Chapter 2

COSMOLOGICAL MODELS WITH OR WITHOUT PRIMORDIAL BLACK HOLES PAIR

2.1 Introduction

The recent advent of new cosmological precision tests, capable of probing physics at very large redshifts, has changed our earlier views of cosmos. It is now understood from the observational data available from the high redshift surveys of Supernovae, the large scale structure and the Wilkinson Microwave Anisotropy Probe (WMAP) that the present universe is accelerating [54]. The usual matter fields of the standard model in the general theory of relativity are not enough to describe the observational facts in the universe. Thus it is a challenging task to develop a consistent theoretical framework for accommodating the present accelerating phase.s

It is also known from COBE that the universe might have emerged from an inflationary phase of expansion in the very early universe. In the last two decades a large volume of literature [4-7] appeared which discussed early universe with an inflationary phase fairly well either in a scalar field theory or in a curvature squared theory of gravitation. Most successful theory developed so far is based on the dynamics of scalar field with a suitable potential which driven inflation and describe the evolution of the early universe. To understand the present acceleration of the late universe different fields and gravitational theories including scalar fields are examined recently. Cosmological models are proposed taking into account matter described by exotic fields that appeared with a new gravitational physics other than scalar field theory e.g., chaplygin gas [55], phantom fields [56], tachyon field [35] etc. In alternative theory of gravity early infiation may be realized by adding a term $\alpha\, R^2$ to the Einstein-Hilbert action. The modification of the Einstein-Hilbert action with higher order terms in curvature invariants that are become effective in the high curvature region lead to modifications of the standard cosmology admitting a de Sitter universe at early times [57]. In analogy to obtain early inflation it may be important to look for a modification of the Einstein gravitational action with terms that are relevant at low curvature region with a motivation to derive the present cosmic acceleration. Carroll *et al.* [58] wanted to construct a viable cosmological model of the universe accommodating the present accelerating phase of the universe by considering a modified gravitational action which is given by

$$I = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{\mu^4}{R} \right], \tag{2.1}$$

where R is the Ricci Scalar, $\kappa^2 = 8\pi G$, and μ is a mass scale of the order of Hubble scale. The modified theory may be useful to construct an alternative theory of dark energy and dark matter [59]. However, it was soon realized that the above higher derivative theory confronts with the success of general relativity in the solar system tests [60]. It is noted that de Sitter solution obtained in the model is unstable. Later, the theory is modified adding a αR^2 term to the above action, which now permits a stable de Sitter solution [61]. The condition for obtaining a stable de Sitter solution in the theory is that the size of the baby universe should be bigger than a size determined by the parameter μ^2 in the action. The above modified theory of gravity may be important because of the presence of both the $1/R$ and R^2 terms in the action. It leads to a better chance of passing solar system tests than the theory based on simple $1/R$ corrections only to the Einstein-Hilbert action [62]. Cosmological models obtained with a modification of the above action have been used to construct an alternative to dark energy and dark matter models [63]. It is also important to explore the cosmological issues in the framework of new theories which are important at the present epoch. One of the important astrophysical objects, for example, black holes need to be investigated in this theory. The mass of these objects may be greater than the solar mass or even less. It is known in stellar physics that a black hole is the ultimate corpse of a collapsing star when its mass exceeds twice the mass of the Sun. Another kind of black holes are also important in cosmology which might have formed due to quantum fluctuations of matter distribution in the early universe. Later kinds of astrophysical objects are termed as topological black holes, having mass many times smaller than the the solar mass and are hot compared to the surrounding at the present epoch. In particular, black holes having mass $\sim 10^{15}g$ have life time comparable to the age of the universe, and they might be evaporating at present leading to structure formation and accounts for Hawking radiation [64]. The mini black holes may provide sources of ultra high energy cosmic rays including extragalactic gamma-rays burst [65] and other astrophysical phenomena whose mechanisms have not yet been solved completely. In addition to that, the relic black holes might contribute to the energy density of the present universe and may solve the dark matter problem [66]. Bousso and Hawking [49] (in short, BH) calculated the probability of the quantum creation of a universe with a pair of primordial black holes (in short, PBH) in $(3 + 1)$ dimensional universe in GTR in the presence of a cosmological constant. To compute the probability measure, BH considered two different Euclidean space-time : (i) a universe with space-like sections with S^3 - topology and (ii) a universe with space-like section with $S^1 \times S^2$- topology, as is obtained in the Schwarzschild- de Sitter solution. The first kind of spatial structure describes an inflationary (de Sitter) universe without black hole while the second kind describes a Nariai universe [67], which describes an inflationary universe with a pair of black holes. BH again examined the issue in a theory with a massive scalar field which provided an effective cosmological constant for a while through a slow-rolling potential (mass-term). In GTR, Chao [68] studied the creation of a primordial black hole (PBH), Green and Malik [69] studied the primordial black holes production during reheating. Paul *et al.* [70] following the approach of BH studied the probability of creation of PBH in a higher derivative gravity with R^2-term in the Einstein action and found that the probability is very much suppressed in the R^2 -theory. Paul and Saha [71] also studied probability of creation of a pair of black hole with higher order Lagrangian i.e., considering higher loop contributions into the effective action that are higher than quadratic in R. In this chapter we study the probability for quantum creation of an inflationary universe with a pair of black hole in a modified theory of gravity with or without a cosmological constant (Λ). The modified theory of gravity is described including the terms ($\delta\ R^{-1}$) along with αR^2. The theory is interesting as it may be regarded as the synthesis of the theories one needs for early inflation and late acceleration, although it remains to be understood how the universe transits from early inflating phase to the present accelerating phase. It is interesting to investigate how the creation rate of primordial black holes depends on the parameters in the action which are important now. Consequently it is important to study the effects of the new term added in the gravitational action in order to get the present accelerating phase for describing the quantum creation of a universe with a pair of PBH.

