CHAPTER-4

A NOTE ON THE MAXIMUM TERMS OF COMPOSITE ENTIRE FUNCTIONS

Chapter 4

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4.1 Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r,f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r,f) = \max_{n\geq 0}(|a_n|\, r^n)$. To start this chapter we just recall the following definitions.

Definition 4.1.1 *The order ρ_f and lower order λ_f of an entire function f is defined as follows:*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.

Definition 4.1.2 *The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of f is defined by*

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

The results of this chapter have been published in International Journal of Contemporary Mathematical Sciences, see [20].

Since for $0 \leq r < R$,

$$\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f),$$

it is easy to see that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log r}, \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log r}.$$

and

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} \mu(r,f)}{\log r}, \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} \mu(r,f)}{\log r}.$$

Somasundaram and Thamizharasi **[60]** introduced the notions of *L−order*, *L−lower order* and *L−type* for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every constant $'a'$. Their definitions are as follows:

Definition 4.1.3 *[60]The L−order ρ_f^L and L−lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

When f is meromorphic, then

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}.$$

Definition 4.1.4 *[60]The L−type σ_f^L of an entire function f with L−order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log M(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

For meromorphic f, the $L-type$ σ_f^L becomes

$$\sigma_f^L = \limsup_{r\to\infty} \frac{T(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

With the help of the notion of maximum terms of entire functions, Definition 4.1.3 and Definition 4.1.4 can be alternatively stated as follows:

Definition 4.1.5 *The $L-$order ρ_f^L and the $L-$lower order λ_f^L of an entire function f are defined as follows:*

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log[rL(r)]} \quad and \quad \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log[rL(r)]}.$$

When f is meromorphic, then ρ_f^L and λ_f^L cannot be defined in the above way.

Definition 4.1.6 *The $L-$type σ_f^L of an entire function f with $L-$order ρ_f^L is defined as*

$$\sigma_f^L = \limsup_{r\to\infty} \frac{\log \mu(r,f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.$$

For meromorphic f, the $L-$type σ_f^L cannot be defined in the above way.

Juneja, Kapoor and Bajpai **[31]** defined the *(p,q)th order* and *(p,q)th lower order* of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r}$$

$$and \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r}.$$

where p, q are positive integers with $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r}$$

$$and \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

With the notion of slowly changing function one can easily define the following :

Definition 4.1.7 *The $L-(p,q)$th order and $L-(p,q)$th lower order of an entire function f are respectively defined as :*

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} [rL(r)]}$$

$$and \quad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]}[rL(r)]}.$$

When f is meromorphic one can easily verify that

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]}$$

$$and \quad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]}[rL(r)]},$$

where p, q are positive integers and $p > q$.

In view of the notion of maximum terms of entire functions, Definition 4.1.7 can be restated in the following way :

Definition 4.1.8 *The $L-(p,q)$th order and $L-(p,q)$th lower order of an entire function f are respectively defined as :*

$$\rho_f^L(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} \mu(r,f)}{\log^{[q]}[rL(r)]}$$

$$and \quad \lambda_f^L(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} \mu(r,f)}{\log^{[q]}[rL(r)]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, then $\rho_f^L(p,q)$ and $\lambda_f^L(p,q)$ cannot be defined in the above way.

The more generalised concept of $L-$ *order* and $L-$ *type* of entire and meromorphic functions are L^*- *order* and L^*- *type* respectively. Their definitions are as follows:

Definition 4.1.9 *The L^*- order, L^*- lower order and L^*- type of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

and

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}, \quad \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

and

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

In view of the notion of maximum terms of entire functions we may state the following definition.

Definition 4.1.10 *The $L^* - (p,q)$th order and $L^* - (p,q)$th lower order of an entire function f are respectively defined as:*

$$\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} \mu(r,f)}{\log^{[q]}\left[re^{L(r)}\right]}$$

$$\textit{and} \quad \lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} \mu(r,f)}{\log^{[q]}\left[re^{L(r)}\right]},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, then $\rho_f^{L^*}(p,q)$ and $\lambda_f^{L^*}(p,q)$ cannot be defined in the above way.

Singh **[61]** proved some theorems on the comparative growth properties of $\log^{[2]} \mu(r, f \circ g)$ with respect to $\log^{[2]} \mu(r^A, f)$ for every positive constant A. In the chapter we further investigate the comparative growths of maximum term of two entire functions with their corresponding left and right factors on the basis of $L - (p,q)$*th order* and $L - (p,q)$*th lower order* where p, q are positive integers and $p > q$.

