Chapter 5

INFLATION WITH HYPERBOLIC POTENTIAL IN THE BRANE-WORLD MODEL

5.1 Introduction

Recent high precision measurements have provided strong evidence that the universe is very close to critical density and that large-scale structure developed through gravitational instability from a primordial spectrum of adiabatic, Gaussian and nearly scale-invariant density perturbations [105]. These observations are consistent with the cornerstone predictions of the simplest class of inflationary models.

In view of these developments, it is important to improve further our understanding of the inflationary scenario from a theoretical perspective. In the last few years considerable interest revived once again to study inflationary models motivated by the successes of superstring and M-theory [106]. The recent successes in superstring/M-theory [107] led us to believe that it is a promising candidate for quantum gravity. These theories require space-time dimensions more than the usual four for their consistent formulation. In recent years there is a paradigm shift in cosmological model building in the higher dimensional theory from that of the previous approach initiated by Kaluza-Klein [108] and others in cosmology [109]. In particular, much attention has been focused on the brane-world scenario, where our observable, four dimensional universe is regarded as a domain wall embedded in a higher dimensional bulk space [110]. In modern higher dimensional scenario, the fields of the standard model are considered to be confined to (3 + 1) dimensional hyper-surface (referred to as 3-Brane) embedded in a higher dimensional space-time but the gravitational field may propagate through the bulk dimension perpendicular to the brane which is referred to as brane-world. According to Randall and Sundrum although the extra dimension is not compact, four dimensional Newtonian gravity is recovered in five dimensional anti-de Sitter spacetime (AdS_5) in the low energy limit [52]. The brane-world scenario has interesting cosmological implications, in particular, the prospects of inflation are enhanced on the brane due to the modifications in the Friedmann equation [111]. The high energy corrections of the Friedmann equation is due to the effect of extra dimension projected on the brane and it changes the nature of expansion dynamics of the early universe [111, 112]. According to Randall-Sundrum brane-world model [52] a spatially isotropic and homogeneous universe may be embedded in a five-dimensional Schwarzschild-Anti-de Sitter(AdS) space with 5D Planck scale, M_5 (M_5 is assumed to be very much smaller than the corresponding 4D effective Planck scale, $M_4 = 1.2 \times 10^{19}$ GeV).

In the literature a large volume of work on cosmological models in the brane-world scenario have been reported, assuming that the brane universe is dominated by a single minimally coupled scalar field which is rolling in a given potential. The potential must be sufficiently steep at the time of inflation to achieve slow-roll conditions in the very high energy limit. Several authors have also studied inflationary dynamics of the universe in the Randall-Sundrum (type II) brane-world model with such relatively steep potentials like exponential and inverse power-law potential [86, 112, 113, 114]. Sami and his collaborators [114] have shown that in the case of inverse power law type of potential, the phantom field may successfully drive the current acceleration with an equation of state parameter $\omega < -1$. This model also fits the supernovae data very well, allowing for $-2.4 < \omega_{\phi} < -1$ at 95% confidence level, where the suffix indicates the phantom field.

In the framework of GTR, the hilltop inflation model is more natural than a very flat potential model as noticed in *Refs.* [115, 116]. The potential covers a range of possibilities to yield analytic formulas for the structure formation and the prediction of spectral index with the observed value. In this paper we study the inflationary dynamics in the Randall-Sundrum brane-world model with an inflaton field in an inverse coshyperbolic potential which is a similar kind of potential that was previously considered in GTR [117, 118]. The potential was used with a phantom for a viable model with an equation of state $\omega_{\phi} < -1$ [117]. In the case of a tachyon rolling on the brane such potential appears in the context of open string field theory for unstable D-brane system in type II superstring theory [118, 119]. Steer and Vernizzi [120] further used the potential to compare a single scalar field inflation predictions with those of an inflationary phase driven by a tachyon field. In this chapter we use inverse coshyperbolic potential as a toy model to determine the evolution of a scalar field in brane-world and estimate the allowed range of values for the spectral index of the scalar perturbation spectrum (n_s) permitted by observational evidences. We also analyze the variation of (n_s) with the number of e-foldings (N) for different initial values of the field taking into account the recent observational constraints to determine the permissible values of N here.

