CHAPTER-1

INTRODUCTION

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Let f be an entire function, *i.e.*, a function of a complex variable regular in the open complex plane. Then it has a Taylor series expansion about $z = 0$ as

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

which can be considered as an extension of a polynomial. The degree of a polynomial, which is equal to the number of zeros, estimates the rate of growth of the polynomial as the independent variable moves without bound. So more zeros, greater is the growth.

To characterise the growth of an entire function and the distribution of its zeros a special growth scale called maximum modulus function of f on $|z| = r$ is introduced as $M(r, f) = \max_{|z|=r} |f(z)|$. Obviously $M(r, f)$ is unbounded for any non-constant entire function f. Also by the maximum modulus theorem $M(r, f)$ increases monotonically as r increases.

Again we know that $\log M(r, f)$ is a continuous, convex and increasing function of $\log r$. In case of transcendental entire function i.e., the function which has an essential singularity at the point at infinity, $M(r, f)$ grows faster than any positive power of r. Thus in order to estimate the growth of transcendental entire functions we choose a comparison function $\exp(r^k)$, $k > 0$.

Set

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r},$$

which is called the order of f. If $0 \leq \rho < \infty$ then f is said to be of finite order. The number

$$\lambda = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

is called the lower order of f. Obviously $\lambda \le \rho$. If $\lambda = \rho$ then f is said to be of regular growth.

With $0 < \rho < \infty$ the growth of an entire function is characterized more precisely by the type of the function. The number τ given by

$$\tau = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}}$$

is called the type of f. Between two functions of same order one can be characterized to be of faster growth if its type is greater.

A function in the open complex plane $\mathbb{C}$ is called a meromorphic function if the only possible singularties in $\mathbb{C}$ are poles. In the present century the modern theory of value distribution of meromorphic functions, developed by Rolf Nevanlinna was one of the most significant achievements in the function theory. He elevated the theory of meromorphic functions to a new height by introducing his first and second fundamental theorems.

The first fundamental theorem provides an upper bound to the number of $a-$points of f whereas the second fundamental theorem deals with the more difficult question of lower bounds for the number of $a-$points of f. The second fundamental theorem is nothing but the extension of Picard's famous theorem that f must assume all values in the complex plane with at most two exceptions.

To state fundamental theorems we need the following notations.

Let f be a meromorphic function in the finite complex plane and $n(r,a;f) \equiv n\left(r, \frac{1}{f-a}\right)$ which is a non negative integer for each r, denotes the number of $a-$ *points* of f in $|z| \le r$, counted with proper multiplicities, for a complex number a, finite or infinite. Obviously $n(r,\infty;f) \equiv n(r,f)$ represents the number of poles of f in $|z| \le r$, counted with proper multiplicities.

We define a function $N(r,a;f)$ as follows:

$$N(r,a;f) = \int_0^r \frac{n(t,a;f) - n(0,a;f)}{t} dt + n(0,a;f)\log r$$

and $N(r,\infty;f) = N(r,f)$.

Let us define

$$log^{+}x = \log x \quad for \quad x \geq 1$$
$$= 0 \quad for \quad 0 \leq x < 1 \; .$$

Then immediately we get the following:

(*i*) $\log^{+} x \geq 0$ if $x \geq 0$,

(*ii*) $\log^{+} x \geq \log x$ if $x > 0$,

(*iii*) $\log^{+} x \geq \log^{+} y$ if $x > y$,

(*iv*) $\log x = \log^{+} x - \log^{+} \frac{1}{x}$ if $x > 0$.

The quantity $m\,(r, f)$ is defined by

$$m\,(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^{+} \left| f\left(re^{i\theta}\right) \right| d\theta$$

and is called the proximity function of f. The term $m\,(r, f)$ is a sort of averaged magnitude of $\log |f\,(z)|$ on arcs of $|z| = r$ where $|f\,(z)|$ is large. We write

$$T\,(r, f) = m\,(r, f) + N\,(r, f) \; .$$

The function $T\,(r, f)$ is called the *Nevanlinna's Characteristic* function of the meromorphic function f which plays an important role in the theory of meromorphic functions.

If f is an entire function then $N(r, f) = 0$ and so $T(r, f) = m(r, f)$. For an entire function f the study of the comparative growth properties of $T\,(r, f)$ and $\log M(r, f)$ is a popular problem for the researchers.

Here we note that for a meromorphic function f, $T(r, f)$ is an increasing convex function of $\log r$ $\{p.9,\ cf$ **[29]**$\}$.

