CHAPTER-6

SOME RESULTS ON THE L(p, q)-TH ORDER OF THE DERIVATIVE OF A MEROMORPHIC FUNCTION

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6.1 Introduction, Definitions and Notations.

It is well known $\{[\mathbf{64}], p.36\}$ that the order of the derivative of an entire function is equal to the order of the function. The same result is proved for a meromorphic function in $\{[\mathbf{9}], [\mathbf{62}], [\mathbf{65}]\}$. Juneja, Kapoor and Bajpai [**31**] defined the *(p, q)th order* and *(p, q)th lower order* of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \quad and \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \quad and \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$, for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

Somasundaram and Thamizharasi [**60**] introduced the notion of *L − order* and *L − type* for entire functions where $L \equiv L(r)$ is a positive

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [**22**].

continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every constant $'a'$. Their definitions are as follows:

Definition 6.1.1 *[60]The $L-$order ρ_f^L and $L-$lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \ \text{and} \ \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]} \ \text{and} \ \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

So with the help of the above notion one can easily define the $L-(p,q)th$ order and $L-(p,q)th$ lower order of entire and meromorphic functions.

Definition 6.1.2 *The $L-(p,q)th$ order $\rho_f^L(p,q)$ and $L-(p,q)th$ lower order $\lambda_f^L(p,q)$ of an entire function f are respectively defined as :*

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}[rL(r)]} \ \text{and} \ \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}[rL(r)]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]} \ \text{and} \ \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]}.$$

where p, q are positive integers and $p > q$.

The more generalised concept of $L-(p,q)th$ *order* and $L-(p,q)th$ *lower order* of entire and meromorphic functions are $L^*-(p,q)th$ *order* and $L^*-(p,q)th$ *lower order* respectively. In order to prove our results we require the following definitions:

Definition 6.1.3 *The L^*-order, L^*-lower order and L^*-type of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\text{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\text{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

Definition 6.1.4 *The $L^* - (p,q)$th order $\rho_f^{L^*}(p,q)$ and $L^* - (p,q)$th lower order $\lambda_f^{L^*}(p,q)$ of an entire function f are defined as*

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

In **[36]** and **[38]** Lahiri proved that the generalised order (generalised lower order) of a meromorphic function f is equal to the generalised order of its derivative f'. In the chapter we establish a relationship between the $L - (p,q)$*th order* of the derivative of a meromorphic function and that of the original function.

6.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 6.2.1 *[38]Let f be a transcendental meromorphic function. Then*

$$T(r,f') \le 2T(2r,f) + o\{T(2r,f)\}$$

for all large values of r.

Lemma 6.2.2 *{Theorem 4.1, [67]; see also Lemma C, [11]}. Let f be a meromorphic function. Then for all large r,*

$$T(r,f) < C\left\{T\left(2r,f'\right)+\log r\right\}$$

where C is a constant which is only dependent on $f(0)$.

6.3 Theorems.

In this section we present the main results of the chapter.

Theorem 6.3.1 *The $L-(p,q)$th order of a meromorphic function f is equal to the $L-(p,q)$th order of its derivative f' where p,q are positive integers and $p>q$.*

Proof. We suppose that f is a transcendental meromorphic function because otherwise the theorem follows easily. From Lemma 6.2.1 we get by taking logarithms $(p-1)$ *times*

$$\log^{[p-1]} T(r,f') \le \log^{[p-1]} T(2r,f) + O(1)$$

which gives that

$$\begin{aligned}\rho^L_{f'}(p,q) &\le \limsup_{r\to\infty}\left\{\frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]}\cdot\frac{1}{1-\frac{\log 2}{\log^{[q]}[rL(r)]}}\right\}\\ &= \limsup_{r\to\infty}\frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]}\cdot\lim_{r\to\infty}\frac{1}{1-\frac{\log 2}{\log^{[q]}[rL(r)]}}\\ &= \rho^L_f(p,q). \qquad (6.1)\end{aligned}$$

Since f is transcendental, we have $\log r = o\{T(r,f)\}$. From Lemma 6.2.2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r,f) + O(1) \le \log^{[p-1]} T(2r,f')$$

which gives

$$\rho^L_f(p,q) \le \limsup_{r\to\infty}\frac{\log^{[p-1]} T(r,f')}{\log^{[q]}[rL(r)]}\cdot\lim_{r\to\infty}\frac{1}{1-\frac{\log 2}{\log^{[q]}[rL(r)]}}$$

$$i.e.,\quad \rho^L_f(p,q) \le \rho^L_{f'}(p,q). \qquad (6.2)$$

Thus the theorem follows from (6.1) and (6.2). ■

Theorem 6.3.2 *The $L-(p,q)$th lower order of a meromorphic function f is equal to the $L-(p,q)$th lower order of its derivative f' where p,q are positive integers and $p>q$.*