We adopt here Busso and Hawking semiclassical approximation technique to evaluate the Euclidean path integrals. The condition that a classical space-time should emerge, to a good approximation, at a large Lorentzian time was selected by a choice of the path of the time parameter τ along the τ^{Re} axis from 0 to $\frac{\pi}{2H}$ and then continues along the τ^{im} axis. The wave-function of the universe in the semiclassical approximation is given by

$$\Psi_o[h_{ij}, \Phi_{\partial M}] \approx \sum_n A_n e^{-I_n} \tag{2.2}$$

where the sum is over the saddle points of the path integral, and I_n denotes the corresponding Euclidean action. The probability measure of the creation of a universe is

$$P[h_{ab}, \Phi_{\partial M}] \sim e^{-2\, I^{Re}}$$

where h_{ab} is the boundary metric and I^{Re} is the real part of the action corresponding to the dominant saddle point, i.e., the classical solution satisfying the Hartle-Hawking (henceforth, HH) boundary conditions [50]. It was believed that all inflationary models lead to $\Omega_o \sim 1$ to a great accuracy. This view was modified after it was discovered that there is a special class of inflaton effective potentials which may lead to a nearly homogeneous open universe with $\Omega_o \leq 1$ at the present epoch. Cornish *et al.* [72, 73] studied the problem of pre-inflationary homogeneity and outlines the possibility of creation of a small, compact, negatively curved universe. We show that a universe with S^3-topology may give birth to an open inflation similar to that obtained in *Ref.* [74].

We explore models of the universe with cosmological issues in the framework of new theories which are also important at the present epoch. Assuming a universe with such primordial black holes we compute [75, 76] here the probability of the quantum creation of a universe in (3 + 1) dimensions considering an action

$$I = \int f(R)\sqrt{g}d^4x$$

where $f(R)$, is a polynomial function in R, R is the Ricci scalar. To study we consider two different Euclidean space-times : (1) a universe with space-like sections with S^3 - topology and (2) a universe with space-like section with $S^1 \times S^2$- topology, as is obtained in the Schwarzschild- de Sitter solution. The first kind of spatial structure describes an inflationary (de Sitter) universe without black hole while the second kind describes a Nariai universe [67], an inflationary universe with a pair of black holes. We found that the probability of a universe with $R \times S^3$ topology turns out to be lower than a universe with topology $R \times S^1 \times S^2$ in some specified restricted conditions in the both modified theories which are physically interesting.

