CHAPTER-2

SOME GROWTH PROPERTIES OF DIFFERENTIAL MONOMIALS GENERATED BY TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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2.1 Introduction, Definitions and Notations.

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [10] proved that

$$\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty.$$

Singh [59] proved some comparative growth properties of $\log T(r,f\circ g)$ and $T(r,f)$. He [59] also raised the question of investigating the comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$, which he was unable to solve. Lahiri [37] proved some results on the comparative growth of $\log T(r,g)$ and $T(r,g)$.

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Let $n_{0j}, n_{1j}, \ldots n_{kj} (k\geq 1)$ be non-negative integers such that for each j, $\sum_{i=0}^{k} n_{ij}\geq 1$. We call

$$M_j[f]=A_j(f)^{n_{0j}}\left(f^{(1)}\right)^{n_{1j}}\ldots\left(f^{(k)}\right)^{n_{kj}}$$

The results of this chapter have been published in International Mathematical Forum, see [16] and [17].

where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers

$$\gamma_{Mj} = \sum_{i=0}^{k} n_{ij} \quad and \quad \Gamma_{Mj} = \sum_{i=0}^{k} (i+1) n_{ij}$$

are called respectively the degree and weight of $M_j[f]$ {[14], [57]}. The expression $P[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers

$$\gamma_P = \max_{1 \leq j \leq s} \gamma_{Mj} \quad and \quad \Gamma_P = \max_{1 \leq j \leq s} \Gamma_{Mj}$$

are called respectively the degree and weight of $P[f]$ {[14], [57]}. Also we call the numbers $\underline{\gamma_P} = \min_{1 \leq j \leq s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_P} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial and we denote by $P_0[f]$ a differential polynomial not containing f i.e. for which $n_{0j} = 0 \ \ for \ \ j = 1, 2, ...s$. In the chapter we establish a relation between the generalised $L-order$ (generalised $L-type$) of

(i) $M_j[f]$ and f

(ii) $P_0[f]$ and f.

Also in this chapter we disscuss about the comparative growth properties of composite entire and meromorphic functions and differential monomials generated by one of the factors.

In the sequel we use the following notation:
$\log^{[k]} x = \log(\log^{[k-1]} x) \ \ for \ \ k = 1, 2, 3, ... \ and \ \log^{[0]} x = x$.
The following definitions are well known.

Definition 2.1.1 *The order ρ_f and lower order λ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad and \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad and \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

Definition 2.1.2 *The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of a meromorphic function f is defined as follows*

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} \quad and \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire, then

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \quad and \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 2.1.3 *[43] Let f be a meromorphic function of order zero. Then the quantities ρ_f^*, λ_f^* and $\bar{\rho}_f^*$, $\bar{\lambda}_f^*$ are defined in the following way :*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}.$$

If f is entire then clearly

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

Definition 2.1.4 *The type σ_f of a meromorphic function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Definition 2.1.5 *[68] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n(r, a; f| = 1)$, the number of simple zeros of $f - a$ in $|z| \leq r$. $N(r, a; f| = 1)$ is defined in terms of $n(r, a; f| = 1)$ in the usual way. We put*

$$\delta_1(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f| = 1)}{T(r, f)},$$

the deficiency of $'a'$ corresponding to the simple $a -$ points of f i.e., simple zeros of $f - a$.

Yi **[70]** and Yang **[67]** proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4$.

Definition 2.1.6 *The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows:*

$$\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

Definition 2.1.7 *[69] For $a \in \mathbb{C} \cup \{\infty\}$ let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is exactly p times and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define*

$$\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)}.$$

Definition 2.1.8 *$P[f]$ is said to be admissible if*

(*i*) $P[f]$ is homogeneous, or
(*ii*) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$.

Definition 2.1.9 *[55] The generalised order ρ_k $(k \geq 2)$ and generalised lower order λ_k $(k \geq 2)$ of a meromorphic function f is defined as*

$$\rho_k = \limsup_{r \to \infty} \frac{\log^{[k-1]} T(r, f)}{\log r} \quad and \quad \lambda_k = \liminf_{r \to \infty} \frac{\log^{[k-1]} T(r, f)}{\log r}.$$

Definition 2.1.10 *The generalised type σ_k $(k \geq 2)$ of a meromorphic function f is defined as*

$$\sigma_k = \limsup_{r \to \infty} \frac{\log^{[k-2]} T(r,f)}{r^{\rho_k}}, \quad 0 < \rho_k < \infty.$$

Somasundaram and Thamizharasi [**60**] introduced the notions of $L-$*order* and $L-$*type* for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant $'a'$. Their definitions are as follows:

Definition 2.1.11 *[60] The generalised $L-$order ${}^{(k)}\rho_f^L$ and the generalised $L-$lower order ${}^{(k)}\lambda_f^L$ of an entire function f are defined as follows:*

$${}^{(k)}\rho_f^L = \limsup_{r \to \infty} \frac{\log^{[k]} M(r,f)}{\log [rL(r)]} \text{ and } {}^{(k)}\lambda_f^L = \liminf_{r \to \infty} \frac{\log^{[k]} M(r,f)}{\log [rL(r)]} \text{ where } k \geq 2.$$

When f is meromorphic, then

$${}^{(k)}\rho_f^L = \limsup_{r \to \infty} \frac{\log^{[k-1]} T(r,f)}{\log [rL(r)]} \text{ and } {}^{(k)}\lambda_f^L = \liminf_{r \to \infty} \frac{\log^{[k-1]} T(r,f)}{\log [rL(r)]} \text{ for } k \geq 2.$$

Definition 2.1.12 *[60] The generalised $L-$type ${}^{(k)}\sigma_f^L$ $(k \geq 2)$ of an entire function f with generalised $L-$order ${}^{(k)}\rho_f^L$ is defined as*

$${}^{(k)}\sigma_f^L = \limsup_{r \to \infty} \frac{\log^{[k-1]} M(r,f)}{[rL(r)]^{{}^{(k)}\rho_f^L}}, \quad 0 < {}^{(k)}\rho_f^L < \infty.$$

