Chapter 4

ANISOTROPIC BIANCHI-I UNIVERSE WITH PHANTOM FIELD

4.1 Introduction

Recent astrophysical data obtained from high redshift surveys of Supernovae, COBE to WMAP predicts that the present universe is passing through an accelerating phase of expansion [54]. An accelerating phase of the universe at a later epoch is permitted with an equation of state $p = \omega\rho$, where $\omega < -1$. Such an equation of state is not permitted with usual matters in the standard model of Particle physics. Thus a modification of the matter sector of the Einstein's equation with new fields, perhaps a new physics, is to be explored. The searches for the requisite parameter demands fields which are popularly termed as exotic fields. In chapter 3, one such exotic field tachyon is considered to explore cosmological model. In this chapter, another exotic field namely phantom field is considered. The appearance of phantom is not yet clear but it has many similarities with the quantum field theory [62]. The interesting feature of the field is that it has unusual kinetic term in the Lagrangian suitable for describing dark energy, which originates in many theories, namely supergravity [82], higher derivative gravity theories [83], brane-world phantom energy [84], etc.. Although such theories are known to be unstable with respect to quantum effects it may be important to explore cosmologies with such fields because of its importance in the early and late era. One interesting aspect of the field is that the universe might end up with a singularity due to the appearance of divergence in scale factor $a(t)$, Hubble parameter and in its first derivative [85]. It is shown that the singularities with phantom matter are different from those of standard matter cosmology. It is also found that the relation between the phantom models and standard matter models are like the duality symmetry of string cosmology [86]. Faraoni [87] studied a spatially flat homogeneous and isotropic universe dominated by a phantom field by the phase space analysis and found that the late time attractors exists for a general phantom potential. Nojiri and Odintsov [88] proposed that early inflation and late time acceleration of the universe may be unified in a single theory based on a phantom field. Although a fundamental quantum phantom is difficult to stabilize [89] nevertheless it is shown that the solution is stable with respect to a small fluctuations of initial data when $m_p^2 \leq \frac{1}{2}$ and small fluctuations of the form of the potential [90]. The phantom field has some strange properties which may be interesting to explore cosmological model even though such theory is known to be unstable with respect to quantum effects. The energy density of phantom increases with increasing scale factor and the phantom energy density becomes infinite at a finite t, known as

"Big Rip" condition [91]. However, the *"Big Rip"* problem may be avoided in some models [92] which meets the current observations fairly well. The finite lifetime of a universe provides an explanation for the apparent coincidence between the current values of the observed matter density and the dark energy density [93]. When the phantom energy becomes strong enough, gravitational instability no longer works and the universe becomes homogeneous. It may be realistic to explore cosmological models with exotic fields, *e.g.* phantom field. Sami and Toporensky [94] examined the nature of future evolution of the universe with potential energy dominated regime of the phantom considering both massive and self interacting phantom potential in addition to matter. It is found that the nature of the future evolution is dependent on the steepness of the field potential. The phantom cosmology has been analyzed adopting phase space analysis technique and found that accelerated universe is an attractor with exponential potential [95]. Singh *et al.* [56] studied the general features of the dynamics of the phantom field. Using inverse coshyperbolic function for the phantom potential it has been demonstrated that it admits $\omega < -1$. In the model it is noted that the de Sitter universe turns out to be the late time attractor. However the time derivative of the field is considered to be zero initially. One of the feature of the phantom field is that it violates the strong energy condition as its kinetic energy is negative. Historically, the pure negative kinetic energy term was first introduced by Hoyle [96] in order to reconcile the homogeneous density based on perfect cosmological principle) by the creation of new matter in voids as a consequence of the expansion of the universe. Later it was reformulated by Hoyle and Narlikar [97] in the context of the steady state theory of the universe which is popularly known

as *"creation" or C-field.* The present objective of introduction of similar field in modern cosmology is to look for an explanation of the present acceleration. One of the disadvantage of the field is that a fundamental quantum phantom is difficult to stabilize [89, 98]. The field has some strange properties, the energy density may increase with time. When the phantom energy becomes strong enough, gravitational instability no longer works and the universe becomes homogeneous. Recently, Kujat *et al.* [99] has studied phantom dark energy models with negative kinetic term and derived the conditions on the potential so that the result is consistent with current cosmological observations and yields a variety of different possible future fates of the universe. Using inverse hyperbolic function for the phantom it was demonstrated that it admits $\omega < -1$ [56]. In the model it is noted that the de Sitter universe turns out to be late time attractor. However the time derivative of the field is considered to be zero initially. We intend to explore the case why $\dot{\phi}$ leads to the usual condition for the accelerating universe in the framework of phantom field. We like to construct a consistent cosmological model out of it.