Proof. Let us suppose that f be a transcendental meromorphic function because otherwise the theorem follows easily. From Lemma 6.2.1 we get by taking logarithms $(p-1)$ *times*

$$\log^{[p-1]} T(r,f') \leq \log^{[p-1]} T(2r,f) + O(1)$$

which gives that

$$\begin{aligned}\lambda_{f'}^{L}(p,q) &\leq \liminf_{r\to\infty}\left\{\frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]}.\frac{1}{1-\frac{\log 2}{\log^{[q]}[rL(r)]}}\right\}\\ &= \liminf_{r\to\infty}\frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]}.\lim_{r\to\infty}\frac{1}{1-\frac{\log 2}{\log^{[q]}[rL(r)]}}\\ &= \lambda_{f}^{L}(p,q). \qquad (6.3)\end{aligned}$$

Since f is transcendental, we have $\log r = o\{T(r,f)\}$. From Lemma 6.2.2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r,f) + O(1) \leq \log^{[p-1]} T(2r,f')$$

which gives

$$\lambda_{f}^{L}(p,q) \leq \liminf_{r\to\infty}\frac{\log^{[p-1]} T(r,f')}{\log^{[q]}[rL(r)]}.\lim_{r\to\infty}\frac{1}{1-\frac{\log 2}{\log^{[q]}[rL(r)]}}$$

$$i.e., \quad \lambda_{f}^{L}(p,q) \leq \lambda_{f'}^{L}(p,q). \qquad (6.4)$$

Thus the theorem follows from (6.3) and (6.4). ■

Theorem 6.3.3 *If f is a transcendental meromorphic function having a finite number of zeros with $f(0)\neq 0,\infty$; $f'(0)\neq 0$ and $\rho_f^L(2,1)<\infty$, then $\rho_{f'}^L(p,q)=\rho_f^L(p,q)$ and $\lambda_{f'}^L(p,q)=\lambda_f^L(p,q)$ where p,q are positive integers and $p>q$.*

Proof. From $\{$*Theorem* 2.2, [**29**], *p*.40$\}$ we know that $m(r,\frac{f'}{f})=O(\log r)$. Also by $\{$*Theorem* 2.3, [**29**], *p*.41$\}$ we obtain in the present case,

$$\log r = o\{T(r,f)\} \quad as \quad r\to\infty.$$

So combining the two

$$m(r,\frac{f'}{f}) = o\{T(r,f)\} \quad as \quad r \to \infty.$$

Since f has a finite number of zeros, it is clear that $N(r,\frac{1}{f}) = O(\log r)$. Hence,

$$N(r,\frac{1}{f}) = o\{T(r,f)\} \ as\ r \to \infty.$$

Now

$$m(r,f') \le m(r,\frac{f'}{f}) + m(r,f)$$

$$i.e.,\quad m(r,f') \le m(r,f) + o\{T(r,f)\} \quad as \quad r \to \infty.$$

Also if f has a pole of order p at z_o, $f'(z)$ has a pole of order $p+1 \le 2p$, so that $N(r,f') \le 2N(r,f)$ $\{p.56, [\mathbf{29}]\}$. Thus by addition we deduce that

$$T(r,f') \le m(r,f) + 2N(r,f) + o\{T(r,f)\}$$

$$i.e.,\quad T(r,f') \le 2T(r,f) + o\{T(r,f)\}$$

$$i.e.,\quad T(r,f') \le \{2+o(1)\}T(r,f) \quad as \quad r \to \infty. \tag{6.5}$$

This gives that

$$\rho_{f'}^{L}(p,q) \le \rho_{f}^{L}(p,q). \tag{6.6}$$

Again we have

$$T(r,f) = m(r,\frac{1}{f}) + N(r,\frac{1}{f}) + O(1)$$

$$i.e.,\quad T(r,f) \le m(r,\frac{1}{f'}) + m(r,\frac{f'}{f}) + N(r,\frac{1}{f}) + O(1)$$

$$i.e.,\quad T(r,f) \le m(r,\frac{1}{f'}) + o\{T(r,f)\}$$

$$i.e.,\quad T(r,f) \le T(r,\frac{1}{f'}) + o\{T(r,f)\}$$

$$i.e.,\quad T(r,f) \le T(r,f') + o\{T(r,f)\} \quad as \quad r \to \infty$$

$$i.e.,\quad \{1+o(1)\}T(r,f) \le T(r,f') \quad as \quad r \to \infty. \tag{6.7}$$

This gives that

$$\rho_{f}^{L}(p,q) \le \rho_{f'}^{L}(p,q). \tag{6.8}$$

Thus the first part of the theorem follows from (6.6) and (6.8).

By (6.5) we may also deduce that

$$\lambda_{f'}^{L}(p,q) \leq \lambda_{f}^{L}(p,q). \tag{6.9}$$

(6.7) also gives that

$$\lambda_{f}^{L}(p,q) \leq \lambda_{f'}^{L}(p,q). \tag{6.10}$$

Thus the second part of the theorem follows from (6.9) and (6.10). ■

Remark 6.3.1 *Theorem* 6.3.3 *can also be proved with a lesser hypothesis* '$N(r, \frac{1}{f}) = O(\log r)$' *than 'having a finite number of zeros'.*

In the chapter we also state a relationship with proof between the $L^* - (p,q)$th order of the derivative of a meromorphic function and that of the original function.

