CHAPTER 1

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Let f be a function of a complex variable defined in the open complex plane $\mathbb{C}$. Then the function f is said to be analytic at a point z_0 if there exists a neighbourhood of z_0 at all points of which $f'(z)$ exists. If f is not analytic at a point z_0 then z_0 is called a singular point or a singularity of f. Now if f be a single valued analytic function on an annulus $D : r_2 < |z - \alpha| < r_1$ then at each point $z \in D$, f can be represented by a series of the form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n + \sum_{n=1}^{\infty} b_n (z - \alpha)^{-n}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz,$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{-n+1}} dz.$$

The above series is called the Laurent's series of f about the point $z = \alpha$. The part $\sum_{n=0}^{\infty} a_n (z - \alpha)^n$ is called the analytic part and the part $\sum_{n=1}^{\infty} b_n (z - \alpha)^{-n}$ is called the principal part of f at $z = \alpha$. If the principal part of Laurent series is terminating then $z = \alpha$ is called a pole of f

and when non-terminating then $z = \alpha$ is called an essential singularity of f.

A function f is said to be an entire or an integral function if it is analytic everywhere in the finite complex plane. The Taylor series expansion (whose generalisation is in fact the Laurent's series expansion) of f about $z = 0$ is given by

$$f = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

which can be considered as an extension of a polynomial. The rate of growth of a polynomial is estimated by the degree of the polynomial, which is equal to the number of zeros, as independent variable moves without bound.

The maximum modulus function of an entire function f on $|z| = r$ is defined as $M(r, f) = \max_{|z|=r} |f(z)|$ which is especially used to characterise the growth of an entire function and the distribution of its zeros. Also $M(r, f)$ is unbounded for any non-constant entire function and by maximum modulus theorem $M(r, f)$ increases monotonically as r increases. The function $\log M(r, f)$ is a continuous, convex and ultimately increasing function of $\log r$ [p.20, [41]].

The order of an entire function f is given by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.$$

If $0 \leq \rho < \infty$ then f is said to be of finite order. Also the number

$$\lambda = \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r}$$

is known as lower order of f. Obviously $\lambda \leq \rho$. The function is said to be of regular growth if $\lambda = \rho$. The number τ given by

$$\tau = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\rho}, 0 < \rho < \infty$$

is called the type of f. By the higher type one function can be characterised to be of faster growth in between two functions of same order.

The quantities ρ, λ and τ are extensively used in the study of growth properties of f. It is well-known that the order and type of an entire function f is equal to those of its derivative f' {Theorem 2.4.1, p.13, [1]}. Let f be an entire function of finite order ρ. When more precise specification of the rate of growth of f is desired, one can use the proximate order, a function $\rho(r)$ with the following properties {p.64, [41]}:

(i) $\rho(r)$ is continuous for $r > r_0$, say.

(ii) $\rho(r)$ is differentiable in adjacent intervals.

(iii) $\limsup\limits_{r\to\infty} \rho(r) = \rho$ and $\liminf\limits_{r\to\infty} \rho(r) \geq \beta$, where $0 \leq \beta \leq \rho$,

(iv) $\lim\limits_{r\to\infty} r\rho'(r)\log r = 0$

and (v) $\limsup\limits_{r\to\infty} \dfrac{\log M(r)}{r^{\rho(r)}} = 1$.

Using some results of Blumenthal , Valiron {p.64-67, [41]} proved the existence of a proximate order for an entire function of finite order. Shah [33] introduced the notion of lower proximate order for an entire function in the following way and proved its existence:

Let f be an entire function of finite lower order λ. The function $\lambda(r)$ is called a lower proximate order of f if it satisfies the following properties:

(i) $\lambda(r)$ is a non-negative continuous function of r for $r > r_0$, say,

(ii) $\lambda(r)$ is differentiable for $r > r_0$ except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist,

(iii) $\lim\limits_{r\to\infty} r\lambda'(r)\log r = 0$,

(iv) $\lim\limits_{r\to\infty} \lambda(r) = \lambda$

and (v) $\liminf\limits_{r\to\infty} \dfrac{\log M(r)}{r^{\lambda(r)}} = 1$.

Using the notion of proximate order and lower proximate order it is sometimes possible to make sharper estimation of the number of zeros of an entire function f within the circle $|z| = r$.

