CHAPTER-10

STUDY OF GROWTH PROPERTIES ON THE BASIS OF RITT ORDER OF ENTIRE DIRICHLET SERIES

Chapter 10

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10.1 Introduction, Definitions and Notations.

During the past decades, several authors {*cf.* [**53**], [**54**], [**56**]} made close investigations on the properties of entire Dirichlet series related to Ritt order. Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{s\lambda_n} \quad where \;\; 0 < \lambda_n < \lambda_{n+1} \; (n \geq 1), \tag{10.1.A}$$

$\lambda_n \to \infty$ as $n \to \infty$ and a_n's are complex constants.

If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1) then clearly $\sigma_c = \sigma_a = \infty$. Let

$$F(\sigma) = \underset{-\infty < t < \infty}{lub} \left| f(\sigma + it) \right|. \tag{10.1.B}$$

Then the Ritt order [**52**] of $f(s)$ denoted by $\rho(f)$ is given by

$$\rho(f) = \limsup_{\sigma \to \infty} \frac{\log \log F(\sigma)}{\sigma} = \limsup_{\sigma \to \infty} \frac{\log^{[2]} F(\sigma)}{\sigma}.$$

The results of this Chapter have been published in **Review Bulletin of the Calcutta Mathematical Society**, see [**27**].

In other words

$$\rho(f) = \inf\{\mu > 0 : \log F(\sigma) < \exp(\sigma\mu) \ for \ all \ \sigma > R(\mu)\}.$$

Similarly the lower Ritt order of $f(s)$ denoted by $\lambda(f)$ may be defined.

Let f and g be two entire functions and $F(r) = \max\{|f(z)| : |z| = r\}$, $G(r) = \max\{|g(z)| : |z| = r\}$. If f is non-constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} F^{-1}(s) = \infty$. Bernal **[2]** introduced the definition of relative order of f with respect to g, denoted as $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^{\mu}) \ for \ all \ r > r_0(\mu) > 0\}$$
$$= \limsup_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

The definition coincides with the classical one if $g(z) = \exp z$. Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log G^{-1}F(r)}{\log r}.$$

The following definition is also well known.

Definition 10.1.1 *The relative hyper order $\bar{\rho}_g(f)$ and relative hyper lower order $\bar{\lambda}_g(f)$ of an entire function f with respect to another entire function g are defined in the following way:*

$$\bar{\rho}_g(f) = \limsup_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log r} \quad and \quad \bar{\lambda}_g(f) = \liminf_{r\to\infty} \frac{\log^{[2]} G^{-1}F(r)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$.

Lahiri and Banerjee **[50]** introduced the following definition.

Definition 10.1.2 *[50]The relative Ritt order of $f(s)$ with respect to an entire function $g(s)$ is given by*

$$\rho_g(f) = \inf\{\mu > 0 : \log F(\sigma) < G(\sigma\mu) \ for \ all \ large \ \sigma\}$$

where $G(r) = \max\{|g(s)| : |s| = r\}$.

Clearly $\rho_g(f) = \rho(f)\ if\ g(s) = e^s$. From Definition 10.1.2 it clearly follows that

$$\rho_g(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma}.$$

Similarly the relative lower Ritt order of $f(s)$ with respect to entire $g(s)$ is defined as

$$\lambda_g(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{\sigma}.$$

Analoguously we can define

Definition 10.1.3 *The relative hyper Ritt order* $\bar{\rho}_g(f)$ *and relative hyper lower Ritt order* $\bar{\lambda}_g(f)$ *of* $f(s)$ *with respect to entire* $g(s)$ *are respectively defined by*

$$\bar{\rho}_g(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma} \quad \text{and} \quad \bar{\lambda}_g(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{\sigma}.$$

Now we state the following definition known as generalised relative Ritt order and generalised relative lower Ritt order.

Definition 10.1.4 *The generalised relative Ritt order* ${}^{(k)}\rho_g(f)$ *and generalised relative lower Ritt order* ${}^{(k)}\lambda_g(f)$ *where* $k = 1, 2, 3, \ldots$ *of* $f(s)$ *with respect to entire* $g(s)$ *are respectively defined by*

$${}^{(k)}\rho_g(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma}$$

and

$${}^{(k)}\lambda_g(f) = \liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[k]} F(\sigma)}{\sigma}.$$

After Bernal [**2**], several papers on relative order of entire functions have appeared in the literature where growing interest of workers on this topic has been noticed {see for example {[**6**], [**7**], [**8**], [**42**], [**45**], [**47**], [**48**] and [**49**]}. Before we pass on, we remark that the papers [**30**], [**32**], [**34**] and [**63**]} contains investigations on relative order (H:K) of entire functions, but Bernal's analysis including subsequent studies after Bernal have little relevance to the studies made in the above papers.

In the chapter we prove some results on the comparative growth properties related to the Ritt order of Entire Dirichlet series.

10.2 Theorems.

In this section we present the main results of the chapter.

