CHAPTER-5

GROWTH ESTIMATES OF ENTIRE FUNCTIONS BASED ON RELATIVE L(p, q)-TH ORDER

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5.1 Introduction, Definitions and Notations.

Let f and g be two entire functions and $F(r) = \max\{|f(z)| : |z| = r\}, G(r) = \max\{|g(z)| : |z| = r\}$. If f is non constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ exists and is such that $\lim_{s\rightarrow\infty} F^{-1}(s) = \infty$.

Bernal [2] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : F(r) < G(r^\mu) \quad for\ all \quad r > r_0(\mu) > 0\} \\ &= \limsup_{r\rightarrow\infty} \frac{\log G^{-1}F(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [63] if $g(z) = \exp z$. Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\rightarrow\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

Somasundaram and Thamizharasi [60] introduced the notions of $L-order$, $L-lower\ order$ and $L-type$ for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \rightarrow \infty$ for every positive constant 'a'. Their definitions are as follows:

The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [21].

Definition 5.1.1 *[60]The $L-$ order ρ_f^L and the $L-$ lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log\left[rL(r)\right]} \quad and \quad \lambda_f^L = \liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log\left[rL(r)\right]},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log\left[rL(r)\right]} \quad and \quad \lambda_f^L = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log\left[rL(r)\right]}.$$

Definition 5.1.2 *[60]The $L-$type σ_f^L of an entire function f with $L-$order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty}\frac{\log M(r,f)}{\left[rL(r)\right]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

For meromorphic f, the $L-type$ σ_f^L becomes

$$\sigma_f^L = \limsup_{r\to\infty}\frac{T(r,f)}{\left[rL(r)\right]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

Juneja, Kapoor and Bajpai **[31]** defined the $(p,q)th$ *order* and the $(p,q)th$ *lower order* of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty}\frac{\log^{[p]} M(r,f)}{\log^{[q]} r}$$
$$and \quad \lambda_f(p,q) = \liminf_{r\to\infty}\frac{\log^{[p]} M(r,f)}{\log^{[q]} r}.$$

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty}\frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r}$$
$$and \quad \lambda_f(p,q) = \liminf_{r\to\infty}\frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

So with the help of the above notion one can easily define the *relative* $L-(p,q)th$ *order* and *relative* $L-(p,q)th$ *lower order* of entire functions.

Definition 5.1.3 *The relative $L-(p,q)$th order and relative $L-(p,q)$th lower order of an entire function f with respect to another entire function g respectively denoted by ${}^{L}\rho_g^f(p,q)$ and ${}^{L}\lambda_g^f(p,q)$ are defined as*

$$ {}^{L}\rho_g^f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}\left[rL(r)\right]}$$

$$\text{and} \quad {}^{L}\lambda_g^f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}\left[rL(r)\right]},$$

where p, q are positive integers and $p > q$.

The more generalised concept of $L-$ *order* and $L-$ *type* of entire and meromorphic functions are L^*- *order* and L^*- *type* respectively. Their definitions are as follows:

Definition 5.1.4 *The L^*- order, L^*- lower order and L^*- type of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\text{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

$$\text{and} \quad \sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

Definition 5.1.5 *The relative $L^*-(p,q)$th order ${}^{L^*}\rho_g^f(p,q)$ and the relative $L^*-(p,q)$th lower order ${}^{L^*}\lambda_g^f(p,q)$ of an entire function f with respect to another entire function g are defined as*

$$ {}^{L^*}\rho_g^f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}\left[re^{L(r)}\right]}$$

$$\text{and} \quad {}^{L^*}\lambda_g^f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

In the chapter we establish some results on the growth properties of entire functions on the basis of *relative* $L-(p,q)$*th order* and *relative* $L-(p,q)$*th lower order* where p, q are positive integers with $p > q$.

5.2 Theorems.

In this section we present the main results of the chapter.

In the following theorems we see the application of *relative* $L-(p,q)$*th order* and *relative* $L-(p,q)$*th lower order* in estimating the growth properties of entire functions where p, q are positive integers with $p > q$.

Theorem 5.2.1 *Let f, g and h be three entire functions such that* $0 < {}^L\lambda_g^f(p,q) \leq {}^L\rho_g^f(p,q) < \infty$ *and* $0 < {}^L\lambda_g^h(m,q) \leq {}^L\rho_g^h(m,q) < \infty$. *Then*

$$\frac{{}^L\lambda_g^f(p,q)}{{}^L\rho_g^h(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^L\lambda_g^f(p,q)}{{}^L\lambda_g^h(m,q)}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^L\rho_g^f(p,q)}{{}^L\lambda_g^h(m,q)},$$

where p, q, m are positive integers with $q < \min\{p, m\}$.