For meromorphic f, the generalised $L-$type ${}^{(k)}\sigma_f^L$ becomes

$${}^{(k)}\sigma_f^L = \limsup_{r \to \infty} \frac{\log^{[k-2]} T(r,f)}{[rL(r)]^{{}^{(k)}\rho_f^L}}, \quad 0 < {}^{(k)}\rho_f^L < \infty.$$

The more extended concept of generalised $L-$*order* and generalised $L-$*type* of entire and meromorphic functions are generalised L^*-*order* and generalised L^*-*type* respectively. Their definitions are as follows:

Definition 2.1.13 *The generalised L^*-order, generalised L^*-lower order and generalised L^*-type of a meromorphic function f are defined by*

$${}^{(k)}\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log^{[k-1]} T(r,f)}{\log \left[re^{L(r)}\right]}, \quad {}^{(k)}\lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log^{[k-1]} T(r,f)}{\log \left[re^{L(r)}\right]}$$

and ${}^{(k)}\sigma_f^{L^*} = \limsup\limits_{r\to\infty} \frac{\log^{[k-2]} T(r,f)}{\left[re^{L(r)}\right]^{{}^{(k)}\rho_f^{L^*}}}$, *where* $0 < {}^{(k)}\rho_f^{L^*} < \infty$ *and* $k \geq 2$.

When f *is entire, one can easily verify that*

$$ {}^{(k)}\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[k]} M(r,f)}{\log\left[re^{L(r)}\right]}, \quad {}^{(k)}\lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[k]} M(r,f)}{\log\left[re^{L(r)}\right]} $$

and ${}^{(k)}\sigma_f^{L^*} = \limsup\limits_{r\to\infty} \frac{\log^{[k-1]} M(r,f)}{\left[re^{L(r)}\right]^{{}^{(k)}\rho_f^{L^*}}}$, *where* $0 < {}^{(k)}\rho_f^{L^*} < \infty$ *and* $k \geq 2$.

2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.2.1 *[10] If* f *and* g *be two entire functions then for all sufficiently large values of* r,

$$ M(r, f \circ g) \geq M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right). $$

Lemma 2.2.2 *[1] If* f *is meromorphic and* g *is entire then for all sufficiently large values of* r,

$$ T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f). $$

Lemma 2.2.3 *[4] Let* f *be meromorphic and* g *be entire and also suppose that* $0 < \mu \leq \rho_g \leq \infty$. *Then for a sequence of values of* r *tending to infinity,*

$$ T(r, f \circ g) \geq T(\exp(r^{\mu}), f). $$

Lemma 2.2.4 *[44] Let* f *be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, *then*

$$ \lim_{r\to\infty} \frac{T(r, P[f])}{T(r,f)} = \{\Gamma_p - (\Gamma_p - \gamma_p)\Theta(\infty; f)\} $$

$$ \textit{where} \quad \Theta(\infty; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)}. $$

Lemma 2.2.5 *If f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, *then the order and lower order of $P[f]$ are same as those of f and the type of $P[f]$ is* $\{\Gamma_p - (\Gamma_p - \gamma_p)\Theta(\infty;f)\}$ *times that of f, i.e.,* $\rho_{P[f]} = \rho_f$, $\lambda_{P[f]} = \lambda_f$ *and*

$$\sigma_{P[f]} = \{\Gamma_p - (\Gamma_p - \gamma_p)\Theta(\infty;f)\}\,\sigma_f.$$

Proof. By Lemma 2.2.4, $\lim_{r\to\infty} \frac{\log T(r,P[f])}{\log T(r,f)}$ exists and is equal to 1. So

$$\begin{aligned}\rho_{P[f]} &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.\lim_{r\to\infty} \frac{\log T(r,P[f])}{\log T(r,f)} \\ &= \rho_f.1 = \rho_f.\end{aligned}$$

In a similar manner,

$$\begin{aligned}\lambda_{P[f]} &= \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.\lim_{r\to\infty} \frac{\log T(r,P[f])}{\log T(r,f)} \\ &= \lambda_f.1 = \lambda_f.\end{aligned}$$

Again

$$\begin{aligned}\sigma_{P[f]} &= \limsup_{r\to\infty} \frac{T(r,P[f])}{r^{\rho_{P[f]}}} \\ &= \lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)}.\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} \\ &= \{\Gamma_p - (\Gamma_p - \gamma_p)\Theta(\infty;f)\}\,\sigma_f.\end{aligned}$$

This proves the lemma. ∎

Lemma 2.2.6 *[28] Let f be meromorphic and g be transcendental entire. If* $\rho_{fog} < \infty$ *then* $\rho_f = 0$.

Lemma 2.2.7 *Let f be meromorphic and g be transcendental entire such that* $\rho_f = 0$ *and* $\rho_g < \infty$. *Then* $\rho_{fog} \le \rho_f^* \rho_g$.

Proof. In view of Lemma 2.2.2 and the inequality $T(r,g) \leq \log^{+} M(r,g)$ we get that

$$\begin{aligned}
\rho_{f \circ g} &= \limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{\log r} \\
&\leq \limsup_{r\to\infty} \frac{\log T(M(r,g), f) + o(1)}{\log r} \\
&\leq \limsup_{r\to\infty} \frac{\log T(M(r,g), f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log r} \\
&= \rho_f^{*} \rho_g .
\end{aligned}$$

This proves the lemma. ■

Lemma 2.2.8 *[5] Let $P_0[f]$ be admissible. If f is of finite order or of non-zero lower order and* $\sum_{a\neq\infty} \Theta(a;f) = 2$ *then*

$$\lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r,f)} = \Gamma_{P_0}.$$

Lemma 2.2.9 *[41] Let f be either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$. *Then for homogeneous $P_0[f]$,*

$$\lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r,f)} = \gamma_{P_0}.$$

2.3 Theorems.

In this section we present the main results of the chapter.