In this chapter we consider a phantom field with its kinetic energy of the order of the anisotropy in an anisotropic Bianchi- I universe. We consider also a cosmological constant in the theory which may appear due to phantom transition and assume that although in the early universe phantom is negligible, it becomes important as the anisotropy gradually decreases at the late epoch. We explore cosmological solutions for the two cases: (i) the anisotropy of the universe is more than the kinetic term and (ii) anisotropy of the universe less than the kinetic energy of the field. It may be pointed out here that in the absence of both anisotropy and cosmological constant, a kinetic energy dominated regime of the phantom field becomes unrealistic in cosmology, which however becomes important in its presence [100], we explore particularly this aspect of the field here to obtain cosmological model.

The chapter is organized as follows: the field equations are written down in sec. 4.2. Section 4.3 is divided into two parts : in the first part (A) we obtain cosmological solutions for kinetically dominated phantom with cosmological constant and the in the second part (B) we study the critical points corresponding to the set of autonomous equations of an axially symmetric Bianchi-I universe. Finally, in sec. 4.4 we give a brief discussion.

4.2 Field equation

The Einstein's field equation is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu} \tag{4.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric and $T_{\mu\nu}$ is the energy momentum tensor. We consider an anisotropic Bianchi-I metric which is given by

$$ds^2 = -dt^2 + \sum_{i=1}^{3} R_i^2(t)(dx^i)^2 \tag{4.2}$$

where R_1, R_2, R_3 represents the directional scale factors for the universe. The energy momentum tensor is

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{4.3}$$

Where $u^\mu u_\mu = -1$, u^μ is the unit fluid velocity of matter and ρ is the energy density and p is the pressure. For a combination of different kinds of fluid model we have

$\rho = \sum_{i=1}^{n} \rho_i$ and $p = \sum_{i=1}^{n} p_i$ The conservation equation is given by

$$\frac{d\rho}{dt} + \Theta(\rho + p) = 0 \tag{4.4}$$

where Θ is the volume expansion rate. The directional Hubble parameters are defined by

$$H_i = \frac{\dot{R}_i}{R_i} \tag{4.5}$$

and the mean scale factor of the universe is $a(t) = (R_1 R_2 R_3)^{\frac{1}{3}}$. The expansion rate is now given by

$$\Theta = 3H = 3\frac{\dot{a}}{a} = \sum_{i=1}^{3} H_i \tag{4.6}$$

the average anisotropy expansion is given by

$$A = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 \tag{4.7}$$

For the Bianchi metric given by (4.2) we get

$$\sigma^{\mu\nu}\sigma_{\mu\nu} = \sum_{i=1}^{3}(H_i - H)^2 = \frac{6K^2}{a^6}, \ \dot{K} = 0 \tag{4.8}$$

where $\sigma_{\mu\nu}$ represents the shear and the anisotropy parameter becomes

$$A = \frac{K^2}{H^2 V^2} \tag{4.9}$$

where $V = R_1 R_2 R_3 = a^3(t)$. The field eq. (4.1) with eqs. (4.7)-(4.9), can be written as,

$$H^2 = \frac{8\pi G}{3}\rho + \frac{K^2}{a^6} \tag{4.10}$$

It may be pointed out here that if one sets $K = 0$, the equations reduces to that obtained for a flat FRW universe. Thus when the universe is sufficiently large it behave like a flat universe. One can write the energy density of the universe as

$\rho = \rho_o + \rho_1 + \rho_2$, where ρ_o represents vacuum energy determined by cosmological constant, ρ_1 represents the field energy and $\rho_2 = \frac{\rho_{rad}}{a^4}$ represents the radiation energy. We now consider a homogeneous field $\phi = \phi(t)$, the energy density and pressure of the field can be written as

$$\rho = \frac{\epsilon}{2}\dot{\phi}^2 + V(\phi), \;\; p = \frac{\epsilon}{2}\dot{\phi}^2 - V(\phi) \tag{4.11}$$

