Chapter 6

VISCOSITY IN BRANE-WORLD

6.1 Introduction

In Chapter 5, the objective and motivation of a Brane-World scenario is discussed. Over the past few years there is a spurt in activities to realize inflation on the brane-world with matter fields. The interest stems from the fact that the Friedmann equation gets modified on the brane-world and some of the outstanding problems of particle physics get resolved. Our observable, four-dimensional universe is considered to be embedded in a higher dimensional bulk space [110]. An important realization of this picture is found in Randall- Sundrum type II (RS) scenario [52], where a spatially isotropic and homogeneous brane propagates in a five dimensional bulk space. One approach to develop the brane-world scenario in a more string theoretic setting is to include higher order curvature invariants in the bulk action [125]. Specifically, the Gauss-Bonnet (GB) term arises as the leading order for quantum correction in the heterotic string effective action [126]. In higher dimensions ($D > 4$) Gauss-Bonnet terms ($GB = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$) where R_{abcd} is the covariant form of Riemann Christofell tensor, R_{ab} is the Ricci tensor and R is the Ricci scalar) plays a crucial role which appears as a low energy limit of string theory. When inflation is driven by an inflaton field, presence of the Gauss-Bonnet term allows the spectral index of the scalar perturbation spectrum to take values in the range between 0.944 and 0.989 [127], thereby bringing the scenario in closer agreement with the most recent observations.

Several investigation on brane cosmological models have been performed with a simplifying assumption of a perfect fluid. But in many cosmological models the simplifying assumption of matter may be inappropriate, especially in the case of matter at very high enough energies. From the physical point of view the inclusion of dissipative terms in the energy-momentum tensor of the cosmological fluid seems to be best motivated generalization of the matter sector of the gravitational field equations on brane. The first attempt to address such imperfect relativistic fluid was proposed by Eckart [128]. However, Eckart theory is not realistic as it suffers from serious shortcomings, viz., causality and stability [129]. Israel and Stewart [53] and Pavon *et al.* [130] developed a fully relativistic formulation of the theory taking into account second order terms in the theory, which is known as extended irreversible thermodynamics (in short, EIT). EIT theory is free from the problems that are seen in Eckart theory. Using the transport equations obtained from EIT, several works in Einstein gravity [21] and in higher derivative gravitational theory have been reported [131].

In this chapter we present exact cosmological solutions in the RS brane-world model with or without GB term in the presence of a causal fluid that are relevant in the early universe. Cosmological solutions are explored here with truncated Israel-

Stewart theory [53]. We also study the stability of the equilibrium points of the dynamical system associated with the evolution of the viscous cosmological fluid in the RS scenario with or without a Gauss-Bonnet term.

6.2 Field Equation in Brane-world:

For a 5D bulk with Einstein-Gauss-Bonnet gravity, containing a 4D brane, the gravitational action is

$$S = \frac{1}{2k_5^2}\int d^5x\ \sqrt{-g_5}\left[-2\Lambda_5 + R + \alpha(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd})\right] - \int_{brane} d^4x\ \sqrt{-g}\ \sigma \tag{6.1}$$

where $\sigma(>0)$ is the brane tension, and $\Lambda_5(<0)$ is the bulk cosmological constant. The fundamental energy scale of gravity is the 5D scale M_5, where $k_5^2 = \frac{8\pi}{M_5^3}$. The Planck Scale $M_4 \sim 10^{16}$ TeV is an effective scale, describing gravity on the brane at low energies, and typically $M_4 >> M_5$. The Gauss-Bonnet (GB) term may be thought of as the lowest-order stringy correction to the 5D Einstein-Hilbert action, with coupling constant $\alpha > 0$. In this case $\alpha \mid R^2 \mid << \mid R \mid$, so that

$$\alpha << l^2 \tag{6.2}$$

where l is the bulk curvature scale, $\mid R \mid \sim l^{-2}$. The RS type models are recovered for $\alpha = 0$.

The modified Friedmann equation on the (spatially flat) brane is [132]

$$k_5^2(\rho + \sigma) = 2\sqrt{H^2 + \mu^2}[3 - 4\alpha\mu^2 + 8\alpha H^2] \tag{6.3}$$

where μ is the energy scale associated with l. The above equation may be rewritten in the useful form [133]

$$H^2 = \frac{1}{4\alpha}\left[(1-4\alpha\mu^2)\cosh\left(\frac{2\chi}{3}-1\right)\right], \tag{6.4}$$

$$k_5^2(\rho+\sigma) = \left[\frac{2(1-4\alpha\mu^2)^3}{\alpha}\right]^{1/2} \sinh\chi \tag{6.5}$$

where χ is a dimensionless measure of the energy density. Expanding eq.(6.4) in χ, one obtains two regimes for the dynamical history of the brane universe during the early period of evolution which are given below :

- In the GB regime,

$$\rho >> m_\alpha^4 \Rightarrow H^2 \approx \left[\frac{k_5^2}{16\alpha}\rho\right]^{2/3} \tag{6.6}$$

which gives, $\rho = \rho_o H^3$, where $\rho_o = \frac{16\alpha}{k_5^2}$.