Proof. From the definition of relative $L-(p,q)$*th order* and *relative* $L-(p,q)$*th lower order* we have for arbitrary positive ϵ and for all large values of r,

$$\log^{[p]} G^{-1}F(r) \geq \left({}^L\lambda_g^f(p,q) - \epsilon\right) \log^{[q]}\left[rL(r)\right] \tag{5.1}$$

$$\text{and} \quad \log^{[m]} G^{-1}H(r) \leq \left({}^L\rho_g^h(m,q) + \epsilon\right) \log^{[q]}\left[rL(r)\right]. \tag{5.2}$$

Now from (5.1) and (5.2) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^L\lambda_g^f(p,q) - \epsilon}{{}^L\rho_g^h(m,q) + \epsilon}.$$

As $\epsilon\ (> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^L\lambda_g^f(p,q)}{{}^L\rho_g^h(m,q)}. \tag{5.3}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \leq ({}^{L}\lambda_g^f(p,q) + \epsilon) \log^{[q]} [rL(r)] \tag{5.4}$$

and for all large values of r,

$$\log^{[m]} G^{-1}H(r) \geq ({}^{L}\lambda_g^h(m,q) - \epsilon) \log^{[q]} [rL(r)] . \tag{5.5}$$

So combining (5.4) and (5.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\lambda_g^f(p,q) + \epsilon}{{}^{L}\lambda_g^h(m,q) - \epsilon}.$$

Since ϵ (> 0) is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\lambda_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}. \tag{5.6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \leq ({}^{L}\lambda_g^h(m,q) + \epsilon) \log^{[q]} [rL(r)] . \tag{5.7}$$

Now from (5.1) and (5.7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L}\lambda_g^f(p,q) - \epsilon}{{}^{L}\lambda_g^h(m,q) + \epsilon}.$$

Choosing $\epsilon \to 0$ we get that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} \geq \frac{{}^{L}\lambda_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}. \tag{5.8}$$

Also for all large values of r,

$$\log^{[p]} G^{-1}F(r) \leq ({}^{L}\rho_g^f(p,q) + \epsilon) \log^{[q]} [rL(r)] . \tag{5.9}$$

So from (5.5) and (5.9) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_g^f(p,q) + \epsilon}{{}^{L}\lambda_g^h(m,q) - \epsilon}.$$

As ϵ (> 0) is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}. \tag{5.10}$$

Thus the theorem follows from (5.3),(5.6),(5.8) and (5.10). ■

Theorem 5.2.2 *Let f, g and h be three entire functions with* $0 < {}^{L}\lambda_{g}^{f}(p,q) \leq {}^{L}\rho_{g}^{f}(p,q) < \infty$ *and* $0 < {}^{L}\rho_{g}^{h}(m,q) < \infty$, *where p, q, m are positive integers with* $q < \min\{p, m\}$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_{g}^{f}(p,q)}{{}^{L}\rho_{g}^{h}(m,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)}.$$

Proof. From the definition of relative $L-(p,q)th$ *order* we get for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \geq \left({}^{L}\rho_{g}^{h}(m,q) - \epsilon\right) \log^{[q]} \left[rL(r)\right]. \tag{5.11}$$

Now from (5.9) and (5.11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_{g}^{f}(p,q) + \epsilon}{{}^{L}\rho_{g}^{h}(m,q) - \epsilon}.$$

As ϵ (> 0) is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L}\rho_{g}^{f}(p,q)}{{}^{L}\rho_{g}^{h}(m,q)}. \tag{5.12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \geq \left({}^{L}\rho_{g}^{f}(p,q) - \epsilon\right) \log^{[q]} \left[rL(r)\right]. \tag{5.13}$$

So combining (5.2) and (5.13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L}\rho_{g}^{f}(p,q) - \epsilon}{{}^{L}\rho_{g}^{h}(m,q) + \epsilon}.$$

Since ϵ (> 0) is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L}\rho_{g}^{f}(p,q)}{{}^{L}\rho_{g}^{h}(m,q)}. \tag{5.14}$$

Thus the theorem follows from (5.12) and (5.14). ■

The following theorem is a natural consequence of Theorem 5.2.1 and Theorem 5.2.2.

Theorem 5.2.3 *Let f, g and h be three entire functions with* $0 < {}^{L}\lambda_g^f(p,q) \leq {}^{L}\rho_g^f(p,q) < \infty$ *and* $0 < {}^{L}\lambda_g^h(m,q) \leq {}^{L}\rho_g^h(m,q) < \infty$ *where p, q, m are positive integers with* $q < \min\{p, m\}$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} \leq \min\left\{\frac{{}^{L}\lambda_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}, \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\rho_g^h(m,q)}\right\}$$
$$\leq \max\left\{\frac{{}^{L}\lambda_g^f(p,q)}{{}^{L}\lambda_g^h(m,q)}, \frac{{}^{L}\rho_g^f(p,q)}{{}^{L}\rho_g^h(m,q)}\right\}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)}.$$

The proof is omitted.

In the following theorems we see some comparative growth properties of entire functions on the basis of relative $L^* - (p,q)$th order and relative $L^* - (p,q)$th lower order where $L \equiv L(r)$ is a slowly changing function and p, q are positive integers with $p > q$.

