_g^{L^*}(f) \leq \rho_g^{L^*}(f) < \infty$ and $0 < \lambda_g^{L^*}(h) \leq \rho_g^{L^*}(h) < \infty$. Then*

$$\begin{aligned} \frac{\lambda_g^{L^*}(f)}{\rho_g^{L^*}(h)} &\leq \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)} \\ &\leq \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f)}{\lambda_g^{L^*}(h)}. \end{aligned}$$

Proof. From the definition of relative L^*-order and relative L^*-lower order we have for arbitrary positive ε and for all large values of r,

$$\log G^{-1}F(r) \geq (\lambda_g^{L^*}(f) - \varepsilon)\log[re^{L(r)}] \tag{4.3.15}$$

$$\text{and } \log G^{-1}H(r) \leq (\rho_g^{L^*}(h) + \varepsilon)\log[re^{L(r)}].$$

Now from (4.3.15) and (4.3.16) it follows for all large values of r,

$$\frac{G^{-1}F(r)}{G^{-1}H(r)} \geq \frac{\lambda_g^{L^*}(f) - \varepsilon}{\rho_g^{L^*}(h) + \varepsilon}. \tag{4.3.16}$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{G^{-1}F(r)}{G^{-1}H(r)} \geq \frac{\lambda_g^{L^*}(f)}{\rho_g^{L^*}(h)}. \tag{4.3.17}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \leq (\lambda_g^{L^*}(f) + \varepsilon)\log[re^{L(r)}] \tag{4.3.18}$$

and for all large values of r,

$$\log G^{-1}H(r) \geq (\lambda_g^{L^*}(h) - \varepsilon)\log[re^{L(r)}]. \tag{4.3.19}$$

So combining (4.3.18) and (4.3.19) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\lambda_g^{L^*}(f) + \varepsilon}{\lambda_g^{L^*}(h) - \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\le\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)}. \tag{4.3.20}$$

Also for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r)\le(\lambda_g^{L^*}(h)+\varepsilon)\log[re^{L(r)}]. \tag{4.3.21}$$

Now from (4.3.15) and (4.3.21) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\ge\frac{\lambda_g^{L^*}(f)-\varepsilon}{\lambda_g^{L^*}(h)+\varepsilon}.$$

Choosing $\varepsilon\to0$ we get that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\ge\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)}. \tag{4.3.22}$$

Also for all large values of r,

$$\log G^{-1}F(r)\le(\rho_g^{L^*}(f)+\varepsilon)\log[re^{L(r)}]. \tag{4.3.23}$$

So from (4.3.19) and (4.3.23) it follows for all large values of r,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\le\frac{\rho_g^{L^*}(f)+\varepsilon}{\lambda_g^{L^*}(h)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}\le\frac{\rho_g^{L^*}(f)}{\lambda_g^{L^*}(h)}. \tag{4.3.24}$$

Thus the theorem follows from (4.3.17), (4.3.20), (4.3.22) and (4.3.24). ■

Theorem 4.3.8 *Let f, g and h be three entire functions with*

$0<\lambda_g^{L^*}(f)\le\rho_g^{L^*}(f)<\infty$ *and* $0<\rho_g^{L^*}(h)<\infty$. *Then*

$$\liminf \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)} \leq \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.$$

Proof. From the definition of relative L^*-order we get for a sequence of values of r tending to infinity,

$$\log G^{-1}H(r) \geq (\rho_g^{L^*}(h) - \varepsilon)\log[re^{L(r)}]. \tag{4.3.25}$$

Now from (4.3.9) and (4.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f) + \varepsilon}{\rho_g^{L^*}(h) - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}. \tag{4.3.26}$$

Again for a sequence of values of r tending to infinity,

$$\log G^{-1}F(r) \geq (\rho_g^{L^*}(f) - \varepsilon)\log[re^{L(r)}]. \tag{4.3.27}$$

So combining (4.3.16) and (4.3.27) we get for a sequence of values of r tending to infinity,

$$\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^{L^*}(f) - \varepsilon}{\rho_g^{L^*}(h) + \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \geq \frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}. \tag{4.3.28}$$

Thus the theorem follows from (4.3.26) and (4.3.28). The following theorem is a natural consequence of Theorem 4.3.7 and Theorem 4.3.8. ■

Theorem 4.3.9 *Let f, g and h be three entire functions with*

$$0 < \lambda_g^{L^*}(f) \leq \rho_g^{L^*}(f) < \infty \text{ and } 0 < \lambda_g^{L^*}(h) \leq \rho_g^{L^*}(h) < \infty. \textit{ Then}$$

$$\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)} \leq \min\{\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)},\frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}\}$$
$$\leq \max\{\frac{\lambda_g^{L^*}(f)}{\lambda_g^{L^*}(h)},\frac{\rho_g^{L^*}(f)}{\rho_g^{L^*}(h)}\}$$
$$\leq \limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log G^{-1}H(r)}.$$

Remark 4.3.5 *Under the same conditions respectively stated in Theorem 4.3.7, Theorem 4.3.8 and Theorem 4.3.9 the conclusions of the theorems can also be deduced by using Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1. We may prove similar results for relative L^*-hyper order and relative L^*-hyper lower order.*

Theorem 4.3.10 *Let f, g and h be three entire functions such that*

$$0 < \overline{\lambda}_g^{L^*}(f) \leq \overline{\rho}_g^{L^*}(f) < \infty \text{ and } 0 < \overline{\lambda}_g^{L^*}(h) \leq \overline{\rho}_g^{L^*}(h) < \infty. \text{ Then}$$

$$\frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)} \leq \liminf_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}.$$

Theorem 4.3.11 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^{L^*}(f) \leq \overline{\rho}_g^{L^*}(f) < \infty \text{ and } 0 < \overline{\rho}_g^{L^*}(h) < \infty. \text{ Then}$$

$$\liminf_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} \leq \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)} \leq \limsup_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}.$$

The following theorem is a natural consequence of Theorem 4.3.10 and Theorem 4.3.11.

Theorem 4.3.12 *Let f, g and h be three entire functions with*

$$0 < \overline{\lambda}_g^{L^*}(f) \leq \overline{\rho}_g^{L^*}(f) < \infty \text{ and } 0 < \overline{\lambda}_g^{L^*}(h) < \infty. \text{ Then}$$

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)} &\leq \min\{\frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}, \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)}\} \\ &\leq \max\{\frac{\overline{\lambda}_g^{L^*}(f)}{\overline{\lambda}_g^{L^*}(h)}, \frac{\overline{\rho}_g^{L^*}(f)}{\overline{\rho}_g^{L^*}(h)}\} \\ &\leq \limsup_{r\to\infty}\frac{\log^{[2]} G^{-1}F(r)}{\log^{[2]} G^{-1}H(r)}.\end{aligned}$$

Remark 4.3.6 *Under the same hypothesis respectively stated in Theorem 4.3.10, Theorem 4.3.11 and Theorem 4.3.12 the conclusions of the theorems can also be drawn by using Lemma 4.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ on a set of logarithmic density 1.*

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Chapter 5

RELATIVE (p,q)TH ORDER AND RELATED GROWTH ESTIMATES OF ENTIRE FUNCTIONS ON THE BASIS OF THEIR MINIMUM MODULUS

CHAPTER

5

RELATIVE (p,q)TH ORDER AND RELATED GROWTH ESTIMATES OF ENTIRE FUNCTIONS ON THE BASIS OF THEIR MINIMUM MODULUS

5.1 Introduction, Definitions and Notations.

Let f and g be two entire functions and

$$F(r) = \max\{|f(z)| : |z| = r\}, G(r) = \max\{|g(z)| : |z| = r\}.$$

If f is non-constant then $F(r)$ is strictly increasing and continuous and its inverse

$$F^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

exists and is such that

$$\lim_{s \to \infty} F^{-1}(s) = \infty.$$

Bernal [3] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\}$$

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [**22**].

$$=\limsup_{r\to\infty}\frac{\log G^{-1}F(r)}{\log r}.$$

The definition coincides with the classical one [65] if

$$g(z)=\exp z$$

Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f)=\liminf_{r\to\infty}\frac{\log G^{-1}F(r)}{\log r}.$$

Juneja, Kapoor and Bajpai [38] defined the (p,q)th order and (p,q)th lower order of an entire function f respectively as follows:

$$\rho_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p+1]}M(r,\ f)}{\log^{[q+1]}r}\text{ and }\lambda_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p+1]}M(r,\ f)}{\log^{[q+1]}r}.$$

where p,q are positive integers and $p>q$. So with the help of the above notion one can easily define the relative (p,q)th order and relative (p,q)th lower order of entire functions.

Definition 5.1.1 *The relative (p,q)th order $\rho_g^f(p,q)$ and the relative (p,q)th lower order $\lambda_g^f(p,q)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}G^{-1}F(r)}{\log^{[q]}r}\ \textit{and}\ \lambda_g^f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p]}G^{-1}F(r)}{\log^{[q]}r},$$

where p,q are positive integers and $p>q$.In the chapter we establish some results on the comparative growth properties of entire functions on the basis of relative (p,q)th order and relative (p,q)th lower order where p,q are positive integers with $p>q$. In the sequel we use the following notations:

$$\log^{[k]}x=\log\left(\log^{[k-1]}x\right)\ \textit{for}\ k=1,2,3,\ldots\ldots\textit{and}\log^{[0]}x=x.$$

In order to develop our results we shall need various kinds of measures and densities for sets of points on the positive axis. Let E be such a set and let $E[a,b]$ denote the part of E for which $a<r<b$. The linear and logarithmic measures of E are defined to be

$$m(E)=\int_E dr\ \textit{and}\ lm(E)=\int_{E(1,\infty)}\frac{dr}{r}\ \textit{respectively.}$$

These may be finite or infinite. We also define the lower and upper densities of E by

$$dens\ E(upper) = \limsup_{r\to\infty} \frac{m(E(0,r))}{r}$$

$$and\ dens\ E(lower) = \liminf_{r\to\infty} \frac{m(E(0,r))}{r}$$

and also the upper and lower logarithmic densities of E by

$$\log\ densE(upper) = \limsup_{r\to\infty} \frac{\lim\,(E(1,r))}{\log r}$$

$$and\ \log\ densE(lower) = \liminf_{r\to\infty} \frac{\lim\,(E(1,r))}{\log r}.$$

$$\text{Also let } f(r) = m(r,f) = \inf_{|z|=r} |f(z)|$$

which is known as the minimum modulus of an entire function f. In this chapter we also estimate some comparative growth properties of composite entire functions in terms of their minimum modulus. In fact all the definitions in this chapter can also be stated in terms of minimum modulus on a set of logarithmic density 1.

5.2 Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 5.2.1 *{[2], [36]}. Let $f(z)$ be an entire function such that*

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^2} \le c < \frac{1}{4e}.$$

If $0 < 4ec < \delta < 1$ then outside a set of upper logarithmic density at most δ,

$$\frac{m(r,f)}{M(r,f)} > k(\delta,c) = \frac{1-2.2\tau}{1+2.2\tau} \text{ where } \tau = \exp\{-\delta/(4ec)\}.$$

If in particular $c = 0$ then

$$\frac{m(r,f)}{M(r,f)} \to 1 \text{ as } r \to \infty$$

on a set of logarithmic density 1.

5.3 Theorems.

In this section we present the main results of this chapter.

In the following theorems we see the application of relative (p,q)th order and relative (p,q)th lower order in the growth properties of entire functions where p, q are positive integers with $p > q$.

Theorem 5.3.1 *Let f, g and h be three entire functions such that*

$$0 < \lambda_g^f(p,q) \leq \rho_g^f(p,q) < \infty$$
$$\text{and } 0 < \lambda_g^h(m,q) \leq \rho_g^h(m,q) < \infty$$

where p, q, m are positive integers with $p > q$ and $m > q$. Then

$$\frac{\lambda_g^f(p,q)}{\rho_g^h(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q)}{\lambda_g^h(m,q)}.$$

Proof. From the definition of relative (p,q)th order and relative (p,q)th lower order we have for arbitrary positive ε and for all large values of r,

$$\log^{[p]} G^{-1}F(r) \geq (\lambda_g^f(p,q) - \varepsilon) \log^{[q]} r \tag{5.3.1}$$

and

$$\log^{[m]} G^{-1}H(r) \leq (\rho_g^h(m,q) + \varepsilon) \log^{[q]} r. \tag{5.3.2}$$

Now from (5.3.1) and (5.3.2) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\lambda_g^f(p,q) - \varepsilon}{\rho_g^h(m,q) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\lambda_g^f(p,q)}{\rho_g^h(m,q)}. \tag{5.3.3}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \leq (\lambda_g^f(p,q) + \varepsilon) \log^{[q]} r \tag{5.3.4}$$

and for all large values of r,

$$\log^{[m]} G^{-1}H(r) \geq (\lambda_g^h(m,q) - \varepsilon)\log^{[q]} r. \qquad (5.3.5)$$

So combining (5.3.4) and (5.3.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\lambda_g^f(p,q) + \varepsilon}{\lambda_g^h(m,q) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}. \qquad (5.3.6)$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \leq (\lambda_g^h(m,q) + \varepsilon)\log^{[q]} r. \qquad (5.3.7)$$

Now from (5.3.1) and (5.3.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\lambda_g^f(p,q) - \varepsilon}{\lambda_g^h(m,q) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}. \qquad (5.3.8)$$

Also for all large values of r,

$$\log^{[p]} G^{-1}F(r) \leq (\rho_g^f(p,q) + \varepsilon)\log^{[q]} r. \qquad (5.3.9)$$

So from (5.3.5) and (5.3.9) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q) + \varepsilon}{\lambda_g^h(m,q) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q)}{\lambda_g^h(m,q)}. \qquad (5.3.10)$$

Thus the theorem follows from (5.3.3), (5.3.6), (5.3.8) and (5.3.10). ∎

Remark 5.3.1 *Under the same conditions stated in Theorem 5.3.1, the conclusion of the theorem can also be drawn with the help of Lemma 5.2.1 in terms of $f(r), g(r)$ and $h(r)$ instead of $F(r), G(r)$ and $H(r)$ respectively on a set of logarithmic density 1.*

Theorem 5.3.2 *Let f, g, h be three entire functions with*

$$0 < \lambda_g^f(p,q) \leq \rho_g^f(p,q) < \infty \text{ and } 0 < \rho_g^h(m,q) < \infty$$

where p, q, m are positive integers with $p > q$ and $m > q$. Then

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)}.$$

Proof. From the definition of relative (p, q)th order we get for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \geq (\rho_g^h(m,q) - \varepsilon) \log^{[q]} r. \tag{5.3.11}$$

Now from (5.3.9) and (5.3.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q) + \varepsilon}{\rho_g^h(m,q) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}. \tag{5.3.12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \geq (\rho_g^f(p,q) - \varepsilon) \log^{[q]} r. \tag{5.3.13}$$

So combining (5.3.2) and (5.3.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\rho_g^f(p,q) - \varepsilon}{\rho_g^h(m,q) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}. \tag{5.3.14}$$

Thus the theorem follows from (5.3.12) and (5.3.14). ■

Remark 5.3.2 *Under the same hypothesis stated in Theorem 5.3.2, the conclusion of the theorem can also be deduced in view of Lemma 5.2.1 in terms of* $f(r), g(r)$ *and* $h(r)$ *instead of* $F(r), G(r)$ *and* $H(r)$ *respectively on a set of logarithmic density 1.The following theorem is a natural consequence of Theorem 5.3.1 and Theorem 5.3.2.*

Theorem 5.3.3 *Let* f, g, h *be three entire functions with*

$$0 < \lambda_g^f(p,q) \leq \rho_g^f(p,q) < \infty \text{ and } 0 < \lambda_g^h(m,q) \leq \rho_g^h(m,q) < \infty$$

where p, q, m *are positive integers with* $p > q$ *and* $m > q$. *Then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} &\leq \min\{\frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}, \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}\}\\ &\leq \max\{\frac{\lambda_g^f(p,q)}{\lambda_g^h(m,q)}, \frac{\rho_g^f(p,q)}{\rho_g^h(m,q)}\}\\ &\leq \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)}.\end{aligned}$$

The proof is omitted.