This chapter is organized as follows: The basic equations of the Randall-Sundrum (type II) brane-world model are presented in sec. 5.2. In sec. 5.3 we obtain inflationary dynamics in the brane-world model with inverse coshyperbolic potential and study detail properties of the model. Finally in sec. 5.4, we give a brief discussions.

5.2 Basic Equations

In Randall-Sundrum (type II) brane-world model Einstein field equation induced on brane is given by [121]:

$$G_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + \left(\frac{8\pi}{M_4^2}\right) T_{\mu\nu} + \left(\frac{8\pi}{M_5^3}\right)^2 \pi_{\mu\nu} - E_{\mu\nu}. \tag{5.1}$$

where $\pi_{\mu\nu}$ and $E_{\mu\nu}$ represent modifications induced on the brane to the standard Einstein field equation. The $\pi_{\mu\nu}$ term comes due to the brane energy-momentum tensor and $E_{\mu\nu}$ term comes due to the 5D Weyl tensor projected on the brane. In the (4+1) dimensional brane-world scenario inspired by the Randall-Sundrum [52] model, the standard Friedmann equation with Robertson-Walker metric is thus modified to [111, 122]:

$$H^2 = \frac{\Lambda_4}{3} + \left(\frac{8\pi}{3M_4^2}\right) \rho + \left(\frac{8\pi}{M_5^3}\right)^2 \rho^2 + \frac{\varepsilon}{a^4}. \tag{5.2}$$

where $H(\equiv \frac{\dot{a}}{a})$ is the Hubble constant and ε is an integration constant which transmits bulk graviton influence onto the brane, and the relationship between the four and five-dimensional Planck masses is given by

$$M_4 = \sqrt{\frac{3}{4\pi}} \left(\frac{M_5^2}{\sqrt{\lambda}}\right) M_5. \tag{5.3}$$

where λ represents the brane tension. The brane tension is also related to the effective four-dimensional cosmological constant Λ_4 to its five-dimensional counterpart Λ as

$$\Lambda_4 = \frac{4\pi}{M_5^3}\left(\Lambda + \frac{4\pi}{3M_5^3}\lambda^2\right). \tag{5.4}$$

If $\Lambda \sim -4\pi\lambda^2/3M_5^3$, then Λ_4 is too small to play an important role in the early universe [111]. The *dark radiation* term $\frac{\varepsilon}{a^4}$ is expected to rapidly disappear once inflation has commenced so that we effectively get [111, 122]:

$$H^2 = \frac{8\pi}{3M_4^2}\, \rho \left[1 + \frac{\rho}{2\lambda}\right], \tag{5.5}$$

the second term in the bracket is the modification of the Friedmann equation on brane. We now consider a universe dominated by a single minimally coupled scalar field moving in a potential $V(\phi)$. Thus for a homogeneous inflaton field the energy density (ρ) is given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{5.6}$$

The equation of motion of the scalar field propagating on the brane is given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0 \tag{5.7}$$

where dot represents the time derivative.

The set of eqs.(5.5) and (5.7) are used to obtain cosmological solution for a known potential by employing slow-roll approximation technique. We now invoke the standard assumption that the energy density on the brane can be separated into two parts, where one of them is the ordinary matter component, ρ and the other contributes when the brane tension, $\lambda > 0$, such that $\sigma = \rho + \lambda$. Steep inflation proceeds in the region of parameter space where $\sigma \approx \rho \gg \lambda$ and naturally comes to an end when $\rho \approx \lambda$. Considering the slow-roll approximation technique, the condition for inflation $\dot{\phi}^2 \ll V$ and $\ddot{\phi} \ll V'$, may be used to re-write eqs.(5.5) and (5.7) as [111, 122]:

$$H^2 \simeq \frac{8\pi}{3M_4^2} V \left[1 + \frac{V}{2\lambda}\right], \qquad \dot{\phi} \simeq -\frac{V'}{3H}. \tag{5.8}$$