4.2 Theorems :

In this section we present the main results of the chapter.

Theorem 4.2.1 *Let f and g be two entire functions such that $0 < \lambda^L_{fog}(p,q) \leq \rho^L_{fog}(p,q) < \infty$ and $0 < \rho^L_g(m,q) < \infty$ where p,q,m are positive integers such that $q < \min\{p,m\}$. Then for any integer A*

$$(i)\quad \liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho^L_{fog}(p,q)}{A\rho^L_g(m,q)} \leq \limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}.$$

Further if $\lambda^L_g(m,q) > 0$ then

$$(ii)\quad \frac{\lambda^L_{fog}(p,q)}{A\rho^L_g(m,q)} \leq \liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,\ f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\lambda^L_{fog}(p,q)}{A\lambda^L_g(m,q)}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho^L_{fog}(p,q)}{A\lambda^L_g(m,q)}.$$

and

$$(iii)\quad \liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \min\left\{\frac{\lambda^L_{fog}(p,q)}{A\lambda^L_g(m,q)}, \frac{\rho^L_{fog}(p,q)}{A\rho^L_g(m,q)}\right\}$$

$$\leq \max\left\{\frac{\lambda^L_{fog}(p,q)}{A\lambda^L_g(m,q)}, \frac{\rho^L_{fog}(p,q)}{A\rho^L_g(m,q)}\right\} \leq \limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}.$$

Proof. (i) From the definition of $L-(p,q)th$ *order* we have for arbitrary positive ϵ and for all large values of r,

$$\log^{[p]}\mu(r,f\circ g) \leq \left(\rho^L_{fog}(p,q)+\epsilon\right)\log^{[q]}\left[rL(r)\right] \tag{4.1}$$

and for a sequence of values of r tending to infinity,

$$\log^{[m]}\mu(r^A,g) \geq A\left(\rho^L_g(m,q)-\epsilon\right)\log^{[q]}\left[rL(r)\right]. \tag{4.2}$$

Now from (4.1) and (4.2) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho^L_{fog}(p,q)+\epsilon}{A\left(\rho^L_g(m,q)-\epsilon\right)}.$$

As $\epsilon\ (>0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho^L_{fog}(p,q)}{A\rho^L_g(m,q)}. \tag{4.3}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} \mu(r, f \circ g) \geq (\rho^L_{fog}(p,q) - \epsilon) \log^{[q]} [rL(r)] . \tag{4.4}$$

Also for all sufficiently large values of r,

$$\log^{[m]} \mu(r^A, g) \leq A \left(\rho^L_g(m,q) + \epsilon\right) \log^{[q]} [rL(r)] . \tag{4.5}$$

So combining (4.4) and (4.5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\rho^L_{fog}(p,q) - \epsilon}{A(\rho^L_g(m,q) + \epsilon)}.$$

Since ϵ (> 0) is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\rho^L_{fog}(p,q)}{A\rho^L_g(m,q)}. \tag{4.6}$$

Thus (i) follows from (4.3) and (4.6). ■

(ii) From the definition of $L-(p,q)th$ *lower order* we have for arbitrary positive ϵ and for all large values of r,

$$\log^{[p]} \mu(r, f \circ g) \geq (\lambda^L_{fog}(p,q) - \epsilon) \log^{[q]} [rL(r)] . \tag{4.7}$$

Now from (4.5) and (4.7) it follows for all large values of r,

$$\frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\lambda^L_{fog}(p,q) - \epsilon}{A(\rho^L_g(m,q) + \epsilon)}.$$

As ϵ (> 0) is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\lambda^L_{fog}(p,q)}{A\rho^L_g(m,q)}. \tag{4.8}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} \mu(r, f \circ g) \leq (\lambda^L_{fog}(p,q) + \epsilon) \log^{[q]} [rL(r)] \tag{4.9}$$

and for all large values of r,

$$\log^{[m]} \mu(r^A, g) \geq A \left(\lambda^L_g(m,q) - \epsilon\right) \log^{[q]} [rL(r)] . \tag{4.10}$$