Here $\epsilon = 1$ corresponds to scalar field and $\epsilon = -1$ corresponds to phantom field. The evolution equation for ϕ is given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{\epsilon}\frac{dV(\phi)}{d\phi} = 0 \tag{4.12}$$

For scalar field we put $\epsilon = 1$, the condition for inflation in the case is realized when $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$. For phantom, we put $\epsilon = -1$, the EOS parameter become

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)} \tag{4.13}$$

It is evident that the conditions $\omega_\phi < -1$ and $\rho_\phi > 0$ could be realized when $0 < \dot{\phi}^2 < 2V(\phi)$ with the phantom also. The condition $\omega_\phi < -1$ is satisfied for phantom only. In the case of a FRW universe with phantom only the essential condition to satisfy is $0 < \dot{\phi}^2 < V(\phi)$, i.e., it requires potential energy to be large compared to kinetic energy [101]. However, in the presence of anisotropy it is evident that the above condition may be relaxed and one can begin with phantom with its kinetic energy domination i.e., $\dot{\phi}^2 > V(\phi)$ provided shear is more than a lower limit $\sigma^2 > \dot{\phi}^2 - V(\phi)$. However, it is important to look for cosmological behaviour for the regime $\dot{\phi}^2 > 2V(\phi)$ and its subsequent evolution. It may be pointed out here that a realistic solution is possible when the kinetic dominated phantom is considered in the presence of a cosmological constant. In this paper considering an anisotropic Bianchi-I universe, cosmological solutions are explored which later may transit to an isotropic universe.

4.3 Cosmological solutions

A. Kinetic Energy Dominated Field

Let us consider a scalar/phantom field in Bianchi-I universe with a kinetic energy dominated epoch in the presence of cosmological constant (Λ). Eq. (4.12) becomes

$$\ddot{\phi} + 3H\dot{\phi} = 0. \tag{4.14}$$

On integrating the above equation we get

$$\dot{\phi} = \pm \frac{C}{a^3} \tag{4.15}$$

where C is an integration constant. We assume an epoch when the kinetic energy ($\sim a^{-6}$) of the field dominates over both the radiation energy density ($\sim a^{-4}$) and matter density ($\sim a^{-3}$). The corresponding Einstein field eq. (4.10) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \beta^2 \left(1 + \frac{\alpha}{a^6}\right) \tag{4.16}$$

where we use the symbols $\alpha = \frac{K^2}{\beta^2} + \frac{\epsilon C^2}{2\Lambda}$ and $\beta = \sqrt{\frac{8\pi G\Lambda}{3}}$. The average scale factor of the universe is obtained on integrating eq. (4.16). In the next section we discuss different cosmological solutions :

- For a scalar field we set $\epsilon = 1$ in eq. (4.12) and on integration (4.16) one obtains

$$a(t) = \alpha^{1/6} \sinh^{1/3}(3\beta t) \tag{4.17}$$

where $\alpha = \frac{K^2}{\beta^2} + \frac{C^2}{2\Lambda} > 0$. The corresponding scalar field evolves as

$$\phi = \phi_o \pm \frac{C}{3\beta\sqrt{\alpha}} ln\ tanh\left(\frac{3\beta}{2}t\right) \tag{4.18}$$

where ϕ_o is a constant. The solution describes a universe from singularity which however transits to an inflationary phase at a later epoch [25]. It has a big bang singularity at $t = 0$, with asymptotic behaviour

$$a(t) \sim t^{1/3} \tag{4.19}$$

and the scalar field evolution near $t \to 0$ goes as

$$\phi(t) = \phi_o - \frac{C}{3\beta\sqrt{\alpha}} ln\left(\frac{3\beta}{2}t\right). \tag{4.20}$$

• For a phantom field, we set $\epsilon = -1$. In this case we discuss different regions determined by the anisotropy parameter.

(i) For $K^2 > \frac{4\pi GC^2}{3}$, the eq. (4.16) admits a hyperbolic solution given by (4.17). But in this case $\alpha_{phantom} < \alpha_{scalar}$ which leads to a smaller universe in case of phantom field filled universe in the early epoch, than a universe filled with scalar field.

