CHAPTER-9

RELATIVE DEFECTS OF A SPECIAL TYPE OF DIFFERENTIAL POLYNOMIAL

Chapter 9

RELATIVE DEFECTS OF A SPECIAL TYPE OF DIFFERENTIAL POLYNOMIAL

9.1 Introduction, Definitions and Notations.

Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. A monomial in f is an expression of the form $M[f] = (f)^{n_0}\left(f^{(1)}\right)^{n_1} \dots \dots \left(f^{(k)}\right)^{n_k}$ where $n_{0,}\ n_{1,}\ n_{2,\dots}n_k$ are non negative integers. $\gamma_M = n_0 + n_1 + \dots + n_k$ is called the degee of the monomial and $\Gamma_M = n_0 + 2n_1 + \dots + (k+1)n_k$ are respectively called the degree and weight of the monomial. Also we call the numbers $\underline{\gamma_Q} = \min\limits_{1 \leq j \leq n} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $Q[f]$ respectively.If $\underline{\gamma_Q} = \gamma_Q$, $Q[f]$ is called a homogeneous differential polynomial. For $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna's deficiency of the value $'a'$. Similarly the Valiron defect of $'a'$ is defined as

$$\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.$$

The results of this chapter have been published in Journal of Mechanics of Continua and Mathematical Sciences, see [26].

The term

$$\delta_R^{(k)}(a;f) = 1 - \limsup_{r\to\infty} \frac{N\left(r,a;f^{(k)}\right)}{T(r,f)} \quad for\ k = 1,2,3,...$$

is called the relative Nevanlinna's defect of $'a'$ with respect to $f^{(k)}$. In a like manner relative Valiron's defect of $'a'$ is defined. Xiong **[66]** has shown various relations between the usual defects and the relative defects for meromorphic functions.

In the chapter we consider $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f and $n = 1,2,3,...$ compare the relative Valiron's defect with the relative Nevanlinna's defect of F. The term $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.

9.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 9.2.1 *[29] Let k be any positive integer and $\psi = \sum_{i=0}^{k} a_i f^{(i)}$, where a_i are meromorphic functions such that $T(r,a_i) = S(r,f)$, for $i = 0,1,2,....k$. Then $m\left(r,\frac{\psi}{f}\right) = S(r,f)$.*

Lemma 9.2.2 *Let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f. If $n \geq 1$ then*

$$\lim_{r\to\infty} \frac{T(r,F)}{T(r,f)} = 1.$$

The proof is omitted.

Lemma 9.2.3 *Let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomial in f. If $n \geq 1$ then for any α,*

$$\delta_R^F(\alpha;f) = \liminf_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}$$

and

$$\Delta_R^F(\alpha;f) = \limsup_{r\to\infty} \frac{m(r,\alpha;F)}{T(r,f)}.$$

Proof. In view of Lemma 9.2.2 we get that

$$\begin{aligned}\delta_R^F(\alpha;f) &= 1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{T(r,f)}\\ &= 1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{T(r,F)}.\lim_{r\to\infty}\frac{T(r,F)}{T(r,f)}\\ &= 1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{T(r,F)}\\ &= \liminf_{r\to\infty}\frac{m(r,\alpha;F)}{T(r,F)}\\ &= \liminf_{r\to\infty}\frac{m(r,\alpha;F)}{T(r,f)}.\lim_{r\to\infty}\frac{T(r,f)}{T(r,F)}\\ &= \liminf_{r\to\infty}\frac{m(r,\alpha;F)}{T(r,f)}.\end{aligned}$$

This proves the first part of the Lemma.

Similarly the second part of Lemma 9.2.3 follows. ■

9.3 Theorems.

In this section we present the main results of the chapter.

Theorem 9.3.1 *Let f be a transcendental meromorphic function of finite order ρ_f and 'a' be any non-zero finite complex number. Then*

$$\delta(0;f)+\Delta_R^F(\infty;f)+\delta(a;f)\le\Delta(\infty;f)+\Delta_R^F(0;f).$$

Proof. Let us consider the following identity

$$\frac{a}{f}=1-\frac{f-a}{F}.\frac{F}{f}.$$

Since $m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{a}{f}\right)+O(1)$, in view of Lemma 9.2.3 we get from the above identity that

$$m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{f-a}{F}\right)+m\left(r,\frac{F}{f}\right)$$

$$\text{i.e.,}\quad m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{f-a}{F}\right)+S(r,f). \tag{9.1}$$