Theorem 6.3.4 *The $L^* - (p,q)$th order of a meromorphic function f is equal to the $L^* - (p,q)$th order of its derivative f' where p, q are positive integers and $p > q$.*

Proof. We suppose that f is a transcendental meromorphic function because otherwise the theorem follows easily. From Lemma 6.2.1 we get by taking logarithms $(p-1)$ *times*

$$\log^{[p-1]} T(r,f') \leq \log^{[p-1]} T(2r,f) + O(1)$$

which gives that

$$\begin{aligned} \rho_{f'}^{L^*}(p,q) &\leq \limsup_{r\to\infty} \left\{ \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \cdot \frac{1}{1 - \frac{\log 2}{\log^{[q]}\left[re^{L(r)}\right]}} \right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} \cdot \lim_{r\to\infty} \frac{1}{1 - \frac{\log 2}{\log^{[q]}\left[re^{L(r)}\right]}} \\ &= \rho_{f}^{L^*}(p,q). \end{aligned} \tag{6.11}$$

Since f is transcendental, we have $\log r = o\{T(r,f)\}$. From Lemma 6.2.2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r,f) + O(1) \leq \log^{[p-1]} T(2r,f')$$

which gives

$$\rho_f^{L^*}(p,q) \le \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f')}{\log^{[q]}\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{1}{1-\frac{\log 2}{\log^{[q]}\left[re^{L(r)}\right]}}$$

$$i.e., \quad \rho_f^{L^*}(p,q) \le \rho_{f'}^{L^*}(p,q). \tag{6.12}$$

Thus the theorem follows from (6.11) and (6.12). ■

Theorem 6.3.5 *The $L^* - (p,q)$th lower order of a meromorphic function f is equal to the $L^* - (p,q)$th lower order of its derivative f' where p,q are positive integers and $p > q$.*

Proof. Let us suppose that f be a transcendental meromorphic function because otherwise the theorem follows easily. From Lemma 6.2.1 we get by taking logarithms $(p-1)$ *times*

$$\log^{[p-1]} T(r,f') \le \log^{[p-1]} T(2r,f) + O(1)$$

which gives that

$$\begin{aligned}
\lambda_{f'}^{L^*}(p,q) &\le \liminf_{r\to\infty} \left\{ \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} . \frac{1}{1-\frac{\log 2}{\log^{[q]}\left[re^{L(r)}\right]}} \right\} \\
&= \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{1}{1-\frac{\log 2}{\log^{[q]}\left[re^{L(r)}\right]}} \\
&= \lambda_f^{L^*}(p,q).
\end{aligned} \tag{6.13}$$

Since f is transcendental, we have $\log r = o\{T(r,f)\}$. From Lemma 6.2.2 we obtain by taking repeated logarithms

$$\log^{[p-1]} T(r,f) + O(1) \le \log^{[p-1]} T(2r,f')$$

which gives

$$\lambda_f^{L^*}(p,q) \le \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f')}{\log^{[q]}\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{1}{1-\frac{\log 2}{\log^{[q]}\left[re^{L(r)}\right]}}$$

$$i.e., \quad \lambda_f^{L^*}(p,q) \le \lambda_{f'}^{L^*}(p,q). \tag{6.14}$$

Thus the theorem follows from (6.13) and (6.14). ■

Theorem 6.3.6 *If f is a transcendental meromorphic function having a finite number of zeros with $f(0) \neq 0, \infty$; $f'(0) \neq 0$ and $\rho_f^{L^*}(2,1) < \infty$, then $\rho_{f'}^{L^*}(p,q) = \rho_f^{L^*}(p,q)$ and $\lambda_{f'}^{L^*}(p,q) = \lambda_f^{L^*}(p,q)$ where p, q are positive integers and $p > q$.*

Proof. From (6.5) we may deduce that

$$\rho_{f'}^{L^*}(p,q) \leq \rho_f^{L^*}(p,q) \text{ and } \quad \lambda_{f'}^{L^*}(p,q) \leq \lambda_f^{L^*}(p,q). \tag{6.15}$$

Also (6.7) gives that

$$\rho_f^{L^*}(p,q) \leq \rho_{f'}^{L^*}(p,q) \text{ and } \lambda_f^{L^*}(p,q) \leq \lambda_{f'}^{L^*}(p,q). \tag{6.16}$$

Thus the theorem follows from (6.15) and (6.16). ∎

Remark 6.3.2 *Theorem* 6.3.6 *can also be proved with a lesser hypothesis '$N(r, \frac{1}{f}) = O(\log r)$' than 'having a finite number of zeros'.*

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