- In the RS regime,

$$m_\alpha^4 >> \rho >> \sigma \equiv m_\sigma^4 \Rightarrow H^2 \approx \frac{k_4^2}{6\sigma}\rho^2 \tag{6.7}$$

which gives $\rho = \rho_o H$, with $\rho_o = \left(\frac{6\sigma}{k_4^2}\right)^{1/2}$. The conservation equation for the matter is

$$\dot{\rho} = -3H(\rho + p). \tag{6.8}$$

To include the effects of viscosity, the perfect fluid pressure should be replaced by an effective pressure p_{eff}, which is given by

$$p_{eff} = p + \Pi \tag{6.9}$$

where p is the perfect fluid contribution and Π is the bulk viscous stress. The causal evolution equation for the bulk viscous pressure Π is given by the equation [130,134]

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{\epsilon}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right). \tag{6.10}$$

where ξ is the coefficient of bulk viscosity, τ is the relaxation coefficient for transient bulk viscous effect and T is the temperature. The parameter ϵ takes the value 0 or 1. $\epsilon = 0$ represents the Truncated Israel-Stewart theory (TIS) and $\epsilon = 1$ represents the full causal theory (FIS). Here we consider only TIS theory for cosmological evolution. We consider an equation of state for the perfect fluid pressure p and the energy density of the usual form

$$p = (\gamma - 1)\rho \tag{6.11}$$

where $\gamma (1 \leq \gamma \leq 2)$ is a constant. Using equations (6.9) and (6.11), eq. (6.8) becomes

$$\dot{\rho} = -3H(\gamma\rho + \Pi). \tag{6.12}$$

An important observational quantity is the deceleration parameter $q = \frac{dH^{-1}}{dt} - 1$. The sign of the deceleration parameter indicates whether the model inflates or not. The positive sign of q corresponds to standard deceleration model whereas the negative sign indicates inflation. The set of eqs. (6.6), (6.10), (6.11), (6.12) will be used to obtain cosmological solutions. As the number of equations are more than the number of unknowns we use some *ad hoc* relations for the viscosity parameters.

In the GB regime, we assume the following simple phenomenological laws for the bulk viscosity coefficient and relaxation time:

$$\xi = \alpha\rho^s = \xi_o H^{3s}, \quad \tau = \frac{\xi}{\rho} = \frac{H^{3(s-1)}}{\tau_o} = \eta\rho^{s-1}$$

where $s \geq 0, \quad \xi \geq 0, \quad \tau \geq 0$ and $\eta \geq 0$ are constants and $\xi_o = \eta\rho_o^s$ and $\tau_o = \frac{\rho_o}{\xi_o}$ The bulk viscous pressure Π in the GB regime may now be obtained from eq. (6.12) in the form

$$\Pi = -H(\dot{H} + 2H^2) \tag{6.13}$$

The bulk viscous evolution eq. (6.10) can be written as

$$\dot{\Pi} + \frac{1}{\tau}\Pi = -3\rho H - \frac{1}{2}\Pi\left(3H - \frac{3}{2}\frac{\dot{\rho}}{\rho}\right). \tag{6.14}$$

For a stiff cosmological fluid on the brane with GB term, given by a second order differential equation

$$\ddot{H} - \frac{5}{4}\frac{\dot{H}^2}{H} + (3H + \tau_o H^{-3(s-1)})\dot{H} + 2\tau_o H^{-3s+5} = 0 \tag{6.15}$$

Using similar relations for the bulk viscosity coefficient and relaxation time above in the RS region the bulk viscous pressure Π can be obtained from eq. (6.12) which is given by

$$\Pi = -\rho_o\left(\frac{\dot{H}}{H} + 6H\right). \tag{6.16}$$

The bulk viscous evolution eq. (6.10) reduces to

$$\dot{\Pi} + \frac{1}{\tau}\Pi = -3\rho H - \frac{\epsilon}{2}\Pi\left(3H - \frac{3}{2}\frac{\dot{\rho}}{\rho}\right). \tag{6.17}$$

and for a stiff cosmological fluid ($\gamma = 2$) on the brane in the RS regime it takes the form

$$\frac{\ddot{H}}{H} - \frac{7}{4}\frac{\dot{H}^2}{H^2} + (3 + \tau_o H^{-s})\dot{H} + 6\tau_o H^{2-s} = 0. \tag{6.18}$$