Theorem 5.2.4 *Let f, g and h be three entire functions such that* $0 < {}^{L^*}\lambda_g^f(p,q) \leq {}^{L^*}\rho_g^f(p,q) < \infty$ *and* $0 < {}^{L^*}\lambda_g^h(m,q) \leq {}^{L^*}\rho_g^h(m,q) < \infty$ *where p, q, m are positive integers with* $q < \min\{p, m\}$. *Then*

$$\frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}.$$

Proof. From the definition of relative $L^* - (p,q)$*th order* and *relative* $L^* - (p,q)$*th lower order* we have for arbitrary positive ϵ and for all large values of r,

$$\log^{[p]} G^{-1}F(r) \geq ({}^{L^*}\lambda_g^f(p,q) - \epsilon) \log^{[q]} \left[re^{L(r)}\right] \tag{5.15}$$
$$\text{and} \quad \log^{[m]} G^{-1}H(r) \leq ({}^{L^*}\rho_g^h(m,q) + \epsilon) \log^{[q]} \left[re^{L(r)}\right]. \tag{5.16}$$

Now from (5.15) and (5.16) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L^*}\lambda_g^f(p,q) - \epsilon}{{}^{L^*}\rho_g^h(m,q) + \epsilon}.$$

As ϵ (> 0) is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)}. \tag{5.17}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \leq ({}^{L^*}\lambda_g^f(p,q)+\epsilon)\log^{[q]}\left[re^{L(r)}\right] \tag{5.18}$$

and for all large values of r,

$$\log^{[m]} G^{-1}H(r) \geq ({}^{L^*}\lambda_g^h(m,q)-\epsilon)\log^{[q]}\left[re^{L(r)}\right]. \tag{5.19}$$

So combining (5.18) and (5.19) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\lambda_g^f(p,q)+\epsilon}{{}^{L^*}\lambda_g^h(m,q)-\epsilon}.$$

Since ϵ (> 0) is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}. \tag{5.20}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \leq ({}^{L^*}\lambda_g^h(m,q)+\epsilon)\log^{[q]}\left[re^{L(r)}\right]. \tag{5.21}$$

Now from (5.15) and (5.21) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L^*}\lambda_g^f(p,q)-\epsilon}{{}^{L^*}\lambda_g^h(m,q)+\epsilon}.$$

Choosing $\epsilon \to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} \geq \frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}. \tag{5.22}$$

Also for all large values of r,

$$\log^{[p]} G^{-1}F(r) \leq ({}^{L^*}\rho_g^f(p,q)+\epsilon)\log^{[q]}\left[re^{L(r)}\right]. \tag{5.23}$$

So from (5.19) and (5.23) it follows for all large values of r,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)+\epsilon}{{}^{L^*}\lambda_g^h(m,q)-\epsilon}.$$

As ϵ (>0) is arbitrary we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}. \tag{5.24}$$

Thus the theorem follows from (5.17),(5.20),(5.22) and (5.24). ■

Theorem 5.2.5 *Let f, g and h be three entire functions with $0 < {}^{L^*}\lambda_g^f(p,q) \leq {}^{L^*}\rho_g^f(p,q) < \infty$ and $0 < {}^{L^*}\rho_g^h(m,q) < \infty$ where p, q, m are positive integers with $q < \min\{p,m\}$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)}.$$

Proof. From the definition of relative $L^*-(p,q)$*th order* we get for a sequence of values of r tending to infinity,

$$\log^{[m]} G^{-1}H(r) \geq ({}^{L^*}\rho_g^h(m,q)-\epsilon)\log^{[q]}\left[re^{L(r)}\right]. \tag{5.25}$$

Now from (5.23) and (5.25) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)+\epsilon}{{}^{L^*}\rho_g^h(m,q)-\epsilon}.$$

As ϵ (>0) is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)}. \tag{5.26}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} G^{-1}F(r) \geq ({}^{L^*}\rho_g^f(p,q)-\epsilon)\log^{[q]}\left[re^{L(r)}\right]. \tag{5.27}$$

So combining (5.16) and (5.27) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L^*}\rho_g^f(p,q)-\epsilon}{{}^{L^*}\rho_g^h(m,q)+\epsilon}.$$

Since ϵ (> 0) is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)}. \tag{5.28}$$

Thus the theorem follows from (5.26) and (5.28). ∎

Theorem 5.2.6 *Let f, g and h be three entire functions such that* $0 < {}^{L^*}\lambda_g^f(p,q) \leq {}^{L^*}\rho_g^f(p,q) < \infty$ *and* $0 < {}^{L^*}\lambda_g^h(m,q) \leq {}^{L^*}\rho_g^h(m,q) < \infty$ *where p, q, m are positive integers with $q < \min\{p, m\}$. Then*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)} &\leq \min\left\{\frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}, \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)}\right\} \\ &\leq \max\left\{\frac{{}^{L^*}\lambda_g^f(p,q)}{{}^{L^*}\lambda_g^h(m,q)}, \frac{{}^{L^*}\rho_g^f(p,q)}{{}^{L^*}\rho_g^h(m,q)}\right\} \\ &\leq \limsup_{r\to\infty}\frac{\log^{[p]} G^{-1}F(r)}{\log^{[q]} G^{-1}H(r)}.\end{aligned}$$

The proof is omitted.

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