Theorem 2.3.1 *Let f be transcendental meromorphic and g be entire satisfying the following conditions:*
(i) ρ_f and ρ_g are both finite, (ii) ρ_f is positive and (iii) $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$.
Then for $\alpha \in (-\infty, \infty)$,

$$\liminf_{r\to\infty} \frac{\{\log T(r, f\circ g)\}^{1+\alpha}}{\log T(\exp(r^p), P[f])} = 0 \quad \textit{if} \quad p > (1+\alpha)\rho_g.$$

Proof. We have for a sequence of values of r tending to infinity and for $\epsilon > 0$,

$$\log T(\exp(r^p), P[g]) > (\rho_{P[g]} - \epsilon)\log(\exp(r^p)). \tag{2.3}$$

Now combining (2.1) and (2.3) and in view of Lemma 2.2.5 we obtain for a sequence of values of r tending to infinity,

$$\frac{\{\log T(r, f \circ g)\}^{1+\alpha}}{\log T(\exp(r^p), P[g])} \leq \frac{r^{(\rho_g+\epsilon)(1+\alpha)}\{(\rho_f+\epsilon)+o(1)\}^{1+\alpha}}{(\rho_g-\epsilon)r^p},$$

from which the theorem follows because we can choose ϵ such that

$$0 < \epsilon < \min\{\rho_g, \frac{p}{1+\alpha} - \rho_g\}.$$

Thus the theorem is established. ■

Theorem 2.3.3 *If f be meromorphic and g be transcendental entire such that $\rho_g < \infty$, $\rho_{fog} = \infty$ and $\sum\limits_{a \in \mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$, then for every $A > 0$,*

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P[g])} = \infty.$$

Proof. If possible, let there exists a constant β such that for all sufficiently large values of r, we have

$$\log T(r, f \circ g) \leq \beta \log T(r^A, P[g]). \tag{2.4}$$

In view of Lemma 2.2.5, for all sufficiently large values of r we get that

$$\log T(r^A, P[g]) \leq (\rho_{P[g]} + \epsilon)A\log r$$
$$i.e., \quad \log T(r^A, P[g]) \leq (\rho_g + \epsilon)A\log r. \tag{2.5}$$

Now combining (2.4) and (2.5) we obtain for all sufficiently large values of r,

$$\log T(r, f \circ g) \leq \beta(\rho_g + \epsilon)A\log r$$
$$i.e., \quad \rho_{fog} \leq \beta A(\rho_g + \epsilon),$$

which contradicts the condition $\rho_{fog} = \infty$. So for a sequence of values of r tending to infinity, it follows that

$$\log T(r, f \circ g) \geq \beta \log T(r^A, P[g]),$$

from which the theorem follows. ■

Corollary 2.3.1 *Under the assumptions of Theorem* 2.3.3,

$$\limsup_{r\to\infty}\frac{T(r,f\circ g)}{T(r^A,P[g])}=\infty.$$

Proof. By Theorem 2.3.3 we obtain for all sufficiently large values of r and for $K>1$,

$$\log T(r,f\circ g)>K\log T(r^A,P[g])$$
$$i.e.,\quad T(r,f\circ g)>\{T(r^A,P[g])\}^K,$$

from which the corollary follows. ■

Remark 2.3.1 *If we take* $\rho_f<\infty$ *and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *instead of* $\rho_g<\infty$ *and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ *respectively, then Theorem* 2.3.3 *and Corollary* 2.3.1 *remain valid with* $P[g]$ *replaced by* $P[f]$ *in the denominator.*

Theorem 2.3.4 *If* f *be transcendental meromorphic and* g *be entire such that* $\rho_f<\infty$, $\rho_{fog}=\infty$ *and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, *then for every* $A>0$,

$$\limsup_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(r^A,P[f])}=\infty.$$

Proof. If possible, let there exists a constant β_0 such that for all sufficiently large values of r, we have

$$\log T(r,f\circ g)\le\beta_0\log T(r^A,P[f]). \tag{2.6}$$

In view of Lemma 2.2.5, for all sufficiently large values of r we get that

$$\log T(r^A,P[f])\le(\rho_{P[f]}+\epsilon)A\log r$$
$$i.e.,\quad \log T(r^A,P[f])\le(\rho_f+\epsilon)A\log r. \tag{2.7}$$

Now combining (2.6) and (2.7) we obtain for all sufficiently large values of r,

$$\log T(r,f\circ g)\le\beta_0(\rho_f+\epsilon)A\log r$$
$$i.e.,\quad \rho_{fog}\le\beta_0A(\rho_f+\epsilon),$$

which contradicts the condition $\rho_{fog}=\infty$. So for a sequence of values of r tending to infinity, it follows that

$$\log T(r,f\circ g)\ge\beta_0\log T(r^A,P[f]),$$

from which the theorem follows. ■

Corollary 2.3.2 *Under the assumptions of Theorem* 2.3.4,

$$\limsup_{r\to\infty}\frac{T(r,f\circ g)}{T(r^A,P[f])}=\infty.$$

Proof. By Theorem 2.3.4 we obtain for all sufficiently large values of r and for $K>1$,

$$\log T(r,f\circ g)>K\log T(r^A,P[f])$$
$$i.e.,\quad T(r,f\circ g)>\{T(r^A,P[f])\}^K,$$

from which the corollary follows. ■

Theorem 2.3.5 *Let f and g be two entire functions with $\lambda_f>0$ and $\rho_f<\lambda_g$. Also let f be transcendental with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Then*