Theorem 10.2.1 *Let f and g be two entire functions such that* $0 < \lambda(f \circ g) \leq \rho(f \circ g) < \infty$ *and* $0 < \lambda(g) \leq \rho(g) < \infty$. *Then*

$$\frac{\lambda(f \circ g)}{\rho(g)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \min\left\{\frac{\lambda(f \circ g)}{\lambda(g)}, \frac{\rho(f \circ g)}{\rho(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda(f \circ g)}{\lambda(g)}, \frac{\rho(f \circ g)}{\rho(g)}\right\} \leq \limsup_{\sigma \to \infty} \frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\rho(f \circ g)}{\lambda(g)}.$$

Proof. From the definition of Ritt order and lower Ritt order of an entire function g, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} G(\sigma) \leq (\rho(g) + \varepsilon)\sigma \tag{10.1}$$

and

$$\log^{[2]} G(\sigma) \geq (\lambda(g) - \varepsilon)\sigma. \tag{10.2}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[2]} G(\sigma) \leq (\lambda(g) + \varepsilon)\sigma \tag{10.3}$$

and

$$\log^{[2]} G(\sigma) \geq (\rho(g) - \varepsilon)\sigma. \tag{10.4}$$

Again from the definition of Ritt order and lower Ritt order of the composite entire function $f \circ g$ we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} F \circ G(\sigma) \leq (\rho(f \circ g) + \varepsilon)\sigma \tag{10.5}$$

and

$$\log^{[2]} F \circ G(\sigma) \geq (\lambda(f \circ g) - \varepsilon)\sigma. \tag{10.6}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[2]} F \circ G(\sigma) \leq (\lambda(f \circ g) + \varepsilon)\sigma \tag{10.7}$$

and

$$\log^{[2]} F \circ G(\sigma) \geq (\rho(f \circ g) - \varepsilon)\sigma. \tag{10.8}$$

Now from (10.1) and (10.6) it follows for all sufficiently large values of r that

$$\frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \geq \frac{(\lambda(f \circ g) - \varepsilon)}{(\rho(g) + \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{\sigma \to \infty} \frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \geq \frac{\lambda(f \circ g)}{\rho(g)}. \tag{10.9}$$

Again combining (10.2) and (10.7) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\lambda(f \circ g) + \varepsilon}{\lambda(g) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary,it follows that

$$\liminf_{\sigma \to \infty} \frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\lambda(f \circ g)}{\lambda(g)}. \tag{10.10}$$

Similarly from (10.4) and (10.5) it follows that for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\rho(f \circ g) + \varepsilon}{\rho(g) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma \to \infty} \frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\rho(f \circ g)}{\rho(g)}. \tag{10.11}$$

Combining (10.9), (10.10) and (10.11) we get that

$$\frac{\lambda(f \circ g)}{\rho(g)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \min\left\{\frac{\lambda(f \circ g)}{\lambda(g)}, \frac{\rho(f \circ g)}{\rho(g)}\right\}. \tag{10.12}$$

Now from (10.3) and (10.6) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} F \circ G(\sigma)}{\log^{[2]} G(\sigma)} \geq \frac{\lambda(f \circ g) - \varepsilon}{\lambda(g) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{\sigma\to\infty} \frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} G(\sigma)} \geq \frac{\lambda(f\circ g)}{\lambda(g)}. \tag{10.13}$$

Again from (10.2) and (10.5) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\rho(f\circ g)+\varepsilon}{\lambda(g)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty} \frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\rho(f\circ g)}{\lambda(g)}. \tag{10.14}$$

Similarly combining (10.1) and (10.8) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} G(\sigma)} \geq \frac{\rho(f\circ g)-\varepsilon}{\rho(g)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} G(\sigma)} \geq \frac{\rho(f\circ g)}{\rho(g)}. \tag{10.15}$$

Therefore combining (10.13), (10.14) and (10.15) we get that

$$\max\left\{\frac{\lambda(f\circ g)}{\lambda(g)}, \frac{\rho(f\circ g)}{\rho(g)}\right\} \leq \limsup_{\sigma\to\infty} \frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} G(\sigma)} \leq \frac{\rho(f\circ g)}{\lambda(g)}. \tag{10.16}$$

Thus the theorem follows from (10.12) and (10.16). ■

Remark 10.2.1 *If we take* $0<\lambda(f)\leq\rho(f)<\infty$ *instead of* $0<\lambda(g)\leq\rho(g)<\infty$ *and the other conditions remain the same then also Theorem* 10.2.1 *holds with* g *replaced by* f *in the denominator as we see in the next theorem.*

Theorem 10.2.2 *Let* f *and* g *be two entire functions such that* $0<\lambda(f\circ g)\leq\rho(f\circ g)<\infty$ *and* $0<\lambda(f)\leq\rho(f)<\infty$. *Then*

$$\frac{\lambda(f\circ g)}{\rho(f)} \leq \liminf_{\sigma\to\infty} \frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)} \leq \min\left\{\frac{\lambda(f\circ g)}{\lambda(f)}, \frac{\rho(f\circ g)}{\rho(f)}\right\}$$
$$\leq \max\left\{\frac{\lambda(f\circ g)}{\lambda(f)}, \frac{\rho(f\circ g)}{\rho(f)}\right\} \leq \limsup_{\sigma\to\infty} \frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)} \leq \frac{\rho(f\circ g)}{\lambda(f)}.$$

Proof. From the definition of Ritt order and lower Ritt order of an entire function f, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[2]} F(\sigma) \leq (\rho(f)+\varepsilon)\sigma \tag{10.17}$$

and

$$\log^{[2]} F(\sigma) \geq (\lambda(f)-\varepsilon)\sigma. \tag{10.18}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[2]} F(\sigma) \leq (\lambda(f)+\varepsilon)\sigma \tag{10.19}$$

and

$$\log^{[2]} F(\sigma) \geq (\rho(f)-\varepsilon)\sigma. \tag{10.20}$$

Now from (10.6) and (10.17) it follows for all sufficiently large values of r that

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)} \geq \frac{(\lambda(f\circ g)-\varepsilon)}{(\rho(f)+\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\liminf_{\sigma\to\infty}\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)} \geq \frac{\lambda(f\circ g)}{\rho(f)}. \tag{10.21}$$

Again combining (10.7) and (10.18) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)} \leq \frac{\lambda(f\circ g)+\varepsilon}{\lambda(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary,it follows that

$$\liminf_{\sigma\to\infty}\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)} \leq \frac{\lambda(f\circ g)}{\lambda(f)}. \tag{10.22}$$