This chapter is organized as follows : in sec. 2.2 the gravitational action for a higher derivative gravity is written and the corresponding dynamical equations are obtained. The gravitational instanton solutions are presented in sec. 2.3, the modified gravitational action is used to estimate the relative probability for the two types of the universes one with PBH and the other without PBH. Finally, in sec. 2.4 we give a brief discussion.

2.2 Gravitational instantons with or without a pair of primordial black holes

We consider the Euclidean action which is given by

$$I_E = -\frac{1}{16\pi}\int d^4x\sqrt{g}\, f(R) - \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}\, Kf'(R) \tag{2.3}$$

where g represents the 4-dimensional Euclidean metric, Λ represents the 4-dimensional cosmological constant, $f(R)$ is a polynomial function in Ricci scalar R. The second term in the action is the gravitational surface term where h_{ij} is the boundary metric and $K = h^{ij}K_{ij}$ is the trace of K_{ij} , the second fundamental form of the boundary ∂M in the given metric g. It may be pointed out here that the second term corresponds to the contribution from $\tau = 0$ back in the gravitational action. The term gives zero contribution for a universe with S^3 - topology, but it gives non zero contribution for $S^1 \times S^2$ - topology.

(A) S^3-Topology, the de Sitter spacetime :

In this section, we first derive the field equations from the Eucliden action and then explore instanton solutions in the modified gravitational theories with spacelike section having S^3 - topology. The corresponding four dimensional Euclidean metric ansatz is

$$ds^2 = d\tau^2 + a^2(\tau)\left[dx_1^2 + sin^2 x_1 \; d\Omega_2^2\right] \tag{2.4}$$

where $a(\tau)$ is the scale factor of a four dimensional universe, τ is the Euclidean time and $d\Omega_2^2$ is a line element on the unit two sphere. The scalar curvature is

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2}{a^2} - \frac{1}{a^2}\right)\right]$$

where an overdot denotes differentiation with respect to τ. We rewrite the action (2.3), using the above constraint through a Lagrangian multiplier β and obtain

$$I_E = -\frac{\pi}{8}\int\left[f(R)a^3 - \beta\left(R + 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2 - 1}{a^2}\right)\right]d\tau - \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}Kf'(R). \tag{2.5}$$

Variation of the action with respect to R and setting it to zero we determine β, which is

$$\beta = a^3 f'(R) \tag{2.6}$$

where prime represents the derivative with respect to R. Substituting β in the action and treating a and R as independent variables we get

$$I_E = -\frac{\pi}{8}\int_{\tau=0}^{T\frac{\pi}{2H}}\left[a^3 f(R) - f'(R)\left(a^3 R - 6a\dot{a}^2 - 6a\right) + 6a^2\dot{a}\dot{R}f''(R)\right]d\tau$$

$$-\frac{3\pi}{4}\left[\dot{a}\; a^2 \; f'(R)\right]_{\tau=0}, \tag{2.7}$$

where we have eliminated $\ddot{a}$ term from the action using integration by parts. The following field equations are now obtained by varying the action with respect to a and R respectively

$$f''(R)\left[R + 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2 - 1}{a^2}\right] = 0, \tag{2.8}$$

$$2f'''(R)\dot{R}^2 + 2f''(R)\left[\ddot{R} + 2\frac{\dot{a}}{a}\dot{R}\right] + f'(R)\left[4\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{2}{a^2} + R\right] - f(R) = 0. \quad (2.9)$$

Let us now consider an action given by the polynomial function in R with $f(R) = R + \frac{\delta}{R} + \alpha R^2 - 2\Lambda$ where α and δ are dimensional coupling parameters, to solve eqs. (2.8)-(2.9). An instanton solution is obtained which is given by

$$a = \frac{1}{H} \sin H\tau \quad (2.10)$$

where $R = 12H^2$ and $H[\delta, \Lambda]$ is determined from the constraint equation which is

$$48H^4 - 16\Lambda H^2 + \delta = 0. \quad (2.11)$$

We note following: (i) A real Hubble parameter is allowed for a positive δ satisfying $\delta \leq \frac{4\Lambda^2}{3}$ (ii) For positive δ, two different classes of instantons are allowed for the region, $0 < \delta < \frac{4\Lambda^2}{3}$ with different H, but only one type of instanton solution is obtained with $H = \sqrt{\frac{\Lambda}{6}}$ for $\delta = \frac{4\Lambda^2}{3}$. However, if $\delta = 0$ one obtains an instanton solution with $H = \sqrt{\frac{\Lambda}{3}}$. (iii) It is also found that now one may obtain an instanton solution even with $\Lambda = 0$ term in the action for a negative δ.