$$\lim_{r\to\infty}\frac{\log^{[2]}M(r,f\circ g)}{\log M(r,P[f])}=\infty.$$

Proof. In view of Lemma 2.2.1, we have for all sufficiently large values of r,

$$M(r,f\circ g)\geq M(\frac{1}{16}M(\frac{r}{2},g),f)$$
$$i.e.,\quad \log^{[2]}M(r,f\circ g)\geq\log^{[2]}M(\frac{1}{16}M(\frac{r}{2},g),f)$$
$$i.e.,\quad \log^{[2]}M(r,f\circ g)\geq(\lambda_f-\epsilon)\log(\frac{1}{16}M(\frac{r}{2},g))$$
$$i.e.,\quad \log^{[2]}M(r,f\circ g)\geq(\lambda_f-\epsilon)\log\frac{1}{16}+(\lambda_f-\epsilon)\log M(\frac{r}{2},g)$$
$$i.e.,\quad \log^{[2]}M(r,f\circ g)\geq O(1)+(\lambda_f-\epsilon)(\frac{r}{2})^{(\lambda_g-\epsilon)}.\tag{2.8}$$

Again for all sufficiently large values of r, we get by Lemma 2.2.5

$$\log M(r,P[f])\leq r^{(\rho_{P_0[f]}+\epsilon)}=r^{(\rho_f+\epsilon)}.\tag{2.9}$$

Now combining (2.8) and (2.9) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]}M(r,f\circ g)}{\log M(r,P[f])}\geq\frac{O(1)+(\lambda_f-\epsilon)(\frac{r}{2})^{(\lambda_g-\epsilon)}}{r^{(\rho_f+\epsilon)}}.\tag{2.10}$$

Since $\rho_f < \lambda_g$, we can choose ϵ (> 0) in such a way that

$$\rho_f + \epsilon < \lambda_g - \epsilon. \tag{2.11}$$

Thus from (2.10) and (2.11) we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, f\circ g)}{\log M(r, P[f])} = \infty,$$

from which the theorem follows. ■

Theorem 2.3.6 *If f be a transcendental meromorphic function and g be entire with $0 < \lambda_f \le \rho_f < \infty$, $\rho_g < \infty$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$, then*

$$\lim_{r\to\infty} \frac{T(r, f\circ g)T(r, P[f])}{T(\exp(r^p), P[f])} = 0, \text{ if } p > \rho_g.$$

Proof. Since $T(r, g) \le log^+ M(r, g)$, for all sufficiently large values of r we get from Lemma 2.2.2,

$$T(r, f\circ g) \le \{1 + o(1)\}T(M(r, g), f)$$
$$i.e., \; T(r, f\circ g) \le \{1 + o(1)\}\exp\{(\rho_f + \epsilon)r^{(\rho_g+\epsilon)}\}. \tag{2.12}$$

Again by Lemma 2.2.5 we obtain for all sufficiently large values of r,

$$T(r, P[f]) \le r^{(\rho_{P[f]}+\epsilon)} = r^{\rho_f+\epsilon}. \tag{2.13}$$

Now combining (2.12) and (2.13) it follows for all sufficiently large values of r,

$$T(r, f\circ g)T(r, P[f]) \le \{1 + o(1)\}r^{\rho_f+\epsilon}\exp\{(\rho_f + \epsilon)r^{(\rho_g+\epsilon)}\}. \tag{2.14}$$

Also in view of Lemma 2.2.5 we have for all sufficiently large values of r,

$$\log T(\exp(r^p), P[f]) \ge (\lambda_{P[f]} - \epsilon)\log\{\exp(r^p)\}$$
$$\text{i.e., } T(\exp(r^p), P[f]) \ge \exp\{(\lambda_f - \epsilon)r^p\}. \tag{2.15}$$

From (2.14) and (2.15) it follows for all sufficiently large values of r,

$$\frac{T(r, f\circ g)T(r, P[f])}{T(\exp(r^p), P[f])} \le \frac{\{1 + o(1)\}r^{\rho_f+\epsilon}\exp\{(\rho_f + \epsilon)r^{(\rho_g+\epsilon)}\}}{\exp\{(\lambda_f - \epsilon)r^p\}}. \tag{2.16}$$

As $p > \rho_g$ so we can choose ϵ (> 0) such that

$$p > \rho_g + \epsilon. \tag{2.17}$$

Thus the theorem follows from (2.16) and (2.17). ■

Theorem 2.3.7 *Let f be a transcendental meromorphic function and g be a transcendental entire function such that $0 < \lambda_f \leq \rho_f < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then for every $A > 0$,*

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P[f])} = \infty.$$

If further, $\rho_g < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. then

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P[g])} = \infty.$$

Proof. Since $\lambda_f > 0, \lambda_{f \circ g} = \infty$ $(cf.$**[3]**$)$. So it follows that for arbitrary large N and for all large values of r

$$\log T(r, f \circ g) > AN \log r. \tag{2.18}$$

Again since $\rho_f < \infty$, for all large values of r we get by Lemma 2.2.5,

$$\log T(r^A, P[f]) < A(\rho_f + 1) \log r. \tag{2.19}$$

Now from (2.18) and (2.19) it follows for all large values of r that

$$\frac{\log T(r, f \circ g)}{\log T(r^A, P[f])} > \frac{AN \log r}{A(\rho_f + 1) \log r}$$

and so

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P[f])} = \infty.$$

Again since $\rho_g < \infty$ then for all large values of r we obtain by Lemma 2.2.5

$$\log T(r^A, P[g]) < A(\rho_g + 1) \log r. \tag{2.20}$$

Now from (2.18) and (2.20) it follows for all large values of r that

$$\frac{\log T(r, f \circ g)}{\log T(r^A, P[g])} > \frac{AN \log r}{A(\rho_g + 1) \log r}. \tag{2.21}$$

Thus the theorem follows from (2.21). ■

Theorem 2.3.8 *Let f be a meromorphic function with $\lambda_f > 0$ and g be transcendental entire satisfying $\rho_f < \infty$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then*

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log T(\exp(r^{\mu}), P[f])} = \infty \quad where \quad 0 < \mu < \rho_g.$$