Remark 5.3.3 *Under the same conditions stated in Theorem 5.3.3, the conclusion of the theorem can also be drawn with the help of Lemma 5.2.1 in terms of* $f(r), g(r)$ *and* $h(r)$ *instead of* $F(r), G(r)$ *and* $H(r)$ *respectively on a set of logarithmic density 1.*

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Chapter 6

SHARING AND $_{(t)}L^{*}$-(p,q)TH ORDER OF MEROMORPHIC AND ENTIRE FUNCTIONS

CHAPTER

6

SHARING AND ${}_{(t)}L^{*} - (p,q)$TH ORDER OF MEROMORPHIC AND ENTIRE FUNCTIONS

6.1 Introduction, Definitions and Notations.

Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbb{C}$ and let $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share the value a CM(counting multiplicities) or IM(ignoring multiplicities) provided $f-a$ and $g-a$ have same zeros CM or IM respectively and f, g share ∞ CM or IM provided that $\frac{1}{f}, \frac{1}{g}$ share 0 CM or IM. In [32] G.G. Gundersen proved the following theorem.

Theorem A. [32] If f and g share three values IM then

$$\frac{1}{3}T(r,g)(1+o(1)) \leq T(r,f) \leq 3T(r,g)(1+o(1)) \text{ as } r \to \infty (r \notin E).$$

G. Brosch [4] improved the result by proving the following theorem.

Theorem B. [4] If f and g share three values CM then

$$\frac{3}{8}T(r,g)(1+o(1)) \leq T(r,f) \leq \frac{8}{3}T(r,g)(1+o(1)) \text{ as } r \to \infty (r \notin E).$$

The results of this chapter have been published in **International Mathematical Forum**, see **[23]**.

In 2001, I.Lahiri [48] introduced the idea of weighted sharing of values and proved some uniqueness theorems for meromorphic functions improving some earlier results. To this end we now explain the notion of weighted sharing as introduced in [48].

Definition 6.1.1 *[48] Let k be a non negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a\,;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.*

Definition 6.1.2 *[48] Let k be a non negative integer or infinity. If for $a \in \mathbb{C} \cup \{\infty\}, E_k(a\,;f) = E_k(a\,;g)$, we say that f, g share the value 'a' with weight k. It is noted that if f, g share a value 'a' with weight 'k' then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n. Also we write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p \leq k$. Also we note that f, g share a value 'a' IM or CM if and only if f, g share $(a, 0)$ or $(0, \infty)$ respectively. If f and g are two non constant entire functions then f, g share (∞, ∞). Throughout the chapter we mean by f, g etc., non constant meromorphic functions defined in the open complex plane $\mathbb{C}$ and $S(r, f)$ any quantity satisfying*

$$S(r, f) = o\{T(r, f)\}(r \to \infty, r \notin E).$$

In the sequel we use the following two notations:

$$\log^{[k]} x = \log(\log^{[k-1]} x) for k = 1, 2, 3, ...and \log^{[0]} x = x$$

$$and\ \exp^{[t]} x = \exp\left(\exp^{[t-1]} x\right)\ for\ t = 1, 2, 3, ...and\ \exp^{[0]} x = x.$$

Somasundaram and Thamizharasi [63] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Their definitions are as follows:

Definition 6.1.3 *[63] The L-order ρ_f^L and the lower L-order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]}\ and\ \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

Juneja, Kapoor and Bajpai [38] defined the (p,q)th order and lower (p,q)th order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q+1]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[q+1]} M(r,f)}{\log^{[q+1]} r}.$$

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r},$$

where p,q are positive integers and $p>q$. So with the help of the above notion one can easily define the $L-(p,q)$th order and $L-(p,q)$th lower order of entire and meromorphic functions.

Definition 6.1.4 *The $L-(p,q)$th order $\rho_f^L(p,q)$ and the lower $L-(p,q)$th order $\lambda_f^L(p,q)$ of an entire f are defined as*

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[rL(r)]} \text{ and } \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[rL(r)]}.$$

When f is meromorphic, one can easily verify that

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[rL(r)]} \text{ and } \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[rL(r)]},$$

where p,q are positive integers and $p>q$.
The more generalised concept of $L-(p,q)$th order and lower $L-(p,q)$th order of entire and meromorphic functions are $L^*-(p,q)$th order and lower $L^*-(p,q)$th order respectively. In order to prove our results the following definitions are necessary:

Definition 6.1.5 *The L^*- order, lower L^*-order and L^*-type of a meromorphic function are defined by*

$$\begin{aligned}\rho_f^{L^*} &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]},\\ \lambda_f^{L^*} &= \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[re^{L(r)}]}\\ \textit{and } \sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{T(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}, 0<\rho_f^{L^*}<\infty.\end{aligned}$$

When f is entire, one can easily verify that

$$\begin{aligned}\rho_f^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]},\\ \lambda_f^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}\\ \textit{and } \sigma_f^{L^*} &= \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)}]^{\rho_f^{L^*}}}.\end{aligned}$$

Definition 6.1.6 *The $L^*-(p,q)$th order $\rho_f^{L^*}(p,q)$ and the lower $L^*-(p,q)$th order $\lambda_f^{L^*}(p,q)$ of an entire function f are defined as*

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[re^{L(r)}]} \textit{ and } \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[re^{L(r)}]}.$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[re^{L(r)}]} \textit{ and } \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[re^{L(r)}]},$$

where p,q are positive integers and $p>q$.
Extending our notion we may obtain the following definition:

Definition 6.1.7 *The $L^*-(p,q)$th order with rate t and the lower $L^*-(p,q)$th order with rate t of an entire function f respectively denoted by ${}_{(t)}\rho_f^{L^*}(p,q)$*

and ${}_{(t)}\lambda_f^{L^*}(p,q)$ *are defined as*

$$ {}_{(t)}\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[r\exp^{[t]} L(r)]} $$

$$ \text{and } {}_{(t)}\lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[r\exp^{[t]} L(r)]}, $$

where t *is any positive integer and* p, q *are positive integers with* $p > q$. *For* $t = 1$, *Definition 6.1.7 reduces to Definition 6.1.6. In the chapter we prove some results on the* $L-(p,q)th$ *order, lower* $L-(p,q)th$ *order and* $L^*-(p,q)th$ *order, lower* $L^*-(p,q)th$ *order and also* ${}_{(t)}L^*-(p,q)th$ *order, lower* ${}_{(t)}L^*-(p,q)th$ *order of entire functions with rate* t *for any positive integer* t *based on the idea of sharing of values of them.*

6.2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 6.2.1 *[8] Let* f *and* g *be two non constant entire functions sharing* (a_1, k_1) *and* (a_2, k_2) *where* $(k_2-1)(k_1k_2-1) > (1+k_2)^2$. *Then*

$$ T(r,f) \le T(r,g)(2+O(1)) \text{ and } T(r,g) \le T(r,f)(2+O(1)). $$

6.3 Theorems.

In this section we present the main results of this chapter.

Theorem 6.3.1 *Let* f *and* g *be two non constant entire functions sharing* (a_1,k_1) *and* (a_2,k_2) *with* $(k_2-1)(k_1k_2-1) > (1+k_2)^2$. *Then* $\rho_f^L(p,q) = \rho_g^L(p,q)$ *where* p, q *are positive integers and* $p > q$.

Proof. By Lemma 6.2.1, we have

$$ \rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[rL(r)]} \le \limsup_{r\to\infty} \frac{\log^{[p]} T(r,g)}{\log^{[q]}[rL(r)]} = \rho_g^L(p,q) $$

and similarly

$$ \rho_g^L(p,q) \le \rho_f^L(p,q). $$

$$ \text{Hence } \rho_f^L(p,q) = \rho_g^L(p,q). $$

This proves the theorem. ■

Theorem 6.3.2 *If f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) with $(k_2-1)(k_1k_2-1) > (1+k_2)^2$ then $\rho_f^{L^*}(p,q) = \rho_g^{L^*}(p,q)$ where $\rho_f^{L^*}(p,q), \rho_g^{L^*}(p,q)$ denote respectively $L^* - (p,q)$th order of f and g and also p, q are any two positive integers with $p > q$.*

Proof. By Lemma 6.2.1, we have

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[re^{L(r)}]} \leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r,g)}{\log^{[q]}[re^{L(r)}]} = \rho_g^{L^*}(p,q)$$

and similarly

$$\rho_g^{L^*}(p,q) \leq \rho_f^{L^*}(p,q).$$

$$\text{Hence } \rho_f^{L^*}(p,q) = \rho_g^{L^*}(p,q).$$

Thus the theorem is established. ■

Theorem 6.3.3 *If f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) with $(k_2 - 1)(k_1k_2 - 1) > (1 + k_2)^2$ then*

$${}_{(t)}\rho_f^{L^*}(p,q) = {}_{(t)}\rho_g^{L^*}(p,q)$$

where ${}_{(t)}\rho_f^{L^*}(p,q)$ and ${}_{(t)}\rho_g^{L^*}(p,q)$ denote respectively ${}_{(t)}L^* - (p,q)$th order of f and g for $t = 1, 2, 3, \ldots$ and also p, q are any two positive integers with $p > q$.

Proof. By Lemma 6.2.1, we have

$${}_{(t)}\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[r\exp^{[t]} L(r)]} \leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r,g)}{\log^{[q]}[r\exp^{[t]} L(r)]} = {}_{(t)}\rho_g^{L^*}(p,q)$$

and similarly

$${}_{(t)}\rho_g^{L^*}(p,q) \leq {}_{(t)}\rho_f^{L^*}(p,q).$$

$$\text{Hence } {}_{(t)}\rho_f^{L^*}(p,q) = {}_{(t)}\rho_g^{L^*}(p,q).$$

This proves the theorem. ■

In the line of Theorem 6.3.1, Theorem 6.3.2 and Theorem 6.3.3 we may state the following three theorems without proof.

Theorem 6.3.4 *Let f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) with $(k_2 - 1)(k_1k_2 - 1) > (1 + k_2)^2$. Then $\lambda_f^L(p,q) = \lambda_g^L(p,q)$ where p, q are positive integers and $p > q$ and $\lambda_f^L(p,q), \lambda_g^L(p,q)$ denote respectively lower $L - (p,q)$th order of f, g.*

Theorem 6.3.5 *If f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) with $(k_2 - 1)(k_1k_2 - 1) > (1 + k_2)^2$. Then $\lambda_f^{L^*}(p, q) = \lambda_g^{L^*}(p, q)$ where p, q are positive integers and $p > q$ and $\lambda_f^{L^*}(p, q), \lambda_g^{L^*}(p, q)$ denote respectively lower $L^* - (p, q)$th order of f, g.*

Theorem 6.3.6 *If f and g be two non constant entire functions sharing (a_1, k_1) and (a_2, k_2) with $(k_2 - 1)(k_1k_2 - 1) > (1 + k_2)^2$ then*

$$_{(t)}\lambda_f^{L^*}(p, q) = {}_{(t)}\lambda_g^{L^*}(p, q)$$

where $_{(t)}\lambda_f^{L^*}(p, q)$ and $_{(t)}\lambda_g^{L^*}(p, q)$ denote respectively lower $_{(t)}L^* - (p, q)$th order of f and g for $t = 1, 2, 3, \ldots$and also p, q are any two positive integers with $p > q$.