If f is regular in $|z| \leq R$ the following result establishes a relation between $T(r, f)$ and $M(r, f)$, which is of frequent use.

Theorem 1.0.1 *{p.18,[29]}If f is regular for $|z| \leq R$ then*

$$T(r, f) \leq \log^{+} M(r, f) \leq \frac{R + r}{R - r} T(R, f), \quad 0 \leq r < R.$$

Since for any positive integer p and complex numbers a_ν $(\nu = 1, 2, ...p)$,

$$\log^+ \left| \prod_{\nu=1}^{p} a_\nu \right| \leq \sum_{\nu=1}^{p} \log^+ |a_\nu|$$
$$\text{and} \quad \log^+ \left| \sum_{\nu=1}^{p} a_\nu \right| \leq \sum_{\nu=1}^{p} \log^+ |a_\nu| + \log p \ ,$$

it is easy to show that for any arbitrary p meromorphic functions $f_1, f_2, \cdots f_p$,

$$m\left(r, \prod_{\nu=1}^{p} f_\nu(z)\right) \leq \sum_{\nu=1}^{p} m\left(r, f_\nu(z)\right)$$
$$\text{and} \quad m\left(r, \sum_{\nu=1}^{p} f_\nu(z)\right) \leq \sum_{\nu=1}^{p} m\left(r, f_\nu(z)\right) + \log p \ .$$

Again

$$N\left(r, \prod_{\nu=1}^{p} f_\nu(z)\right) \leq \sum_{\nu=1}^{p} N\left(r, f_\nu(z)\right)$$
$$\text{and} \quad N\left(r, \sum_{\nu=1}^{p} f_\nu(z)\right) \leq \sum_{\nu=1}^{p} N\left(r, f_\nu(z)\right) \ .$$

From these we get that

$$T\left(r, \prod_{\nu=1}^{p} f_\nu(z)\right) \leq \sum_{\nu=1}^{p} T\left(r, f_\nu(z)\right)$$
$$\text{and} \quad T\left(r, \sum_{\nu=1}^{p} f_\nu(z)\right) \leq \sum_{\nu=1}^{p} T\left(r, f_\nu(z)\right) + \log p.$$

We now state Nevanlinna's first fundamental theorem in the following form.

Theorem 1.0.2 {*p.6,[29]*}*If f is a meromorphic function in $|z| < \infty$ and a be any complex number, finite or infinite, then*

$$m\left(r, a; f\right) + N\left(r, a; f\right) = T\left(r, f\right) + O\left(1\right) \ .$$

The result shows remarkable symmetry exhibited by a meromorphic function in its behaviour relative to different complex numbers, finite or infinite. For different values of a, the sum $m(r,a;f)+N(r,a;f)$ maintains a total, given by $T(r,f)$ which is invariant apart from a bounded additive term.

The first term $m(r,a;f)$ of this invariant sum is termed as the mean value of $\log^+\left|\frac{1}{f-a}\right|$ $\left(or\ \log^+|f|\ if\ a=\infty\right)$ on the circle $|z|=r$, contributes remarkably for those arcs on the circle where the functional values differ very little from the given value $'a'$. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle $|z|=r$ of the functional value f from the value a.

The term $N(r,a;f)$ indicates the density of the average distribution of the roots of the equation $f(z)=a$ in the disc $|z|<r$. The larger the number of $a-$points the faster this counting function for $a-$points grows with r.

At this junction we introduce some definitions.

Definition 1.0.1 {p.16, *[29]*}*Let $S(r)$ be a real and non-negative increasing function for $r_0 \leq r < \infty$, $r_0 > 0$. The order k and lower order λ of the function $S(r)$ are defined as*

$$k=\limsup_{r\to\infty}\frac{\log S(r)}{\log r}$$

and

$$\lambda=\liminf_{r\to\infty}\frac{\log S(r)}{\log r}.$$

respectively. Moreover if $0<k<\infty$, we set

$$c=\limsup_{r\to\infty}\frac{S(r)}{r^k}$$

and distinguish the following possibilities:

(a) $S(r)$ has maximal type if $c=+\infty$;

(b) $S(r)$ has mean type if $0<c<+\infty$;

(c) $S(r)$ has minimal type if $c=0$ and

(d) $S(r)$ has convergence class if

$$\int_r^\infty \frac{S(t)}{t^{k+1}}dt \text{ converges .}$$

The following theorem can be proved easily.

