Chapter 9

RELATIVE L-RITT ORDER AND RELATED COMPARATIVE GROWTH PROPERTIES OF ENTIRE DIRICHLET SERIES

CHAPTER

9

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9.1 Introduction, Definitions and Notations.

During the past decades, several authors {cf. [54], [57], [58] and[61]} made close investigations on the properties of entire Dirichlet series related to Ritt order. Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by everywhere absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$...(9.1.$A$) where $0 < \lambda_n < \lambda_{n+1} (n \geq 1)$, $\lambda_n \to \infty$ as $n \to \infty$ and a_n's are complex constants. If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (9.1.A) then in this clearly $\sigma_c = \sigma_a = \infty$.

$$\text{Let } F(\sigma) = \underset{-\infty<t<\infty}{lub} |f(\sigma + it)|.$$

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [**26**].

Then the Ritt order[59] of $f(s)$ denoted by $\rho(f)$ is given by

$$\rho(f) = \limsup_{\sigma\to\infty} \frac{\log\log F(\sigma)}{\sigma} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma}.$$

In other words

$$\rho(f) = \inf\{\mu > 0 : \log F(\sigma) < \exp(\sigma\mu) \text{ for all } \sigma > R(\mu)\}.$$

Similarly the lower Ritt order of $f(s)$ denoted by $\lambda(f)$ may be defined. In this chapter we prove some results on the comparative growth properties related to the L-Ritt order of entire Dirichlet series where $L \equiv L(\sigma)$ is a positive continuous function increasing slowly i.e. $L(a\sigma) \sim L(\sigma)$ as $\sigma \to \infty$ for every constant 'a'. The following definitions are well known.

Definition 9.1.1 *The L-Ritt order $\rho_f^L \equiv \rho^L(f)$ and the L-Ritt lower order (or equivalently lower L-Ritt order) $\lambda_f^L \equiv \lambda^L(f)$ of $f(s)$ are defined as follows respectively*

$$\rho^L(f) = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma L(\sigma)} \text{ and } \lambda^L(f) = \liminf_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma L(\sigma)}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$. Similarly one can define the relative L-Ritt order and relative lower L-Ritt order of $f(s)$.

Definition 9.1.2 *The relative L-Ritt order $\rho_g^L(f)$ and the relative lower L-Ritt order $\lambda_g^L(f)$ of $f(s)$ with respect to entire $g(s)$ are respectively defined as*

$$\rho_g^L(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma L(\sigma)} \text{ and } \lambda_g^L(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma L(\sigma)}.$$

Analogously one can define the following.

Definition 9.1.3 *The hyper L-Ritt order $\overline{\rho}_f^L \equiv \overline{\rho}^L(f)$ and the hyper L-Ritt lower order (or equivalently hyper lower L-Ritt order) $\overline{\lambda}_f^L \equiv \lambda^L(f)$ of $f(s)$ are defined respectively as follows*

$$\overline{\rho}^L(f) = \limsup_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma L(\sigma)} \text{ and } \overline{\lambda}^L(f) = \liminf_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma L(\sigma)}.$$

Definition 9.1.4 *The relative hyper L-Ritt order* $\overline{\rho}_g^L(f)$ *and the relative hyper lower L-Ritt order* $\overline{\lambda}_g^L(f)$ *of* $f(s)$ *with respect to entire* $g(s)$ *are respectively defined as*

$$\overline{\rho}_g^L(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma L(\sigma)} \text{ and } \overline{\lambda}_g^L(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma L(\sigma)}.$$

The more generalised concept of L-Ritt order (lower L-Ritt order) and relative L-Ritt order (relative lower L-Ritt order) are respectively L^**-Ritt order (lower* L^**-Ritt order) and relative* L^**-Ritt order (relative lower* L^**-Ritt order). We may now state the following definitions.*