So combining (4.9) and (4.10) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\leq\frac{\lambda^L_{fog}(p,q)+\epsilon}{A(\lambda^L_g(m,q)-\epsilon)}.$$

Since ϵ (>0) is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\leq\frac{\lambda^L_{fog}(p,q)}{A\lambda^L_g(m,q)}. \tag{4.11}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]}\mu(r^A,g)\leq A\left(\lambda^L_g(m,q)+\epsilon\right)\log^{[q]}\left[rL(r)\right]. \tag{4.12}$$

Now from (4.7) and (4.12) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\geq\frac{\lambda^L_{fog}(p,q)-\epsilon}{A(\lambda^L_g(m,q)+\epsilon)}.$$

As ϵ (>0) is arbitrary we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\geq\frac{\lambda^L_{fog}(p,q)}{A\lambda^L_g(m,q)}. \tag{4.13}$$

Again from (4.1) and (4.10) it follows for all large values of r,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\leq\frac{\rho^L_{fog}(p,q)+\epsilon}{A(\lambda^L_g(m,q)-\epsilon)}.$$

As ϵ (>0) is arbitrary we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\leq\frac{\rho^L_{fog}(p,q)}{A\lambda^L_g(m,q)}. \tag{4.14}$$

Thus (*ii*) follows from (4.8),(4.11),(4.13) and (4.14).

(*iii*) Combining (*i*) and (*ii*) of Theorem 4.2.1, (*iii*) follows.

Theorem 4.2.2 *If f and g be two entire functions with $\rho^L_g(m,q)<\infty$ and $\rho^L_{fog}(p,q)=\infty$, then for every positive number A,*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}=\infty,$$

where p,q,m are positive integers with $q<\min\{p,m\}$.

Proof. Let us assume that the conclusion of Theorem 4.2.2 do not hold. Then there exists a constant $B > 0$ such that for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq B \log^{[m]} \mu(r^A, g). \tag{4.15}$$

Again from the definition of $\rho_g^L(m, q)$ it follows that

$$\log^{[m]} \mu(r^A, g) \leq (\rho_g^L(m, q) + \epsilon) A \log^{[q]} [rL(r)] \tag{4.16}$$

holds for all large values of r. So from (4.15) and (4.16) we obtain for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_g^L(m, q) + \epsilon) AB \log^{[q]} [rL(r)]. \tag{4.17}$$

From (4.17) it follows that $\rho_{fog}^L(p, q) < \infty$.

So we arrive at a contradiction. This proves the theorem. ■

Remark 4.2.1 *If we take $\rho_f^L(m, q) < \infty$ instead of $\rho_g^L(m, q) < \infty$ in Theorem 4.2.2 and the other conditions remain the same then the theorem remains valid with g replaced by f in the denominator as we see in the following theorem.*

Theorem 4.2.3 *If f and g be two entire functions with $\rho_f^L(m, q) < \infty$ and $\rho_{fog}^L(p, q) = \infty$, then for every positive number A,*

$$\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, f)} = \infty,$$

where p, q, m are positive integers with $q < \min\{p, m\}$.

Proof. Let us assume that the conclusion of Theorem 4.2.3 do not hold.Then there exists a constant $B_0 > 0$ such that for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq B_0 \log^{[m]} \mu(r^A, f). \tag{4.18}$$

Again from the definition of $\rho_f^L(m, q)$ it follows that

$$\log^{[m]} \mu(r^A, f) \leq (\rho_f^L(m, q) + \epsilon) A \log^{[q]} [rL(r)] \tag{4.19}$$

holds for all large values of r. So from (4.18) and (4.19) we obtain for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_f^L(m, q) + \epsilon) AB_0 \log^{[q]} [rL(r)]. \tag{4.20}$$

From (4.20) it follows that $\rho^{L}_{fog}(p,q) < \infty$.

Thus we arrive at a contradiction. This proves the theorem. ■

In the line of Theorem 4.2.1 and Theorem 4.2.2 we may respectively state the following two theorems whose proofs are given below.