Theorem 1.0.3 $\{p.18,[29]\}$*If f is an entire function then the order k of the functions $S_1(r) = \log^+ M(r,f)$ and $S_2(r) = T(r,f)$ is the same. Further if $0 < k < \infty$, $S_1(r)$ and $S_2(r)$ belong to the same classes (a), (b), (c) or (d). Also lower order of $S_1(r)$ and $S_2(r)$ are same.*

A meromorphic function f is said to have order ρ, lower order λ and maximal, minimal, mean type or convergence class if the function $T(r,f)$ has this property. For an entire function these coincides by the above theorem with the corresponding definition in terms of $\log^+ M(r,f)$ which is classical. The quantity τ defined by

$$\tau = \limsup_{r\to\infty} \frac{T(r,f)}{r^\rho}$$

where ρ is the positive finite order of f, is called the type of f.

Nevanlinna's second fundamental theorem is the following :

Theorem 1.0.4 $\{p.34,[29]\}$*Suppose that f is a non-constant meromorphic function in $|z| \le r$ and $f(0) \ne 0, \infty$, $f'(0) \ne 0$. Let $a_1, a_2, \cdots a_q$ where $q \ge 2$, be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| \ge \delta$ $(0 < \delta < 1)$ for $1 \le \mu < \nu \le q$. Then*

$$m(r,\infty;f) + \sum_{\nu=1}^{q} m(r,a_\nu,f) \le 2T(r,f) - N_1(r) + S(r,f),$$

where

$$N_1(r) = N\left(r,\frac{1}{f'}\right) + 2N(r,f) - N(r,f') \text{ and}$$

$$S(r,f) = m\left(r,\frac{f'}{f}\right) + m\left\{r,\sum_{\nu=1}^{q}\frac{f'}{f-a_\nu}\right\} + q\log^+\frac{3q}{\delta} + \log 2 + \log\frac{1}{|f'(0)|}.$$

The cases $f(0) = 0$ or ∞ , $f'(0) = 0$ can be dealt with suitable modification.

The quantity $S(r,f)$ will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.

We can estimate $S(r,f)$ from the following theorem :

Theorem 1.0.5 *{p.34,[29]}Let f be a non constant meromorphic function defined in $|z| < R_0 \leq \infty$ and that $S(r,f)$ is as in the above theorem. Then we have the following:*

(*i*) *If $R_0 = +\infty$, $S(r,f) = O\{\log T(r,f)\} + O(\log r)$, as $r \to \infty$ through all values if f has finite order and as $r \to \infty$ outside a set E of finite linear measure otherwise.*

(*ii*) *If $0 < R_0 < \infty$,*

$$S(r,f) = O\left\{\log^+ T(r,f) + \log \frac{1}{R_0 - r}\right\} \text{ as } r \to R_0$$

outside a set E such that $\int_E \frac{dr}{R_0 - r} < +\infty$.

Further there is a point r outside E for which $\rho < r < \rho'$ provided that $0 < R - \rho' < e^{-2}(R - \rho)$.

As a consequence we get the following theorem immediately :

Theorem 1.0.6 *{p.41,[29]}Let f be a non-constant meromorphic function in $|z| < R_0$. Then*

$$\frac{S(r,f)}{T(r,f)} \to 0 \tag{1.A}$$

as $r \to R_0$ with the following provisions :

(*a*) *(1.A) holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.*

(*b*) *If f has infinite order in the plane and $R_0 = +\infty$, (1.A) still holds as $r \to \infty$ outside a certain exceptional set E of finite length. Here E depends only on f.*

(*c*) *If $R_0 < +\infty$ and*

$$\limsup_{r \to R_0} \frac{T(r,f)}{\log\{\frac{1}{R_0 - r}\}} = +\infty ,$$

then (1.A) holds as $r \to R_0$ through a suitable sequence r_n, which depends on f only.

This theorem indicates why $S(r,f)$ plays the role of an unimportant error term.

Let f be a non constant meromorphic function in the plane. By $S(r,f)$ we shall denote any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set r of finite linear measure.

We shall now present an important consequence of Nevanlinna's second fundamental theorem, which is called Nevanlinna's theorem on deficient values. To rewrite this, the following definitions are useful.