It is evident that the instanton solution (2.10) obtained here satisfies the HH no boundary conditions viz., $a(0) = 0$, $\dot{a}(0) = 1$. One can choose a path along the τ^{Re} axis to $\tau = \frac{\pi}{2H}$, the solution describes half of the Euclidean de Sitter instanton S^3. Analytic continuation of the metric (2.4) to Lorentzian region, $x_1 \rightarrow \frac{\pi}{2} + i\sigma$, gives

$$ds^2 = d\tau^2 + a^2(\tau)\left[-d\sigma^2 + \cosh^2\sigma \, d\Omega_2^2\right]$$

which is a spatially inhomogeneous de Sitter like metric. However, if one sets $\tau = it$ and $\sigma = i\frac{\pi}{2} + \chi$, the metric becomes

$$ds^2 = -dt^2 + b^2(t)\,[d\chi^2 + \sinh^2\chi \, d\Omega_2^2] \quad (2.12)$$

where $b(t) = - i\ a(it)$. The line element (2.12) now describes an open inflationary universe. The real part of the Euclidean action corresponding to the solution calculated by following the complex contour of τ suggested by BH is given by

$$I_{S^3}^{Re} = -\frac{\pi}{8}\left[\frac{1728\alpha H^6 + 144H^4 - 24\Lambda H^2 + \delta}{18H^6}\right]. \tag{2.13}$$

With the chosen path for τ, the solution describes half the de Sitter instanton with S^4 topology, joined to a real Lorentzian hyperboloid of topology $R^1 \times S^3$. It can be joined to any boundary satisfying the condition $a_{\partial M} > 0$. For $a_{\partial M} > H^{-1}$, the wave function oscillates and predicts a classical space-time. We note the following cases :

(i) S^3 solution obtained by BH is recovered when $\delta = 0$, $\alpha = 0$.

(ii) $\delta \neq 0$, $\alpha \neq 0$ and $\Lambda = 0$, in this case the action becomes $I = - 4\pi\left[3\ \alpha + \frac{2}{\sqrt{-3\ \delta}}\right]$. It is evident that a real value of the action demands $\delta < 0$ which supports the action considered by Carrol *et al.* [58].

(iii) if one begins with a positive δ and $\delta = \frac{4\Lambda^2}{3}$ say then the action becomes $I = -12\ \pi\alpha - \frac{2\pi}{\Lambda}$. For a positive α, the Euclidean action is more negative compared to a negative α when the cosmological constant is positive definite.

(B) $S^1 \times S^2$-Topology, the Nariai spacetime:

In this section we first derive the field equation from the Euclidean action and then explore instanton solutions in the modified gravitational theories. Next, we look for a solution with spacelike sections, $S^1 \times S^2$ - topology, which accommodates a pair of black holes in an inflationary universe scenario. We choose an ansatz for (1 + 1 + 2) dimensions which is given by

$$ds^2 = d\tau^2 + a^2(\tau)\ dx^2 + b^2(\tau)\ d\Omega_2^2 \tag{2.14}$$

where $a(\tau)$ is the scale factor of S^1-surface and $b(\tau)$ is the scale factor of the two sphere. The metric for the two-sphere is given by

$$d\Omega_2^2 = d\theta^2 + sin^2\theta \; d\phi^2.$$

The scalar curvature corresponding to the metric (2.14) is

$$R = -\left[2\frac{\ddot{a}}{a} + 4\frac{\ddot{b}}{b} + 2\left(\frac{\dot{b}^2}{b^2} - \frac{1}{b^2}\right) + 4\frac{\dot{a}\dot{b}}{ab}\right]. \tag{2.15}$$