6.3 Cosmological Solutions with TIS ($\epsilon = 0$)

Case A. In the GB regime:

In this case, eq. (6.10) for TIS becomes

$$\tau\dot{\Pi} + \Pi = -3\xi H \tag{6.19}$$

Here we use $\xi = \eta\rho^s$ and $\tau = \eta\rho^{s-1}$. In the next section we explore viscous cosmological model for a known behaviour of the scale factor of the universe.

(i) Power-law model :

Let us consider a power-law behaviour for the scale factor of the universe given by

$$a(t) = a_o t^D \tag{6.20}$$

where a_o and D are constants. From eq.(6.6) we get the energy density

$$\rho = \rho_o' t^{-3} \tag{6.21}$$

where $\rho_o' = \frac{D^3}{C^3}$ and $C = \left(\frac{k_5^2}{16\alpha}\right)^{1/3}$. The bulk viscous stress is obtained from eq.(6.13)

$$\Pi = \Pi_o t^{-3} \tag{6.22}$$

where $\Pi_o = \left(\frac{\rho_o'^{2/3}}{C} - \gamma\rho_o'\right) = \frac{D^2}{C^3}(1-\gamma D)$. Using transport eq. (6.19) one obtains

$$A_1 t^{-3} + A_2 t^{-3s-1} = 0 \tag{6.23}$$

where $A_1 = \frac{D^2}{C^3}(1-\gamma D)$ and $A_2 = 3\eta\frac{D^{3s}}{C^{3s}}\left(D - \frac{1}{D} + \gamma\right)$. We note the following :

(a) when $A_1 = 0$ one obtains $D = \frac{1}{\gamma}$, power law inflation is permitted here for $\gamma < 1$; (b) when $A_2 = 0$, it leads to either $\eta = 0$, *i.e.*, no viscosity or $\gamma = \frac{1-D^2}{D}$. In this case $D = 1$ corresponds to a matter with viscosity and the universe evolves as $a \sim t$ which is a new solution on brane. For $s = \frac{2}{3}$, one obtains $A_1 + A_2 = 0$ which yields $\eta = \frac{\gamma D - 1}{3C\left(D - \frac{1}{D} + \gamma\right)}$.

(ii) Exponential model :

In this case, we choose $H = H_o$. The energy density is constant which is given by

$$\rho = \frac{H_o^3}{C^3} \tag{6.24}$$

The bulk viscous stress is given by

$$\Pi = -\frac{\gamma}{C^3} H_o^3 \tag{6.25}$$

and the coefficient of bulk viscosity is

$$\xi = \frac{\gamma}{3C^3} H_o^2 \tag{6.26}$$

which yields

$$\eta = \frac{\gamma C^{3(s-1)}}{3} H_o^{2-3s} \tag{6.27}$$

For $s = \frac{2}{3}$, we find

$$\eta = \frac{\gamma}{3C} \tag{6.28}$$

The scale factor of the universe is

$$a(t) = a_o \exp(H_o t) \tag{6.29}$$

For sufficient inflation to solve the cosmological problems, one requires $H_o t > 65$.

Case B. In the RS regime:

In this case, eq. (6.10) for TIS becomes

$$\tau \dot{\Pi} + \Pi = -3\xi H \tag{6.30}$$

Here we use $\xi = \eta \rho^s$ and $\tau = \eta \rho^{s-1}$.

(i) **Power-law model :**

We consider a power-law for the scale factor of the universe

$$a(t) = a_o t^D \tag{6.31}$$

where a_o and D are constants. From eq.(6.7) we get the energy density

$$\rho = \rho_o'' t^{-1} \tag{6.32}$$

where $\rho''_o = \frac{D}{C'}$ and $C' = \left(\frac{k_4^2}{6\sigma}\right)^{1/2}$. The bulk viscous stress is obtained from eq.(6.16)

$$\Pi = \Pi'_o t^{-1} \tag{6.33}$$

where $\Pi'_o = \frac{1}{3C'} - \gamma\frac{D}{C'}$. From transport eq. (6.30) we obtain

$$A'_1 t^{-1} + A'_2 t^{-s-1} = 0 \tag{6.34}$$

where $A'_1 = \frac{1}{3C'}(1 - 3\gamma D)$ and $A_2 = \eta\frac{D^s}{C'^s}\left(\frac{3D}{C'} - \frac{1}{3D} + \gamma\right)$. We note the following :