Definition 9.1.5 *The* L^**-Ritt order* $\rho_f^{L^*} \equiv \rho^{L^*}(f)$ *and the* L^**-Ritt lower order (or equivalently lower* L^**-Ritt order)* $\lambda_f^{L^*} \equiv \lambda^{L^*}(f)$ *of* $f(s)$ *are defined respectively as follows*

$$\rho_f^{L^*} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \lambda_f^{L^*} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)}.$$

Definition 9.1.6 *The relative* L^**-Ritt order* $\rho_g^{L^*}(f)$ *and the relative lower* L^**-Ritt order* $\lambda_g^{L^*}(f)$ *of* $f(s)$ *with respect to entire* $g(s)$ *are respectively defined as*

$$\rho_g^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \lambda_g^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma \exp L(\sigma)}.$$

Definition 9.1.7 *The hyper* L^**-Ritt order* $\overline{\rho}_f^{L^*} \equiv \overline{\rho}^{L^*}(f)$ *and the hyper* L^**-Ritt lower order (hyper lower* L^**-Ritt order)* $\overline{\lambda}_f^{L^*} \equiv \lambda^{L^*}(f)$ *of* $f(s)$ *are defined respectively as follows*

$$\overline{\rho}^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \lambda^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{\log^{[3]} F(\sigma)}{\sigma \exp L(\sigma)}.$$

Definition 9.1.8 *The relative hyper* L^**-Ritt order* $\overline{\rho}_g^{L^*}(f)$ *and the relative hyper lower* L^**-Ritt order* $\overline{\lambda}_g^{L^*}(f)$ *of* $f(s)$ *with respect to entire* $g(s)$ *are respectively defined as*

$$\overline{\rho}_g^{L^*}(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } \overline{\lambda}_g^{L^*}(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma \exp L(\sigma)}.$$

Generalising our notion we may state the following definitions.

Definition 9.1.9 *The generalised L-Ritt order* ${}^{(k)}\rho_f^L \equiv {}^{(k)}\rho^L(f)$ *and the generalised L-Ritt lower order (generalised lower L-Ritt order)* ${}^{(k)}\lambda_f^L \equiv {}^{(k)}\lambda^L(f)$ *are defined respectively as follows.*

$$ {}^{(k)}\rho^L(f) = \limsup_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma L(\sigma)} \text{ and } {}^{(k)}\lambda^L(f) = \liminf_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma L(\sigma)} $$

where $k = 2, 3, \ldots$

Definition 9.1.10 *The generalised* L^**-Ritt order* ${}^{(k)}\rho_f^{L^*} \equiv {}^{(k)}\rho^{L^*}(f)$ *and the generalised* L^**-Ritt lower order (or equivalently generalised lower* L^**-Ritt order)* ${}^{(k)}\lambda_f^L \equiv {}^{(k)}\lambda^L(f)$ *are respectively defined as*

$$ {}^{(k)}\rho^{L^*}(f) = \limsup_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } {}^{(k)}\lambda^{L^*}(f) = \liminf_{\sigma\to\infty}\frac{\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} $$

where $k = 2, 3, \ldots$

Definition 9.1.11 *The generalised relative L-Ritt order* ${}^{(k)}\rho_g^L(f)$ *and the generalised relative lower L-Ritt order* ${}^{(k)}\lambda_g^L(f)$ *of* $f(s)$ *with respect to entire* $g(s)$ *are respectively defined as*

$$ {}^{(k)}\rho_g^L(f) = \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma L(\sigma)} \text{ and } {}^{(k)}\lambda_g^L(f) = \liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma L(\sigma)} $$

where $k = 1, 2, 3, \ldots$

Definition 9.1.12 *The generalised relative* L^**-Ritt order* ${}^{(k)}\rho_g^{L^*}(f)$ *and the generalised relative lower* L^**-Ritt order* ${}^{(k)}\lambda_g^{L^*}(f)$ *of* $f(s)$ *with respect to* $g(s)$ *are respectively defined as follows*

$$ {}^{(k)}\rho_g^{L^*}(f) = \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} \text{ and } {}^{(k)}\lambda_g^{L^*}(f) = \liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma \exp L(\sigma)} $$

where $k = 1, 2, 3, \ldots$

9.2 Theorems.

In this section we present the main results of this chapter.