A function in the open complex plane $\mathbb{C}$ is called a meromorphic function if the only possible singularities in $\mathbb{C}$ are poles. The meromorphic function having an essential singularity at the point at infinity is called a transcendental meromorphic function. Now we discuss the value distribution theory of meromorphic functions developed by Rolf Nevanlinna [31] where he introduced his first and second fundamental theorem which are of frequent use.

Let f be a non-constant meromorphic function defined in the finite complex plane. For any complex number a, finite or infinite, $n(r, a; f) = n\left(r, \frac{1}{f-a}\right)$ denotes the number of roots, counted with proper multiplicities, of the equation $f = a$ in $|z| \leq r$ and $n(r, \infty; f) = n(r, f)$ denotes the number of poles of f in $|z| \leq r$. The quantities $n(r, a; f)$ and $n(r, f)$ are non-negative integers for each r. The quantity $N(r, a; f)$ is defined by

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r$$

and $N(r, \infty; f) = N(r, f)$.

We define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta$$

where

$$\begin{aligned} \log^+ x &= \log x, \text{ if } x \geq 1 \\ &= 0, \quad \text{ if } 0 \leq x < 1. \end{aligned}$$

The term $m(r, f)$ is called the proximity function of f and it is a sort of averaged magnitude of $\log |f(z)|$ on arcs of $|z| = r$ where $|f|$ is large. We write

$$T(r, f) = m(r, f) + N(r, f).$$

The function $T(r, f)$ is called the Nevanlinna's characteristic function of f and it plays a significant role in the theory of meromorphic functions. Note that for a meromorphic function f, $T(r, f)$ is a convex increasing function of $\log r$ [p.9, [24]]. If f is an entire function then $N(r, f) = 0$ and so $T(r, f) = m(r, f)$.

We note that if $a_1, a_2, \cdots, a_p$ are any complex numbers then

$$\log^+ \left| \sum_{\nu=1}^{p} a_\nu \right| \leq \sum_{\nu=1}^{p} \log^+ |a_\nu| + \log p$$

and

$$\log^+ \left| \prod_{\nu=1}^{p} a_\nu \right| \leq \sum_{\nu=1}^{p} \log^+ |a_\nu| .$$

With these inequalities and using the definition of $m(r, f)$ we obtain for p meromorphic functions $f_1, f_2, \cdots, f_p$ that

$$m\left(r, \sum_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} m(r, f_\nu) + \log p$$

and

$$m\left(r, \prod_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} m(r, f_\nu) .$$

Also

$$N\left(r, \sum_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} N(r, f_\nu)$$

and

$$N\left(r, \prod_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} N(r, f_\nu) .$$

Using the definition of $T(r, f)$ we get that

$$T\left(r, \sum_{\nu=1}^{p} f_\nu\right) \leq \sum_{\nu=1}^{p} T(r, f_\nu) + \log p$$

and

$$T\left(r,\prod_{\nu=1}^{p}f_{\nu}\right)\leq\sum_{\nu=1}^{p}T\left(r,f_{\nu}\right).$$

We now state Nevanlinna's first fundamental theorem.

Theorem 1.0.1. (p.5, [24]) If a is any complex number then

$$m\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-a}\right)=T\left(r,f\right)-\log\left|f\left(0\right)-a\right|+\varepsilon\left(a,r\right)$$

where

$$\left|\varepsilon\left(a,r\right)\right|\leq\log^{+}\left|a\right|+\log 2.$$

If we allow r to vary, then the first fundamental theorem can be written simply as (p.6, [24])

$$m\left(r,a;f\right)+N\left(r,a;f\right)=T\left(r,f\right)+O\left(1\right).$$

where a is any complex number, finite or infinite.

The term $m\left(r,a;f\right)$ refers to the average smallness in a certain sense of $f-a$ on the circle $|z|=r$. The second term $N\left(r,a;f\right)$ signifies the density of the average distribution of the roots of the equation $f=a$ in $|z|<r$. For any complex number a, the sum of $m\left(r,a;f\right)$ and $N\left(r,a;f\right)$ which is given by $T\left(r,f\right)$ is the same apart from a bounded term.

Now we state the following definition.

Definition 1.0.1. (p.16, [24]) Let $S\left(r\right)$ be a real and non-negative function increasing for $r_0<r<\infty$ where $r_0>0$. The order k and lower order λ of the function $S\left(r\right)$ are defined as

$$k=\limsup_{r\to\infty}\frac{\log S\left(r\right)}{\log r}\text{ and }\lambda=\liminf_{r\to\infty}\frac{\log S\left(r\right)}{\log r}.$$

The order k and the lower order λ of a function always satisfy the relation $0\leq\lambda\leq k\leq\infty$.