Using the above relation as constraint, we rewrite the Euclidean action (2.3) using Lagrange's undetermined multiplier β, which becomes

$$I_E = -\frac{\pi}{2}\int\left[f(R)ab^2 - \beta\left(R + 2\frac{\ddot{a}}{a} + 4\frac{\ddot{b}}{b} + 4\frac{\dot{a}\dot{b}}{ab} + 2\frac{\dot{b}^2}{b^2} - \frac{2}{b^2}\right)\right]d\tau$$

$$-\frac{1}{8\pi}\int_{\partial M}\sqrt{h}\; d^3x \; K f'(R). \tag{2.16}$$

Adopting similar technique as is used in the previous section, β can be determined, making use of this we get

$$I_{S^1\times S^2} = -\frac{\pi}{2}\int_{\tau=0}^{\tau_{\partial M}}\left[f(R) - f'(R)\left(R - 4\frac{\dot{a}\dot{b}}{ab} - 2\frac{\dot{b}^2}{b^2} - \frac{2}{b^2}\right) + 2f''(R)\dot{R}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\right]$$

$$ab^2 d\tau - \pi\left[\left(\dot{a}b^2 + 2ab\dot{b}\right)f'(R)\right]_{\tau=0}. \tag{2.17}$$

Variation of the action with respect to R, a and b respectively are given by the following equations

$$f''(R)\left[R + 2\frac{\ddot{a}}{a} + 4\frac{\ddot{b}}{b} + 4\frac{\dot{a}\dot{b}}{ab} + 2\frac{\dot{b}^2}{b^2} - \frac{2}{b^2}\right] = 0, \tag{2.18}$$

$$2f'''(R)\dot{R}^2 + 2f''(R)\left[\ddot{R} + 2\dot{R}\frac{\dot{b}}{b}\right] + f'(R)\left[R + 4\frac{\ddot{b}}{b} + 2\frac{\dot{b}^2 - 1}{b^2}\right] - f(R) = 0, \tag{2.19}$$

$$2f'''(R)\dot{R}^2 + 2f''(R)\left[\dot{R}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) + \ddot{R}\right] + f'(R)\left[R + 2\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} + 2\frac{\dot{a}\dot{b}}{ab}\right]$$

$$- f(R) = 0. \tag{2.20}$$

To solve the eqs. (2.18)-(2.20), we consider a polynomial function in R : $f(R) = R + \frac{\delta}{R} + \alpha\ R^2 - 2\Lambda$ as considered in the previous section. The set of eqs. (2.18)-(2.20) admit an instanton solution which is given by

$$a = \frac{1}{H_o}\ sin(H_o\tau), \quad b = H_o^{-1},$$

$$R = 4H_o^2 \tag{2.21}$$

where H_o satisfies the constraint equation

$$16H_o^4 - 16\Lambda H_o^2 + 3\delta = 0. \tag{2.22}$$

We note the following : (i) For a positive δ, we get two classes of instanton solutions when $0 < \delta < \frac{4\Lambda^2}{3}$, and only one kind when $\delta = 0$ with $H_0^2 = \Lambda$. (ii) However, for a positive δ with $\delta = \frac{4}{3}\Lambda^2$ one obtains an instanton solution with $H_o^2 = \frac{\Lambda}{2}$, (iii) For a negative δ, instanton solutions are obtained even without a cosmological constant with $H_o^2 = \sqrt{-\frac{3\delta}{16}}$.

It is evident that the instanton solution (2.21) satisfies the HH boundary conditions $a(0) = 0$, $\dot{a}(0) = 1, b(0) = b_o$, $\dot{b}(0) = 0$. Analytic continuation of the metric (2.14) to Lorentzian region, i.e., $\tau \longrightarrow it$ and $x \longrightarrow \frac{\pi}{2} + i\sigma$ yields

$$ds^2 = -dt^2 + c^2(t)\ d\sigma^2 + H_o^{-2}\ d\Omega_2^2, \tag{2.23}$$

where $c(t) = -\ i\ a(it)$. In this case the analytic continuation of time and space do not give an open inflationary universe. The corresponding Lorentzian solution is