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Chapter 7

GENERALISED L^*-(p,q)TH ORDER OF THE DERIVATIVE OF A MEROMORPHIC FUNCTION

CHAPTER

7

GENERALISED $L^*-(p,q)$TH ORDER OF THE DERIVATIVE OF A MEROMORPHIC FUNCTION

7.1 Introduction, Definitions and Notations.

We know {cf.[66], p.36} that the order of the derivative of an entire function is equal to the order of the function. The same result is proved for a meromorphic function in {cf.[12], [64], [67]}. In [42] and [43] Lahiri proved that the generalised order (generalised lower order) of a meromorphic function f is equal to the generalised order of its derivative f'. Using the notion of (p,q)th order ((p,q)th lower order) for any two positive integers with $p>q$ of an entire function introduced by Juneja, Kapoor and Bajpai [38] and the notion of slowly changing functions investigated by Somasundaram and Thamizharasi [63], Datta and Mondal [18] established a relationship between the $L-(p,q)$th order of the derivative of a meromorphic function and that of the original function where $L\equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar)\sim L(r)$ as $r\rightarrow\infty$ for every constant 'a' and

The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences**, see [24].

p, q are any two positive integers with $p > q$. In this chapter we generalise the results of Datta and Mondal [18] and for this we introduce the following definition:

Definition 7.1.1 *The generalised* $L^* - (p,q)th$ *order with rate* t, ${}^{(t)}\rho_f^{L^*}(p,q)$ *and generalised* $L^* - (p,q)th$ *lower order with rate* t, ${}^{(t)}\lambda_f^{L^*}(p,q)$ *of an entire function* f *are defined as*

$$
\begin{aligned}
{}^{(t)}\rho_f^{L^*}(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[r\exp^{[t]} L(r)]} \\
\textit{and } {}^{(t)}\lambda_f^{L^*}(p,q) &= \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[r\exp^{[t]} L(r)]}
\end{aligned}
$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$ *and* $\exp^{[t]} x = \exp(\exp^{[t-1]} x)$ *for* $t = 1, 2, 3, \ldots$ *and* $\exp^{[0]} x = x$ *and also* p, q *are any two positive integers with* $p > q$. *When* f *is meromorphic, one can easily verify that*

$$
\begin{aligned}
{}^{(t)}\rho_f^{L^*}(p,q) &= \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[r\exp^{[t]} L(r)]} \\
\textit{and } {}^{(t)}\lambda_f^{L^*}(p,q) &= \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[r\exp^{[t]} L(r)]}.
\end{aligned}
$$

7.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 7.2.1 *[43] Let* f *be a transcendental meromorphic function. Then*

$$T(r, f') \le 2T(2r, f) + o\{T(2r, f)\} \textit{ for all large values of } r.$$

Lemma 7.2.2 *{ Theorem 4.1, [68]; see also Lemma C, [17]} Let* f *be a meromorphic function. Then for all large* r,

$$T(r, f) < C\{T(2r, f') + \log r\}$$

where C *is a constant which is only dependent on* $f(0)$.

7.3 Theorems.

In this section we present the main results of this chapter.

Theorem 7.3.1 *The generalised $L^* - (p,q)$th order with rate t of a meromorphic function f is equal to the generalised $L^* - (p,q)$th order of its derivative f' where p, q are positive integers and $p > q$ with $t = 1, 2, 3, \ldots$*

Proof. We suppose that f is a transcendental meromorphic function because otherwise the theorem follows easily. From Lemma 7.2.1 we get by taking logarithms $(p-1)$times

$$\log^{[p-1]} T(r, f') \leq \log^{[p-1]} T(2r, f) + O(1)$$

which gives that

$$\begin{aligned}
{}^{(t)}\rho_{f'}^{L^*}(p,q) &\leq \limsup_{r\to\infty} \left\{ \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]} \cdot \frac{1}{1 - \frac{\log 2}{\log^{[q]}[r \exp^{[t]} L(r)]}} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]} \cdot \lim_{r\to\infty} \frac{1}{1 - \frac{\log^{[2]}}{\log^{[q]}[r \exp^{[t]} L(r)]}} \\
&= {}^{(t)}\rho_{f}^{L^*}(p,q). \qquad (7.3.1)
\end{aligned}$$

Since f is transcendental, we have

$$\log r = o\{T(r, f)\}.$$

From Lemma 7.2.2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r, f) + O(1) \leq \log^{[p-1]} T(2r, f')$$

which gives that

$${}^{(t)}\rho_{f}^{L^*}(p,q) \leq \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f')}{\log^{[q]}[r \exp^{[t]} L(r)]} \cdot \lim_{r\to\infty} \frac{1}{1 - \frac{\log 2}{\log^{[q]}[r \exp^{[t]} L(r)]}}$$

$$\text{i.e., } {}^{(t)}\rho_{f}^{L^*}(p,q) \leq {}^{(t)}\rho_{f'}^{L^*}(p,q). \qquad (7.3.2)$$

Thus the theorem follows from (7.3.1) and (7.3.2). ■

Remark 7.3.1 *Theorem 7.3.1 is a generalisation of Theorem 1 [18].*

Theorem 7.3.2 *The generalised $L^*-(p,q)$th lower order with rate t of a meromorphic function f is equal to the generalised $L^*-(p,q)$th lower order of its derivative f' where p, q are positive integers and $p > q$ with $t = 1, 2, 3, \ldots$*

Proof. Let us suppose that f is a transcendental meromorphic function because otherwise the theorem follows easily. From Lemma 7.2.1 we obtain by taking logarithms $(p-1)$ times

$$\log^{[p-1]} T(r, f') \le \log^{[p-1]} T(2r, f) + O(1)$$

which gives that

$$^{(t)}\lambda_f^{L^*}(p,q) \le \liminf_{r\to\infty} \{\frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]} . \frac{1}{1 - \frac{\log 2}{\log^{[q]}[r \exp^{[t]} L(r)]}}\}$$

■

$$= \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]} . \lim_{r\to\infty} \frac{1}{1 - \frac{\log 2}{\log^{[q]}[r \exp^{[t]} L(r)]}}$$

$$= \; ^{(t)}\lambda_f^{L^*}(p,q). \tag{7.3.3}$$

Since f is transcendental, we have $\log r = o\{T(r,f)\}$. From Lemma 7.2.2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r, f) + O(1) \le \log^{[p-1]} T(2r, f')$$

which gives

$$^{(t)}\lambda_f^{L^*}(p,q) \le \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f')}{\log^{[q]}[r \exp^{[t]} L(r)]} . \lim_{r\to\infty} \frac{1}{1 - \frac{\log 2}{\log^{[q]}[r \exp^{[t]} L(r)]}}$$

$$\text{i.e., } ^{(t)}\lambda_f^{L^*}(p,q) \le \; ^{(t)}\lambda_{f'}^{L^*}(p,q). \tag{7.3.4}$$

Thus the theorem follows from (7.3.3) and (7.3.4).

Remark 7.3.2 *Theorem 7.3.2 is a generalisation of Theorem 2 [18].*

Theorem 7.3.3 *If f is a transcendental meromorphic function having a finite number of zeros with*

$$f(0) \neq 0, \infty, \ f'(0) \neq 0 \text{ and } {}^{(t)}\rho_f^{L^*}(2,1) < \infty \text{ then } {}^{(t)}\rho_{f'}^{L^*}(p,q) = {}^{(t)}\rho_f^{L^*}(p,q)$$

$$\text{and } {}^{(t)}\lambda_{f'}^{L^*}(p,q) = {}^{(t)}\lambda_f^{L^*}(p,q)$$

where p, q are positive integers and $p > q$ with $t = 1, 2, 3, \ldots$

Proof. From {Theorem 2.2, [34], p.40} we know that

$$m(r, \frac{f'}{f}) = O(\log r).$$

Also by {Theorem 2.3, [34], p.41} we obtain in the present case,

$$\log r = o\{T(r,f)\} \text{ as } r \to \infty.$$

So combining the two we get that

$$m(r, \frac{f'}{f}) = o\{T(r,f)\} \text{ as } r \to \infty.$$

Since f has a finite number of zeros, it is clear that

$$N(r, \frac{1}{f}) = O(\log r).$$

$$\text{Hence } N(r, \frac{1}{f}) = o\{T(r,f)\} \text{ as } r \to \infty.$$

$$\text{Now } m(r, f') \leq m(r, \frac{f'}{f}) + m(r, f)$$

$$\text{i.e., } m(r, f') \leq m(r, f) + o\{T(r,f)\} \text{ as } r \to \infty.$$

Also if f has a pole of order p at z_0, $f'(z)$ has a pole of order $p + 1 \leq 2p$, so that

$$N(r, f') \leq 2N(r, f) \text{ \{p.56, [4]\}}.$$

Thus by addition we deduce that

$$T(r, f') \leq m(r, f) + 2N(r, f) + o\{T(r,f)\}$$

$$\text{i.e., } T(r, f') \leq 2T(r, f) + o\{T(r,f)\}$$

$$\text{i.e., } T(r,f') \leq \{2+o(1)\}T(r,f) \text{ as } r \to \infty. \tag{7.3.5}$$

This gives that

$$^{(t)}\rho_{f'}^{L^*}(p,q) \leq \ ^{(t)}\rho_f^{L^*}(p,q). \tag{7.3.6}$$

Again we have

$$\begin{aligned}
T(r,f) &= m(r,\frac{1}{f}) + N(r,\frac{1}{f}) + O(1) \\
\text{i.e., } T(r,f) &\leq m(r,\frac{1}{f'}) + m(r,\frac{f'}{f}) + N(r,\frac{1}{f}) + O(1) \\
\text{i.e., } T(r,f) &\leq m(r,\frac{1}{f'}) + o\{T(r,f)\} \\
\text{i.e., } T(r,f) &\leq T(r,\frac{1}{f'}) + o\{T(r,f)\} \\
\text{i.e., } T(r,f) &\leq T(r,f') + o\{T(r,f)\} \text{ as } r \to \infty
\end{aligned}$$

$$\text{i.e., } \{1+o(1)\}T(r,f) \leq T(r,f') \text{ as } r \to \infty. \tag{7.3.7}$$

This gives that

$$^{(t)}\rho_f^{L^*}(p,q) \leq \ ^{(t)}\rho_{f'}^{L^*}(p,q). \tag{7.3.8}$$

Thus the first part of the theorem follows from (7.3.6) and (7.3.8). Similarly,

$$^{(t)}\lambda_{f'}^{L^*}(p,q) = \ ^{(t)}\lambda_f^{L^*}(p,q).$$

This proves the theorem. ■

Remark 7.3.3 *Theorem 7.3.3 is a generalisation of Theorem 3 [18].*

Remark 7.3.4 *Theorem 7.3.3 can also be proved with a lesser hypothesis*

$$N(r,\frac{1}{f}) = O(\log r)$$

than 'having a finite number of zeros'.

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Chapter 8

A NOTE ON RELATIVE L-ORDER AND RELATIVE L^*-ORDER OF ENTIRE FUNCTIONS

CHAPTER

8

A NOTE ON RELATIVE L-ORDER AND RELATIVE L^*-ORDER OF ENTIRE FUNCTIONS

8.1 Introduction, Definitions and Notations.

Let f and g be two entire functions and

$$F(r) = \max\{|f(z)| : |z| = r\} \text{ and } G(r) = \max\{|g(z)| : |z| = r\}.$$

If f is non constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that

$$\lim_{s\to\infty} F^{-1}(s) = \infty.$$

The order ρ_f of f {cf. [35],[65]} is given by

$$\begin{aligned}\rho_f &= \inf\{\mu > 0 : F(r) < \exp(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log\log F(r)}{\log r}\end{aligned}$$

The results of this chapter have been published in **International Mathematical Forum**, see **[25]**.

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$. The lower order of f denoted by λ_f is defined as

$$\lambda_f = \liminf_{r \to \infty} \frac{\log \log F(r)}{\log r}.$$

If ρ_f and λ_f are equal then f is said to be of regular growth {cf. [66]}. Recently Bernal [3] introduced the idea of relative order of f with respect to g, denoted by $\rho_g(f)$, as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r \to \infty} \frac{\log G^{-1} F(r)}{\log r} \end{aligned}$$

where g is a non-constant entire function g. The definition coincides with the classical one if $g(z) = \exp z$. If f is non-constant and $g = f$ then $\rho_g(f) = 1$. As in the classical case we define the relative lower order of f with respect to a non-constant entire function g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r \to \infty} \frac{\log G^{-1} F(r)}{\log r}.$$

If $\rho_g(f) = \lambda_g(f)$ then f is said to be of regular relative growth with respect to g. Therefore if f is of regular relative growth with respect to a non-constant entire function g, we have

$$\rho_g(f) = \lim_{r \to \infty} \frac{\log G^{-1} F(r)}{\log r}.$$

Clearly if f is of regular relative growth with respect to $g(z) = \exp z$ then f is also of regular growth. Somasundaram and Thamizharasi [63] introduced the notions of L-order and L^*-order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. With the help of the above notion we may define relative L-order and relative L^*-order. The following definitions are then obvious.

Definition 8.1.1 *[63] The L-order ρ_f^L and the L-lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]}.$$

Definition 8.1.2 *[63] The L^*-order $\rho_f^{L^*}$ and L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as follows:*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)})]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log M(r,f)}{[re^{L(r)})]}.$$

Definition 8.1.3 *The relative L-order and relative lower L-order of an entire function f with respect to an entire function g respectively denoted by $\rho_g^L(f)$ and $\lambda_g^L(f)$ are defined as*

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]} \text{ and } \lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[rL(r)]}.$$

Definition 8.1.4 *The relative L-order and relative lower L-order of an entire function f with respect to an entire function g respectively denoted by $\rho_g^{L^*}(f)$ and $\lambda_g^{L^*}(f)$ are defined as*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[re^{L(r)}]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[re^{L(r)}]}.$$

In fact Definition8.1.2 and Definition 8.1.4 are more generalised than Definition 8.1.1 and Definition 8.1.3 respectively. The following definition is most generalised.

Definition 8.1.5 *The relative L^*-order and relative lower L^*-order of an entire function f with respect to an entire function g with rate t respectively denoted by ${}^{(t)}\rho_g^{L^*}(f)$ and ${}^{(t)}\lambda_g^{L^*}(f)$ are defined as*

$${}^{(t)}\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[r\exp^{[t]} L(r)]} \text{ and } {}^{(t)}\lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log[r\exp^{[t]} L(r)]}$$

where $t = 1, 2, 3, \ldots$
In this chapter we prove a few theorems on the relationship between $\rho_g^L(f)$ and ρ_f^L. Throughout the chapter we assume f, g etc. as non-constant functions, unless otherwise stated.

8.2 Theorems.

In this section we present the main results of this chapter.