If $0 < k < \infty$ we set

$$C = \limsup_{r \to \infty} \frac{S(r)}{r^k}$$

and distinguish the following possibilities

(a) $S(r)$ has maximal type if $C = +\infty$

(b) $S(r)$ has mean type if $0 < C < +\infty$

(c) $S(r)$ has minimal type if $C = 0$

(d) $S(r)$ has convergence class if

$$\int_{r_0}^{\infty} \frac{S(t)}{t^{k+1}} dt \text{ converges.}$$

Note that if $S(r)$ is of order k, where $0 < k < \infty$ and $\varepsilon > 0$ then

$$\begin{aligned} S(r) &< r^{k+\varepsilon} \text{ for all large } r, \\ \text{and } S(r) &> r^{k-\varepsilon} \text{ for some large } r. \end{aligned}$$

We next state the following fundamental inequality between $T(r, f)$ and $M(r, f)$ when f is regular in $|z| \leq R$.

Theorem 1.0.2. (p.18, [24]) If f is regular in $|z| \leq R$ and $M(r, f) = \max_{|z|=r} |f(z)|$, then

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f), \; 0 \leq r < R.$$

We deduce immediately the following:

Theorem 1.0.3. (p.18, [24]) If f is an entire function then the order k of the functions $S_1(r) = \log^+ M(r, f)$ and $S_2(r) = T(r, f)$ are the same. Further if $0 < k < \infty$, $S_1(r)$ and $S_2(r)$ belong to the same classes (a), (b), (c) or (d). Also lower order of $S_1(r)$ and $S_2(r)$ are same.

In the sequel we can say that a meromorphic function f has order ρ, lower order λ and maximal, minimal, mean type or convergence class if the function $T(r,f)$ has this property. For entire function these coincide by the above theorem with the corresponding definition in terms of $\log^{+} M(r,f)$ which is classical. If ρ is the positive finite order of f, the quantity τ is defined by

$$\tau = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho}}$$

is called the type of f.

The notion of order and type of an entire function and of a meromorphic function has been generalised by a number of authors in various directions, some of which may be seen in [2], [3], [22] and [34].

Nevanlinna's second fundamental theorem is the following:

Theorem 1.0.4. (p.31, [24]) Suppose that f is a non-constant meromorphic function in $|z| \le r$. Let $a_1, a_2, \cdots, a_q$ where $q > 2$, be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| \ge \delta$ for $1 \le \mu < \nu \le q$. Then

$$m(r,\infty;f) + \sum_{\nu=1}^{q} m(r,a_\nu;f) \le 2T(r,f) - N_1(r) + S(r,f),$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r,f) - N(r,f').$$

and

$$S(r,f) = m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{\nu=1}^{q} \frac{f'}{(f-a_\nu)}\right) - q\log^{+}\frac{3q}{\delta} + \log 2 + \log\frac{1}{|f'(0)|}$$

with modifications if $f(0) = \infty$ or $f'(0) = 0$.

The second fundamental theorem is a consequence of the following theorem by which we can estimate $S(r,f)$.

Theorem 1.0.5. ($p.34$, [24]) Suppose that f is meromorphic and not constant in $|z| < R_0 \leq +\infty$ and that $S(r, f)$ is defined as in the above theorem. Then we have

(i) If $R_0 = +\infty$,

$$S(r, f) = O\{\log T(r, f)\} + O(\log r),$$

as $r \to \infty$ through all values if f has finite order and as $r \to \infty$ outside a set E of finite linear measure otherwise.

(ii) If $0 < R_0 < +\infty$,

$$S(r, f) = O\left\{\log^+ T(r, f) + \log \frac{1}{R_0 - r}\right\}$$

as $r \to R_0$ outside a set E such that

$$\int_E \frac{dr}{R_0 - r} < +\infty.$$

Further there is a point r outside E for which $\rho < r < \rho'$ provided that $0 < R - \rho' < e^{-2}(R - \rho)$.

The next theorem is an immediate consequence of the above theorem.