Similarly from (10.5) and (10.20) it follows that for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)} \leq \frac{\rho(f\circ g)+\varepsilon}{\rho(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{\sigma\to\infty}\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\leq\frac{\rho(f\circ g)}{\rho(f)}. \tag{10.23}$$

Now combining (10.21), (10.22) and (10.23) we get that

$$\frac{\lambda(f\circ g)}{\rho(f)}\leq\liminf_{\sigma\to\infty}\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\leq\min\left\{\frac{\lambda(f\circ g)}{\lambda(f)},\frac{\rho(f\circ g)}{\rho(f)}\right\}. \tag{10.24}$$

Now from (10.6) and (10.19) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\geq\frac{\lambda(f\circ g)-\varepsilon}{\lambda(f)+\varepsilon}.$$

Choosing $\varepsilon\to 0$ we get that

$$\limsup_{\sigma\to\infty}\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\geq\frac{\lambda(f\circ g)}{\lambda(f)}. \tag{10.25}$$

Again from (10.5) and (10.18) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\leq\frac{\rho(f\circ g)+\varepsilon}{\lambda(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty}\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\leq\frac{\rho(f\circ g)}{\lambda(f)}. \tag{10.26}$$

Similarly combining (10.8) and (10.17) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\geq\frac{\rho(f\circ g)-\varepsilon}{\rho(f)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty}\frac{\log^{[2]} F\circ G(\sigma)}{\log^{[2]} F(\sigma)}\geq\frac{\rho(f\circ g)}{\rho(f)}. \tag{10.27}$$

Therefore combining (10.25), (10.26) and (10.27) we get that

$$\max\left\{\frac{\lambda(f\circ g)}{\lambda(f)},\frac{\rho(f\circ g)}{\rho(f)}\right\}\le\limsup_{\sigma\to\infty}\frac{\log^{[2]}F\circ G(\sigma)}{\log^{[2]}F(\sigma)}\le\frac{\rho(f\circ g)}{\lambda(f)}. \tag{10.28}$$

Thus the theorem follows from (10.24) and (10.28). ■

Now we prove two theorems on hyper Ritt order and hyper lower Ritt order.

Theorem 10.2.3 *Let f and g be two entire functions such that $0<\bar{\lambda}(f\circ g)\le\bar{\rho}(f\circ g)<\infty$ and $0<\bar{\lambda}(g)\le\bar{\rho}(g)<\infty$. Then*

$$\frac{\bar{\lambda}(f\circ g)}{\bar{\rho}(g)}\le\liminf_{\sigma\to\infty}\frac{\log^{[3]}F\circ G(\sigma)}{\log^{[3]}G(\sigma)}\le\min\left\{\frac{\bar{\lambda}(f\circ g)}{\bar{\lambda}(g)},\frac{\bar{\rho}(f\circ g)}{\bar{\rho}(g)}\right\}$$
$$\le\max\left\{\frac{\bar{\lambda}(f\circ g)}{\bar{\lambda}(g)},\frac{\bar{\rho}(f\circ g)}{\bar{\rho}(g)}\right\}\le\limsup_{\sigma\to\infty}\frac{\log^{[3]}F\circ G(\sigma)}{\log^{[3]}G(\sigma)}\le\frac{\bar{\rho}(f\circ g)}{\bar{\lambda}(g)}.$$

Proof. From the definition of hyper Ritt order and hyper lower Ritt order of an entire function g, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[3]}G(\sigma)\le(\bar{\rho}(g)+\varepsilon)\sigma \tag{10.29}$$

and

$$\log^{[3]}G(\sigma)\ge(\bar{\lambda}(g)-\varepsilon)\sigma. \tag{10.30}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[3]}G(\sigma)\le(\bar{\lambda}(g)+\varepsilon)\sigma \tag{10.31}$$

and

$$\log^{[3]}G(\sigma)\ge(\bar{\rho}(g)-\varepsilon)\sigma. \tag{10.32}$$

Again from the definition of hyper Ritt order and hyper lower Ritt order of the composite entire function $f\circ g$ we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[3]}F\circ G(\sigma)\le(\bar{\rho}(f\circ g)+\varepsilon)\sigma \tag{10.33}$$

and

$$\log^{[3]} F \circ G(\sigma) \geq (\bar{\lambda}(f \circ g) - \varepsilon)\sigma. \tag{10.34}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[3]} F \circ G(\sigma) \leq (\bar{\lambda}(f \circ g) + \varepsilon)\sigma \tag{10.35}$$

and

$$\log^{[3]} F \circ G(\sigma) \geq (\bar{\rho}(f \circ g) - \varepsilon)\sigma. \tag{10.36}$$

Now from (10.29) and (10.34) it follows for all sufficiently large values of r that

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \geq \frac{(\bar{\lambda}(f \circ g) - \varepsilon)}{(\bar{\rho}(g) + \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \geq \frac{\bar{\lambda}(f \circ g)}{\bar{\rho}(g)}. \tag{10.37}$$

Again combining (10.30) and (10.35) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \leq \frac{\bar{\lambda}(f \circ g) + \varepsilon}{\bar{\lambda}(g) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary,it follows that

$$\liminf_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \leq \frac{\bar{\lambda}(f \circ g)}{\bar{\lambda}(g)}. \tag{10.38}$$

Similarly from (10.32) and (10.33) it follows that for a sequence of values of r tending to infinity,

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \leq \frac{\bar{\rho}(f \circ g) + \varepsilon}{\bar{\rho}(g) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \leq \frac{\bar{\rho}(f \circ g)}{\bar{\rho}(g)}. \tag{10.39}$$