Theorem 9.2.1 *Let f and g be two entire functions such that* $0 < {}^{(k)}\lambda^{L^*}(fog) \leq {}^{(k)}\rho^{L^*}(fog) < \infty$ *and* $0 < {}^{(k)}\lambda^{L^*}(g) \leq {}^{(k)}\rho^{L^*}(g) < \infty$. *Then*

$$\begin{aligned}
\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)} &\leq \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \\
&\leq \min\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}\} \\
&\leq \max\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}\} \\
&\leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}
\end{aligned}$$

where $k = 1, 2, 3, \ldots$

Proof. From the definition of generalised L^*-Ritt order and generalised L^*-lower Ritt order of entire g we have for arbitrary positive ε and for all sufficiently large values of σ,

$$\log^{[k]} G(\sigma) \leq ({}^{(k)}\rho^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.1}$$

$$\text{and } \log^{[k]} G(\sigma) \geq ({}^{(k)}\lambda^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.2}$$

Also for a sequence of values of σ tending to infinity,

$$\log^{[k]} G(\sigma) \leq ({}^{(k)}\lambda^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.3}$$

$$\text{and } \log^{[k]} G(\sigma) \geq ({}^{(k)}\rho^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.4}$$

Now again from the definition of generalised L^*-Ritt order and generalised L^*-lower Ritt order of the composite function fog we have for arbitrary positive ε and for all sufficiently large values of σ,

$$\log^{[k]} FoG(\sigma) \leq ({}^{(k)}\rho^{L^*}(fog) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.5}$$

$$\text{and } \log^{[k]} FoG(\sigma) \geq ({}^{(k)}\lambda^{L^*}(fog) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.6}$$

Again for a sequence of values of σ tending to infinity

$$\log^{[k]} FoG(\sigma) \leq ({}^{(k)}\lambda^{L^*}(fog) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.7}$$

$$\text{and } \log^{[k]} FoG(\sigma) \geq ({}^{(k)}\rho^{L^*}(fog) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.8}$$

Now from (9.2.1) and (9.2.6) it follows for all sufficiently large values of σ,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog) - \varepsilon}{{}^{(k)}\rho^{L^*}(g) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma \to \infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}. \tag{9.2.9}$$

Again combining (9.2.2) and (9.2.7) we get for a sequence of values of σ tending to infinity,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\lambda^{L^*}(fog) + \varepsilon}{{}^{(k)}\lambda^{L^*}(g) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{\sigma \to \infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.10}$$

Similarly from (9.2.4) and (9.2.5) it follows for a sequence of values of σ tending to infinity that

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog) + \varepsilon}{{}^{(k)}\rho^{L^*}(g) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma \to \infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}. \tag{9.2.11}$$

Now combining (9.2.9), (9.2.10) and (9.2.11) we get that

$$\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)}$$

$$\leq \min\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}\}. \tag{9.2.12}$$

Now from (9.2.3) and (9.2.6) we obtain for a sequence of values of σ tending to infinity,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog) - \varepsilon}{{}^{(k)}\lambda^{L^*}(g) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.13}$$

Again from (9.2.2) and (9.2.5) it follows for all sufficiently large values of σ,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog) + \varepsilon}{{}^{(k)}\lambda^{L^*}(g) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.14}$$

Similarly combining (9.2.1) and (9.2.8) we get for a sequence of values of σ tending to infinity,

$$\frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\rho^{L^*}(fog) - \varepsilon}{{}^{(k)}\rho^{L^*}(g) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \geq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(g)}. \tag{9.2.15}$$

Therefore combining (9.2.13), (9.2.14) and (9.2.15) we get that

$$\max\left\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}\right\} \leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(g)}. \tag{9.2.16}$$