Theorem 4.2.4 *Let f and g be two entire functions such that $0 < \lambda^{L^*}_{fog}(p,q) \leq \rho^{L^*}_{fog}(p,q) < \infty$ and $0 < \rho^{L^*}_{g}(m,q) < \infty$ where p,q,m are positive integers such that $q < \min\{p,m\}$. Then for any integer A*

$$(i)\quad \liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho^{L^*}_{fog}(p,q)}{A\rho^{L^*}_{g}(m,q)} \leq \limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}.$$

Further if $\lambda^{L^*}_{g}(m,q) > 0$ then

$$(ii)\quad \frac{\lambda^{L^*}_{fog}(p,q)}{A\rho^{L^*}_{g}(m,q)} \leq \liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\lambda^{L^*}_{fog}(p,q)}{A\lambda^{L^*}_{g}(m,q)}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho^{L^*}_{fog}(p,q)}{A\lambda^{L^*}_{g}(m,q)}$$

and

$$(iii)\quad \liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \min\left\{\frac{\lambda^{L^*}_{fog}(p,q)}{A\lambda^{L^*}_{g}(m,q)}, \frac{\rho^{L^*}_{fog}(p,q)}{A\rho^{L^*}_{g}(m,q)}\right\}$$

$$\leq \max\left\{\frac{\lambda^{L^*}_{fog}(p,q)}{A\lambda^{L^*}_{g}(m,q)}, \frac{\rho^{L^*}_{fog}(p,q)}{A\rho^{L^*}_{g}(m,q)}\right\} \leq \limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}.$$

Proof. (i) From the definition of $L^* - (p,q)th$ *order* we have for arbitrary positive ϵ and for all large values of r,

$$\log^{[p]}\mu(r,f\circ g) \leq \left(\rho^{L^*}_{fog}(p,q)+\epsilon\right)\log^{[q]}\left[re^{L(r)}\right] \tag{4.21}$$

and for a sequence of values of r tending to infinity,

$$\log^{[m]}\mu(r^A,g) \geq A\left(\rho^{L^*}_{g}(m,q)-\epsilon\right)\log^{[q]}\left[re^{L(r)}\right]. \tag{4.22}$$

Now from (4.21) and (4.22) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho^{L^*}_{fog}(p,q)+\epsilon}{A\left(\rho^{L^*}_{g}(m,q)-\epsilon\right)}.$$

As ϵ (>0) is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\leq\frac{\rho^{L^*}_{fog}(p,q)}{A\rho^{L^*}_g(m,q)}. \tag{4.23}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]}\mu(r,f\circ g)\geq(\rho^{L^*}_{fog}(p,q)-\epsilon)\log^{[q]}\left[re^{L(r)}\right]. \tag{4.24}$$

Also for all sufficiently large values of r,

$$\log^{[m]}\mu(r^A,g)\leq A\left(\rho^{L^*}_g(m,q)+\epsilon\right)\log^{[q]}\left[re^{L(r)}\right]. \tag{4.25}$$

So combining (4.24) and (4.25) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\geq\frac{\rho^{L^*}_{fog}(p,q)-\epsilon}{A(\rho^{L^*}_g(m,q)+\epsilon)}.$$

Since ϵ (>0) is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\geq\frac{\rho^{L^*}_{fog}(p,q)}{A\rho^{L^*}_g(m,q)}. \tag{4.26}$$

Thus (i) follows from (4.23) and (4.26). ■

(ii) From the definition of $L^*-(p,q)$*th lower order* we have for arbitrary positive ϵ and for all large values of r,

$$\log^{[p]}\mu(r,f\circ g)\geq(\lambda^{L^*}_{fog}(p,q)-\epsilon)\log^{[q]}\left[re^{L(r)}\right]. \tag{4.27}$$

Now from (4.25) and (4.27) it follows for all large values of r,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\geq\frac{\lambda^{L^*}_{fog}(p,q)-\epsilon}{A(\rho^{L^*}_g(m,q)+\epsilon)}.$$

As ϵ (>0) is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)}\geq\frac{\lambda^{L^*}_{fog}(p,q)}{A\rho^{L^*}_g(m,q)}. \tag{4.28}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]}\mu(r,f\circ g)\leq(\lambda^{L^*}_{fog}(p,q)+\epsilon)\log^{[q]}\left[re^{L(r)}\right] \tag{4.29}$$

and for all large values of r,

$$\log^{[m]}\mu(r^A,g) \geq A\left(\lambda_g^{L^*}(m,q)-\epsilon\right)\log^{[q]}\left[re^{L(r)}\right]. \tag{4.30}$$