Proof. Let $0 < \mu < \rho_g$.Then in view of Lemma 2.2.3, we get for a sequence of values of r tending to infinity,

$$\begin{aligned} & \log T(r, f \circ g) \geq \log T(\exp(r^{\mu}), f) \\ i.e., \quad & \log T(r, f \circ g) \geq (\lambda_f - \epsilon) \log\{\exp(r^{\mu})\} \\ i.e., \quad & \log T(r, f \circ g) \geq (\lambda_f - \epsilon) r^{\mu} \\ i.e., \quad & \log^{[2]} T(r, f \circ g) \geq O(1) + \mu \log r. \end{aligned}$$

So for a sequence of values of r tending to infinity,

$$\begin{aligned} & \log^{[2]} T(\exp(r^{\rho_g}), f \circ g) \geq O(1) + \mu \log\{\exp(r^{\rho_g})\} \\ i.e., \quad & \log^{[2]} T(\exp(r^{\rho_g}), f \circ g) \geq O(1) + \mu r^{\rho_g}. \end{aligned} \tag{2.22}$$

Again in view of Lemma 2.2.5, we obtain for all sufficiently large values of r,

$$\begin{aligned} & \log T(\exp(r^{\mu}), P[f]) \leq (\rho_{P[f]} + \epsilon) \log\{\exp(r^{\mu})\} \\ i.e., \quad & \log T(\exp(r^{\mu}), P[f]) \leq (\rho_f + \epsilon) r^{\mu}. \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log T(\exp(r^{\mu}), P[f])} \geq \frac{O(1) + \mu r^{\rho_g}}{(\rho_f + \epsilon) r^{\mu}}. \tag{2.24}$$

Since $\mu < \rho_g$, we get from (2.24) that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log T(\exp(r^{\mu}), P[f])} = \infty.$$

This proves the theorem. ■

In the line of Theorem 2.3.8 we may prove the following for the right factor g.

Theorem 2.3.9 *Let f be a meromorphic function with $\lambda_f > 0$ and g be transcendental entire satisfying $\rho_g < \infty$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. Then for all sufficiently large values of r,*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f\circ g)}{\log T(\exp(r^{\mu}), P[g])} = \infty \quad \textit{where} \quad 0 < \mu < \rho_g.$$

Proof. Again in view of Lemma 2.2.5, we obtain that

$$\begin{aligned} &\log T(\exp(r^{\mu}), P[g]) \leq (\rho_{P[g]} + \epsilon) \log\{\exp(r^{\mu})\} \\ i.e., \quad &\log T(\exp(r^{\mu}), P[g]) \leq (\rho_g + \epsilon) r^{\mu}. \end{aligned} \tag{2.25}$$

Combining (2.22) and (2.25) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), f\circ g)}{\log T(\exp(r^{\mu}), P[g])} \geq \frac{O(1) + \mu r^{\rho_g}}{(\rho_g + \epsilon) r^{\mu}}. \tag{2.26}$$

Since $\mu < \rho_g$, we get from (2.26) that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f\circ g)}{\log T(\exp(r^{\mu}), P[g])} = \infty.$$

Thus the theorem is established. ■

Theorem 2.3.10 *Let f be meromorphic and g be transcendental entire such that (i) $0 < \rho_g < \infty$, (ii) $\sigma_g > 0$, (iii) $0 < \rho_{f\circ g} < \infty$, (iv) $\sigma_{f\circ g} < \infty$, (v) $\rho_f^* < 1$ and (vi) $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. Then*

$$\liminf_{r\to\infty} \frac{T(r, f\circ g)}{T(r, P[g])} = 0.$$

Proof. From the definition of type, we have for arbitrary positive ϵ and for sufficiently large values of r,

$$T(r, f\circ g) \leq (\sigma_{f\circ g} + \epsilon) r^{\rho_{f\circ g}}. \tag{2.27}$$

Again in view of Lemma 2.2.5 we get for a sequence of values of r tending to infinity,

$$\begin{aligned} &T(r, P[g]) \geq (\sigma_{P[g]} - \epsilon) r^{\rho_{P[g]}} \\ i.e., \quad &T(r, P[g]) \geq [\{\Gamma_p - (\Gamma_p - \gamma_p)\Theta(\infty; g)\}\, \sigma_g - \epsilon] r^{\rho_g}. \end{aligned} \tag{2.28}$$

exists and is equal to 1. Thus we get that

$$\begin{aligned}
{}^{(k)}\rho^{L}_{P_0[f]} &= \limsup_{r\to\infty}\frac{\log^{[k-1]}T(r,P_0[f])}{\log[rL(r)]}\\
&= \limsup_{r\to\infty}\left\{\frac{\log^{[k-1]}T(r,f)}{\log[rL(r)]}.\frac{\log^{[k-1]}T(r,P_0[f])}{\log^{[k-1]}T(r,f)}\right\}\\
&= \limsup_{r\to\infty}\frac{\log^{[k-1]}T(r,f)}{\log[rL(r)]}.\lim_{r\to\infty}\frac{\log^{[k-1]}T(r,P_0[f])}{\log^{[k-1]}T(r,f)}\\
&= {}^{(k)}\rho^{L}_{f}.1 = {}^{(k)}\rho^{L}_{f}.
\end{aligned}$$

This proves the theorem. ■

Theorem 2.3.14 *Let f be a meromorphic function of finite order or of non-zero lower order. Also let $\sum\limits_{a\neq\infty}\Theta(a;f)=2$. Then the generalised $L-$ lower order of homogeneous $P_0[f]$ and f are equal.*