Now by Nevanlinna's First Fundamental theorem and by Lemma 9.2.1 it follows from (9.1) that

$$m\left(r,\frac{1}{f}\right)\leq T\left(r,\frac{f-a}{F}\right)-N\left(r,\frac{f-a}{F}\right)+S\left(r,f\right)$$

$$i,e.,\ m\left(r,\frac{1}{f}\right)\leq T\left(r,\frac{F}{f-a}\right)-N\left(r,\frac{f-a}{F}\right)+S\left(r,f\right)$$

$$i.e.,\ m\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{F}{f-a}\right)+m\left(r,\frac{F}{f-a}\right)-N\left(r,\frac{f-a}{F}\right)+S\left(r,f\right)$$

$$i.e.,\ m\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{F}{f-a}\right)-N\left(r,\frac{f-a}{F}\right)+S\left(r,f\right). \tag{9.2}$$

In view of $\{p.34,$ **[29]**$\}$ it follows from(9.2) that

$$m\left(r,\frac{1}{f}\right)\leq N\left(r,F\right)+N\left(r,\frac{1}{f-a}\right)-N\left(r,f-a\right)-N\left(r,\frac{1}{F}\right)+S\left(r,f\right)$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{T\left(r,f\right)}\leq\liminf_{r\to\infty}\left\{\frac{N\left(r,F\right)}{T\left(r,f\right)}-\frac{N\left(r,f\right)}{T\left(r,f\right)}-\frac{N\left(r,\frac{1}{F}\right)}{T\left(r,f\right)}\right\}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f}\right)}{T\left(r,f\right)}\leq\liminf_{r\to\infty}\frac{N\left(r,F\right)}{T\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,f\right)}{T\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{F}\right)}{T\left(r,f\right)}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)}$$

$$i.e.,\ \delta\left(0;f\right)\leq\left\{1-\Delta_R^F\left(\infty;f\right)\right\}-\left\{1-\Delta\left(\infty;f\right)\right\}-\left\{1-\Delta_R^F\left(0;f\right)\right\}+\left\{1-\delta\left(a;f\right)\right\}$$

$$i.e.,\ \delta\left(0;f\right)+\Delta_R^F\left(\infty;f\right)+\delta\left(a;f\right)\leq\Delta\left(\infty;f\right)+\Delta_R^F\left(0;f\right).$$

This proves the theorem. ■

Remark 9.3.1 *The sign* $'\leq'$ *in Theorem* 9.3.1 *cannot be replaced by* $'<'$ *only as we see in the following examples.*

Example 9.3.1 *Let* $f = \exp z$,

$$n_0 = 1,\ n_1 = n_2 = ... = n_k = 0 \quad and \quad n = 1.$$

Then

$$F = f^n Q[f] = (\exp z)(\exp z) = \exp^{[2]} z.$$

Also

$$\rho_f = 1, \quad \delta(0, f) = 1, \quad \Delta_R^F(\infty; f) = 1,$$
$$\Delta(\infty; f) = 1 \quad and \quad \Delta_R^F(0; f) = 1.$$

Since

$$\delta(0; f) = \delta(\infty; f) = 1,$$

we get that

$$\delta(a; f) = 0 \quad for \quad a \neq 0, \infty.$$

Thus

$$\delta(0, f) + \Delta_R^F(\infty; f) + \delta(a; f) = 2 = \Delta(\infty; f) + \Delta_R^F(0; f),$$

which contradicts Theorem 9.3.1.

Remark 9.3.2 *The condition that a is any non-zero finite complex number in Theorem* 9.3.1 *is essential as we see in the following two examples.*

Example 9.3.2 *Let* $f = \exp z$,

$$n_0 = 1,\ n_1 = n_2 = ... = n_k = 0 \quad and \quad n = 1.$$

Then

$$F = f^n Q[f] = (\exp z)(\exp z) = \exp^{[2]} z.$$

Also

$$\rho_f = 1, \quad \delta(0, f) = 1, \quad \Delta_R^F(\infty; f) = 1,$$
$$\Delta(\infty; f) = 1 \quad and \quad \Delta_R^F(0; f) = 1.$$

Also let $a = 0$.

Thus

$$\begin{aligned}&\delta(0,f)+\Delta_R^F(\infty;f)+\delta(a;f)\\&=2\delta(0,f)+\Delta_R^F(\infty;f)\\&=2.1+1=3,\end{aligned}$$

but

$$\Delta(\infty;f)+\Delta_R^F(0;f)=1+1=2.$$

Example 9.3.3 *Let* $f=\exp z$,

$$n_0=1,\ n_1=n_2=...=n_k=0\ \ \textit{and}\ \ n=1.$$

Then

$$F=f^nQ[f]=(\exp z)(\exp z)=\exp^{[2]}z.$$

Also

$$\begin{aligned}\rho_f=1,\ \ \delta(0,f)=1,\ \ \Delta_R^F(\infty;f)=1,\\ \Delta(\infty;f)=1\ \ \textit{and}\ \ \Delta_R^F(0;f)=1.\end{aligned}$$