Combining (10.37), (10.38) and (10.39) we get that

$$\frac{\bar{\lambda}(f \circ g)}{\bar{\rho}(g)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \leq \min \left\{ \frac{\bar{\lambda}(f \circ g)}{\bar{\lambda}(g)}, \frac{\bar{\rho}(f \circ g)}{\bar{\rho}(g)} \right\}. \tag{10.40}$$

Now from (10.31) and (10.34) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \geq \frac{\bar{\lambda}(f \circ g) - \varepsilon}{\bar{\lambda}(g) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \geq \frac{\bar{\lambda}(f \circ g)}{\bar{\lambda}(g)}. \tag{10.41}$$

Again from (10.30) and (10.33) it follows for all sufficiently large values of r,

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \leq \frac{\bar{\rho}(f \circ g) + \varepsilon}{\bar{\lambda}(g) - \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\limsup_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \leq \frac{\bar{\rho}(f \circ g)}{\bar{\lambda}(g)}. \tag{10.42}$$

Similarly combining (10.29) and (10.36) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \geq \frac{\bar{\rho}(f \circ g) - \varepsilon}{\bar{\rho}(g) + \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, it follows that

$$\limsup_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} G(\sigma)} \geq \frac{\bar{\rho}(f \circ g)}{\bar{\rho}(g)}. \tag{10.43}$$

Therefore combining (10.41), (10.42) and (10.43) we get that

$$\max\left\{\frac{\bar{\lambda}(f\circ g)}{\bar{\lambda}(g)},\frac{\bar{\rho}(f\circ g)}{\bar{\rho}(g)}\right\}\leq\limsup_{\sigma\to\infty}\frac{\log^{[3]}F\circ G(\sigma)}{\log^{[3]}G(\sigma)}\leq\frac{\bar{\rho}(f\circ g)}{\bar{\lambda}(g)}. \quad (10.44)$$

Thus the theorem follows from (10.40) and (10.44). ■

Theorem 10.2.4 *Let f and g be two entire functions such that $0<\bar{\lambda}(f\circ g)\leq\bar{\rho}(f\circ g)<\infty$ and $0<\bar{\lambda}(f)\leq\bar{\rho}(f)<\infty$. Then*

$$\frac{\bar{\lambda}(f\circ g)}{\bar{\rho}(f)}\leq\liminf_{\sigma\to\infty}\frac{\log^{[3]}F\circ G(\sigma)}{\log^{[3]}F(\sigma)}\leq\min\left\{\frac{\bar{\lambda}(f\circ g)}{\bar{\lambda}(f)},\frac{\bar{\rho}(f\circ g)}{\bar{\rho}(f)}\right\}$$
$$\leq\max\left\{\frac{\bar{\lambda}(f\circ g)}{\bar{\lambda}(f)},\frac{\bar{\rho}(f\circ g)}{\bar{\rho}(f)}\right\}\leq\limsup_{\sigma\to\infty}\frac{\log^{[3]}F\circ G(\sigma)}{\log^{[3]}F(\sigma)}\leq\frac{\bar{\rho}(f\circ g)}{\bar{\lambda}(f)}.$$

Proof. From the definition of hyper Ritt order and hyper lower Ritt order of an entire function f, we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[3]}F(\sigma)\leq(\bar{\rho}(f)+\varepsilon)\sigma \quad (10.45)$$

and

$$\log^{[3]}F(\sigma)\geq(\bar{\lambda}(f)-\varepsilon)\sigma. \quad (10.46)$$

Also for a sequence of values of r tending to infinity,

$$\log^{[3]}F(\sigma)\leq(\bar{\lambda}(f)+\varepsilon)\sigma \quad (10.47)$$

and

$$\log^{[3]}F(\sigma)\geq(\bar{\rho}(f)-\varepsilon)\sigma. \quad (10.48)$$

Now from (10.34) and (10.45) it follows for all sufficiently large values of r that

$$\frac{\log^{[3]}F\circ G(\sigma)}{\log^{[3]}F(\sigma)}\geq\frac{(\bar{\lambda}(f\circ g)-\varepsilon)}{(\bar{\rho}(f)+\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\liminf_{\sigma\to\infty}\frac{\log^{[3]}F\circ G(\sigma)}{\log^{[3]}F(\sigma)}\geq\frac{\bar{\lambda}(f\circ g)}{\bar{\rho}(f)}. \quad (10.49)$$

Again combining (10.35) and (10.46) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \frac{\bar{\lambda}(f \circ g) + \varepsilon}{\bar{\lambda}(f) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \frac{\bar{\lambda}(f \circ g)}{\bar{\lambda}(f)}. \tag{10.50}$$

Similarly from (10.33) and (10.48) it follows that for a sequence of values of r tending to infinity,

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \frac{\bar{\rho}(f \circ g) + \varepsilon}{\bar{\rho}(f) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \frac{\bar{\rho}(f \circ g)}{\bar{\rho}(f)}. \tag{10.51}$$

Now combining (10.49), (10.50) and (10.51) we get that

$$\frac{\bar{\lambda}(f \circ g)}{\bar{\rho}(f)} \leq \liminf_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \min\left\{\frac{\bar{\lambda}(f \circ g)}{\bar{\lambda}(f)}, \frac{\bar{\rho}(f \circ g)}{\bar{\rho}(f)}\right\}. \tag{10.52}$$

Now from (10.34) and (10.47) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} F(\sigma)} \geq \frac{\bar{\lambda}(f \circ g) - \varepsilon}{\bar{\lambda}(f) + \varepsilon}.$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{\sigma \to \infty} \frac{\log^{[3]} F \circ G(\sigma)}{\log^{[3]} F(\sigma)} \geq \frac{\bar{\lambda}(f \circ g)}{\bar{\lambda}(f)}. \tag{10.53}$$