Definition 1.0.2 *For a complex number 'a' we put*

$$\delta(a;f)=\liminf_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}$$
$$=1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}$$

and

$$\Theta(a;f)=1-\limsup_{r\to\infty}\frac{\bar{N}(r,a;f)}{T(r,f)},$$

where $\bar{N}(r,a;f)$ is the counting function for distinct $a-$points of f. Also

$$\theta(a;f)=\liminf_{r\to\infty}\frac{N(r,a;f)-\bar{N}(r,a;f)}{T(r,f)}.$$

The quantities $\delta(a;f)$, $\Theta(a;f)$ and $\theta(a;f)$ are called respectively the deficiency, the ramification index and the index of multiplicity of the value 'a'. Evidently $\delta(a;f)$ is positive only if there are relatively few roots of the equation $f(z)=a$. Maximum value of the deficiency is 1 when the roots of the equation $f(z)=a$ is very sparsely distributed, in particular when the value 'a' is a Picard's exceptional value. In general $0\leq m(r,a;f)\leq T(r,f)$ so we have $0\leq\delta(a;f)\leq 1$. Thus the quantity gives us a very accurate measure for relative density of the points where the function f assumes the value 'a'. We shall call a value normal if $\delta(a;f)$ vanishes. Again $\Theta(a;f)$ is positive only when $f(z)=a$ has relatively many multiple roots.

Let $\varepsilon(>0)$ be an arbitrary small number. Then from the above definitions for sufficiently large values of r we have

$$N(r,a;f)-\bar{N}(r,a;f)>\{\theta(a;f)-\varepsilon\}T(r,f),$$

$$N(r,a;f) < \{1-\delta(a;f)+\varepsilon\}T(r,f)$$

and so

$$\bar{N}(r,a;f) < \{1-\delta(a;f)-\Theta(a;f)+2\varepsilon\}T(r,f)$$

so that

$$\Theta(a;f) \geq \delta(a;f)+\theta(a;f).$$

Now we state Nevanlinna's deficient value theorem.

Theorem 1.0.7 *{p.43,[29]}Let f be a non constant meromorphic function defined on the plane. Then the set of values a for which $\theta(a;f) > 0$ is countable and summing over all such values a*

$$\sum_a \{\delta(a;f)+\theta(a;f)\} \leq \sum_a \Theta(a;f) \leq 2\ .$$

Now Picard's theorem, that a non-constant meromorphic function can have at most two Picard exceptional values, follows easily from Nevanlinna's deficient value theorem because for a Picard's exceptional value a, $\delta(a;f) = 1$.

Now we state Milloux's theorem which plays an important role in studing the properties of meromorphic derivatives.

Theorem 1.0.8 *{p.55,[29]}Let l be a positive integer and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$, where $T(r,a_\nu) = S(r,f), \nu = 0,1,...l$. Then $m\left(r,\frac{\psi}{f}\right) = S(r,f)$ and*

$$T(r,\psi) \leq (l+1)T(r,f)+S(r,f).$$

Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of $f(z) = a$ by roots of the equation $\psi(z) = b$, where ψ is as above. The following theorem explains this fact.

Theorem 1.0.9 *{p.57,[29]}Let f be meromorphic and non constant and $\psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$, l is a positive integer, be non-constant in the complex plane. Then*

$$T(r,f) \leq \overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{\psi-1}\right)-N_0\left(r,\frac{1}{\psi'}\right)+S(r,f),$$

where in $N_0\left(r, \frac{1}{\psi'}\right)$ *only zeros of* ψ' *not corresponding to the repeated roots of* $\psi(z) = 1$ *are to be considered.*

An entire function $f(z)$ has a Taylor's expansion about any point a in the complex plane of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Since this series is absolutely convergent everywhere in the plane, the terms $|a_n|$ must approach zero. Consequently, there exists for each a, an index $n_o = n(a)$ for which $|a_n|$ is a maximal coefficient. Some researchers raised the problem of characterising entire functions for which $n(a)$ is bounded which are called functions of bounded index.

Definition 1.0.3 *An entire function* f *is said to be of bounded index if and only if there exists an integer* N*, such that for all* z

$$\max\left(|f|, \left|f^{(1)}\right|, \frac{\left|f^{(1)}\right|}{2!}, ..., \frac{\left|f^{(N)}\right|}{N!}\right) \geq \frac{\left|f^{(j)}\right|}{j!} \tag{1.B}$$

where $j = 0, 1, 2, 3, ...$ *and* $f^{(0)}$ *denotes* f*. We shall say that* f *is of index* N*, if* N *is the smellest integer for which* $(1.B)$ *holds. An entire function which is not of bounded index is said to be of unbounded index.*

A function of bounded index satisfies

$$\sum_{i=0}^{N} \frac{\left|f^{(i)}\right|}{i!} \geq \frac{\left|f^{(j)}\right|}{j!} \tag{1.C}$$

for $j = 0, 1, 2, 3, ...$

Furthermore, if $(1.C)$ holds then

$$\max\left(|f|, \left|f^{(1)}\right|, \frac{\left|f^{(1)}\right|}{2!}, ..., \frac{\left|f^{(N)}\right|}{N!}\right) \geq \frac{1}{(N+1)} \cdot \frac{\left|f^{(j)}\right|}{j!}$$

for $j = 0, 1, 2, 3, ...$ The function $f(z) = \exp z$ is absolutely of bounded index.