Proof. Let ${}^{(k)}\lambda^{L}_{f}$ and ${}^{(k)}\lambda^{L}_{P_0[f]}$ $(k\geq 2)$ be the generalised $L-$ *lower orders* of f and $P_0[f]$ respectively.
Now in view of Lemma 2.2.8,

$$\lim_{r\to\infty}\frac{\log^{[k-1]}T(r,P_0[f])}{\log^{[k-1]}T(r,f)}$$

exists and is equal to 1. Thus we get

$$\begin{aligned}
{}^{(k)}\lambda^{L}_{P_0[f]} &= \liminf_{r\to\infty}\frac{\log^{[k-1]}T(r,P_0[f])}{\log[rL(r)]}\\
&= \liminf_{r\to\infty}\left\{\frac{\log^{[k-1]}T(r,f)}{\log[rL(r)]}.\frac{\log^{[k-1]}T(r,P_0[f])}{\log^{[k-1]}T(r,f)}\right\}\\
&= \liminf_{r\to\infty}\frac{\log^{[k-1]}T(r,f)}{\log[rL(r)]}.\lim_{r\to\infty}\frac{\log^{[k-1]}T(r,P_0[f])}{\log^{[k-1]}T(r,f)}\\
&= {}^{(k)}\lambda^{L}_{f}.1 = {}^{(k)}\lambda^{L}_{f}.
\end{aligned}$$

This proves the theorem. ■

Theorem 2.3.15 *Let f be either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$. Then the generalised $L-$ orders of homogeneous $P_0[f]$ and f are same.*

Proof. Let ${}^{(k)}\rho_f^L$ and ${}^{(k)}\rho_{P_0[f]}^L$ $(k \geq 2)$ be the generalised $L-orders$ of f and $P_0[f]$ respectively.
Now in view of Lemma 2.2.9,

$$\lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)}$$

exists and is equal to 1. Thus we get

$$\begin{aligned}
{}^{(k)}\rho_{P_0[f]}^L &= \limsup_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log[rL(r)]} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log^{[k-1]} T(r, f)}{\log[rL(r)]} . \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[k-1]} T(r, f)}{\log[rL(r)]} . \lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \\
&= {}^{(k)}\rho_f^L . 1 = {}^{(k)}\rho_f^L .
\end{aligned}$$

Thus the theorem is established. ∎

Theorem 2.3.16 *Let f be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$. Then the generalised $L-$lower order of homogeneous $P_0[f]$ is same as that of f.*

Proof. Let ${}^{(k)}\lambda_f^L$ and ${}^{(k)}\lambda_{P_0[f]}^L$ $(k \geq 2)$ be the generalised $L-lower\ orders$ of f and $P_0[f]$ respectively.
Now in view of Lemma 2.2.9,

$$\lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)}$$

exists and is equal to 1. Thus we obtain that

$$\begin{aligned}
{}^{(k)}\lambda_{P_0[f]}^L &= \liminf_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log[rL(r)]} \\
&= \liminf_{r\to\infty} \left\{ \frac{\log^{[k-1]} T(r, f)}{\log[rL(r)]} . \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \right\} \\
&= \liminf_{r\to\infty} \frac{\log^{[k-1]} T(r, f)}{\log[rL(r)]} . \lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \\
&= {}^{(k)}\lambda_f^L . 1 = {}^{(k)}\lambda_f^L .
\end{aligned}$$

This proves the theorem. ■

Corollary 2.3.3 *Let f be a meromorphic function of finite order or of non-zero lower order such that $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Then for every homogeneous $P_0[f]$, the generalised $L-$ orders (generalised $L-$ lower orders) of $P_0[f]$ and f are equal.*

Proof. Since f has more than one deficient value, its lower order is either positive or infinite. Also $\delta(\infty; f) = \sum \delta(a; f) = 1$ implies

$$\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1.$$

So the corollary follows from Theorem 2.3.15 and Theorem 2.3.16. ■

Theorem 2.3.17 *Let f be of finite order or of non-zero lower order. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Then the generalised $L-$ type of $M_j[f]$ is same as that of f.*

Proof. ${}^{(k)}\sigma_f^L$ and ${}^{(k)}\sigma_{M_j[f]}^L$ be the generalised $L-$ *types* of f and $M_j[f]$ respectively where $k > 2$.
Now by Lemma 2.2.4,

$$\lim_{r\to\infty} \frac{\log^{[k-2]} T(r, M_j[f])}{\log^{[k-2]} T(r, f)}$$

exists and is equal to 1. Then we obtain that

$$\begin{aligned}
{}^{(k)}\sigma_{M_j[f]}^L &= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r, M_j[f])}{[rL(r)]^{{}^{(k)}\rho_{M_j[f]}^L}} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log^{[k-2]} T(r, f)}{[rL(r)]^{{}^{(k)}\rho_f^L}} . \frac{\log^{[k-2]} T(r, M_j[f])}{\log^{[k-2]} T(r, f)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r, f)}{[rL(r)]^{{}^{(k)}\rho_f^L}} . \lim_{r\to\infty} \frac{\log^{[k-2]} T(r, M_j[f])}{\log^{[k-2]} T(r, f)} \\
&= {}^{(k)}\sigma_f^L . 1 = {}^{(k)}\sigma_f^L .
\end{aligned}$$

Thus the theorem is established. ■

Remark 2.3.2 *For $k = 2$, the $L-$type of $M_j[f]$ is $\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}$ times that of f.*

In the next two theorems we see that the generalised $L-$ types of $P_0[f]$ and f are same under different conditions.