The factor $\left[1 + \frac{V}{2\lambda}\right]$ in the above field equation is a modification of the energy density on brane when we compare it to the standard Einstein equation using slow-roll approximation. However, the standard expression in GTR is recovered by taking the brane tension to a limiting value $\lambda \to \infty$. The slow-roll parameters on the brane [111, 122] are given below :

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{M_4^2}{4\pi} \left(\frac{V'}{V}\right)^2 \left[\frac{(\lambda(\lambda + V)}{(2\lambda + V)^2}\right], \tag{5.9}$$

$$\eta = \frac{V''}{3H^2} = \frac{M_4^2}{4\pi} \left(\frac{V''}{V}\right)^2 \left[\frac{\lambda}{2\lambda + V}\right]. \tag{5.10}$$

The number of e-folds is an important quantity which indicates the proper multiplication of the size of the universe as the inflation ends to get rid of the problems encountered by a standard Big bang model. Using slow-roll approximation we get

$$N \simeq -\frac{8\pi}{M_4^2} \int_{\phi_i}^{\phi_f} \frac{V}{V'} \left(1 + \frac{V}{2\lambda}\right) d\phi \tag{5.11}$$

where ϕ_i and ϕ_f denotes the initial and final values of inflaton field respectively. The scalar amplitude of density perturbations $A_S^2 = H^4/(25\pi^2 \dot{\phi}^2)$ in this case can be written as

$$A_s^2 = \frac{512\pi}{75M_4^6}\left(\frac{V^3}{V'^2}\right)^2\left[\frac{2\lambda+V}{2\lambda}\right]^3\Bigg|_{k=aH} \tag{5.12}$$

In the high energy limit, it becomes

$$A_s^2 \simeq \frac{64\pi}{75\lambda^3 M_4^6}\left(\frac{V^6}{V'^2}\right). \tag{5.13}$$

The contribution of the gravitational wave relative to density perturbation is suppressed in the high energy limit and the relative amplitude [47, 122, 123] is given by

$$\frac{A_T^2}{A_S^2} = 6\epsilon\frac{\lambda}{\rho}$$

i.e.

$$\frac{A_T^2}{A_S^2} = \frac{3M_4^2}{2\pi}\frac{\alpha^2}{M_P^2\beta^2}\left[\frac{\lambda}{V}\right]^2. \tag{5.14}$$

The spectral index of the scalar spectrum, $n_s - 1 \equiv \frac{d(lnA_S^2)}{d(lnk)}$ in terms of slow-roll parameters obtained above are related as

$$n_s = 1 - 6\epsilon + 2\eta. \tag{5.15}$$

At high energies, $V/\lambda \to \infty$ leads to a value of the spectral index, n_s approaching 1. In the next section we study the dynamics of an inflaton field in an inverse coshyperbolic potential in the Randall-Sundrum (type II) brane-world model.

5.3 Inflationary dynamics with inverse coshyperbolic potential

In the brane-world scenario, at high energy limit where $V \gg \lambda$, the modified terms in the Friedmann equation are important and therefore the field equations are given by

$$H^2 \simeq \left(\frac{4\pi}{3M_5^3}\right) V^2, \quad \dot{\phi} \simeq -\frac{V'}{V}\left(\frac{M_5^3}{4\pi}\right). \tag{5.16}$$

The slow-roll parameters defined in the previous section become :

$$\epsilon = \frac{M_4^2}{16\pi}\left(\frac{V'}{V}\right)^2\left[\frac{4\lambda}{V}\right], \tag{5.17}$$

$$\eta = \frac{M_4^2}{8\pi}\left(\frac{V''}{V}\right)\left[\frac{2\lambda}{V}\right]. \tag{5.18}$$

and the number of e-foldings becomes

$$N \simeq -\frac{4\pi}{\lambda M_4^2}\int_{\phi_i}^{\phi_f}\frac{V^2}{V'}d\phi \tag{5.19}$$

where ϕ_i and ϕ_f denotes the initial and final values of inflaton field respectively.