The following definition is also necessary.

Definition 1.0.4 *An entire function $f(z)$ is said to be of non uniform bounded index if and only if there exist integers N_j, such that*

$$\sum_{i=0}^{N} \frac{\left|f^{(i)}(z)\right|}{i!} \geq C.\frac{\left|f^{(j)}(z)\right|}{j!} \quad for \quad |z| > N_j$$

where $j = 0, 1, 2, 3, \ldots$ and C is any fixed constant.

Let K be a positive integer. The entire function $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(Kn)!}$ is of non uniform bounded index.

Apart from **Chapter 1** the thesis consists of nine other chapters.

- In **Chapter 2** we study the comparative growth properties of composite entire or meromorphic functions and differential monomials generated by one of the factors. In this chapter we also discuss the relationship between the generalised $L-$order (generalised $L-$type) of a transcendental meromorphic function and that of a differential monomial generated by it. We also establish some theorems on the relationship between the generalised $L-$order (generalised $L-$type) of a meromorphic function and that of a differential polynomial generated by it under different conditions. The results of this chapter have been published in **International Mathematical Forum,** see **[16]** and **[17].**

- In **Chapter 3** we discuss the comparative growth of composite entire or meromorphic functions and Wronskians generated by one of the factors. In this chapter we also study the relationship between the $L-order$ ($L-type$) of a transcendental meromorphic function and that of a wronskian generated by it. Some portion of the results of this chapter have been published in **International Journal of Mathematical Analysis** see **[18]** and the remaining have been published in **International Journal of Contemporary Mathematical Sciences,** see **[19].**

- In **Chapter 4** we compare the maximum term of composition of two entire functions with their corresponding left and right factors on the basis of $L-(p,q)th\ order$ where $L \equiv L(r)$ is a slowly changing function and p, q are positive integers with $p > q$. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences,** see **[20].**

- In **Chapter 5** we study the comparative growth properties of entire functions on the basis of relative $L-(p,q)$th order where p,q are positive integers with $p > q$ and $L \equiv L(r)$ is a slowly changing function. The results of this chapter have been published in **International Journal of Mathematical Analysis** see [21].

- In **Chapter 6** we will show that the $L-(p,q)$th order of the derivative of a meromorphic function is the same as that of the original function where p,q are positive integers with $p > q$ and $L \equiv L(r)$ is a slowly changing function. The results of this chapter have been published in **International Journal of Mathematical Analysis,** see [22].

- In **Chapter 7** we introduce the definition of the weak type of a meromorphic function of lower order zero or lower order infinity and obtain its integral representation. Some growth properties related to the weak type of meromorphic and entire functions are also established with examples. Some portion of the results of this chapter have been published in **International Journal of Mathematical Sciences and Engineering Applications,** see [24] and the remaining have been published in **International Journal of Contemporary Mathematical Sciences,** see [25].

- In **Chapter 8** we investigate the comparative growth of composite entire functions which satisfy second order linear differential equations. The results of this chapter have been published in **International Mathematical Forum,** see [23].

- In **Chapter 9** we compare the Valiron defect with the relative Nevanlinna defect of a special type of differential polynomial generated by a transcendental meromorphic function. The results of this chapter have been published in **Journal of Mechanics of Continua and Mathematical Sciences,** see [26].

- In **Chapter 10** we discuss about the comparative growth properties related to the Ritt order of entire Dirichlet series. The results of this chapter have been published in **Review Bulletin of the Calcutta Mathematical Society,** see [27].

From **Chapter 2** onwards when we write **Theorem** $a.b.c$ (or **Corollary** $a.b.c$ etc.) where a, b and c are positive integers, we mean the c**-th theorem** (or c**-th corollary** etc.) of the b**-th section** in the a**-th chapter**. Also by **equation number** $(a.b)$ we mean the b**-th equation** in the a**-th chapter** for positive integers a and b. Individual chapters have been presented in such a manner that they are almost independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

The author of the thesis is thankful to the authors of various papers and books which have been consulted during the preparation of the entire thesis.

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