Theorem 2.3.18 *Let f be a meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\neq\infty}\Theta(a; f) = 2$. Also let $P_0[f]$ be admissible. Then the generalised $L-$ type of $P_0[f]$ is same as that of f.*

Proof. ${}^{(k)}\sigma_f^L$ and ${}^{(k)}\sigma_{P_0[f]}^L$ be the generalised $L-types$ of f and $P_0[f]$ respectively where $k > 2$.
Now by Lemma 2.2.8,

$$\lim_{r\to\infty}\frac{\log^{[k-2]}T(r, P_0[f])}{\log^{[k-2]}T(r, f)}$$

exists and is equal to 1. Then we obtain that

$$\begin{aligned}
{}^{(k)}\sigma_{P_0[f]}^L &= \limsup_{r\to\infty}\frac{\log^{[k-2]}T(r, P_0[f])}{[rL(r)]^{{}^{(k)}\rho_{P_0[f]}^L}} \\
&= \limsup_{r\to\infty}\left\{\frac{\log^{[k-2]}T(r, f)}{[rL(r)]^{{}^{(k)}\rho_f^L}}.\frac{\log^{[k-2]}T(r, P_0[f])}{\log^{[k-2]}T(r, f)}\right\} \\
&= \limsup_{r\to\infty}\frac{\log^{[k-2]}T(r, f)}{[rL(r)]^{{}^{(k)}\rho_f^L}}.\lim_{r\to\infty}\frac{\log^{[k-2]}T(r, P_0[f])}{\log^{[k-2]}T(r, f)} \\
&= {}^{(k)}\sigma_f^L.1 = {}^{(k)}\sigma_f^L.
\end{aligned}$$

Thus the theorem is established. ■

Remark 2.3.3 *The $L-$ type of $P_0[f]$ is Γ_{P_0} times that of f.*

Theorem 2.3.19 *Let f be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum\limits_{a\neq\infty}\delta_p(a; f) = 1$. Then for homogeneous $P_0[f]$, the generalised $L-$ types of $P_0[f]$ and f are equal.*

Proof. ${}^{(k)}\sigma_f^L$ and ${}^{(k)}\sigma_{P_0[f]}^L$ be the generalised $L-types$ of f and $P_0[f]$ respectively where $k > 2$.
Now by Lemma 2.2.9,

$$\lim_{r\to\infty}\frac{\log^{[k-2]}T(r, P_0[f])}{\log^{[k-2]}T(r, f)}$$

exists and is equal to 1. Then we obtain that

$$\begin{aligned}
{}^{(k)}\sigma^{L}_{P_0[f]} &= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r, P_0[f])}{[rL(r)]^{{}^{(k)}\rho^{L}_{P_0[f]}}} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log^{[k-2]} T(r, f)}{[rL(r)]^{{}^{(k)}\rho^{L}_{f}}} . \frac{\log^{[k-2]} T(r, P_0[f])}{\log^{[k-2]} T(r, f)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r, f)}{[rL(r)]^{{}^{(k)}\rho^{L}_{f}}} . \lim_{r\to\infty} \frac{\log^{[k-2]} T(r, P_0[f])}{\log^{[k-2]} T(r, f)} \\
&= {}^{(k)}\sigma^{L}_{f} .1 = {}^{(k)}\sigma^{L}_{f} .
\end{aligned}$$

Thus the theorem is established. ■

Remark 2.3.4 *The $L-$ type of $P_0[f]$ is γ_{P_0} times that of f.*

Remark 2.3.5 *The conclusion of Theorem* 2.3.19 *can also be drawn under the condition $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$ instead of*

$$\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1.$$

Theorem 2.3.20 *Let f be transcendental meromorphic of positive finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Then the generalised L^*- order of $M_j[f]$ is same as that of f.*

Proof. Let ${}^{(k)}\rho^{L^*}_{f}$ and ${}^{(k)}\rho^{L^*}_{M_j[f]}$ $(k \geq 2)$ be the generalised $L^* -$ *orders* of f and $M_j[f]$ respectively.
Now in view of Lemma 2.2.4,

$$\lim_{r\to\infty} \frac{\log^{[k-1]} T(r, M_j[f])}{\log^{[k-1]} T(r, f)}$$

Proof. Let ${}^{(k)}\rho_f^{L^*}$ and ${}^{(k)}\rho_{P_0[f]}^{L^*}$ $(k \geq 2)$ be the generalised $L^* -$ *orders* of f and $P_0[f]$ respectively.
Now in view of Lemma 2.2.8,

$$\lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)}$$

exists and is equal to 1. Thus we get

$$\begin{aligned}
{}^{(k)}\rho_{P_0[f]}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log\left[re^{L(r)}\right]} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log^{[k-1]} T(r, f)}{\log\left[re^{L(r)}\right]} . \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[k-1]} T(r, f)}{\log\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \\
&= {}^{(k)}\rho_f^{L^*}.1 = {}^{(k)}\rho_f^{L^*}.
\end{aligned}$$

This proves the theorem. ■

Theorem 2.3.23 *Let f be a meromorphic function of finite order or of non-zero lower order. If $\sum_{a\neq\infty}\Theta(a; f) = 2$, then the generalised $L^* -$ lower order of homogeneous $P_0[f]$ and f are equal.*

Proof. Let ${}^{(k)}\lambda_f^{L^*}$ and ${}^{(k)}\lambda_{P_0[f]}^{L^*}$ $(k \geq 2)$ be the generalised $L^* -$ *lower orders* of f and $P_0[f]$ respectively.
Now in view of Lemma 2.2.8,

$$\lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)}$$

exists and is equal to 1. Thus we get

$$\begin{aligned}
{}^{(k)}\lambda_{P_0[f]}^{L^*} &= \liminf_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log\left[re^{L(r)}\right]} \\
&= \liminf_{r\to\infty} \left\{ \frac{\log^{[k-1]} T(r, f)}{\log\left[re^{L(r)}\right]} . \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \right\} \\
&= \liminf_{r\to\infty} \frac{\log^{[k-1]} T(r, f)}{\log\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \\
&= {}^{(k)}\lambda_f^{L^*}.1 = {}^{(k)}\lambda_f^{L^*}.
\end{aligned}$$

This proves the theorem. ■

Theorem 2.3.24 *Let f be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$. Then the generalised L^*- orders of homogeneous $P_0[f]$ and f are same.*