$$a(\tau^{Im})|_{\tau^{Re} = \frac{\pi}{2H_o}} = H_o^{-1} \cosh H_o\tau^{Im},$$

$$b(\tau^{Im})|_{\tau^{Re}=\frac{\pi}{2H_o}} = H_o^{-1}. \tag{2.24}$$

Its space like sections can be visualized as three spheres of radius H_o^{-1} with a *hole* of radius $b = H_o^{-1}$ punched through the north and south poles. The physical interpretation of the solution is that of two - spheres containing two black holes at opposite ends. The black holes have the radius H_o^{-1} which accelerates away from each other with the expansion of the universe. This describes half of a Lorentzian Nariai universe. The real part of the action can be determined following the contour suggested by BH [49], and it is given by

$$I^{Re}_{S^1\times S^2} = -\frac{\pi}{8H_o^2}\left[64\alpha H_o^6 + 16H_o^4 - 8\Lambda H_0^2 + \delta\right]. \tag{2.25}$$

The Lorentzian solution given by eq.(2.24) describes a universe with two black holes at the poles of a two sphere. It may be pointed out here that the contribution of the integrand in the action (2.17) for the instanton vanishes and a non-zero contribution of the action here arises just from the boundary term only. We note the following :

(i) $\delta = 0$, $\alpha = 0$, corresponds to the action obtained by BH [49].

(ii) $\delta \neq 0$, $\alpha \neq 0$, we get

(a) for $\delta = \frac{4\Lambda^2}{3}$, the action becomes

$$I^{Re}_{S^1\times S^2} = \frac{2}{3}\left(-12\pi\alpha - \frac{2\pi}{\Lambda}\right) = \frac{2}{3} I^{Re}_{S^3}$$

(b) for $\Lambda = 0$, the action is

$$I^{Re}_{S^1\times S^2} = -\frac{8\pi}{3}\left(3\alpha + \frac{2}{\sqrt{-3\,\delta}}\right) = \frac{2}{3} I^{Re}_{S^3}$$

Thus it is evident that a physically realistic solution is obtained for $\delta < 0$.

2.3 Evaluation of probability for the primordial black holes pair creation

In the previous sections the gravitational action for an inflationary universe with or without a pair of black holes are calculated. It is now possible to compare the probability measures in the two cases. The probability for creation of a de Sitter universe is determined from the action (2.7). The de Sitter universe does not accommodate a pair of black holes. The probability for nucleation of an inflationary universe without a pair of Black holes becomes

$$P_{S^3} \sim e^{\frac{\pi}{9}\left[\frac{216\alpha H^4 + 12H^2 - \Lambda}{H^4}\right]} \tag{2.26}$$

However, for an inflationary universe with a pair of black holes the corresponding probability of nucleation can be computed from the action (2.17) which becomes :

$$P_{S^1 \times S^2} \sim e^{\frac{2\pi}{3}\left[\frac{24\alpha H_o^4 + 4H_0^2 - \Lambda}{H_0^4}\right]} \tag{2.27}$$

The following physically interesting results emerged:

(i) The probabilities obtained by Bousso and Hawking [49] is recovered when both δ and α are set zero. The probabilities are :

$$P_{S^3} \sim e^{\frac{3\pi}{\Lambda}}, \quad P_{S^1 \times S^2} \sim e^{\frac{2\pi}{\Lambda}}. \tag{2.28}$$

In this case the probability for a universe with PBH is less than that without PBH. The instanton solution does not admit a negative cosmological constant.

(ii) For $\alpha \neq 0$ with a positive δ the probabilities for the creation of a universe without PBH pair and with PBH pair are given by

$$P_{S^3} \sim e^{\frac{4\pi}{\Lambda} + 24\pi\alpha}, \quad P_{S^1 \times S^2} \sim e^{\frac{8\pi}{3\Lambda} + 16\pi\alpha} \tag{2.29}$$

respectively when $\delta = \frac{4\Lambda^2}{3}$. It is evident that a positive cosmological constant and $\alpha > 0$ lead to a case where de Sitter universe is more probable than a universe with a pair of black holes. However, for $\alpha < -\frac{1}{6\Lambda}$, we note that creation of a universe with a pair of PBH is more probable. However, the solution is not interesting as the Friedmann universe is unstable in this case.