Again from (10.33) and (10.46) it follows for all sufficiently large values of r,

$$\frac{\log^{[3]} F\circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \frac{\bar{\rho}(f\circ g)+\varepsilon}{\bar{\lambda}(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{\sigma\to\infty}\frac{\log^{[3]} F\circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \frac{\bar{\rho}(f\circ g)}{\bar{\lambda}(f)}. \tag{10.54}$$

Similarly combining (10.36) and (10.45) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[3]} F\circ G(\sigma)}{\log^{[3]} F(\sigma)} \geq \frac{\bar{\rho}(f\circ g)-\varepsilon}{\bar{\rho}(f)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty}\frac{\log^{[3]} F\circ G(\sigma)}{\log^{[3]} F(\sigma)} \geq \frac{\bar{\rho}(f\circ g)}{\bar{\rho}(f)}. \tag{10.55}$$

Therefore combining (10.53), (10.54) and (10.55) we get that

$$\max\left\{\frac{\bar{\lambda}(f\circ g)}{\bar{\lambda}(f)},\frac{\bar{\rho}(f\circ g)}{\bar{\rho}(f)}\right\} \leq \limsup_{\sigma\to\infty}\frac{\log^{[3]} F\circ G(\sigma)}{\log^{[3]} F(\sigma)} \leq \frac{\bar{\rho}(f\circ g)}{\bar{\lambda}(f)}. \tag{10.56}$$

Thus the theorem follows from (10.52) and (10.56). ■

Theorem 10.2.5 *Let f, g and h be three entire functions such that $0<\rho_g(f)<\infty$ and $0<\rho_h(f)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \leq \frac{\rho_h(f)}{\rho_g(f)} \leq \limsup_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)}.$$

Proof. From the definition of relative Ritt order of an entire function with respect to another entire function we get for a sequence of values of r tending to infinity,

$$H^{-1}\log F(\sigma) \geq (\rho_h(f)-\varepsilon)\sigma \tag{10.57}$$

and for all sufficiently large values of r,

$$H^{-1}\log F(\sigma) \leq (\rho_h(f)+\varepsilon)\sigma. \tag{10.58}$$

Also for all sufficiently large values of r,

$$G^{-1}\log F(\sigma) \leq (\rho_g(f)+\varepsilon)\sigma. \tag{10.59}$$

Again for a sequence of values of r tending to infinity,

$$G^{-1}\log F(\sigma) \geq (\rho_g(f)-\varepsilon)\sigma. \tag{10.60}$$

Now from (10.57) and (10.59) it follows for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \geq \frac{\rho_h(f)-\varepsilon}{\rho_g(f)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\limsup_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \geq \frac{\rho_h(f)}{\rho_g(f)}. \tag{10.61}$$

Again combining (10.58) and (10.60) we get for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \leq \frac{\rho_h(f)+\varepsilon}{\rho_g(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \leq \frac{\rho_h(f)}{\rho_g(f)}. \tag{10.62}$$

Thus the theorem follows from (10.61) and (10.62). ■

Theorem 10.2.6 *Let f, g and h be three entire functions such that $0<\lambda_g(f)<\infty$ and $0<\lambda_h(f)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \leq \frac{\lambda_h(f)}{\lambda_g(f)} \leq \limsup_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)}.$$

Proof. From the definition of relative lower Ritt order of an entire function with respect to another entire function we get for all sufficiently large values of r,

$$H^{-1}\log F(\sigma) \geq (\lambda_h(f)-\varepsilon)\sigma \tag{10.63}$$

and for a sequence of values of r tending to infinity,

$$H^{-1}\log F(\sigma) \leq (\lambda_h(f)+\varepsilon)\sigma. \tag{10.64}$$

Also for a sequence of values of r tending to infinity,

$$G^{-1}\log F(\sigma) \leq (\lambda_g(f)+\varepsilon)\sigma. \tag{10.65}$$

Again for all sufficiently large values of r,

$$G^{-1}\log F(\sigma) \geq (\lambda_g(f)-\varepsilon)\sigma. \tag{10.66}$$

Now from (10.63) and (10.65) it follows for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \geq \frac{\lambda_h(f)-\varepsilon}{\lambda_g(f)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\limsup_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \geq \frac{\lambda_h(f)}{\lambda_g(f)}. \tag{10.67}$$

Again combining (10.64) and (10.66) we get for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \leq \frac{\lambda_h(f)+\varepsilon}{\lambda_g(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log F(\sigma)}{G^{-1}\log F(\sigma)} \leq \frac{\lambda_h(f)}{\lambda_g(f)}. \tag{10.68}$$

Thus the theorem follows from (10.67) and (10.68). ■

Analogous theorems can also be proved for relative hyper Ritt order and relative hyper lower Ritt order.