Theorem 8.2.1 *If f be the L-regular growth and of L-regular relative growth with respect to g and $\rho_g^L(f) = \rho_f^L > 0$ then g is of L-regular growth of L-order one. Conversely if g is of L-regular growth of order one then $\rho_g^L(f) = \rho_f^L$ for every entire f with L-regular relative growth.*

Proof. Let us first suppose that

$$\rho_g^L(f) = \rho_f^L = \rho > 0.$$

Also let $0 < \varepsilon < 1$. Let us set $\varepsilon_1 = \dfrac{\rho\varepsilon}{2+\varepsilon}$.

So $\varepsilon_1 < \rho$.

Then there exists $r_0 > 0$ such that for $r \geq r_0$

$$F(r) < \exp[\{rL(r)\}^{\rho+\varepsilon_1}] \text{ and } F(r) > \exp[\{rL(r)\}^{\rho-\varepsilon_1}]. \tag{8.2.1}$$

$$\text{Also } F(r) < G(r^{\rho+\varepsilon_1}) \text{ and } F(r) > G(r^{\rho-\varepsilon_1}). \tag{8.2.2}$$

From (8.2.1) and (8.2.2) we get for $r \geq r_0$

$$G(r^{\rho-\varepsilon_1}) < F(r) < \exp[\{rL(r)\}^{\rho+\varepsilon_1}]$$

and therefore for $r \geq r_0^{\rho}$ we obtain from above that

$$G(r) < \exp[\{rL(r)\}^{\frac{\rho+\varepsilon_1}{\rho-\varepsilon_1}}] = \exp[\{(rL(r)\}^{1+\frac{2\varepsilon_1}{\rho-\varepsilon_1}}].$$

$$\text{So, } G(r) < \exp[\{(rL(r)\}^{1+\varepsilon}] \text{ for } r \geq r_0^{\rho}. \tag{8.2.3}$$

Similarly from (8.2.1) and (8.2.2) we obtain that

$$\exp[\{(rL(r)\}^{1-\varepsilon}] < G(r) \text{ for } r \geq r_0^{2\rho}. \tag{8.2.4}$$

So from (8.2.3) and (8.2.4), for $r \geq r_0^{2\rho}$

$$\exp[\{rL(r)\}^{1-\varepsilon}] < G(r) < \exp[\{rL(r)\}^{1+\varepsilon}].$$

So $g(z)$ is of L-regular growth of order one. Conversely for $\varepsilon > 0$ there exists $r_1 > 0$ such that for $r \geq r_1$,

$$\exp[\{rL(r)\}^{1-\varepsilon}] < G(r) < \exp[\{rL(r)\}^{1+\varepsilon}]. \tag{8.2.5}$$

Also from the definition of $\rho_g^L(f)$, there exists $r_2 > 0$ such that for $r \geq r_2$,

$$G(r^{\rho_g^L(f)-\varepsilon}) < F(r) < G(r^{\rho_g^L(f)+\varepsilon}). \tag{8.2.6}$$

From (8.2.5) and (8.2.6), we have for $r \geq r_3 = \max(r_1, r_2)$,

$$\exp[\{rL(r)\}^{\rho_g^L(f)-\varepsilon(1+\rho_g^L(f)-\varepsilon)}] < F(r) < \exp[\{rL(r)\}^{\rho_g^L(f)+\varepsilon(1+\rho_g^L(f)+\varepsilon)}]. \tag{8.2.7}$$

Since $\varepsilon > 0$ is arbitrary, from (8.2.7) we obtain that

$$\rho_f^L = \lim_{r\to\infty} \frac{\log\log F(r)}{\log[rL(r)]} = \rho_g^L(f).$$

This proves the theorem. ■

In the next theorem we see the more generalisation of Theorem 8.2.1.

Theorem 8.2.2 *If f be of L^*-regular growth and L^*-regular relative growth with respect to g and $\rho_g^{L^*}(f) = \rho_f^{L^*} > 0$ then g is of L^*-regular growth of L^*-order one. Conversely if g is of L^*-regular growth of L^*-order one then $\rho_g^{L^*}(f) = \rho_f^{L^*}$ for every entire f with L^*-regular relative growth.*

Proof. Let us first suppose that

$$\rho_g^{L^*}(f) = \rho_f^{L^*} = \rho > 0.$$

Also let $0 < \varepsilon < 1$.

$$\text{Let us set } \varepsilon_1 = \frac{\rho\varepsilon}{2+\varepsilon}.$$

$$\text{So } \varepsilon_1 < \rho.$$

Then there exists $r_0 > 0$ such that for $r \geq r_0$

$$F(r) < \exp[\{re^{L(r)}\}^{\rho+\varepsilon_1}] \text{ and } F(r) > \exp[\{re^{L(r)}\}^{\rho-\varepsilon_1}]\,. \tag{8.2.8}$$

From (8.2.2) and (8.2.8) for $r \geq r_0$,

$$G(r^{\rho-\varepsilon_1}) < F(r) < \exp[\{re^{L(r)}\}^{\rho+\varepsilon_1}]$$

and therefore for $r \geq r_0^\rho$ we obtain from above that

$$G(r) < \exp[\{re^{L(r)}\}^{\frac{\rho+\varepsilon_1}{\rho-\varepsilon_1}}] = \exp[\{re^{L(r)}\}^{\frac{2\varepsilon_1}{\rho-\varepsilon_1}}].$$

$$\text{So, } G(r) < \exp[\{re^{L(r)}\}^{1+\varepsilon}] \text{ for } r \geq r_0^\rho. \tag{8.2.9}$$

Similarly from (8.2.2) and (8.2.8) we obtain that

$$\exp[\{re^{L(r)}\}^{1-\varepsilon}] < G(r) \text{ for } r \geq r_0^{2\rho}. \tag{8.2.10}$$

So from (8.2.9) and (8.2.10), for $r \geq r_0^{2\rho}$

$$\exp[\{re^{L(r)}\}^{1-\varepsilon}] < G(r) < \exp[\{re^{L(r)}\}^{1+\varepsilon}].$$

So g is of L^*-regular growth of order one. Conversely for $\varepsilon > 0$ there exists $r_1 > 0$ such that for $r \geq r_1$,

$$\exp[\{re^{L(r)}\}^{1-\varepsilon}] < G(r) < \exp[\{re^{L(r)}\}^{1+\varepsilon}]. \tag{8.2.11}$$

Also from the definition of $\rho_g^{L^*}(f)$, there exists $r_2 > 0$ such that for $r \geq r_2$,

$$G(r^{\rho_g^{L^*}(f)-\varepsilon}) < F(r) < G(r^{\rho_g^{L^*}(f)+\varepsilon}). \tag{8.2.12}$$

From (8.2.11) and (8.2.12), we have for $r \geq r_3 = \max(r_1, r_2)$,

$$\exp[\{re^{L(r)}\}^{\rho_g^{L^*}(f)-\varepsilon(1+\rho_g^{L^*}(f)-\varepsilon)}] < F(r)$$

$$< \exp[re^{L(r)}\}^{\rho_g^{L^*}(f)+\varepsilon(1+\rho_g^{L^*}(f)+\varepsilon)}]. \tag{8.2.13}$$

Since $\varepsilon > 0$ is arbitrary, from (8.2.13) we obtain that

$$\rho_f^{L^*} = \lim_{r\to\infty} \frac{\log^{[2]} F(r)}{\log[re^{L(r)}]} = \rho_g^{L^*}(f).$$

Thus the theorem is established. ■

Theorem 8.2.3 *If f be of L^*-regular growth and of L^*-regular relative growth with respect to g with rate t for $t = 1, 2, 3, \ldots$and ${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_f^{L^*}$ then g is of L^*-regular growth of L^*-order one with rate t in each case. Conversely if g is of L^*-regular growth of L^*-order one with rate t in each case then ${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_f^{L^*}$ for every entire f with L^*-regular relative growth with rate t.*

Proof. Let us first suppose that

$${}^{(t)}\rho_g^{L^*}(f) = {}^{(t)}\rho_f^{L^*} = \rho > 0.$$

Also let $0 < \varepsilon < 1$.

Let us set $\varepsilon_1 = \dfrac{\rho\varepsilon}{2+\varepsilon}$.

So $\varepsilon_1 < \rho$.

Then there exists $r_0 > 0$ such that for $r \geq r_0$

$$F(r) < \exp[\{r \exp^{[t]} L(r)\}^{\rho+\varepsilon_1}] \text{ and } F(r) > \exp[\{r \exp^{[t]} L(r)\}^{\rho-\varepsilon_1}]. \quad (8.2.14)$$

From (8.2.2) and (8.2.14) for $r \geq r_0$,

$$G(r^{\rho-\varepsilon_1}) < F(r) < \exp[\{r \exp^{[t]} L(r)\}^{\rho+\varepsilon_1}]$$

and therefore for $r \geq r_0^{\rho}$ we obtain from above that

$$G(r) < \exp[\{r \exp^{[t]} L(r)\}^{\frac{\rho+\varepsilon_1}{\rho-\varepsilon_1}}] = \exp[\{r \exp^{[t]} L(r)\}^{1+\frac{2\varepsilon_1}{\rho-\varepsilon_1}}].$$

$$\text{So, } G(r) < \exp[\{r \exp^{[t]} L(r)\}^{1+\varepsilon}] \text{ for } r \geq r_0^{\rho}. \quad (8.2.15)$$

Similarly from (8.2.2) and (8.2.14) we obtain that

$$\exp[\{r \exp^{[t]} L(r)\}^{1-\varepsilon}] < G(r) \text{ for } r \geq r_0^{2\rho}. \quad (8.2.16)$$

So, from (8.2.15) and (8.2.16), for $r \geq r_0^{2\rho}$

$$\exp[\{r \exp^{[t]} L(r)\}^{1-\varepsilon}] < G(r) < \exp[\{r \exp^{[t]} L(r)\}^{1+\varepsilon}].$$

So g is of L^*-regular growth of order one with rate t. Conversely for $\varepsilon > 0$ there exists $r_2 > 0$ such that for $r \geq r_1$,

$$\exp[\{r \exp^{[t]} L(r)\}^{1-\varepsilon}] < G(r) < \exp[\{r \exp^{[t]} L(r)\}^{1+\varepsilon}]. \quad (8.2.17)$$

Also from the definition of ${}^{(t)}\rho_g^{L^*}(f)$, there exists $r_2 > 0$ such that for $r \geq r_2$,

$$G(r^{{}^{(t)}\rho_g^{L^*}(f)-\varepsilon}) < F(r) < G(r^{{}^{(t)}\rho_g^{L^*}(f)+\varepsilon}). \quad (8.2.18)$$

From (8.2.17) and (8.2.18), we have for $r \geq r_3 = \max(r_1, r_2)$,

$$\exp[\{r \exp^{[t]} L(r)\}^{{}^{(t)}\rho_g^{L^*}(f)-\varepsilon(1+{}^{(t)}\rho_g^{L^*}(f)-\varepsilon)}] < F(r)$$

$$< \exp[\{r \exp^{[t]} L(r)\}^{{}^{(t)}\rho_g^{L^*}(f)+\varepsilon(1+{}^{(t)}\rho_g^{L^*}(f)+\varepsilon)}]. \quad (8.2.19)$$

Since $\varepsilon(> 0)$ is arbitrary, from (8.2.19) we obtain that

$${}^{(t)}\rho_f^{L^*} = \lim_{r\to\infty} \frac{\log^{[2]} F(r)}{\log[r \exp^{[t]} L(r)]} = {}^{(t)}\rho_g^{L^*}(f).$$

This proves the theorem. ■

Remark 8.2.1 *For $t = 1$, Theorem 8.2.3 coincides with Theorem 8.2.2.*

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Chapter 9

RELATIVE L-RITT ORDER AND RELATED COMPARATIVE GROWTH PROPERTIES OF ENTIRE DIRICHLET SERIES

CHAPTER

9

RELATIVE L-RITT ORDER AND RELATED COMPARATIVE GROWTH PROPERTIES OF ENTIRE DIRICHLET SERIES

9.1 Introduction, Definitions and Notations.

During the past decades, several authors {cf. [54], [57], [58] and[61]} made close investigations on the properties of entire Dirichlet series related to Ritt order. Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by everywhere absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$...(9.1.$A$) where $0 < \lambda_n < \lambda_{n+1} (n \geq 1)$, $\lambda_n \to \infty$ as $n \to \infty$ and a_n's are complex constants. If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (9.1.A) then in this clearly $\sigma_c = \sigma_a = \infty$.

$$\text{Let } F(\sigma) = \underset{-\infty<t<\infty}{lub} |f(\sigma + it)|.$$

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see **[26]**.

Then the Ritt order[59] of $f(s)$ denoted by $\rho(f)$ is given by

$$\rho(f) = \limsup_{\sigma\to\infty} \frac{\log\log F(\sigma)}{\sigma} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma}.$$

In other words

$$\rho(f) = \inf\{\mu > 0 : \log F(\sigma) < \exp(\sigma\mu) \text{ for all } \sigma > R(\mu)\}.$$

Similarly the lower Ritt order of $f(s)$ denoted by $\lambda(f)$ may be defined. In this chapter we prove some results on the comparative growth properties related to the L-Ritt order of entire Dirichlet series where $L \equiv L(\sigma)$ is a positive continuous function increasing slowly i.e. $L(a\sigma) \sim L(\sigma)$ as $\sigma \to \infty$ for every constant 'a'. The following definitions are well known.

Definition 9.1.1 *The L-Ritt order $\rho_f^L \equiv \rho^L(f)$ and the L-Ritt lower order (or equivalently lower L-Ritt order) $\lambda_f^L \equiv \lambda^L(f)$ of $f(s)$ are defined as follows respectively*

$$\rho^L(f) = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma L(\sigma)} \text{ and } \lambda^L(f) = \liminf_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma L(\sigma)}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$. Similarly one can define the relative L-Ritt order and relative lower L-Ritt order of $f(s)$.

Definition 9.1.2 *The relative L-Ritt order $\rho_g^L(f)$ and the relative lower L-Ritt order $\lambda_g^L(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as*

$$\rho_g^L(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma L(\sigma)} \text{ and } \lambda_g^L(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma L(\sigma)}.$$

Analogously one can define the following.