(iii) If the parameter in the modified gravitational action δ is considered negative, then instanton solutions are permitted even in the absence of a cosmological constant. The evaluated probabilities are :

$$P_{S^3} \sim e^{24\pi\alpha - \frac{16\pi}{\sqrt{-3\delta}}}, \quad P_{S^1 \times S^2} \sim e^{16\pi\alpha - \frac{32\pi}{3\sqrt{-3\delta}}}. \tag{2.30}$$

Thus, a universe without a PBH is more probable for a positive value of the parameter α and a negative δ. In other case, we note that it is less probable if one considers a negative α satisfying the inequality $\alpha < -\frac{2}{3}\frac{1}{\sqrt{-3\delta}}$, which gets determined by a negative δ also.

2.4 Discussions

The probability for creation of an inflationary universe with or without a pair of primordial black holes (PBH) in a modified theory of gravity are studied. The modified gravitational action considered here contains higher order terms including a term inverse power in curvature scalar. Although the inverse scalar curvature term is important in describing late universe and is not important in the quantum creation era of the universe, we examined here how the parameter δ in the action may play an important role in deciding a universe with PBH pair in the inflationary universe scenario. In section 2.2, gravitational instanton solutions are found in the two cases

: (A) a universe with S^3 - topology and (B) a universe with $S^1 \times S^2$ - topology respectively. The Euclidean action is then evaluated corresponding to the instanton solutions using BH technique [58]. It is found that the probability of a universe with S^3 topology turns out to be lower than a universe with topology $S^1 \times S^2$ in some specified restricted values of the parameters in the action. It may be mentioned here that one gets a regular instanton in S^4 - topology if there is no black hole. The existence of black hole restricts such a regular topology. The evaluated values obtained here on the probability of creation of a universe with a pair of primordial black holes are found to be strongly suppressed depending on the parameters α, δ in the action. Although, δ is small in the early epoch, its sign and magnitude become important which decides the topology of the universe to begin with. It is noted that the probability for a universe with PBH is strongly suppressed for small values of δ lying in the sector $0 < \delta < \frac{4\Lambda^2}{3}$, in the presence of a positive cosmological constant.

An interesting solution in the framework of a modified gravitational action with inverse power of R -theory, which admits de Sitter instantons with S^3 and $S^1 \times S^2$ topologies is obtained even without a cosmological constant. It is found that if $\delta = \frac{4\Lambda^2}{3}$, the probability of a universe with a pair of PBH is suppressed in case (a) $\Lambda < -\frac{1}{6\alpha}$ with a positive α and (b) $\Lambda > \frac{1}{6|\alpha|}$ for a negative α. However, a negative δ is required if one considers a theory without a cosmological constant which supports the action considered by Carroll *et al.* [58] to obtain an accelerating universe. Unfortunately the above theory confronts with the success of GR in the solar system, the sign of the coupling constant with inverse power of the scalar curvature gets further support from our study.

Another interesting solution is that a de Sitter universe is less probable if $\alpha < -\frac{1}{6\Lambda}$. The result obtained earlier with curvature square term in the action in Ref. [70] without a δ-term in the action now can be compared with that when δ is non zero. The probability of a universe with a pair of PBH is suppressed when $\alpha > 0$ and enhances when $\alpha < 0$, which is however gets determined by the value of $\delta(\Lambda)$ term in the action. If $\Lambda = 0$ and $\delta < 0$, probability of a universe with a pair of PBH is suppressed for a positive α, however for a negative α there is a lower bound imposed by the parameter δ ($|\alpha| < \frac{2}{3\sqrt{3|\delta|}}$). It is interesting to note here that analytic continuation of a $R \times S^3$ metric considered here to Lorentzian region leads to an open 3 - space. One obtains Hawking-Turok [74] type open inflationary universe in this case. In the other spatial section, analytic continuation of $R \times S^1 \times S^2$-topology an open inflationary universe is never attained. Thus in a modified Lagrangian with inverse power in R-theory in addition to R^2 -term, quantum creation of PBH seems to be suppressed in the minisuperspace cosmology for some values of the parameters in the action, which are determined here. Another interesting result obtained here is that gravitational instanton solution may be obtained in this case even with a negative cosmological constant which is however not permitted in the case considered by BH [49].