Theorem 1.0.6. ($p.41$, [24]) Let f be a non-constant meromorphic function defined in $|z| < R_0$. Then

$$\frac{S(r, f)}{T(r, f)} \to 0 \text{ as } r \to R_0 \tag{1.1}$$

with the following provisions:

(a) (1.1) holds without restrictions if $R_0 = +\infty$ and f is meromorphic of finite order in the plane.

(b) If f has infinite order in the plane (1.1) still holds as $r \to \infty$ outside a certain exceptional set E_0 of finite length. Here E_0 depends on f but not on the a_ν or on q.

(c) If $R_0 < +\infty$ and

$$\limsup_{r \to R_0} \frac{T(r, f)}{\log\left\{\frac{1}{R_0 - r}\right\}} = +\infty, \tag{1.2}$$

then (1.1) holds as $r \to R_0$ through a suitable sequence r_n, which depends on f but not on the a_ν or on q.

In general the quantity $S(r, f)$ plays the role of an unimportant error term which can be designated by the above theorem. In the sequel we shall say that f is admissible in $|z| < R_0$ if either $R_0 < +\infty$ and (1.2) holds, or if $R_0 = +\infty$ and f is non-constant. With this (1.1) holds, at least as $r \to R_0$ through a suitable sequence of values, and only then the second fundamental theorem can be used effectively.

For any non-constant meromorphic function f, by $S(r, f)$ we shall denote any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set E of finite linear measure otherwise.

In the consequence of Nevanlinna's second fundamental theorem we shall now discuss the Nevanlinna's theory of deficient values. The quantity $N(r, a; f)$ is already defined in the form of $n(r, a; f)$ where $n(r, a; f)$ denotes the number of roots of the equation $f = a$ in $|z| \leq r$, multiple roots being counted with proper multiplicities. Similarly $\bar{n}(r, a; f)$ denotes the number of distinct roots of $f = a$ in $|z| \leq r$ and $\bar{N}(r, a; f)$ is defined as

$$\bar{N}(r, a; f) = \int_0^r \frac{\bar{n}(t, a; f) - \bar{n}(0, a; f)}{t} dt + \bar{n}(0, a; f) \log r.$$

Now we define the following quantities.

Definition 1.0.2. (*p*.42, [24]) We set

$$\begin{aligned} \delta(a; f) &= 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}, \\ \Theta(a; f) &= 1 - \limsup_{r \to \infty} \frac{\bar{N}(r, a; f)}{T(r, f)} \\ \text{and } \theta(a; f) &= \liminf_{r \to \infty} \frac{N(r, a; f) - \bar{N}(r, a; f)}{T(r, f)} \end{aligned}$$

where a is any complex number.

Evidently, for any arbitrary small number $\varepsilon\,(>0)$, we have for all sufficiently large values of r from the above definition

$$\begin{aligned} N(r,a;f)-\bar{N}(r,a;f) &> \{\theta(a;f)-\varepsilon\}\,T(r,f)\,. \\ N(r,a;f) &< \{1-\delta(a;f)+\varepsilon\}\,T(r,f) \end{aligned}$$

and hence

$$\bar{N}(r,a;f)<\{1-\delta(a;f)-\theta(a;f)+2\varepsilon\}\,T(r,f)$$

so that

$$\Theta(a;f)\geq\delta(a;f)+\theta(a;f)\,.$$

The quantity $\delta(a;f)$ is called the deficiency of the value a and $\Theta(a;f)$ and $\theta(a;f)$ are called the ramification index and the index of multiplicity respectively. Clearly $\delta(a;f)$ is positive only if there are relatively few roots of the equation $f=a$ while $\theta(a;f)$ is positive if there are relatively many multiple roots.

In general $0\leq m(r,a;f)\leq T(r,a;f)=T(r,f)+O(1)$, so we have $0\leq\delta(a;f)\leq 1$. Maximum value of $\delta(a;f)$ is 1 when the roots of the equation $f=a$ are very sparsely distributed, in particular when the value a is a Picard's exceptional value.

Now we state Nevanlinna's theorem on deficient values.

Theorem 1.0.7. (*p.*43, [24]) Let f be a meromorphic function admissible in $|z|<R_0$. Then the set of values a for which $\Theta(a;f)>0$ is countable and on summing over all such values a we have

$$\sum_a\{\delta(a;f)+\theta(a;f)\}\leq\sum_a\Theta(a;f)\leq 2.$$

So Picard's theorem, that a non-constant meromorphic function can have at most two Picard's exceptional values, follows easily from Nevanlinna's deficient value theorem because for a Picard's exceptional value a, $\delta(a;f)=1$.