(a) when $A'_1 = 0$ one gets $D = \frac{1}{3\gamma}$, one obtains power law inflation with $D > 1$ if $\gamma < \frac{1}{3}$ which is a different result that from GB case; (b) when $A'_2 = 0$ one gets $\eta = 0$, *i.e.*, no viscosity or $\gamma = \frac{C'-9D^2}{3C'D}$. When viscosity and matter coexist one gets $D = \sqrt{C'/9}$, where C' can be chosen to make $D > 1$ *i.e.* power law inflation which is a new result on brane. However, if $s = 0$, it leads to $A'_1 + A'_2 = 0$ which yields $\eta = \frac{D(3\gamma D-1)}{9D^2+C'(3\gamma D-1)}$.

(ii) Exponential model :

In this case, we choose $H = H_o$. The energy density is constant which is given by

$$\rho = \frac{H_o}{C'} \tag{6.35}$$

The bulk viscous stress is given by

$$\Pi = -\frac{\gamma}{C'}H_o \tag{6.36}$$

and the coefficient of bulk viscosity is

$$\xi = \frac{\gamma}{3C'} \tag{6.37}$$

which yields

$$\eta = \frac{\gamma C'^{s-1}}{3}H_o^{-s} \tag{6.38}$$

For $s = 0$, we find

$$\eta = \frac{\gamma}{3C'} \tag{6.39}$$

The scale factor of the universe is

$$a(t) = a_o \exp(H_o t) \tag{6.40}$$

For sufficient inflation to solve the cosmological problems, we require $H_o t > 65$.

6.4 Stability Analysis of the equilibrium points

Case A. GB regime:

The general evolution equation of the bulk viscous cosmological fluid on the brane with GB term is given by eq. (6.15). From mathematical point of view it is a second order non-linear differential equation of the form $\ddot{H} + R(H, \dot{H}) = 0$ with

$$R(H, \dot{H}) = -\frac{5}{4}\frac{\dot{H}^2}{H} + (3H + \tau_o H^{-3(s-1)})\dot{H} + 2\tau_o H^{-3s+5} \tag{6.41}$$

In order to study the stability of the equilibrium points of the evolution equation of the viscous cosmological fluid on the brane with GB term, we shall rewrite eq. (6.41) in the form of an autonomous dynamical system, by introducing a new variable $X = \dot{H}$:

$$P(H, X) = \frac{dH}{dt} = X \tag{6.42}$$

$$Q(H, X) = \frac{dX}{dt} = \frac{5}{4}\frac{X^2}{H} - (3H + \tau_o H^{-3(s-1)})X - 2\tau_o H^{-3s+5} \tag{6.43}$$

The Jacobian matrix, L = $\begin{pmatrix} P_H & P_X \\ Q_H & Q_X \end{pmatrix}$ where

$$P_H = \frac{\partial H}{\partial t} = 0 \tag{6.44}$$

$$P_X = \frac{\partial P}{\partial X} = 1 \tag{6.45}$$

$$Q_H = \frac{\partial Q}{\partial H} = -\frac{5}{4}\frac{X^2}{H^2} - \left(3 - \tau_o 3(s-1)H^{-3s+2}\right) X - 2\tau_o(-3s+5)H^{-3s+4} \tag{6.46}$$

$$Q_X = \frac{\partial Q}{\partial X} = \frac{5}{2}\frac{X}{H} - \left(3H + \tau_o H^{-3(s-1)}\right) \tag{6.47}$$

The secular determinant determines the eigenvalue λ,

$$\begin{vmatrix} 0-\lambda & 1 \\ Q_H & Q_X - \lambda \end{vmatrix} = 0$$

i.e.

$$\lambda^2 - \lambda Q_X - Q_H = 0 \tag{6.48}$$

The elements of the matrix must be evaluated at the equilibrium points (h_i, X_i), which are found by solving the system, $P(h_i, X_i) = Q(h_i, X_i) = 0$. Now, $P(h_i, X_i) = 0$ leads to $X = 0$ and $H = 0$ or h_o (constant). Therefore, critical points are $P_o(0,0)$ and $P_o(h_o, 0)$.

After linearization we introduce the standard notation:

$$p = Q_X \tag{6.49}$$

$$q = -Q_H \tag{6.50}$$

and

$$\Delta = p^2 - 4q = Q_X^2 + 4Q_H \tag{6.51}$$

Since p, q determine the roots, these also determine the behaviour of solution near critical point P_o. We note the following:

(i) P_o is stable and attractive if $p < 0$ and $q > 0$, which is permitted when $h_o > \left(-\frac{\tau_o}{3}\right)^{\frac{1}{3s-2}}$ for $s = \frac{5}{3}$ at the point $(h_o, 0)$.