Theorem 10.2.7 *Let f, g and h be three entire functions such that $0<\bar{\rho}_g(f)<\infty$ and $0<\bar{\rho}_h(f)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)} \leq \frac{\bar{\rho}_h(f)}{\bar{\rho}_g(f)} \leq \limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)}.$$

Proof. From the definition of relative hyper Ritt order of an entire function with respect to another entire function we get for a sequence of values of r tending to infinity,

$$H^{-1}\log^{[2]} F(\sigma) \geq (\bar{\rho_h}(f) - \varepsilon)\sigma \tag{10.69}$$

and for all sufficiently large values of r,

$$H^{-1}\log^{[2]} F(\sigma) \leq (\bar{\rho_h}(f) + \varepsilon)\sigma. \tag{10.70}$$

Also for all sufficiently large values of r,

$$G^{-1}\log^{[2]} F(\sigma) \leq (\bar{\rho_g}(f) + \varepsilon)\sigma. \tag{10.71}$$

Again for a sequence of values of r tending to infinity,

$$G^{-1}\log^{[2]} F(\sigma) \geq (\bar{\rho_g}(f) - \varepsilon)\sigma. \tag{10.72}$$

Now from (10.69) and (10.71) it follows for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} F(\sigma)} \geq \frac{\bar{\rho_h}(f) - \varepsilon}{\bar{\rho_g}(f) + \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\limsup_{\sigma\to\infty} \frac{H^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} F(\sigma)} \geq \frac{\bar{\rho_h}(f)}{\bar{\rho_g}(f)}. \tag{10.73}$$

Again combining (10.70) and (10.72) we get for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} F(\sigma)} \leq \frac{\bar{\rho_h}(f) + \varepsilon}{\bar{\rho_g}(f) - \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\liminf_{\sigma\to\infty} \frac{H^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} F(\sigma)} \leq \frac{\bar{\rho_h}(f)}{\bar{\rho_g}(f)}. \tag{10.74}$$

Thus the theorem follows from (10.73) and (10.74). ■

Theorem 10.2.8 *Let f, g and h be three entire functions such that $0<\bar{\lambda}_g(f)<\infty$ and $0<\bar{\lambda}_h(f)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)}\leq\frac{\bar{\lambda}_h(f)}{\bar{\lambda}_g(f)}\leq\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)}.$$

Proof. From the definition of relative hyper lower Ritt order of an entire function with respect to another entire function we get for all sufficiently large values of r,

$$H^{-1}\log^{[2]}F(\sigma)\geq(\bar{\lambda}_h(f)-\varepsilon)\sigma \tag{10.75}$$

and for a sequence of values of r tending to infinity,

$$H^{-1}\log^{[2]}F(\sigma)\leq(\bar{\lambda}_h(f)+\varepsilon)\sigma. \tag{10.76}$$

Also for a sequence of values of r tending to infinity,

$$G^{-1}\log^{[2]}F(\sigma)\leq(\bar{\lambda}_g(f)+\varepsilon)\sigma. \tag{10.77}$$

Again for all sufficiently large values of r,

$$G^{-1}\log^{[2]}F(\sigma)\geq(\bar{\lambda}_g(f)-\varepsilon)\sigma. \tag{10.78}$$

Now from (10.75) and (10.77) it follows for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)}\geq\frac{\bar{\lambda}_h(f)-\varepsilon}{\bar{\lambda}_g(f)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)}\geq\frac{\bar{\lambda}_h(f)}{\bar{\lambda}_g(f)}. \tag{10.79}$$

Again combining (10.76) and (10.78) we get for a sequence of values of r tending to infinity,

$$\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)}\leq\frac{\bar{\lambda}_h(f)+\varepsilon}{\bar{\lambda}_g(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[2]}F(\sigma)}{G^{-1}\log^{[2]}F(\sigma)}\leq\frac{\overline{\lambda}_h(f)}{\overline{\lambda}_g(f)}. \tag{10.80}$$

Thus the theorem follows from (10.79) and (10.80). ■

Theorem 10.2.9 *Let f, g and h be three entire functions such that $0<\rho_g(f)<\infty$ and $0<\rho_g(h)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)}\leq\frac{\rho_g(f)}{\rho_g(h)}\leq\limsup_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)}.$$

Proof. From the definition of relative Ritt order we get for a sequence of values of r tending to infinity,

$$G^{-1}\log H(\sigma)\geq(\rho_g(h)-\varepsilon)\sigma. \tag{10.81}$$

Now from (10.59) and (10.81) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)}\leq\frac{\rho_g(f)+\varepsilon}{\rho_g(h)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that.

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)}\leq\frac{\rho_g(f)}{\rho_g(h)}. \tag{10.82}$$

Again for all large values of r,

$$G^{-1}\log H(\sigma)\leq(\rho_g(h)+\varepsilon)\sigma. \tag{10.83}$$

So combining (10.60) and (10.83) we get for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)}\geq\frac{\rho_g(f)-\varepsilon}{\rho_g(h)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)}\geq\frac{\rho_g(f)}{\rho_g(h)}. \tag{10.84}$$

Thus the theorem follows from (10.82) and (10.84). ■

Theorem 10.2.10 *Let f, g and h be three entire functions such that $0 < \lambda_g(f) < \infty$ and $0 < \lambda_g(h) < \infty$. Then*

$$\liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)} \le \frac{\lambda_g(f)}{\lambda_g(h)} \le \limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)}.$$

Proof. From the definition of relative lower Ritt order we get for all sufficiently large values of r,

$$G^{-1}\log H(\sigma) \ge (\lambda_g(h) - \varepsilon)\sigma. \tag{10.85}$$

Now from (10.65) and (10.85) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)} \le \frac{\lambda_g(f) + \varepsilon}{\lambda_g(h) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)} \le \frac{\lambda_g(f)}{\lambda_g(h)}. \tag{10.86}$$

Again for a sequence of values of r tending to infinity,

$$G^{-1}\log H(\sigma) \le (\lambda_g(h) + \varepsilon)\sigma. \tag{10.87}$$

So combining (10.66) and (10.87) we get for a sequence of values of r tending to infinity that

$$\frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)} \ge \frac{\lambda_g(f) - \varepsilon}{\lambda_g(h) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{G^{-1}\log H(\sigma)} \ge \frac{\lambda_g(f)}{\lambda_g(h)}. \tag{10.88}$$