We now consider scalar field in an inverse coshyperbolic potential [117-120] which is given by

$$V = V_o\left[cosh\frac{\alpha\phi}{M_p}\right]^{-1} = V_o sech(\gamma\phi) \tag{5.20}$$

where $M_p = \frac{M_4}{\sqrt{8\pi}}$ and $\gamma = \frac{\alpha}{M_p}$, the time evolution of the field is obtained by integrating eq.(5.16) which is

$$\phi(t) = \frac{1}{\gamma}sinh^{-1}\exp\left(Bt + C\right) \tag{5.21}$$

where $B = \frac{M_5^3\gamma^2}{4\pi}$ and $C = \ln\left[sinh(\gamma\phi_i)\right] - Bt_o$, are constants, and $\phi = \phi_i$ when the inflation starts, $t = t_i$. We obtain the scale factor by integrating eq.(5.16), which is given by

$$a(t) = a_i\, e^{\left[\frac{A}{B}\left(tanh^{-1}\frac{1}{\sqrt{e^{2(Bt+C)}+1}}\right)\right]} \tag{5.22}$$

where $A = \sqrt{\frac{4\pi V_o}{3M_5^3}}$. In the model inflation ends when

$$t_{end} = \frac{\ln\left[\frac{\sqrt{2}\sqrt{A^2+\sqrt{A^4+4B^2A^2}}}{2B}\right] - C}{B}. \tag{5.23}$$

Thus one can determine the value of the inflaton field at the end of inflation, which is given by

$$\phi_{end} = \frac{1}{\gamma}\ sinh^{-1}\left[\frac{\sqrt{2}\sqrt{A^2+\sqrt{A^4+4B^2A^2}}}{2B}\right]. \tag{5.24}$$

Consequently one can determine the magnitude of the potential at the end of inflation which is

$$V_{end} = V_o\ sech\left[sinh^{-1}\left(\frac{\sqrt{2}\sqrt{A^2+\sqrt{A^4+4B^2A^2}}}{2B}\right)\right] \tag{5.25}$$

The expression for the number of e-foldings is given by

$$N(t) = \ln\frac{a(t)}{a_o} = \left[\frac{A}{B}\left(tanh^{-1}\frac{1}{\sqrt{(\exp(Bt+C))^2+1}}\right)\right]_{t_i}^{t_{end}}. \tag{5.26}$$

The number of e-foldings is related to the initial and end value of the potential V_i and V_{end} respectively which is

$$V_{end} = V_o\ tanh\left[tanh^{-1}\left(\frac{V_i}{V_o}\right) - \left(\frac{B}{A}\right)N\right]. \tag{5.27}$$

We now derive the analytical expressions of the relevant inflationary parameters as a function of the field. With the inverse coshyperbolic potential given by eq.(5.20), the slow-roll parameters given by eqs. (5.17) and (5.18) become

$$\epsilon = \beta\left(\frac{tanh^2 x}{sech x}\right), \tag{5.28}$$

$$\eta = \beta\left(\frac{tanh^2 x - sech^2 x}{sech x}\right) \tag{5.29}$$

where $\beta = \frac{3\alpha^2 M_5^6}{2\pi M_4^2 V_o}$ and $x = \gamma\phi$. At the end of inflation, since $\epsilon \sim 1$ we get

$$\beta = \frac{sech x_{end}}{tanh^2 x_{end}} \tag{5.30}$$

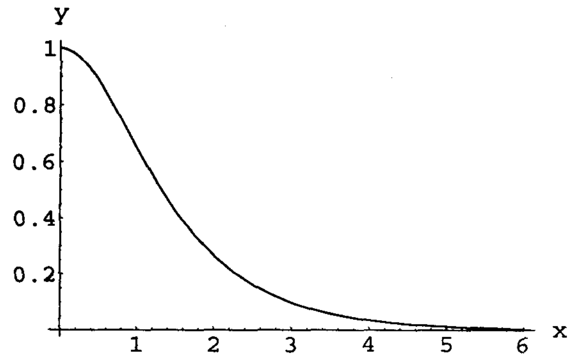

Figure 5.1: Illustrating the variation of the potential ($y = \frac{V}{V_o}$) with the field (x) which has maximum at $x \approx 0.9$

The number of e-foldings can be obtained using eq.(5.19), which is

$$N = \frac{1}{\beta}\left[-ln\left(cosh\frac{x}{2}\right) + ln\left(sinh\ \frac{x}{2}\right)\right]_{x_i}^{x_{end}} \tag{5.31}$$

where x_{end} denotes the value of x when inflation ends.