Definition 9.1.3 *The hyper L-Ritt order $\overline{\rho}_f^L \equiv \overline{\rho}^L(f)$ and the hyper L-Ritt lower order (or equivalently hyper lower L-Ritt order) $\overline{\lambda}_f^L \equiv \lambda^L(f)$ of $f(s)$ are defined respectively as follows*

$$\overline{\rho}^L(f) = \limsup_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma L(\sigma)} \text{ and } \overline{\lambda}^L(f) = \liminf_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma L(\sigma)}.$$

Definition 9.1.4 *The relative hyper L-Ritt order $\overline{\rho}_g^L(f)$ and the relative hyper lower L-Ritt order $\overline{\lambda}_g^L(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as*

$$\overline{\rho}_g^L(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma L(\sigma)} \text{ and } \overline{\lambda}_g^L(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma L(\sigma)}.$$

The more generalised concept of L-Ritt order (lower L-Ritt order) and relative L-Ritt order (relative lower L-Ritt order) are respectively L^*-Ritt order (lower L^*-Ritt order) and relative L^*-Ritt order (relative lower L^*-Ritt order). We may now state the following definitions.

Definition 9.1.5 *The L^*-Ritt order $\rho_f^{L^*} \equiv \rho^{L^*}(f)$ and the L^*-Ritt lower order (or equivalently lower L^*-Ritt order) $\lambda_f^{L^*} \equiv \lambda^{L^*}(f)$ of $f(s)$ are defined respectively as follows*

$$\rho_f^{L^*} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \lambda_f^{L^*} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)}.$$

Definition 9.1.6 *The relative L^*-Ritt order $\rho_g^{L^*}(f)$ and the relative lower L^*-Ritt order $\lambda_g^{L^*}(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as*

$$\rho_g^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \lambda_g^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma \exp L(\sigma)}.$$

Definition 9.1.7 *The hyper L^*-Ritt order $\overline{\rho}_f^{L^*} \equiv \overline{\rho}^{L^*}(f)$ and the hyper L^*-Ritt lower order (hyper lower L^*-Ritt order) $\overline{\lambda}_f^{L^*} \equiv \lambda^{L^*}(f)$ of $f(s)$ are defined respectively as follows*

$$\overline{\rho}^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \lambda^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma \exp L(\sigma)}.$$

Definition 9.1.8 *The relative hyper L^*-Ritt order $\overline{\rho}_g^{L^*}(f)$ and the relative hyper lower L^*-Ritt order $\overline{\lambda}_g^{L^*}(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as*

$$\overline{\rho}_g^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \overline{\lambda}_g^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)}.$$

Generalising our notion we may state the following definitions.

Definition 9.1.9 *The generalised L-Ritt order* ${}^{(k)}\rho_f^L \equiv {}^{(k)}\rho^L(f)$ *and the generalised L-Ritt lower order (generalised lower L-Ritt order)* ${}^{(k)}\lambda_f^L \equiv {}^{(k)}\lambda^L(f)$ *are defined respectively as follows.*

$$ {}^{(k)}\rho^L(f) = \limsup_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma L(\sigma)} \quad \textit{and} \quad {}^{(k)}\lambda^L(f) = \liminf_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma L(\sigma)} $$

where $k = 2, 3, \ldots$

Definition 9.1.10 *The generalised* L^**-Ritt order* ${}^{(k)}\rho_f^{L^*} \equiv {}^{(k)}\rho^{L^*}(f)$ *and the generalised* L^**-Ritt lower order (or equivalently generalised lower* L^**-Ritt order)* ${}^{(k)}\lambda_f^L \equiv {}^{(k)}\lambda^L(f)$ *are respectively defined as*

$$ {}^{(k)}\rho^{L^*}(f) = \limsup_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} \quad \textit{and} \quad {}^{(k)}\lambda^{L^*}(f) = \liminf_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} $$

where $k = 2, 3, \ldots$

Definition 9.1.11 *The generalised relative L-Ritt order* ${}^{(k)}\rho_g^L(f)$ *and the generalised relative lower L-Ritt order* ${}^{(k)}\lambda_g^L(f)$ *of* $f(s)$ *with respect to entire* $g(s)$ *are respectively defined as*

$$ {}^{(k)}\rho_g^L(f) = \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma L(\sigma)} \quad \textit{and} \quad {}^{(k)}\lambda_g^L(f) = \liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma L(\sigma)} $$

where $k = 1, 2, 3, \ldots$

Definition 9.1.12 *The generalised relative* L^**-Ritt order* ${}^{(k)}\rho_g^{L^*}(f)$ *and the generalised relative lower* L^**-Ritt order* ${}^{(k)}\lambda_g^{L^*}(f)$ *of* $f(s)$ *with respect to* $g(s)$ *are respectively defined as follows*

$$ {}^{(k)}\rho_g^{L^*}(f) = \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} \quad \textit{and} \quad {}^{(k)}\lambda_g^{L^*}(f) = \liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} $$

where $k = 1, 2, 3, \ldots$

9.2 Theorems.

In this section we present the main results of this chapter.

Theorem 9.2.1 *Let f and g be two entire functions such that* $0 < {}^{(k)}\lambda^{L^*}(fog) \leq {}^{(k)}\rho^{L^*}(fog) < \infty$ *and* $0 < {}^{(k)}\lambda^{L^*}(g) \leq {}^{(k)}\rho^{L^*}(g) < \infty$. *Then*

$$\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)} \leq \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)}$$
$$\leq \min\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}\}$$
$$\leq \max\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}\}$$
$$\leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}$$

where $k = 1, 2, 3, ...$

Proof. From the definition of generalised L^*-Ritt order and generalised L^*-lower Ritt order of entire g we have for arbitrary positive ε and for all sufficiently large values of σ,

$$\log^{[k]} G(\sigma) \leq ({}^{(k)}\rho^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.1}$$

$$\text{and } \log^{[k]} G(\sigma) \geq ({}^{(k)}\lambda^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.2}$$

Also for a sequence of values of σ tending to infinity,

$$\log^{[k]} G(\sigma) \leq ({}^{(k)}\lambda^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.3}$$

$$\text{and } \log^{[k]} G(\sigma) \geq ({}^{(k)}\rho^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.4}$$

Now again from the definition of generalised L^*-Ritt order and generalised L^*-lower Ritt order of the composite function fog we have for arbitrary positive ε and for all sufficiently large values of σ,

$$\log^{[k]} FoG(\sigma) \leq ({}^{(k)}\rho^{L^*}(fog) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.5}$$

$$\text{and } \log^{[k]} FoG(\sigma) \geq ({}^{(k)}\lambda^{L^*}(fog) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.6}$$

Again for a sequence of values of σ tending to infinity

$$\log^{[k]} FoG(\sigma) \leq ({}^{(k)}\lambda^{L^*}(fog) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.7}$$

$$\text{and } \log^{[k]} FoG(\sigma) \geq ({}^{(k)}\rho^{L^*}(fog) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.8}$$

Now from (9.2.1) and (9.2.6) it follows for all sufficiently large values of σ,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog) - \varepsilon}{{}^{(k)}\rho^{L^*}(g) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}. \tag{9.2.9}$$

Again combining (9.2.2) and (9.2.7) we get for a sequence of values of σ tending to infinity,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\lambda^{L^*}(fog) + \varepsilon}{{}^{(k)}\lambda^{L^*}(g) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.10}$$

Similarly from (9.2.4) and (9.2.5) it follows for a sequence of values of σ tending to infinity that

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog) + \varepsilon}{{}^{(k)}\rho^{L^*}(g) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}. \tag{9.2.11}$$

Now combining (9.2.9), (9.2.10) and (9.2.11) we get that

$$\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)} \leq \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)}$$

$$\leq \min\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}\}. \tag{9.2.12}$$

Now from (9.2.3) and (9.2.6) we obtain for a sequence of values of σ tending to infinity,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog) - \varepsilon}{{}^{(k)}\lambda^{L^*}(g) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.13}$$

Again from (9.2.2) and (9.2.5) it follows for all sufficiently large values of σ,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog) + \varepsilon}{{}^{(k)}\lambda^{L^*}(g) - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.14}$$

Similarly combining (9.2.1) and (9.2.8) we get for a sequence of values of σ tending to infinity,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\rho^{L^*}(fog) - \varepsilon}{{}^{(k)}\rho^{L^*}(g) + \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}. \tag{9.2.15}$$

Therefore combining (9.2.13), (9.2.14) and (9.2.15) we get that

$$\max\left\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}\right\} \leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.16}$$

Thus the theorem follows from (9.2.12) and (9.2.16). ■

Remark 9.2.1 *If we take* $0 < {}^{(k)}\lambda^{L^*}(f) \leq {}^{(k)}\rho^{L^*}(f) < \infty$ *instead of* $0 < {}^{(k)}\lambda^{L^*}(g) \leq {}^{(k)}\rho^{L^*}(g) < \infty$ *and the other conditions remain the same then also Theorem 9.2.1 holds with* g *replaced by* f *in the denominator as we see in the next theorem.*

Theorem 9.2.2 *Let f and g be two entire functions such that* $0 < {}^{(k)}\lambda^{L^*}(fog) \leq {}^{(k)}\rho^{L^*}(fog) < \infty$ *and* $0 < {}^{(k)}\lambda^{L^*}(f) \leq {}^{(k)}\rho^{L^*}(f) < \infty$. *Then*

$$\begin{aligned}\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(f)} &\leq \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)} \\ &\leq \min\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(f)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(f)}\} \\ &\leq \max\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(f)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(f)}\} \\ &\leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(f)}\end{aligned}$$

where $k = 1, 2, 3, \ldots$
In fact, Theorem 9.2.1 and Theorem 9.2.2 are the more generalised concept of Theorem 9.2.3 and Theorem 9.2.4 respectively.

Theorem 9.2.3 *Let f and g be two entire functions such that* $0 < {}^{(k)}\lambda^{L}(fog) \leq {}^{(k)}\rho^{L}(fog) < \infty$ *and* $0 < {}^{(k)}\lambda^{L}(g) \leq {}^{(k)}\rho^{L^*}(g) < \infty$. *Then*

$$\begin{aligned}\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\rho^{L}(g)} &\leq \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \\ &\leq \min\{\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\lambda^{L}(g)}, \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\rho^{L}(g)}\} \\ &\leq \max\{\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\lambda^{L}(g)}, \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\rho^{L}(g)}\} \\ &\leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\lambda^{L}(g)}\end{aligned}$$

where $k = 1, 2, 3, \ldots$
The proof is omitted.

Theorem 9.2.4 *Let f and g be two entire functions such that* $0 < {}^{(k)}\lambda^{L}(fog) \leq$

${}^{(k)}\rho^{L}(fog) < \infty$ *and* $0 < {}^{(k)}\lambda^{L}(f) \le {}^{(k)}\rho^{L}(f) < \infty$. *Then*

$$\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\rho^{L}(f)} \le \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)}$$
$$\le \min\{\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\lambda^{L}(f)}, \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\rho^{L}(f)}\}$$
$$\le \max\{\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\lambda^{L}(f)}, \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\rho^{L}(f)}\}$$
$$\le \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)} \le \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\lambda^{L}(f)}$$

where $k = 1, 2, 3, ...$
The proof is omitted.
Lahiri and Banerjee [54] studied on relative Ritt order of entire Dirichlet series and proved some basic theorems. In the subsequent theorems we prove something more.

Theorem 9.2.5 *Let* f, g *and* h *be three entire functions with* $0 < {}^{(k)}\lambda_h(f) \le {}^{(k)}\rho_h(f) < \infty$ *and* $0 < {}^{(k)}\lambda_h(g) \le {}^{(k)}\rho_h(g) < \infty$. *Then*

$$(i)\ \liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \le \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)} \le \limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)}$$

$$\text{and } (ii)\ \liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \le \min\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}, \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}$$
$$\le \max\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}, \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}$$
$$\le \limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)}.$$

Proof. From the definition of generalised relative L^*-Ritt order and generalised relative L^*-lower Ritt order of entire g with respect to entire h we have for arbitrary positive ε and for all sufficiently large values of σ,

$$H^{-1}\log^{[k]} G(\sigma) \le ({}^{(k)}\rho_h^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.17}$$

$$\text{and } H^{-1}\log^{[k]} G(\sigma) \ge ({}^{(k)}\lambda_h^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.18}$$

Also for a sequence of values of σ tending to infinity,

$$H^{-1}\log^{[k]} G(\sigma) \leq ({}^{(k)}\lambda_h^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.19}$$

$$\text{and } H^{-1}\log^{[k]} G(\sigma) \geq ({}^{(k)}\rho_h^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.20}$$

Now again from the definition of generalised relative L^*-Ritt order and generalised relative L^*-lower Ritt order of entire f with respect to entire h we have for arbitrary positive ε and for all sufficiently large values of σ,

$$H^{-1}\log^{[k]} F(\sigma) \leq ({}^{(k)}\rho_h^{L^*}(f) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.21}$$

$$\text{and } H^{-1}\log^{[k]} F(\sigma) \geq ({}^{(k)}\lambda_h^{L^*}(f) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.22}$$

Also for a sequence of values of σ tending to infinity,

$$H^{-1}\log^{[k]} F(\sigma) \leq ({}^{(k)}\lambda_h^{L^*}(f) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.23}$$

$$\text{and } H^{-1}\log^{[k]} F(\sigma) \geq ({}^{(k)}\rho_h^{L^*}(f) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.24}$$

Now from (9.2.17) and (9.2.22) it follows for all sufficiently large values of σ,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g) + \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.25}$$

Again combining (9.2.18) and (9.2.23) we get for a sequence of values of σ tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g) - \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.26}$$

Similarly from (9.2.20) and (9.2.21) it follows for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\rho_h^{L^*}(g) - \varepsilon}{{}^{(k)}\rho_h^{L^*}(f) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}. \tag{9.2.27}$$

Now from (9.2.19) and (9.2.22) we obtain for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g) + \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) - \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.28}$$

Again from (9.2.18) and (9.2.23) it follows for all sufficiently large values of r,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g) - \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.29}$$

Similarly combining (9.2.17) and (9.2.24) we get for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g) + \varepsilon}{{}^{(k)}\rho_h^{L^*}(f) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}. \tag{9.2.30}$$

Combining (9.2.28) and (9.2.29) we obtain that

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}\leq\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}\leq\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}.$$

This proves the first part of the theorem. Again combining (9.2.26) and (9.2.27) it follows that

$$\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}\geq\max\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)},\frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}.\tag{9.2.31}$$

Now combining (9.2.28) and (9.2.30) we get that

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}\leq\min\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)},\frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}.\tag{9.2.32}$$

Thus from (9.2.31) and (9.2.32) the second part of the theorem follows. ■

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Chapter 10

GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS OF TWO COMPLEX VAIABLES

CHAPTER

10

GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

10.1 Introduction, Definitions and Notations.

Let $f(z_1, z_2)$ be a non-constant entire function of two complex variables z_1 and z_2, holomorphic in the closed polydisc

$$\{(z_1, z_2) : |z_j| \le r_j,\ j = 1, 2 \text{ for all } r_1 \ge 0, r_2 \ge 0\}.$$

$$\text{Let } F(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_j| \le r_j,\ j = 1, 2\}.$$

Then by the Hartogs theorem and maximum principle $\{[30], p.21, p.51\}$, $F(r_1, r_2)$ is an increasing function of r_1, r_2. The order $\rho = \rho(f)$ of $f(z_1, z_2)$ is defined $\{[30]$, p.338$\}$ as the infimum of all positive numbers μ for which

$$F(r_1, r_2) < \exp[(r_1 r_2)^{\mu}] \tag{10.1.A}$$

holds for all sufficiently large value of r_1 and r_2.In other words

$$\rho(f) = \inf\{\mu > 0 : F(r_1, r_2) < \exp[(r_1 r_2)^{\mu}] \text{ for all } r_1 \ge R(\mu), r_2 \ge R(\mu)\}.$$

The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences**, see [27].