Thus the theorem follows from (10.86) and (10.88). ■

Theorem 10.2.11 *Let f, g and h be three entire functions such that $0 < \bar{\rho}_g(f) < \infty$ and $0 < \bar{\rho}_g(h) < \infty$. Then*

$$\liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \le \frac{\bar{\rho}_g(f)}{\bar{\rho}_g(h)} \le \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)}.$$

Proof. From the definition of relative hyper Ritt order we get for a sequence of values of r tending to infinity,

$$G^{-1}\log^{[2]} H(\sigma) \geq (\bar{\rho}_g(h) - \varepsilon)\sigma. \tag{10.89}$$

Now from (10.71) and (10.89) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \leq \frac{\bar{\rho}_g(f) + \varepsilon}{\bar{\rho}_g(h) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \leq \frac{\bar{\rho}_g(f)}{\bar{\rho}_g(h)}. \tag{10.90}$$

Again for all large values of r,

$$G^{-1}\log^{[2]} H(\sigma) \leq (\bar{\rho}_g(h) + \varepsilon)\sigma. \tag{10.91}$$

So combining (10.72) and (10.91) we get for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \geq \frac{\bar{\rho}_g(f) - \varepsilon}{\bar{\rho}_g(h) + \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \geq \frac{\bar{\rho}_g(f)}{\bar{\rho}_g(h)}. \tag{10.92}$$

Thus the theorem follows from (10.90) and (10.92). ■

Theorem 10.2.12 *Let f, g and h be three entire functions such that $0 < \bar{\lambda}_g(f) < \infty$ and $0 < \bar{\lambda}_g(h) < \infty$. Then*

$$\liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \leq \frac{\bar{\lambda}_g(f)}{\bar{\lambda}_g(h)} \leq \limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)}.$$

Proof. From the definition of relative hyper lower Ritt order we get for all sufficiently large values of r,

$$G^{-1}\log^{[2]} H(\sigma) \geq (\overline{\lambda}_g(h) - \varepsilon)\sigma. \tag{10.93}$$

Now from (10.77) and (10.93) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \leq \frac{\overline{\lambda}_g(f) + \varepsilon}{\overline{\lambda}_g(h) - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\liminf_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \leq \frac{\overline{\lambda}_g(f)}{\overline{\lambda}_g(h)}. \tag{10.94}$$

Again for a sequence of values of r tending to infinity,

$$G^{-1}\log^{[2]} H(\sigma) \leq (\overline{\lambda}_g(h) + \varepsilon)\sigma. \tag{10.95}$$

So combining (10.78) and (10.95) we get for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \geq \frac{\overline{\lambda}_g(f) - \varepsilon}{\overline{\lambda}_g(h) + \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty} \frac{G^{-1}\log^{[2]} F(\sigma)}{G^{-1}\log^{[2]} H(\sigma)} \geq \frac{\overline{\lambda}_g(f)}{\overline{\lambda}_g(h)}. \tag{10.96}$$

Thus the theorem follows from (10.94) and (10.96). ■

Theorem 10.2.13 *Let f, g, h and k be four entire functions such that $0 < \rho_g(f) < \infty$ and $0 < \rho_k(h) < \infty$. Then*

$$\liminf_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \leq \frac{\rho_g(f)}{\rho_k(h)} \leq \limsup_{\sigma\to\infty} \frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)}.$$

Proof. From the definition of relative Ritt order we get for a sequence of values of r tending to infinity,

$$K^{-1}\log H(\sigma) \geq (\rho_k(h) - \varepsilon)\sigma \tag{10.97}$$

and for all sufficiently large values of r,

$$K^{-1}\log H(\sigma) \leq (\rho_k(h) + \varepsilon)\sigma. \tag{10.98}$$

Now from (10.59) and (10.97) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \leq \frac{\rho_g(f) + \varepsilon}{\rho_k(h) - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that.

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \leq \frac{\rho_g(f)}{\rho_k(h)}. \tag{10.99}$$

Again combining (10.60) and (10.98) we get for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \geq \frac{\rho_g(f) - \varepsilon}{\rho_k(h) + \varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \geq \frac{\rho_g(f)}{\rho_k(h)}. \tag{10.100}$$

Thus the theorem follows from (10.99) and (10.100). ∎

Theorem 10.2.14 *Let f, g, h and k be four entire functions such that $0 < \lambda_g(f) < \infty$ and $0 < \lambda_k(h) < \infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \leq \frac{\lambda_g(f)}{\lambda_k(h)} \leq \limsup_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)}.$$

Proof. From the definition of relative lower Ritt order we get for all sufficiently large values of r that

$$K^{-1}\log H(\sigma) \geq (\lambda_k(h) - \varepsilon)\sigma \tag{10.101}$$

and for a sequence of values of r tending to infinity,

$$K^{-1}\log H(\sigma) \leq (\lambda_k(h) + \varepsilon)\sigma. \tag{10.102}$$

Now from (10.65) and (10.101) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \leq \frac{\lambda_g(f)+\varepsilon}{\lambda_k(h)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \leq \frac{\lambda_g(f)}{\lambda_k(h)}. \tag{10.103}$$

Again combining (10.66) and (10.102) we get for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \geq \frac{\lambda_g(f)-\varepsilon}{\lambda_k(h)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{K^{-1}\log H(\sigma)} \geq \frac{\lambda_g(f)}{\lambda_k(h)}. \tag{10.104}$$

Thus the theorem follows from (10.103) and (10.104). ■

Theorem 10.2.15 *Let f, g, h and k be four entire functions such that $0 < \bar{\rho}_g(f) < \infty$ and $0 < \bar{\rho}_k(h) < \infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)} \leq \frac{\bar{\rho}_g(f)}{\bar{\rho}_k(h)} \leq \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}.$$