For $N \approx 70$, is the minimum number of e-foldings required to solve outstanding problems of cosmology, using this N, eqs.(5.29) and (5.30) determines $x_{end} = 5.10$ and $\beta = 0.012$, assuming $x_i = 0.9$ with a consideration that $\epsilon \sim 1$ at the end of inflation. We have verified that the variation of the slope of the potential attains a maximum at $x \approx 0.9$, one obtains a sufficient inflation. This is achieved if the field starts with an initial $x \geq 0.9$ where $V/V_o \leq 0.7$ (fig. 5.1).

The amplitude of scalar perturbations is given by

$$A_S^2 = \left(\frac{64\pi^2}{45\sqrt{8\pi}}\right)^2 \frac{M_4^2}{M_5^{18}}\frac{V_o^4}{\alpha^2}\left[\frac{sech^4 x_{cobe}}{tanh^2 x_{cobe}}\right] \tag{5.32}$$

For our model, we choose $N_{COBE} \approx 55$, $x_i = x_{COBE} = 0.9$, using eqs. (5.30) and (5.31), we obtain

$$\beta = 0.015, \tag{5.33}$$

$$x_{end} = 4.866. \tag{5.34}$$

Using β, one can determine,

$$V_o = 31.004\, \alpha^2 \left(\frac{M_5^6}{M_4^2}\right) \tag{5.35}$$

which gets fixed for a known α. Using COBE normalization, $A_S^2 = 4 \times 10^{-10}$, we get,

$$\alpha = 2.22 \times 10^{-3} \left(\frac{M_4}{M_5}\right) \tag{5.36}$$

and the corresponding ϕ_{COBE} and V_o become

$$\phi_{cobe} = 80.867 M_5, \tag{5.37}$$

$$V_o = 1.53 \times 10^{-4} M_5^4 \tag{5.38}$$

Since $M_4 > M_5$, we get a limiting value of α from eq.(5.36) which is

$$\alpha > 2.22 \times 10^{-3} \tag{5.39}$$

It is possible to determine the spectral index of the scalar spectrum from eq.(5.15) using the values of slow-roll parameters for a given number of e-foldings. Let us now take $N = 70$, the corresponding spectral index becomes

$$n_s = 0.947 \tag{5.40}$$

which lies within the bounds resulting from the CMB data [124] :

$$0.8 < n_s < 1.05 \tag{5.41}$$

Repeating the above calculations with $x_i = 0.9$, but with different number of e-foldings, N, one can determine the spectral index which are presented in the Table 5.1. From the table 1 we see that as N increases, β decreases but $x_f = x_{end}$ and

$x_i = 0.9$			
N	x_f	β	n_s
10	3.24	0.079	0.657
20	3.89	0.041	0.822
30	4.28	0.028	0.879
40	4.55	0.021	0.909
50	4.77	0.017	0.926
60	4.95	0.014	0.939
70	5.10	0.012	0.947

Table 5.1: Showing variation of x_f, β and n_s with N for $x_i = 0.9$.