Equivalent formula for $\rho(f)$ is {[30], p.339; [1]}

$$\rho(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} F(r_1, r_2)}{\log(r_1 r_2)}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$and $\log^{[0]} x = x$.

Similarly the lower order $\lambda = \lambda(f)$ of $f(z_1, z_2)$ is defined as

$$\lambda(f) = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} F(r_1, r_2)}{\log(r_1 r_2)}.$$

Extending our notion we can easily define the hyper order (and hyper lower order), generalised order (and generalised lower order) and (p, q)th order (and (p, q)th lower order) of entire functions of two complex variables where p and q are any two positive integers with $p > q$.

Definition 10.1.1 *The hyper order $\overline{\rho}(f)$ and hyper lower order $\overline{\lambda}(f)$ of an entire function f of two complex variables are defined as*

$$\overline{\rho}(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[3]} F(r_1, r_2)}{\log(r_1 r_2)} \text{ and } \overline{\lambda}(f) = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[3]} F(r_1, r_2)}{\log(r_1 r_2)}.$$

Definition 10.1.2 *The generalised order $^{(k)}\rho(f)$ and generalised lower order $^{(k)}\lambda(f)$ of an entire function f of two complex variables are defined as*

$$^{(k)}\rho(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[k+1]} F(r_1, r_2)}{\log(r_1 r_2)} \text{ and } ^{(k)}\lambda(f) = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[k+1]} F(r_1, r_2)}{\log(r_1 r_2)}$$

where $k = 1, 2, 3, \ldots$

Definition 10.1.3 *The (p, q)th order $\rho_q^p(f)$ and lower (p, q)th order $\lambda_q^p(f)$ of an entire function f of two complex variables are defined as follows*

$$\rho_q^p(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[p+1]} F(r_1, r_2)}{\log^{[q]}(r_1 r_2)} \text{ and } \lambda_q^p(f) = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[p+1]} F(r_1, r_2)}{\log^{[q]}(r_1 r_2)}$$

where p, q are any two positive integers with $p > q$. Using the above notion, in this chapter we discuss some comparative growth properties of composite entire functions of two complex variables.

10.2 Theorems.

In this section we present the main results of this chapter.

Theorem 10.2.1 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \lambda(f) \leq \rho(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\lambda(fog)}{A\lambda(f)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\lambda(f)}.$$

Proof. From the definition of $\rho(f)$ and $\lambda(f)$ we have for arbitrary positive ε and for sufficiently large values of r_1, r_2

$$\log^{[2]} FoG(r_1,r_2) \geq [\lambda(fog) - \varepsilon] \log(r_1 r_2) \qquad (10.2.1)$$

$$\text{and } \log^{[2]} F(r_1^A,r_2^A) \leq [A\rho(fog) + \varepsilon] \log(r_1 r_2). \qquad (10.2.2)$$

Now from (10.2.1) and (10.2.2) it follows for all sufficiently large values of r_1, r_2 that

$$\frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \geq \frac{\lambda(fog) - \varepsilon}{A\rho(fog) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \geq \frac{\lambda(fog)}{A\rho(fog)}. \qquad (10.2.3)$$

Again for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} FoG(r_1,r_2) \leq [\lambda(fog) + \varepsilon] \log(r_1 r_2) \qquad (10.2.4)$$

and for all sufficiently large values of r_1, r_2

$$\log^{[2]} F(r_1^A,r_2^A) \geq [A\lambda(f) - \varepsilon] \log(r_1 r_2). \qquad (10.2.5)$$

Combining (10.2.4) and (10.2.5) we get for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\lambda(fog) + \varepsilon}{A\lambda(f) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\lambda(fog)}{A\lambda(f)}. \tag{10.2.6}$$

Also for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} F(r_1^A, r_2^A) \leq [A\lambda(f) + \varepsilon] \log(r_1 r_2). \tag{10.2.7}$$

Now from (10.2.1) and (10.2.7) we obtain for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \geq \frac{\lambda(fog) - \varepsilon}{A\lambda(f) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we get that

$$\limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \geq \frac{\lambda(fog)}{A\lambda(f)}. \tag{10.2.8}$$

Also for all sufficiently large values of r_1, r_2

$$\log^{[2]} FoG(r_1, r_2) \leq [\rho(fog) + \varepsilon] \log(r_1 r_2). \tag{10.2.9}$$

From (10.2.5) and (10.2.9) it follows for all sufficiently large values of r_1, r_2

$$\frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\rho(fog) + \varepsilon}{A\lambda(f) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\rho(fog)}{A\lambda(f)}. \tag{10.2.10}$$

Thus the theorem follows from (10.2.3), (10.2.6), (10.2.8) and (10.2.10). ■

Theorem 10.2.2 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \le \rho(fog) < \infty$ and $0 < \rho(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \le \frac{\rho(fog)}{A\rho(f)} \le \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)}.$$

Proof. From the definition of order of an entire function of two variables we get for a sequence of values of r_1 tending to infinity and also for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} F(r_1^A,r_2^A) \ge [A\rho(f)-\varepsilon]\log(r_1r_2). \tag{10.2.11}$$

Now from (10.2.9) and (10.2.11) it follows for a sequence of values of r_1 tending to infinity and for a sequence of values r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \le \frac{\rho(fog)+\varepsilon}{A\rho(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \le \frac{\rho(fog)}{A\rho(f)}. \tag{10.2.12}$$

Again for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} FoG(r_1,r_2) \ge [\rho(fog)-\varepsilon]\log(r_1r_2). \tag{10.2.13}$$

So combining (10.2.2) and (10.2.13)we get for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \ge \frac{\rho(fog)-\varepsilon}{A\rho(f)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \ge \frac{\rho(fog)}{A\rho(f)}. \tag{10.2.14}$$

Thus the theorem follows from (10.2.12) and (10.2.14). The following theorem is a natural consequence of Theorem 10.2.1 and Theorem 10.2.2. ∎

Theorem 10.2.3 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \lambda(f) \leq \rho(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \min\{\frac{\lambda(fog)}{A\lambda(f)},\frac{\rho(fog)}{A\rho(f)}\}$$

$$\leq \max\{\frac{\lambda(fog)}{A\lambda(f)},\frac{\rho(fog)}{A\rho(f)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\lambda(f)}.$$

The proof is omitted.

Remark 10.2.1 *If we take $0 < \lambda(g) \leq \rho(g) < \infty$ instead of $0 < \lambda(f) \leq \rho(f) < \infty$ and the other conditions remain the same then Theorem10.2.1, Theorem 10.2.2 and Theorem 10.2.3 are still valid with $G(r_1^A,r_2^A)$ in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 10.2.4, Theorem 10.2.5 and Theorem 10.2.6 respectively.*

Theorem 10.2.4 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \lambda(g) \leq \rho(g) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \leq \frac{\lambda(fog)}{A\lambda(g)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\lambda(g)}.$$

The proof of Theorem 10.2.4 is omitted because it can be carried out in the line of Theorem 10.2.1. In the line of Theorem 10.2.2 we may prove the following theorem.

Theorem 10.2.5 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \rho(g) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\rho(g)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)}.$$

The proof is omitted. The following theorem is a natural consequence of Theorem 10.2.4 and Theorem 10.2.5.

Theorem 10.2.6 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \le \rho(fog) < \infty$ and $0 < \lambda(g) \le \rho(g) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(g)} \le \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \le \min\{\frac{\lambda(fog)}{A\lambda(g)}, \frac{\rho(fog)}{A\rho(g)}\}$$

$$\le \max\{\frac{\lambda(fog)}{A\lambda(g)}, \frac{\rho(fog)}{A\rho(g)}\} \le \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \le \frac{\rho(fog)}{A\lambda(g)}.$$

Using Definition 10.1.1, we may obtain the following theorems.

Theorem 10.2.7 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \le \overline{\rho}(fog) < \infty$ and $0 < \overline{\lambda}(f) \le \overline{\rho}(f) < \infty$. Then for any positive number A,*

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(f)} \le \liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A,r_2^A)} \le \frac{\overline{\lambda}(fog)}{A\overline{\lambda}(f)}$$

$$\le \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A,r_2^A)} \le \frac{\overline{\rho}(fog)}{A\overline{\lambda}(f)}.$$

Theorem 10.2.8 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \le \overline{\rho}(fog) < \infty$ and $0 < \overline{\rho}(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A,r_2^A)} \le \frac{\overline{\rho}(fog)}{A\overline{\rho}(f)} \le \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A,r_2^A)}.$$

The following theorem is a natural consequence of Theorem 10.2.7 and Theorem 10.2.8.

Theorem 10.2.9 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \le \overline{\rho}(fog) < \infty$ and $0 < \overline{\lambda}(f) \le \overline{\rho}(f) < \infty$. Then for any positive number A,*

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(f)} \le \liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A,r_2^A)} \le \min\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(f)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(f)}\}$$

$$\le \max\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(f)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(f)}\} \le \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A,r_2^A)} \le \frac{\overline{\rho}(fog)}{A\overline{\lambda}(f)}.$$

Remark 10.2.2 *If we consider* $0 < \overline{\lambda}(g) \leq \overline{\rho}(g) < \infty$ *instead of* $0 < \overline{\lambda}(f) \leq \overline{\rho}(f) < \infty$ *and the other conditions remain the same then Theorem 10.2.7, Theorem 10.2.8 and Theorem 10.2.9 are still valid with* $G(r_1^A, r_2^A)$ *in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 10.2.10, Theorem 10.2.11 and Theorem 10.2.12 respectively.*

Theorem 10.2.10 *Let* f *and* g *be two non-constant entire functions of two complex variables such that* $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ *and* $0 < \overline{\lambda}(g) \leq \overline{\rho}(g) < \infty$. *Then for any positive number* A,

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(g)} \leq \liminf_{r_1, r_2 \to \infty} \frac{\log^{[3]} FoG(r_1, r_2)}{\log^{[3]} G(r_1^A, r_2^A)} \leq \frac{\overline{\lambda}(fog)}{A\overline{\lambda}(g)}$$

$$\leq \limsup_{r_1, r_2 \to \infty} \frac{\log^{[3]} FoG(r_1, r_2)}{\log^{[3]} G(r_1^A, r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\lambda}(g)}.$$

Theorem 10.2.11 *Let* f *and* g *be two non-constant entire functions of two complex variables such that* $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ *and* $0 < \overline{\rho}(g) < \infty$. *Then for any positive number* A,

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[3]} FoG(r_1, r_2)}{\log^{[3]} G(r_1^A, r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\rho}(g)} \leq \limsup_{r_1, r_2 \to \infty} \frac{\log^{[3]} FoG(r_1, r_2)}{\log^{[3]} G(r_1^A, r_2^A)}.$$

The following theorem is a natural consequence of Theorem 10.2.10 and Theorem 10.2.11.

Theorem 10.2.12 *Let* f *and* g *be two non-constant entire functions of two complex variables such that* $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ *and* $0 < \overline{\lambda}(g) \leq \overline{\rho}(g) < \infty$. *Then for any positive number* A,

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(g)} \leq \liminf_{r_1, r_2 \to \infty} \frac{\log^{[3]} FoG(r_1, r_2)}{\log^{[3]} G(r_1^A, r_2^A)} \leq \min\left\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(g)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(g)}\right\}$$

$$\leq \max\left\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(g)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(g)}\right\} \leq \limsup_{r_1, r_2 \to \infty} \frac{\log^{[3]} FoG(r_1, r_2)}{\log^{[3]} G(r_1^A, r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\lambda}(g)}.$$

Using Definition 10.1.2, we may obtain the next three theorems.

Theorem 10.2.13 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(f) \leq {}^{(k)}\rho(f) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(f)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(f)}$$

where $k = 1, 2, 3, \ldots$

Theorem 10.2.14 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\rho(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(f)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)}$$

where $k = 1, 2, 3, \ldots$ The following theorem is a natural consequence of Theorem 10.2.13 and Theorem 10.2.14.