Proof. From the definition of relative hyper Ritt order we get for a sequence of values of r tending to infinity,

$$K^{-1}\log^{[2]}H(\sigma) \geq (\bar{\rho}_k(h)-\varepsilon)\sigma \tag{10.105}$$

and for all sufficiently large values of r,

$$K^{-1}\log^{[2]}H(\sigma) \leq (\bar{\rho}_k(h)+\varepsilon)\sigma. \tag{10.106}$$

Now from (10.71) and (10.105) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)} \leq \frac{\bar{\rho}_g(f)+\varepsilon}{\bar{\rho}_k(h)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\leq\frac{\bar{\rho}_g(f)}{\bar{\rho}_k(h)}. \tag{10.107}$$

Again combining (10.72) and (10.106) we get for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\geq\frac{\bar{\rho}_g(f)-\varepsilon}{\bar{\rho}_k(h)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\geq\frac{\bar{\rho}_g(f)}{\bar{\rho}_k(h)}. \tag{10.108}$$

Thus the theorem follows from (10.107) and (10.108). ■

Theorem 10.2.16 *Let f, g, h and k be four entire functions such that $0<\bar{\lambda}_g(f)<\infty$ and $0<\bar{\lambda}_k(h)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\leq\frac{\bar{\lambda}_g(f)}{\bar{\lambda}_k(h)}\leq\limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}.$$

Proof. From the definition of relative hyper lower Ritt order we get for all sufficiently large values of r that

$$K^{-1}\log^{[2]}H(\sigma)\geq(\bar{\lambda}_k(h)-\varepsilon)\sigma \tag{10.109}$$

and for a sequence of values of r tending to infinity,

$$K^{-1}\log^{[2]}H(\sigma)\leq(\bar{\lambda}_k(h)+\varepsilon)\sigma. \tag{10.110}$$

Now from (10.77) and (10.109) it follows for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\leq\frac{\bar{\lambda}_g(f)+\varepsilon}{\bar{\lambda}_k(h)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\leq\frac{\overline{\lambda}_g(f)}{\overline{\lambda}_k(h)}. \tag{10.111}$$

Again combining (10.78) and (10.110) we get for a sequence of values of r tending to infinity,

$$\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\geq\frac{\overline{\lambda}_g(f)-\varepsilon}{\overline{\lambda}_k(h)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[2]}F(\sigma)}{K^{-1}\log^{[2]}H(\sigma)}\geq\frac{\overline{\lambda}_g(f)}{\overline{\lambda}_k(h)}. \tag{10.112}$$

Thus the theorem follows from (10.111) and (10.112). ■

The subsequent theorems can analogously be established by using Definition 10.1.4.

Theorem 10.2.17 *Let f,g and h be three entire functions such that $0<\ {}^{(k)}\rho_g(f)<\infty$ and $0<\ {}^{(k)}\rho_h(f)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}F(\sigma)}\leq\frac{{}^{(k)}\rho_h(f)}{{}^{(k)}\rho_g(f)}\leq\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}F(\sigma)}\ \textit{where}\ k=1,2,3,...$$

Theorem 10.2.18 *Let f,g and h be three entire functions such that $0<\ {}^{(k)}\lambda_g(f)<\infty$ and $0<\ {}^{(k)}\lambda_h(f)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}F(\sigma)}\leq\frac{{}^{(k)}\lambda_h(f)}{{}^{(k)}\lambda_g(f)}\leq\limsup_{\sigma\to\infty}\frac{H^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}F(\sigma)}\ \textit{where}\ k=1,2,3,...$$

Theorem 10.2.19 *Let f,g and h be three entire functions such that $0<\ {}^{(k)}\rho_g(f)<\infty$ and $0<\ {}^{(k)}\rho_g(h)<\infty$. Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}H(\sigma)}\leq\frac{{}^{(k)}\rho_g(f)}{{}^{(k)}\rho_g(h)}\leq\limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}H(\sigma)}\ \textit{where}\ k=1,2,3,...$$

Theorem 10.2.20 *Let f, g and h be three entire functions such that* $0 < {}^{(k)}\lambda_g(f) < \infty$ *and* $0 < {}^{(k)}\lambda_g(h) < \infty$. *Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}H(\sigma)} \leq \frac{{}^{(k)}\lambda_g(f)}{{}^{(k)}\lambda_g(h)} \leq \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[k]}F(\sigma)}{G^{-1}\log^{[k]}H(\sigma)} \quad \textit{where } k = 1,2,3,...$$

Theorem 10.2.21 *Let f, g, h and k be four entire functions such that* $0 < {}^{(l)}\rho_g(f) < \infty$ *and* $0 < {}^{(l)}\rho_k(h) < \infty$. *Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[l]}F(\sigma)}{K^{-1}\log^{[l]}H(\sigma)} \leq \frac{{}^{(l)}\rho_g(f)}{{}^{(l)}\rho_k(h)} \leq \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[l]}F(\sigma)}{K^{-1}\log^{[l]}H(\sigma)} \quad \textit{where } k = 1,2,3,...$$

Theorem 10.2.22 *Let f, g, h and k be four entire functions such that* $0 < {}^{(l)}\lambda_g(f) < \infty$ *and* $0 < {}^{(l)}\lambda_k(h) < \infty$. *Then*

$$\liminf_{\sigma\to\infty}\frac{G^{-1}\log^{[l]}F(\sigma)}{K^{-1}\log^{[l]}H(\sigma)} \leq \frac{{}^{(l)}\lambda_g(f)}{{}^{(l)}\lambda_k(h)} \leq \limsup_{\sigma\to\infty}\frac{G^{-1}\log^{[l]}F(\sigma)}{K^{-1}\log^{[l]}H(\sigma)} \quad \textit{where } k = 1,2,3,...$$

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