Another solution is permitted in the absence of cosmological constant ($\Lambda = 0$). The scale factor follows a power law expansion.

$$a(t) \sim t^{1/3}, \tag{4.21}$$

the corresponding scalar field evolves slowly as :

$$\phi = \phi_o \pm \frac{C}{3\alpha} lnt \tag{4.22}$$

The anisotropy decreases as

$$A = \frac{K^2}{t} \tag{4.23}$$

leading to an isotropic universe. Thus even if initial anisotropy is large compared to the kinetic energy of the field, the anisotropic universe transits to an isotropic universe. In this case the universe is decelerating.

(ii) For $K^2 < \frac{4\pi GC^2}{3}$, the field eq. (4.16) admits a non-singular solution which is

$$a(t) = \tilde{\alpha}^{1/6} \cosh^{1/3}(3\beta t) \tag{4.24}$$

where $\tilde{\alpha} = -\alpha > 0$. The universe originates from a non-singular state. The corresponding evolution of the phantom field is given by

$$\phi = \phi_o \pm \frac{2C}{3\beta\sqrt{\tilde{\alpha}^3}} \tan^{-1}\left[\tanh\left(\frac{3\beta}{2}t\right)\right]. \tag{4.25}$$

The anisotropy in this epoch decreases as

$$A = \frac{4K^2}{\tilde{\alpha}\beta^2 \sinh^2(3\beta t)} \tag{4.26}$$

The universe quickly transits to an inflationary phase with a potential energy dominated phase as the kinetic part decreases rapidly. The asymptotic behaviour of the scalar field at early epoch is

$$\phi_{\pm} = \phi_o \pm \frac{C}{\sqrt{\tilde{\alpha}}}\, t \tag{4.27}$$

The above solution is obtained either with an increasing or a decreasing mode of the field. The solution is new and interesting as it admits an accelerating late universe.

(iii) For $K^2 = \frac{4\pi GC^2}{3}$, the kinetic energy density of the phantom field gets canceled with the anisotropic contribution in the field equation. In this case we note :

- $\Lambda \neq 0$, one obtains de Sitter expansion with kinetic dominated phantom determined by the cosmological constant given by

$$a(t) = a_o e^{\beta t} \tag{4.28}$$

where $\beta = \sqrt{\frac{8\pi GV(\phi)}{3}}$. The solution permits a zero kinetic energy ($\dot{\phi} = 0$) phantom released at a distance from the origin which was considered in [56] to describe cosmological evolution with an inverse coshyperbolic potential with potential energy dominated epoch.

• $\Lambda = 0$, one obtains an interesting solution considering coexistence of two fluids radiation and phantom field. The evolution of the universe is given by

$$a(t) = \left(\frac{32\pi G\rho_{rad}}{3}\right)^{1/4} \sqrt{t} \tag{4.29}$$

and the corresponding field evolves as

$$\phi = \phi_o \pm \frac{\phi_1}{\sqrt{t}} \tag{4.30}$$

which however attains a constant value at a later epoch. In the next section we consider phantom in an exponential potential.

B. Autonomous System with Phantom in an Exponential potential

The field eq. (4.1) for anisotropic Bianchi-I metric can be written as a set of first order non-linear partial differential equations by treating the shear as a massless scalar field [99]. The corresponding field equations with phantom are

$$\dot{H} = -\,4\pi G\left(2\sigma^2 - \dot{\phi}^2\right), \tag{4.31}$$

$$\ddot{\phi} = -3H\dot{\phi} + \frac{d\phi}{dt}, \tag{4.32}$$

$$\dot{\sigma} = -3H\sigma \tag{4.33}$$

together with the constraint equation

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\sigma^2 - \frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{4.34}$$

where dot denotes derivative with respect to cosmic time, shear $\sigma = \frac{1}{\sqrt{24\pi G}}\left(\frac{\dot{R}_1}{R_1} - \frac{\dot{R}_2}{R_2}\right)$ (taking $R_3 = R_2$ axially symmetric Bianchi-I metric). We consider here a phantom field in an exponential potential of the form $V(\phi) = V_o e^{-\sqrt{8\pi G}\,\lambda\phi}$, where V_o and λ are free parameters. It is evident from eq. (4.31) that an exact exponential expansion $\dot{H} = 0$ is admissible in Bianchi-I universe with phantom. Moreover, the field equations may admit $\dot{H} > 0$, a different result when the source consists of a self interacting scalar field [102, 103]. Now making use of the following dimensionless variables

$$x = \sqrt{\frac{4\pi G\dot{\phi}^2}{3H^2}}, \quad y = \sqrt{\frac{8\pi GV}{3H^2}}. \tag{4.35}$$