Theorem 10.2.15 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(f) \leq {}^{(k)}\rho(f) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \leq \min\left\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(f)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(f)}\right\}$$

$$\leq \max\left\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(f)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(f)}\right\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(f)}$$

where $k = 1, 2, 3, \ldots$

Remark 10.2.3 *If we consider $0 < {}^{(k)}\lambda(g) < {}^{(k)}\rho(g) < \infty$ instead of $0 < {}^{(k)}\lambda(f) < {}^{(k)}\rho(f) < \infty$ and the other conditions remain the same then Theorem 10.2.13, Theorem 10.2.14 and Theorem 10.2.15 are still valid with $G(r_1^A, r_2^A)$ in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 10.2.16, Theorem 10.2.17 and Theorem 10.2.18.*

Theorem 10.2.16 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(g) \leq {}^{(k)}\rho(g) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(g)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(g)}$$

where $k = 1, 2, 3, \ldots$

Theorem 10.2.17 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\rho(g) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(g)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)}$$

where $k = 1, 2, 3, \ldots$ The following theorem is a natural consequence of Theorem 10.2.16 and Theorem 10.2.17.

Theorem 10.2.18 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(g) \leq {}^{(k)}\rho(g) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \leq \min\left\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(g)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(g)}\right\}$$

$$\leq \max\left\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(g)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(g)}\right\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(g)}$$

where $k = 1, 2, 3, \ldots$

Theorem 10.2.19 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ and $0 < \lambda_q^m(f) \leq \rho_q^m(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A,r_2^A)} \leq \frac{\lambda_q^p(fog)}{A\lambda_q^m(f)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A, r_2^A)} \leq \frac{\rho_q^p(fog)}{A\lambda_q^m(f)}$$

where p, q and m are any three positive integers such that $p > q$ and $m > q$ i.e., $q < \min\{p, m\}$.

Theorem 10.2.20 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ and $0 < \rho_q^m(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A, r_2^A)} \leq \frac{\rho_q^p(fog)}{A\rho_q^m(f)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A, r_2^A)}$$

where p, q and m are any three positive integers such that $p > q$ and $m > q$ i.e., $q < \min\{p, m\}$. The following theorem is a natural consequence of Theorem 10.2.19 and Theorem 10.2.20.

Theorem 10.2.21 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ and $0 < \lambda_q^m(f) \leq \rho_q^m(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A, r_2^A)} \leq \min\left\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(f)}, \frac{\rho_q^p(fog)}{A\rho_q^m(f)}\right\}$$

$$\leq \max\left\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(f)}, \frac{\rho_q^p(fog)}{A\rho_q^m(f)}\right\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A, r_2^A)} \leq \frac{\rho_q^p(fog)}{A\lambda_q^m(f)},$$

where p, q and m are any three positive integers such that $p > q$ and $m > q$ i.e., $q < \min\{p, m\}$.

Remark 10.2.4 *If we consider $0 < \lambda_q^m(g) \leq \rho_q^m(g) < \infty$ instead of $0 < \lambda_q^m(f) \leq \rho_q^m(f) < \infty$ and the other conditions remain the same then Theorem 10.2.19, Theorem 10.2.20 and Theorem 10.2.21 are still valid with $G(r_1^A, r_2^A)$ in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 10.2.22, Theorem 10.2.23 and Theorem 10.2.24.*

Theorem 10.2.22 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda_q^p(fog) < \infty$ and $0 < \lambda_q^m(g) \leq \rho_q^m(g) < \infty$. Then for any positive number A,*

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A, r_2^A)} \leq \frac{\lambda_q^p(fog)}{A\lambda_q^m(g)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A,r_2^A)} \leq \frac{\rho_q^m(fog)}{A\lambda_q^m(g)}$$

where p,q and m are any three positive integers such that $p>q$ and $m>q$ i.e., $q<\min\{p,m\}$.

Theorem 10.2.23 *Let f and g be two non-constant entire functions of two complex variables such that $0<\lambda_q^p(fog)\leq\rho_q^p(fog)<\infty$ and $0<\rho_q^m(g)<\infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A,r_2^A)} \leq \frac{\rho_q^p(fog)}{A\rho_q^m(g)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A,r_2^A)}$$

where p,q and m are any three positive integers such that $p>q$ and $m>q$ i.e., $q<\min\{p,m\}$.

Theorem 10.2.24 *Let f and g be two non-constant entire functions of two complex variables such that $0<\lambda_q^p(fog)\leq\rho_q^p(fog)<\infty$ and $0<\lambda_q^m(g)\leq\rho_q^m(g)<\infty$. Then for any positive number A,*

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A,r_2^A)} \leq \min\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(g)}, \frac{\rho_q^p(fog)}{A\rho_q^m(g)}\}$$

$$\leq \max\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(g)}, \frac{\rho_q^p(fog)}{A\rho_q^m(g)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A,r_2^A)} \leq \frac{\rho_q^p(fog)}{A\lambda_q^m(g)}$$

where p,q and m are any three positive integers such that $p>q$ and $m>q$ i.e., $q<\min\{p,m\}$.

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Chapter 11

STUDY OF GROWTH PROPERTIES ON THE BASIS OF GOL'DBERG ORDER OF COMPOSITE ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

CHAPTER

11

STUDY OF GROWTH PROPERTIES ON THE BASIS OF GOL'DBERG ORDER OF COMPOSITE ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

11.1 Introduction, Definitions and Notations.

We denote complex and real n-space by $C^n$and R^n respectively and indicate the point $(z_{1,} ....z_n), (m_{1,} ...m_n\ )$ of C^n or I^n by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. The modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + ... + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^m will denote $z_1^{m_1}...z_n^{m_n}$ and $||m|| = m_1 + ... + m_n$.
Let $D \subset C^n$ be an arbitrary bounded complete n-circular domain with center at the origin of coordinates. Then for the analytic function f and $R > 0$, $M_{f,_D}(R) = \sup\limits_{z\epsilon D_R} |f(z)|$, where a point $z\epsilon D_R$ if and only if $\frac{z}{R}\epsilon D$.

The results of this chapter have been published in **International Journal of Mathematical Analysis,** see [**28**].

Definition 11.1.1 *{[30], p. 339} The Gol'dberg order (briefly G-order) ρ^f_D of f with respect to the domain D is defined as*

$$\rho_D(f) \equiv \rho^f_D = \limsup_{R\to\infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$.

The lower Gol'dberg order λ^f_D of f with respect to the domain D is defined as

$$\lambda_D(f) \equiv \lambda^f_D = \liminf_{R\to\infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}.$$

We say that f is of regular growth if $\rho^f_D = \lambda^f_D$.

Definition 11.1.2 *[56] Let f and g be two entire functions of n variables and D be a bounded complete n-circular domain with centre at the origin in C^n. Then the relative G-order $\rho_{g,D}(f)$ of f with respect to g and the domain D is defined by*

$$\rho_{g,D}(f) = \inf\{\mu > 0;\ M_{f,D}(R) < M_{g,D}(R^{\mu}),\ \textit{for all } R > R_0(\mu) > 0\}.$$

If f is a non-constant entire function, then $M_{f,D}(R)$ is a strictly increasing and continuous function of R and its inverse function $M^{-1}_{f,D} : (|f(0)|, \infty) \to (0, \infty)$ exists. It then easily follows that

$$\rho_{g,D}(f) = \limsup_{R\to\infty} \frac{\log M^{-1}_{g,D}(M_{f,D}(R))}{\log R}.$$

Similarly, the relative lower order $\lambda_{g,D}(f)$ of f with respect to g and the domain D is defined by

$$\lambda_{g,D}(f) = \liminf_{R\to\infty} \frac{\log M^{-1}_{g,D}(M_{f,D}(R))}{\log R}.$$

Throughout this chapter we shall measure the growth of entire functions relative to the entire function g and D will represent a bounded complete n-circular domain. Unless otherwise stated all the entire functions under consideration will be transcendental. Extending the notions of Definition 11.1.1 and Definition 11.1.2 we may give the following definitions:

Definition 11.1.3 *The hyper Gol'dberg order (briefly hyper G-order) $\overline{\rho}_D^f$ of f with respect to the domain D is defined by*

$$\overline{\rho}_D(f) \equiv \overline{\rho}_D^f = \limsup_{R\to\infty} \frac{\log^{[3]} M_{f,D}(R)}{\log R}.$$

The hyper lower Gol'dberg order (briefly hyper lower G-order) of f with respect to the domain D is defined by

$$\overline{\lambda}_D(f) \equiv \overline{\lambda}_D^f = \liminf_{R\to\infty} \frac{\log^{[3]} M_{f,D}(R)}{\log R}.$$

Definition 11.1.4 *Let f and g be two entire functions of n variables and D be a bounded complete n-circular domain with centre at origin in C^n. Then the relative hyper Gol'dberg order $\overline{\rho}_{g,D}(f)$ of f with respect to g and the domain D is defined as*

$$\overline{\rho}_{g,D}(f) \equiv \overline{\rho}_{g,D}^f = \limsup_{R\to\infty} \frac{\log^{[2]} M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.$$

Similarly the relative hyper lower order $\overline{\lambda}_{g,D}(f)$ of f with respect to g and the domain D is defined by

$$\overline{\lambda}_{g,D}(f) \equiv \overline{\lambda}_{g,D}^f = \liminf_{R\to\infty} \frac{\log^{[2]} M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.$$

Generalising our notion we may get the following definition.

Definition 11.1.5 *The generalised Gol'dberg order (briefly generalised G-order) $\rho_D^{(k)}(f)$ of f with respect to the domain D is defined by*

$$\rho_D^{(k)}(f) = \limsup_{R\to\infty} \frac{\log^{[k]} M_{f,D}(R)}{\log R} \text{ where } k = 1, 2, 3, \ldots$$

The generalised lower Gol'dberg order (briefly generalised lower G-order) of f with respect to the domain D is defined by

$$\lambda_D^{(k)}(f) = \liminf_{R\to\infty} \frac{\log^{[k]} M_{f,D}(R)}{\log R} \text{ where } k = 1, 2, 3, \ldots$$

Definition 11.1.6 *Let f and g be two entire functions of n variables and D be a bounded complete n-circular domain with centre at origin in C^n. Then the generalised relative Gol'dberg order $\rho^{(k)}_{g,D}(f)$ of f with respect to g and the domain D is defined by*

$$\rho^{(k)}_{g,D}(f) = \limsup_{R\to\infty} \frac{\log^{[k]} M^{-1}_{g,D}(M_{f,D}(R))}{\log R}, \text{ for } k = 1,2,3,...$$

Similarly, the generalised relative lower Gol'dberg order $\lambda^{(k)}_{g,D}(f)$ of f with respect to g and the domain D is defined as

$$\lambda^{(k)}_{g,D}(f) = \liminf_{R\to\infty} \frac{\log^{[k]} M^{-1}_{g,D}(M_{f,D}(R))}{\log R}, \text{ for } k = 1,2,3,...$$

In this chapter we establish some results on the comparative growth properties related to Gol'dberg order (lower Gol'dberg order) and relative Gol'dberg order (relative lower Gol'dberg order) of entire functions.

11.2 Theorems.

In this section we present the main results of this chapter.

Theorem 11.2.1 *Let f and g be two entire functions of n variables and D be a bounded complete n-circular domain with centre at origin in C^n. Also let $0 < \lambda_D(fog) \le \rho_D(fog) < \infty$ and $0 < \lambda_D(g) \le \rho_D(g) < \infty$. Then*

$$\frac{\lambda_D(fog)}{\rho_D(g)} \le \liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \le \min\{\frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)}\}$$

$$\le \max\{\frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)}\} \le \limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \le \frac{\rho_D(fog)}{\lambda_D(g)}.$$

Proof. From the definition of Gol'dberg order and lower Gol'dberg order of an entire function g we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[2]} M_{g,D}(R) \le (\rho_D(g) + \varepsilon) \log R \qquad (11.2.1)$$

$$\text{and } \log^{[2]} M_{g,D}(R) \ge (\lambda_D(g) - \varepsilon) \log R. \qquad (11.2.2)$$

Also for a sequence of values of R, tending to infinity,

$$\log^{[2]} M_{g,D}(R) \leq (\lambda_D(g) + \varepsilon) \log R \tag{11.2.3}$$

$$\text{and } \log^{[2]} M_{g,D}(R) \geq (\rho_D(g) - \varepsilon) \log R. \tag{11.2.4}$$

Again from the definition of Gol'dberg order and lower Gol'dberg order of the composite entire function fog we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[2]} M_{fog,D}(R) \leq (\rho_D(fog) + \varepsilon) \log R \tag{11.2.5}$$

$$\text{and } \log^{[2]} M_{fog,D}(R) \geq (\lambda_D(fog) - \varepsilon) \log R. \tag{11.2.6}$$

Again for a sequence of values of R tending to infinity,

$$\log^{[2]} M_{fog,D}(R) \leq (\lambda_D(fog) + \varepsilon) \log R \tag{11.2.7}$$

$$\text{and } \log^{[2]} M_{fog,D}(R) \geq (\rho_D(fog) - \varepsilon) \log R. \tag{11.2.8}$$

Now from (11.2.1) and (11.2.6) it follows for all sufficiently large values of R that

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \geq \frac{\lambda_D(fog) - \varepsilon}{\rho_D(g) + \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \geq \frac{\lambda_D(fog)}{\rho_D(g)}. \tag{11.2.9}$$

Again combining (11.2.2) and (11.2.7) we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \frac{\lambda_D(fog) + \varepsilon}{\lambda_D(g) - \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \frac{\lambda_D(fog)}{\lambda_D(g)}. \tag{11.2.10}$$

Similarly from (11.2.4) and (11.2.5) it follows for a sequence of values of R tending to infinity that

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \frac{\rho_D(fog) + \varepsilon}{\rho_D(g) - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \frac{\rho_D(fog)}{\rho_D(g)}. \qquad (11.2.11)$$

Now combining (11.2.9), (11.2.10) and (11.2.11) we get that

$$\frac{\lambda_D(fog)}{\rho_D(g)} \leq \liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \min\{\frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)}\}. \qquad (11.2.12)$$

Now from (11.2.3) and (11.2.6) we obtain for a sequence of values of R tending to infinity that

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \geq \frac{\lambda_D(fog)-\varepsilon}{\lambda_D(g)+\varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \geq \frac{\lambda_D(fog)}{\lambda_D(g)}. \qquad (11.2.13)$$

Again from (11.2.2) and (11.2.5) it follows for all sufficiently large values of R that

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \frac{\rho_D(fog)+\varepsilon}{\lambda_D(g)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(g)}. \qquad (11.2.14)$$