(ii) P_o is stable if $p \leq 0$ and $q > 0$, which leads to $h_o \geq \left(-\frac{\tau_o}{3}\right)^{\frac{1}{3s-2}}$ for $s = \frac{5}{3}$ at the point $(h_o, 0)$

(iii) P_o is a node point if $q > 0$ and $\Delta \geq 0$, which gives $h_o \geq \frac{1}{54\tau_o}$ for $s = \frac{5}{3}$.

(iv) P_o is a saddle point if $q < 0$, which gives $s = \frac{5}{3}$.

(v) P_o is a spiral point if $p \neq 0$ and $\Delta < 0$, which is permitted when $\tau_o > \frac{1}{2}$ for $s = \frac{2}{3}$.

We further note that no center or unstable point exist here.

Case B. RS regime:

The general evolution equation of the bulk viscous cosmological fluid on the brane in the RS scenario is given by eq.(6.18). From mathematical point of view it is a second order non-linear differential equation of the form $\ddot{H} + R(H, \dot{H}) = 0$ with

$$R(H, \dot{H}) = -\frac{7}{4}\frac{\dot{H}^2}{H} + (3H + \tau_o H^{1-s})\dot{H} + 6\tau_o H^{3-s} \tag{6.52}$$

In order to study the stability of the equilibrium points of the evolution equation of the viscous cosmological fluid on the brane, we shall rewrite eq. (6.52) in the form of an autonomous dynamical system, by introducing a new variable $X = \dot{H}$:

$$P(H, X) = \frac{dH}{dt} = X \tag{6.53}$$

$$Q(H, X) = \frac{dX}{dt} = \frac{7}{4}\frac{X^2}{H} - (3H + \tau_o H^{1-s})X - 6\tau_o H^{3-s} \tag{6.54}$$

The critical points in this case are $P_o(0,0)$ and $P_o(h_o,0)$. After linearization we introduce the standard notation as was done earlier:

$$p = Q_X \tag{6.55}$$

$$q = -Q_H \tag{6.56}$$

and

$$\Delta = p^2 - 4q = Q_X^2 + 4Q_H \tag{6.57}$$

We note the following:

(i) P_o is stable and attractive when $h_o > \left(-\frac{\tau_O}{3}\right)^{\frac{1}{s}}$ with $s = 3$ at the point $(h_o, 0)$

(ii) P_o is stable if $h_o \geq \left(-\frac{\tau_O}{3}\right)^{\frac{1}{s}}$ for $s = 3$ at the point $(h_o, 0)$

(iii) P_o is a node point if $h_o \geq \left(-\frac{\tau_O}{3}\right)^{\frac{1}{3}}$.

(iv) P_o is a saddle point when $s = 3$.

(v) P_o is a spiral point if $2.75 < s < 3$.

In this case also no center and unstable points exist.

6.5 Discussions

In this chapter, we explore both power law and exponential solutions in the Randall-Sundrum type II (RS) brane-world scenario with or without Gauss-Bonnet (GB) term in the presence of a causal viscous fluid. The truncated Israel-Stewart theory is employed to describe the viscous fluid here. It is found that the effect of viscosity in general is to increase the rate of expansion of the universe. In this exercise we study the stability of the equilibrium points of the dynamical system associated with the evolution of the viscous cosmological fluid in the RS brane-world scenario with or without a GB-term. We do not get a viscous universe with power law behavior in the GB- regime for $s = \frac{2}{3}$ and $D = \frac{1}{\gamma}$ and in the RS regime there is no viscous universe for $s = 0$ and $D = \frac{1}{3\gamma}$ in TIS theory. Thus for $\gamma = 1$ we note $a(t) \sim t$ in GB but $a(t) \sim t^{1/3}$ without GB. Thus the rate of expansion is more in the presence of GB term than that when GB-term is absent in RS-brane. The coefficient of viscosity η is determined in terms of γ. For an exponential solution ($H = H_o$) in TIS theory it is noted that both in the GB regime and in the RS regime it admit for any s. In the GB regime the equilibrium point is stable and attractive if $h_o > \left(-\frac{T_o}{3}\right)^{\frac{1}{3}}$ for $s = \frac{5}{3}$ and in the RS regime the equilibrium point is stable and attractive if $h_o > \left(-\frac{T_o}{3}\right)^{\frac{1}{3}}$ for $s = 3$ at the point $(h_o, 0)$. There is no unstable point both in the GB regime and in the RS regime.