Thus the theorem follows from (9.2.12) and (9.2.16). ■

Remark 9.2.1 *If we take* $0 < {}^{(k)}\lambda^{L^*}(f) \leq {}^{(k)}\rho^{L^*}(f) < \infty$ *instead of* $0 < {}^{(k)}\lambda^{L^*}(g) \leq {}^{(k)}\rho^{L^*}(g) < \infty$ *and the other conditions remain the same then also Theorem 9.2.1 holds with* g *replaced by* f *in the denominator as we see in the next theorem.*

Theorem 9.2.2 *Let f and g be two entire functions such that $0 < {}^{(k)}\lambda^{L^*}(fog) \leq {}^{(k)}\rho^{L^*}(fog) < \infty$ and $0 < {}^{(k)}\lambda^{L^*}(f) \leq {}^{(k)}\rho^{L^*}(f) < \infty$. Then*

$$\begin{aligned}\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(f)} &\leq \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)} \\ &\leq \min\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(f)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(f)}\} \\ &\leq \max\{\frac{{}^{(k)}\lambda^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(f)}, \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\rho^{L^*}(f)}\} \\ &\leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho^{L^*}(fog)}{{}^{(k)}\lambda^{L^*}(f)}\end{aligned}$$

where $k = 1, 2, 3, \ldots$
In fact, Theorem 9.2.1 and Theorem 9.2.2 are the more generalised concept of Theorem 9.2.3 and Theorem 9.2.4 respectively.

Theorem 9.2.3 *Let f and g be two entire functions such that $0 < {}^{(k)}\lambda^{L}(fog) \leq {}^{(k)}\rho^{L}(fog) < \infty$ and $0 < {}^{(k)}\lambda^{L}(g) \leq {}^{(k)}\rho^{L^*}(g) < \infty$. Then*

$$\begin{aligned}\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\rho^{L}(g)} &\leq \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \\ &\leq \min\{\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\lambda^{L}(g)}, \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\rho^{L}(g)}\} \\ &\leq \max\{\frac{{}^{(k)}\lambda^{L}(fog)}{{}^{(k)}\lambda^{L}(g)}, \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\rho^{L}(g)}\} \\ &\leq \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} G(\sigma)} \leq \frac{{}^{(k)}\rho^{L}(fog)}{{}^{(k)}\lambda^{L}(g)}\end{aligned}$$

where $k = 1, 2, 3, \ldots$
The proof is omitted.

Theorem 9.2.4 *Let f and g be two entire functions such that $0 < {}^{(k)}\lambda^{L}(fog) \leq$*

${}^{(k)}\rho^L(fog) < \infty$ *and* $0 < {}^{(k)}\lambda^L(f) \le {}^{(k)}\rho^L(f) < \infty$. *Then*

$$\frac{{}^{(k)}\lambda^L(fog)}{{}^{(k)}\rho^L(f)} \le \liminf_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)}$$
$$\le \min\{\frac{{}^{(k)}\lambda^L(fog)}{{}^{(k)}\lambda^L(f)}, \frac{{}^{(k)}\rho^L(fog)}{{}^{(k)}\rho^L(f)}\}$$
$$\le \max\{\frac{{}^{(k)}\lambda^L(fog)}{{}^{(k)}\lambda^L(f)}, \frac{{}^{(k)}\rho^L(fog)}{{}^{(k)}\rho^L(f)}\}$$
$$\le \limsup_{\sigma\to\infty} \frac{\log^{[k]} FoG(\sigma)}{\log^{[k]} F(\sigma)} \le \frac{{}^{(k)}\rho^L(fog)}{{}^{(k)}\lambda^L(f)}$$

where $k = 1, 2, 3, \dots$
The proof is omitted.
Lahiri and Banerjee [54] studied on relative Ritt order of entire Dirichlet series and proved some basic theorems. In the subsequent theorems we prove something more.