Similarly combining (11.2.1) and (11.2.8) we get for a sequence of values of R tending to infinity that

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \geq \frac{\rho_D(fog)-\varepsilon}{\rho_D(g)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \geq \frac{\rho_D(fog)}{\rho_D(g)}. \qquad (11.2.15)$$

Therefore combining (11.2.13), (11.2.14) and (11.2.15) we get that

$$\max\{\frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)}\} \leq \limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{g,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(g)}. \quad (11.2.16)$$

Thus the theorem follows from (11.2.12) and (11.2.16). ■

Example 11.2.1 *Considering $f = z, g = \exp z$ and $n = 1$ one can easily verify that the sign '$\leq$'in Theorem 11.2.1 cannot be replaced by '$<$' only.*

Remark 11.2.1 *If we take $0 < \lambda_D(f) \leq \rho_D(f) < \infty$ instead of $0 < \lambda_D(g) \leq \rho_D(g) < \infty$ and the other conditions remain the same then also Theorem 11.2.1 holds with g replaced by f in the denominator as we see in the next theorem.*

Theorem 11.2.2 *Let f and g be two entire functions of n variables and D be a bounded complete n-circular domain with centre at origin in C^n. Also let $0 < \lambda_D(fog) \leq \rho_D(fog) < \infty$ and $0 < \lambda_D(f) \leq \rho_D(f) < \infty$. Then*

$$\frac{\lambda_D(fog)}{\rho_D(f)} \leq \liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \min\{\frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)}\}$$

$$\leq \max\{\frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)}\} \leq \limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(f)}.$$

Proof. From the definition of Gol'dberg order and lower Gol'dberg order of an entire function f we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[2]} M_{f,D}(R) \leq (\rho_D(f) + \varepsilon) \log R \quad (11.2.17)$$

$$\text{and } \log^{[2]} M_{f,D}(R) \geq (\lambda_D(f) - \varepsilon) \log R. \quad (11.2.18)$$

Also for a sequence of values of R tending to infinity,

$$\log^{[2]} M_{f,D}(R) \leq (\lambda_D(f) + \varepsilon) \log R \quad (11.2.19)$$

$$\text{and } \log^{[2]} M_{f,D}(R) \geq (\rho_D(f) - \varepsilon) \log R. \quad (11.2.20)$$

Again from the definition of Gol'dberg order and lower Gol'dberg order of the composite entire function (or, composition of two entire functions f and

g) fog we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[2]} M_{fog,D}(R) \leq (\rho_D(fog) + \varepsilon) \log R \tag{11.2.21}$$

$$\text{and } \log^{[2]} M_{fog,D}(R) \geq (\lambda_D(fog) - \varepsilon) \log R. \tag{11.2.22}$$

Again for a sequence of values of R tending to infinity,

$$\log^{[2]} M_{fog,D}(R) \leq (\lambda_D(fog) + \varepsilon) \log R \tag{11.2.23}$$

$$\text{and } \log^{[2]} M_{fog,D}(R) \geq (\rho_D(fog) - \varepsilon) \log R\,. \tag{11.2.24}$$

Now from (11.2.17) and (11.2.22) it follows for all sufficiently large values of R that

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \geq \frac{\lambda_D(fog) - \varepsilon}{\rho_D(f) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \geq \frac{\lambda_D(fog)}{\rho_D(f)}. \tag{11.2.25}$$

Again combining (11.2.18) and (11.2.23) we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\lambda_D(fog) + \varepsilon}{\lambda_D(f) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\lambda_D(fog)}{\lambda_D(f)}. \tag{11.2.26}$$

Similarly from (11.2.20) and (11.2.21) it follows for a sequence of values of R tending to infinity that

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\rho_D(fog) + \varepsilon}{\rho_D(f) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\rho_D(f)}. \tag{11.2.27}$$

Now combining (11.2.25), (11.2.26) and (11.2.27) we get that

$$\frac{\lambda_D(fog)}{\rho_D(f)} \leq \liminf_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \min\{\frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)}\}. \quad (11.2.28)$$

Now from (11.2.19) and (11.2.22) we obtain for a sequence of values of R tending to infinity,

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \geq \frac{\lambda_D(fog) - \varepsilon}{\lambda_D(f) + \varepsilon}.$$

Choosing $\varepsilon(> 0)$ we get that

$$\limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \geq \frac{\lambda_D(fog)}{\lambda_D(f)}. \quad (11.2.29)$$

Again from (11.2.18) and (11.2.21) it follows for all sufficiently large values of R,

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\rho_D(fog) + \varepsilon}{\lambda_D(f) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(f)}. \quad (11.2.30)$$

Similarly combining (11.2.17) and (11.2.24) we get for a sequence of values of R tending to infinity

$$\frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \geq \frac{\rho_D(fog) - \varepsilon}{\rho_D(f) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \geq \frac{\rho_D(fog)}{\rho_D(f)}. \quad (11.2.31)$$

Therefore combining (11.2.29), (11.2.30) and (11.2.31) we get that

$$\max\{\frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)}\} \leq \limsup_{R\to\infty} \frac{\log^{[2]} M_{fog,D}(R)}{\log^{[2]} M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(f)}. \quad (11.2.32)$$

Thus the theorem follows from (11.2.28) and (11.2.32). ■

Example 11.2.2 *Taking $f = \exp z, g = z$ and $n = 1$ one can easily verify that the sign '$\leq$' in Theorem 11.2.2 cannot be replaced by '$<$' only.*
Extending the notion we may prove the subsequent theorems using hyper Gol'dberg order.

Theorem 11.2.3 *Let f and g be two entire functions of n variables and D be a bounded complete n-circular domain with centre at origin in C^n. Also let*

$$0 < \overline{\lambda}_D(fog) \leq \overline{\rho}_D(fog) < \infty \text{ and } 0 < \overline{\lambda}_D(g) \leq \overline{\rho}_D(g) < \infty.$$

$$\text{Then } \frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(g)} \leq \liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \min\left\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)}\right\}$$

$$\leq \max\left\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)}\right\} \leq \limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(g)}.$$

Proof. From the definition of hyper Gol'dberg order and hyper Gol'dberg lower order of an entire function g we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[3]} M_{g,D}(R) \leq (\overline{\rho}_D(g) + \varepsilon) \log R \tag{11.2.33}$$

$$\text{and } \log^{[3]} M_{g,D}(R) \geq (\overline{\lambda}_D(g) - \varepsilon) \log R. \tag{11.2.34}$$

Also for a sequence of values of R tending to infinity,

$$\log^{[3]} M_{g,D}(R) \leq (\overline{\lambda}_D(g) + \varepsilon) \log R \tag{11.2.35}$$

$$\text{and } \log^{[3]} M_{g,D}(R) \geq (\overline{\rho}_D(g) - \varepsilon) \log R. \tag{11.2.36}$$

Again from the definition of hyper Gol'dberg order and hyper lower Gol'dberg order of the composite entire function fog we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[3]} M_{fog,D}(R) \leq (\overline{\rho}_D(fog) + \varepsilon) \log R \tag{11.2.37}$$

$$\text{and } \log^{[3]} M_{fog,D}(R) \geq (\overline{\lambda}_D(fog) - \varepsilon) \log R. \tag{11.2.38}$$

Again for a sequence of values of R tending to infinity,

$$\log^{[3]} M_{fog,D}(R) \leq (\overline{\lambda}_D(fog) + \varepsilon) \log R \tag{11.2.39}$$

$$\text{and } \log^{[3]} M_{fog,D}(R) \geq (\overline{\rho}_D(fog) - \varepsilon) \log R. \tag{11.2.40}$$

Now from (11.2.33) and (11.2.38) it follows for all sufficiently large values of R that

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \geq \frac{\overline{\lambda}_D(fog) - \varepsilon}{\overline{\rho}_D(g) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \geq \frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(g)}. \tag{11.2.41}$$

Again combining (11.2.34) and (11.2.39) we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\lambda}_D(fog) + \varepsilon}{\overline{\lambda}_D(g) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}. \tag{11.2.42}$$

Similarly from (11.2.36) and (11.2.37) it follows for a sequence of values of R tending to infinity that

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\rho}_D(fog) + \varepsilon}{\overline{\rho}_D(g) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)}. \tag{11.2.43}$$

Now combining (11.2.41), (11.2.42) and (11.2.43) we get that

$$\frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(g)} \leq \liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \min\left\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)}\right\}. \tag{11.2.44}$$

Now from (11.2.35) and (11.2.38) we obtain for a sequence of values of R tending to infinity that

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \geq \frac{\overline{\lambda}_D(fog) - \varepsilon}{\overline{\lambda}_D(g) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \geq \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}. \tag{11.2.45}$$

Again from (11.2.34) and (11.2.37) it follows for all sufficiently large values of R,

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\rho}_D(fog)+\varepsilon}{\overline{\lambda}_D(g)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(g)}. \tag{11.2.46}$$

Similarly combining (11.2.33) and (11.2.40) we get for a sequence of values of R tending to infinity that

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \geq \frac{\overline{\rho}_D(fog)-\varepsilon}{\overline{\rho}_D(g)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \geq \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)}. \tag{11.2.47}$$

Therefore combining (11.2.45), (11.2.46) and (11.2.47) we get that

$$\max\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)}\} \leq \limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(g)}. \tag{11.2.48}$$

Thus the theorem follows from (11.2.44) and (11.2.48). ■

Remark 11.2.2 *If we take* $0 < \overline{\lambda}_D(f) \leq \overline{\rho}_D(f) < \infty$ *instead of* $0 < \overline{\lambda}_D(g) \leq \overline{\rho}_D(g) < \infty$ *and the other conditions remain the same then also Theorem 11.2.3 holds with g replaced by f in the denominator as we see in the next theorem.*

Example 11.2.3 *Let* $f = z, g = \exp^{[2]} z = \exp(\exp z)$ *and* $n = 1$. *Then it can be easily shown that the sign '$\leq$' in Theorem 11.2.3 cannot be replaced by '$<$' only.*

Theorem 11.2.4 *Let f and g be two entire functions of n variables and D be a bounded complete n-circular domain with centre at origin in C^n. Also let $0 < \overline{\lambda}_D(fog) \leq \overline{\rho}_D(fog) < \infty$ and $0 < \overline{\lambda}_D(f) \leq \overline{\rho}_D(f) < \infty$. Then*

$$\frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(f)} \leq \liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{g,D}(R)} \leq \min\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)}\}$$

$$\leq \max\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)}\} \leq \limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(f)}.$$

Proof. From the definition of hyper Gol'dberg order and hyper lower Gol'dberg order of an entire function f we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[3]} M_{f,D}(R) \leq (\overline{\rho}_D(f) + \varepsilon) \log R \tag{11.2.49}$$

$$\text{and } \log^{[3]} M_{f,D}(R) \geq (\overline{\lambda}_D(f) - \varepsilon) \log R. \tag{11.2.50}$$

Also for a sequence of values of R tending to infinity,

$$\log^{[3]} M_{f,D}(R) \leq (\overline{\lambda}_D(f) + \varepsilon) \log R \tag{11.2.51}$$

$$\text{and } \log^{[3]} M_{f,D}(R) \geq (\overline{\rho}_D(f) - \varepsilon) \log R. \tag{11.2.52}$$

Again from the definition of hyper Gol'dberg order and hyper lower Gol'dberg order of the composite entire function fog we have for arbitrary positive ε and for all sufficiently large values of R,

$$\log^{[3]} M_{fog,D}(R) \leq (\overline{\rho}_D(fog) + \varepsilon) \log R \tag{11.2.53}$$

$$\text{and } \log^{[3]} M_{fog,D}(R) \geq (\overline{\lambda}_D(fog) - \varepsilon) \log R. \tag{11.2.54}$$

Again for a sequence of values of R tending to infinity,

$$\log^{[3]} M_{fog,D}(R) \leq (\overline{\lambda}_D(fog) + \varepsilon) \log R \tag{11.2.55}$$

$$\text{and } \log^{[3]} M_{fog,D}(R) \geq (\overline{\rho}_D(fog) - \varepsilon) \log R. \tag{11.2.56}$$

Now from (11.2.49) and (11.2.54) it follows for all sufficiently large values of R that

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\lambda}_D(fog) - \varepsilon}{\overline{\rho}_D(f) + \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(f)}. \tag{11.2.57}$$

Again combining (11.2.50) and (11.2.55) we get for a sequence of values of R tending to infinity,

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\lambda}_D(fog)+\varepsilon}{\overline{\lambda}_D(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}. \tag{11.2.58}$$

Similarly from (11.2.52) and (11.2.53) it follows for a sequence of values of R tending to infinity that

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\rho}_D(fog)+\varepsilon}{\overline{\rho}_D(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)}. \tag{11.2.59}$$

Now combining (11.2.57), (11.2.58) and (11.2.59) we get that

$$\frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(f)} \leq \liminf_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \min\left\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)}\right\} \tag{11.2.60}$$

Now from (11.2.51) and (11.2.54) we obtain for a sequence of values of R tending to infinity,

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\lambda}_D(fog)-\varepsilon}{\overline{\lambda}_D(f)+\varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}. \tag{11.2.61}$$

Again from (11.2.50) and (11.2.53) it follows for all sufficiently large values of R,

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\rho}_D(fog)+\varepsilon}{\overline{\lambda}_D(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(f)}. \tag{11.2.62}$$

Similarly combining (11.2.49) and (11.2.56) we get for a sequence of values of R tending to infinity that

$$\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\rho}_D(fog)-\varepsilon}{\overline{\rho}_D(f)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)}. \tag{11.2.63}$$

Therefore combining (11.2.61), (11.2.62) and (11.2.63) we get that

$$\max\{\frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)}\} \leq \limsup_{R\to\infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(f)}. \tag{11.2.64}$$

Thus the theorem follows from (11.2.60) and (11.2.64). ∎

Example 11.2.4 *Considering $f = \exp^{[2]} z$, $g = z$ and $n = 1$ one can easily verify that the sign '$\leq$' in Theorem 11.2.4 cannot be replaced by '$<$'only.*

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