The following result of Nevanlinna is very useful in the study of deficient values.

Theorem 1.0.8. (p.47, [24]) If f is meromorphic on the plane and a_1, a_2, a_3 are distinct meromorphic functions satisfying for $\nu = 1, 2$ and 3

$$T(r, a_\nu) = o\{T(r, f)\} \text{ as } r \to \infty,$$

$$\text{then} \quad \{1 + o(1)\} T(r, f) \leq \sum_{\nu=1}^{3} \bar{N}\left(r, \frac{1}{f - a_\nu}\right) + S(r, f) \text{ as } r \to \infty.$$

To estimate the distribution of the values of meromorphic functions and their derivatives, a basic role will be played by the following theorem of Milloux.

Theorem 1.0.9. (p.55, [24]) Let l be a positive integer and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$, where a_ν are meromorphic functions such that $T(r, a_\nu) = S(r, f)$ as $r \to \infty$ for $\nu = 0, 1, 2, \cdots, l$. Then

$$m\left(r, \frac{\psi}{f}\right) = S(r, f)$$

$$\text{and} \quad T(r, \psi) \leq (l + 1) T(r, f) + S(r, f).$$

It was proved by Milloux that in the second fundamental theorem we can replace the counting functions for certain roots of $f = a$ by the roots of the equation $\psi = b$ where ψ is as above. In this connection we have the following theorem.

Theorem 1.0.10. (p.57, [24]) Suppose that f is meromorphic and non-constant in the plane and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$ is not constant where l is a positive integer. Then

$$T(r, f) < \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r, f),$$

where in $N_0\left(r, \frac{1}{\psi'}\right)$ only zeros of ψ' not corresponding to the repeated roots of $\psi(z) = 1$ are to be considered.

Note that this result reduces to the second fundamental theorem if we consider $\psi = f$ and $q = 3$.

Apart from **Chapter 1** the thesis contains four chapters which are structured as follows:

In **Chapter 2** we establish some results on relative defects of meromorphic functions by means of their proximate deficiency and the proximate deficiency of meromorphic functions on the basis of sharing of values of them. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [11]; **International Journal of Pure and Applied Mathematics**, see [12] and **International Journal of Contemporary Mathematical Sciences**, see [17].

In **Chapter 3** we consider several meromorphic functions having common roots and find some relations involving their relative defects. In this chapter we also study the deficiencies of two non constant meromorphic functions sharing $0, 1, \infty$ CM(counting multiplicities). Some portion of the results of this chapter has been published in **International Mathematical Forum**, see [9], [18]; **International Journal of Mathematical Sciences and Engineering Applications**, see [14] and the remaining portion has been published in **Review Bulletin of the Calcutta Mathematical Society**, see [19].

In **Chapter 4** we compare the maximum term of composition of two entire functions with their corresponding left and right factors on the basis of slowly changing functions. Some portion of the results of this chapter has been published in **International Journal of Contemporary Mathematical Sciences**, see [16].

In **Chapter 5** we study the comparative growth properties of composite entire and meromorphic functions using L-(p,q) th order and L^*-(p,q) th order improving some earlier results where $L = L(r)$ is a slowly changing function and p, q are positive integers and $p > q$. In this chapter we also establish the relationship between the L^*-order (L^*-type) of two meromorphic functions based on their sharing of values. Further we prove some results on the equality of (p,q)th order ((p,q)th lower order), L-(p, q)th order (L-(p, q)th lower order) and L^*-(p, q)th order (L^*-(p, q)th lower order) of entire and meromorphic functions based on relative sharing of values of them. Some portion of the results of this chapter has been published in **International Journal of Pure and Applied Mathematics,** see [13], [15] and the remaining portion has been accepted for publication and to appear in **Bulletin of the Calcutta Mathematical Society,** see [20] and **International Journal of Mathematical Analysis,** see [21].

From Chapter 2 onwards when we write Theorem $a.b.c$ (Corollary $a.b.c$ etc.) where a, b and c are positive integers, we mean the c-th Theorem (c-th Corollary etc.) of the b-th section in the a-th chapter. Also by equation number $(a.b)$ we mean the b-th equation in the a-th chapter for positive integers a and b. Individual chapters have been presented in such a manner that they are almost independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

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