Theorem 9.2.5 *Let f, g and h be three entire functions with* $0 < {}^{(k)}\lambda_h(f) \le {}^{(k)}\rho_h(f) < \infty$ *and* $0 < {}^{(k)}\lambda_h(g) \le {}^{(k)}\rho_h(g) < \infty$. *Then*

$$(i)\ \liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \le \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)} \le \limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)}$$

$$\text{and } (ii)\ \liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \le \min\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}, \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}$$
$$\le \max\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}, \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}$$
$$\le \limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)}.$$

Proof. From the definition of generalised relative L^*-Ritt order and generalised relative L^*-lower Ritt order of entire g with respect to entire h we have for arbitrary positive ε and for all sufficiently large values of σ,

$$H^{-1}\log^{[k]} G(\sigma) \le ({}^{(k)}\rho_h^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.17}$$

$$\text{and } H^{-1}\log^{[k]} G(\sigma) \ge ({}^{(k)}\lambda_h^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.18}$$

Also for a sequence of values of σ tending to infinity,

$$H^{-1}\log^{[k]} G(\sigma) \leq ({}^{(k)}\lambda_h^{L^*}(g) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.19}$$

$$\text{and } H^{-1}\log^{[k]} G(\sigma) \geq ({}^{(k)}\rho_h^{L^*}(g) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.20}$$

Now again from the definition of generalised relative L^*-Ritt order and generalised relative L^*-lower Ritt order of entire f with respect to entire h we have for arbitrary positive ε and for all sufficiently large values of σ,

$$H^{-1}\log^{[k]} F(\sigma) \leq ({}^{(k)}\rho_h^{L^*}(f) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.21}$$

$$\text{and } H^{-1}\log^{[k]} F(\sigma) \geq ({}^{(k)}\lambda_h^{L^*}(f) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.22}$$

Also for a sequence of values of σ tending to infinity,

$$H^{-1}\log^{[k]} F(\sigma) \leq ({}^{(k)}\lambda_h^{L^*}(f) + \varepsilon)\sigma \exp L(\sigma) \tag{9.2.23}$$

$$\text{and } H^{-1}\log^{[k]} F(\sigma) \geq ({}^{(k)}\rho_h^{L^*}(f) - \varepsilon)\sigma \exp L(\sigma). \tag{9.2.24}$$

Now from (9.2.17) and (9.2.22) it follows for all sufficiently large values of σ,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g) + \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above that

$$\liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.25}$$

Again combining (9.2.18) and (9.2.23) we get for a sequence of values of σ tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g) - \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) + \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.26}$$

Similarly from (9.2.20) and (9.2.21) it follows for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\rho_h^{L^*}(g) - \varepsilon}{{}^{(k)}\rho_h^{L^*}(f) + \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}. \tag{9.2.27}$$

Now from (9.2.19) and (9.2.22) we obtain for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g) + \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) - \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.28}$$

Again from (9.2.18) and (9.2.23) it follows for all sufficiently large values of r,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g) - \varepsilon}{{}^{(k)}\lambda_h^{L^*}(f) + \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \geq \frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}. \tag{9.2.29}$$

Similarly combining (9.2.17) and (9.2.24) we get for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g) + \varepsilon}{{}^{(k)}\rho_h^{L^*}(f) - \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[k]} G(\sigma)}{H^{-1}\log^{[k]} F(\sigma)} \leq \frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}. \tag{9.2.30}$$

Combining (9.2.28) and (9.2.29) we obtain that

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}\le\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)}\le\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}.$$

This proves the first part of the theorem. Again combining (9.2.26) and (9.2.27) it follows that

$$\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}\ge\max\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)},\frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}. \tag{9.2.31}$$

Now combining (9.2.28) and (9.2.30) we get that

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}G(\sigma)}{H^{-1}\log^{[k]}F(\sigma)}\le\min\{\frac{{}^{(k)}\lambda_h^{L^*}(g)}{{}^{(k)}\lambda_h^{L^*}(f)},\frac{{}^{(k)}\rho_h^{L^*}(g)}{{}^{(k)}\rho_h^{L^*}(f)}\}. \tag{9.2.32}$$

Thus from (9.2.31) and (9.2.32) the second part of the theorem follows. ■

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