So combining (4.29) and (4.30) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\lambda_{fog}^{L^*}(p,q)+\epsilon}{A(\lambda_g^{L^*}(m,q)-\epsilon)}.$$

Since ϵ (>0) is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\lambda_{fog}^{L^*}(p,q)}{A\lambda_g^{L^*}(m,q)}. \tag{4.31}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]}\mu(r^A,g) \leq A\left(\lambda_g^{L^*}(m,q)+\epsilon\right)\log^{[q]}\left[re^{L(r)}\right]. \tag{4.32}$$

Now from (4.27) and (4.32) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \geq \frac{\lambda_{fog}^{L^*}(p,q)-\epsilon}{A(\lambda_g^{L^*}(m,q)+\epsilon)}.$$

As ϵ (>0) is arbitrary we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \geq \frac{\lambda_{fog}^{L^*}(p,q)}{A\lambda_g^{L^*}(m,q)}. \tag{4.33}$$

Again from (4.21) and (4.30) it follows for all large values of r,

$$\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho_{fog}^{L^*}(p,q)+\epsilon}{A(\lambda_g^{L^*}(m,q)-\epsilon)}.$$

As ϵ (>0) is arbitrary we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,f\circ g)}{\log^{[m]}\mu(r^A,g)} \leq \frac{\rho_{fog}^{L^*}(p,q)}{A\lambda_g^{L^*}(m,q)}. \tag{4.34}$$

Thus (*ii*) follows from (4.28),(4.31),(4.33) and (4.34).

(*iii*) Combining (*i*) and (*ii*) of Theorem 4.2.4, (*iii*) follows.

Theorem 4.2.5 *If f and g be two entire functions with $\rho_g^{L^*}(m,q) < \infty$ and $\rho_{fog}^{L^*}(p,q) = \infty$, then for every positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} = \infty,$$

where p, q, m are positive integers with $q < \min\{p, m\}$.

Proof. Let us assume that the conclusion of Theorem 4.2.5 does not hold.Then there exists a constant $C > 0$ such that for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq C \log^{[m]} \mu(r^A, g). \tag{4.35}$$

Again from the definition of $\rho_g^{L^*}(m,q)$ it follows that

$$\log^{[m]} \mu(r^A, g) \leq (\rho_g^{L^*}(m,q) + \epsilon) A \log^{[q]} \left[re^{L(r)} \right] \tag{4.36}$$

holds for all large values of r. So from (4.35) and (4.36) we obtain for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_g^{L^*}(m,q) + \epsilon) AC \log^{[q]} \left[re^{L(r)} \right]. \tag{4.37}$$

From (4.37) it follows that $\rho_{fog}^{L^*}(p,q) < \infty$.

So we arrive at a contradiction.This proves the theorem. ■

Remark 4.2.2 *If we take $\rho_f^{L^*}(m,q) < \infty$ instead of $\rho_g^{L^*}(m,q) < \infty$ in Theorem 4.2.5 and the other conditions remain the same then the theorem remains valid with g replaced by f in the denominator as we see in the following theorem.*

Theorem 4.2.6 *If f and g be two entire functions with $\rho_f^{L^*}(m,q) < \infty$ and $\rho_{fog}^{L^*}(p,q) = \infty$, then for every positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, f)} = \infty,$$

where p, q, m are positive integers with $q < \min\{p, m\}$.

Proof. Let us assume that the conclusion of Theorem 4.2.6 do not hold.Then there exists a constant $C_0 > 0$ such that for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq C_0 \log^{[m]} \mu(r^A, f). \qquad (4.38)$$

Again from the definition of $\rho_f^{L^*}(m, q)$ it follows that

$$\log^{[m]} \mu(r^A, f) \leq (\rho_g^{L^*}(m, q) + \epsilon) A \log^{[q]} \left[re^{L(r)} \right] \qquad (4.39)$$

holds for all large values of r. So from (4.38) and (4.39) we obtain for all sufficiently large values of r,

$$\log^{[p]} \mu(r, f \circ g) \leq (\rho_g^{L^*}(m, q) + \epsilon) A C_0 \log^{[q]} \left[re^{L(r)} \right]. \qquad (4.40)$$

From (4.40) it follows that $\rho_{f\circ g}^{L^*}(p, q) < \infty$.

So we arrive at a contradiction. Thus the theorem is established. ■

$* * * * * * X * * * * * *$