We obtain a set of plane autonomous system using eqs. (3.31)-(3.34),

$$x' = -3x + 3x(1-y^2) - \sqrt{\frac{3}{2}}\lambda y^2, \tag{4.36}$$

$$y' = -\sqrt{\frac{3}{2}}\lambda xy + 3y(1-y^2) \tag{4.37}$$

where a prime represents derivative with respect to logarithm of the scale factor ($N = \ln a$). The constraint eq. (4.34) now becomes

$$1 = \frac{8\pi G\sigma^2}{3H^2} - x^2 + y^2. \tag{4.38}$$

The phantom density parameter Ω_ϕ and the effective phantom equation of state γ_ϕ are

$$\Omega_\phi = -x^2 + y^2, \quad \Omega_\phi\gamma_\phi = -2x^2. \tag{4.39}$$

It is evident that $\gamma_\phi < 0$, if $x^2 < y^2$.

Critical points and Stability :

We note the following :

- $(0, 0)$ is a critical point. It is saddle if $\Omega_\phi = 0$ with $\omega_\phi = undefined$.
- $\left(-\frac{\lambda}{\sqrt{6}}, \sqrt{1+\frac{\lambda^2}{6}}\right)$ is a critical point. It is stable for $\Omega_\phi = 1$ with $\omega_\phi = -\frac{\lambda^2}{3}$. It is a saddle point if λ is a complex number satisfying the inequality $\lambda^2 < -6$.

Figure 4.1: Variation of ω is shown with N (for $\lambda = \sqrt{2}$ with continuous hne and $\lambda = \sqrt{3}$ with broken line)

4.4 Discussions

We obtain cosmological models with phantom in an anisotropic Bianchi-I universe with or without a cosmological constant. We explore kinetic dominated phantom. In the case of a vanishing cosmological constant, two fluid model (phantom and radiation) is required for a physically relevant solution. We found that a singular solution obtained by Gron [104] (for $\epsilon = 1$) is revisited with phantom for a restricted domain of initial anisotropy. But in the case of scalar field such a solution is permitted for any value of initial anisotropy. In the case of phantom we obtain a new and interesting solution which accommodates a late accelerating universe provided the kinetic energy of the phantom exceeds a lower limit determined by the anisotropy of an anisotropic universe. A pure de Sitter universe is obtained here when the kinetic energy of the phantom is of the order of the anisotropy *i.e.*, at $K^2 = \frac{4\pi GC^2}{3}$. However, in the absence of cosmological constant and for a kinetically dominated phantom the evolution of the universe at $K^2 = \frac{4\pi GC^2}{3}$, is in accordance with the matter content in the universe. In another case we consider radiation with phantom. In that case it leads to a late evolution satisfying the conditions necessary for inflationary universe with inverse coshyperbolic potential as was taken up by Singh *et al.* [56], which may be realized even in an anisotropic background with non zero anisotropy of the universe. We consider phantom in an exponential potential in an axially symmetric Bianchi-I type universe and obtain the set of autonomous equation. It is found that there exist one stable critical point which is interesting. We note that exponential potential admits $a(t) \sim t^{\frac{1}{3}}$ in the presence of shear. However, one obtains power law inflation $a(t) \sim t^D$ with $D > 1$ for $V_o > \frac{1}{4\pi G\lambda^2}$ when $\sigma \to 0$. During this regime the field evolves slowly as $\phi = \sqrt{\frac{1}{2\pi G\lambda^2}} \ln t$. As the anisotropic universe transits to an isotropic universe, the evolution of the universe may be determined by the phantom field. To understand this we plot ω against N in fig. 1 for $\lambda = \sqrt{2}$ (shown by continuous line) and $\lambda = \sqrt{3}$ (shown by broken line) for $y = 1$. It is evident that the value of ω settles to a de Sitter value, admitting late acceleration of the universe. In the literature [56, 89] viable cosmological models with phantom in various potentials have been explored in isotropic FRW model. It may be interesting to work with those potentials in anisotropic universe in future project.