Proof. Let ${}^{(k)}\rho_f^{L^*}$ and ${}^{(k)}\rho_{P_0[f]}^{L^*}$ $(k \geq 2)$ be the generalised $L^* -$ *orders* of f and $P_0[f]$ respectively.
Now in view of Lemma 2.2.9,

$$\lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)}$$

exists and is equal to 1. Thus we obtain that

$$\begin{aligned}
{}^{(k)}\rho_{P_0[f]}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log\left[re^{L(r)}\right]} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log^{[k-1]} T(r, f)}{\log\left[re^{L(r)}\right]} \cdot \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[k-1]} T(r, f)}{\log\left[re^{L(r)}\right]} \cdot \lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)} \\
&= {}^{(k)}\rho_f^{L^*} .1 = {}^{(k)}\rho_f^{L^*} .
\end{aligned}$$

Thus the theorem is established. ■

Theorem 2.3.25 *Let f be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$. Then the generalised L^*-lower order of homogeneous $P_0[f]$ is same as that of f.*

Proof. Let ${}^{(k)}\lambda_f^{L^*}$ and ${}^{(k)}\lambda_{P_0[f]}^{L^*}$ $(k \geq 2)$ be the generalised $L^* -$ *lower orders* of f and $P_0[f]$ respectively.
Now in view of Lemma 2.2.9,

$$\lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r, f)}$$

exists and is equal to 1. Thus we get

$$\begin{aligned}
{}^{(k)}\lambda^{L^*}_{P_0[f]} &= \liminf_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log\left[re^{L(r)}\right]} \\
&= \liminf_{r\to\infty} \left\{ \frac{\log^{[k-1]} T(r,f)}{\log\left[re^{L(r)}\right]} . \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r,f)} \right\} \\
&= \liminf_{r\to\infty} \frac{\log^{[k-1]} T(r,f)}{\log\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{\log^{[k-1]} T(r, P_0[f])}{\log^{[k-1]} T(r,f)} \\
&= {}^{(k)}\lambda^{L^*}_f .1 = {}^{(k)}\lambda^{L^*}_f .
\end{aligned}$$

This proves the theorem. ∎

Corollary 2.3.4 *Let f be a meromorphic function of finite order or of non-zero lower order such that $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Then for every homogeneous $P_0[f]$, the generalised L^* – orders (generalised L^* – lower orders) of $P_0[f]$ and f are equal.*

Proof. Since f has more than one deficient value, its lower order is either positive or infinite. Also $\delta(\infty; f) = \delta(a; f) = 1$ implies

$$\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1.$$

So the corollary follows from Theorem 2.3.24 and Theorem 2.3.25. ∎

Theorem 2.3.26 *Let f be finite order or of non-zero lower order. Also let $\sum_{a\epsilon\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ then the generalised L^* – type of $M_j[f]$ is same as that of f.*

Proof. Let ${}^{(k)}\sigma^{L^*}_f$ and ${}^{(k)}\sigma^{L^*}_{M_j[f]}$ $(k > 2)$ be the generalised L^* – *types* of f and $M_j[f]$ respectively.
Now by Lemma 2.2.4,

$$\lim_{r\to\infty} \frac{\log^{[k-2]} T(r, M_j[f])}{\log^{[k-2]} T(r,f)}$$

exists and is equal to 1. Then we obtain that

$$
\begin{aligned}
{}^{(k)}\sigma_{M_j[f]}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r, M_j[f])}{[rL(r)]^{{}^{(k)}\rho_{M_j[f]}^{L^*}}} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log^{[k-2]} T(r,f)}{[rL(r)]^{{}^{(k)}\rho_f^{L^*}}} . \frac{\log^{[k-2]} T(r, M_j[f])}{\log^{[k-2]} T(r,f)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r,f)}{[rL(r)]^{{}^{(k)}\rho_f^{L^*}}} . \lim_{r\to\infty} \frac{\log^{[k-2]} T(r, M_j[f])}{\log^{[k-2]} T(r,f)} \\
&= {}^{(k)}\sigma_f^{L^*}.1 = {}^{(k)}\sigma_f^{L^*}.
\end{aligned}
$$

Thus the theorem is established. ■

Remark 2.3.6 *For $k = 2$, the L^*–type of $M_j[f]$ is $\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}$ times that of f.*

In the next two theorems we see that the generalised $L^* -$ *types* of $P_0[f]$ and f are same under different conditions.

Theorem 2.3.27 *Let f be a meromorphic function of finite order or of non-zero lower order and $\sum_{a\neq\infty} \Theta(a; f) = 2$. Also let $P_0[f]$ be admissible. Then the generalised $L^* -$ type of $P_0[f]$ is same as that of f.*

Proof. Let ${}^{(k)}\sigma_f^{L^*}$ and ${}^{(k)}\sigma_{P_0[f]}^{L^*}$ $(k > 2)$ be the generalised $L^* -$ *types* of f and $P_0[f]$ respectively.
Now by Lemma 2.2.8,

$$
\lim_{r\to\infty} \frac{\log^{[k-2]} T(r, P_0[f])}{\log^{[k-2]} T(r,f)}
$$

exists and is equal to 1. Then we obtain that

$$
\begin{aligned}
{}^{(k)}\sigma_{P_0[f]}^{L^*} &= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r, P_0[f])}{[re^{L(r)}]^{{}^{(k)}\rho_{P_0[f]}^{L^*}}} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log^{[k-2]} T(r,f)}{[re^{L(r)}]^{{}^{(k)}\rho_f^{L^*}}} . \frac{\log^{[k-2]} T(r, P_0[f])}{\log^{[k-2]} T(r,f)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log^{[k-2]} T(r,f)}{[re^{L(r)}]^{{}^{(k)}\rho_f^{L^*}}} . \lim_{r\to\infty} \frac{\log^{[k-2]} T(r, P_0[f])}{\log^{[k-2]} T(r,f)} \\
&= {}^{(k)}\sigma_f^{L^*}.1 = {}^{(k)}\sigma_f^{L^*}.
\end{aligned}
$$