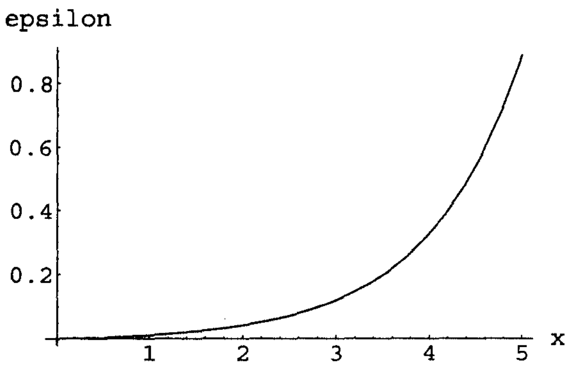

Figure 5.2: Illustrating the variation of the slow-roll parameter, ϵ with x_i.

n_s increases. The constraints from CMB puts a limit on N. We get $N > 20$ and $\beta < 0.04$.

Table 5.2 shows the variation of x_f, β and n_s with x_i for $N = 70$. We see that as x_i increases, starting from $x_i = 0.9$, x_f increases but β and n_s decreases. It is also found that the spectral index n_s is not altered significantly with x_i.

Repeating the same calculations for other N, we obtain Table 5.3. It is found that for higher N, the variation of n_s decreases with x_i and the allowed values of spectral index, n_s, is weakly dependent on N, for $N \geq 50$.

$N = 70$			
x_i	x_f	β	n_s
0.9	5.10	0.012	0.947
1.1	5.32	0.010	0.946
1.3	5.54	0.008	0.946
1.5	5.75	0.006	0.945
1.7	5.95	0.005	0.945
1.9	6.16	0.004	0.944
2.1	6.36	0.003	0.944
2.3	6.56	0.003	0.944
2.5	6.76	0.002	0.944

Table 5.2: Showing variation of x_f, β and n_s with x_i, for $x_i \geq 0.9$ and $N = 70$.

N	10	20	30	40	50	60	70
n_s	0.64-0.66	0.81-0.82	0.87-0.88	0.90 -0.91	0.92-0.93	0.93-0.94	0.94-0.95

Table 5.3: Showing variation of n_s with N, for $5.1 \geq x_i \geq 0.9$.

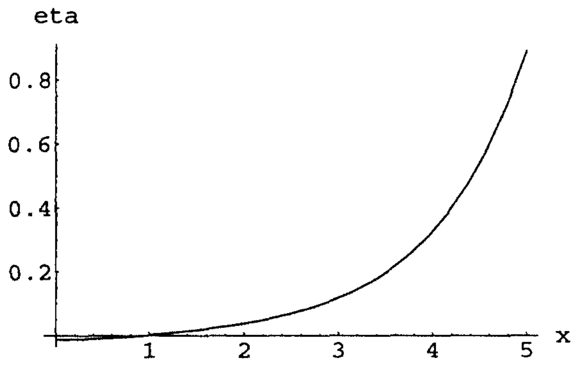

Figure 5.3: Illustrating the variation of the slow-roll parameter, η with x_i.

The contribution of gravitational wave relative to density perturbation can now be obtained using the values of the parameters determined above, which is

$$\frac{A_T^2}{A_S^2} \approx 6.6 \times \left(\frac{\lambda}{V}\right)^2. \tag{5.42}$$

The r-parameter is given by

$$r \approx 4\pi \frac{A_T^2}{A_S^2} \tag{5.43}$$

It is evident that (i) for $\frac{\lambda}{V} \sim \frac{1}{10}$, $r \approx 0.8$, (ii) for $\frac{\lambda}{V} \sim \frac{1}{20}$, $r \approx 0.2$. The measurement of r-parameter is significant, which however may be verified by the future Planck satellite for the suitability of the model.

5.4 Discussions

The inflationary dynamics of a scalar field in an inverse coshyperbolic potential in brane-world scenario is studied in this chapter. We note that sufficient inflation may be obtained in the Randall-Sundrum(type II) brane-world model at high energy limit even with an inverse coshyperbolic potential for a restricted domain of the initial values of the scalar field. In the high energy limit cosmological solutions obtained in the framework of brane-world are interesting. We determine the allowed range of values of spectral index of the scalar perturbation spectrum and determine its variation with the variation of the number of e-foldings. The inflationary parameters that are predicted in the model are in good agreement with the current observations. In fact it is found that the measurement of r-parameter will put an effective limit on the high energy scale in brane-world scenario with such an unusual potential.