Also let $a=\infty$.
Thus

$$\begin{aligned}&\delta(0,f)+\Delta_R^F(\infty;f)+\delta(a;f)\\&=\delta(0,f)+\Delta_R^F(\infty;f)+\delta(\infty;f)\\&=1+1+1=3\end{aligned}$$

but

$$\Delta(\infty;f)+\Delta_R^F(0;f)=1+1=2.$$

Theorem 9.3.2 *Let* $a,b\neq 0,\infty$ *be any two distinct complex numbers. Then for any transcendental meromorphic function* f *of finite order* ρ_f,

$$2\delta(a;f)+\delta(b;f)+2\Delta_R^F(\infty;f)\leq 2\Delta(\infty;f)+2\Delta_R^F(0;f).$$

Proof. Considering the identity

$$\frac{b-a}{f-a}=\frac{F}{f-a}\left\{\frac{f-a}{F}-\frac{f-b}{F}\right\},$$

we obtain in view of Lemma 9.2.1 that

$$m\left(r,\frac{b-a}{f-a}\right)\leq m\left(r,\frac{f-a}{F}\right)+m\left(r,\frac{f-b}{F}\right)+m\left(r,\frac{F}{f-a}\right)$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{b-a}{f-a}\right)\leq{} & T\left(r,\frac{f-a}{F}\right)-N\left(r,\frac{f-a}{F}\right)+T\left(r,\frac{f-b}{F}\right)\\ & -N\left(r,\frac{f-b}{F}\right)+S\left(r,f\right). \end{aligned} \tag{9.3}$$

Since $m\left(r,\frac{1}{f-a}\right)\leq m\left(r,\frac{b-a}{f-a}\right)+O\left(1\right)$ and $T\left(r,f\right)=T\left(r,\frac{1}{f}\right)+O\left(1\right)$, it follows from (9.3) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right)\leq{} & T\left(r,\frac{F}{f-a}\right)-N\left(r,\frac{f-a}{F}\right)+T\left(r,\frac{F}{f-b}\right)\\ & -N\left(r,\frac{f-b}{F}\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f-a}\right)\leq{} & N\left(r,\frac{F}{f-a}\right)+m\left(r,\frac{F}{f-a}\right)-N\left(r,\frac{f-a}{F}\right)\\ & +N\left(r,\frac{F}{f-b}\right)+m\left(r,\frac{F}{f-b}\right)-N\left(r,\frac{f-b}{F}\right)\\ & +S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f-a}\right)\leq{} & N\left(r,\frac{F}{f-a}\right)-N\left(r,\frac{f-a}{F}\right)+N\left(r,\frac{F}{f-b}\right)\\ & -N\left(r,\frac{f-b}{F}\right)+S\left(r,f\right)+O\left(1\right). \end{aligned} \tag{9.4}$$

In view of $\{p.34,[29]\}$ we get from (9.4) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right)\leq{} & N\left(r,F\right)+N\left(r,\frac{1}{f-a}\right)-N\left(r,f-a\right)\\ & -N\left(r,\frac{1}{F}\right)+N\left(r,F\right)+N\left(r,\frac{1}{f-b}\right)\\ & -N\left(r,f-b\right)-N\left(r,\frac{1}{F}\right)+S\left(r,f\right) \end{aligned}$$

$$i.e.,\ m\left(r,\frac{1}{f-a}\right)\leq 2N\left(r,F\right)-2N\left(r,f\right)-2N\left(r,\frac{1}{F}\right)$$
$$+N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-b}\right)+S\left(r,f\right)+O(1)$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)}\leq 2\liminf_{r\to\infty}\left\{\frac{N\left(r,F\right)}{T\left(r,f\right)}-\frac{N\left(r,f\right)}{T\left(r,f\right)}-\frac{N\left(r,\frac{1}{F}\right)}{T\left(r,f\right)}\right\}$$
$$+\limsup_{r\to\infty}\left\{\frac{N\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)}+\frac{N\left(r,\frac{1}{f-b}\right)}{T\left(r,f\right)}\right\}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)}\leq 2\left\{\liminf_{r\to\infty}\frac{N\left(r,F\right)}{T\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,f\right)}{T\left(r,f\right)}-\liminf_{r\to\infty}\frac{N\left(r,\frac{1}{F}\right)}{T\left(r,f\right)}\right\}$$
$$+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)}+\limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{T\left(r,f\right)}$$

$$i.e.,\ \delta\left(a;f\right)\leq 2\left\{1-\Delta_R^F\left(\infty;f\right)\right\}-2\left\{1-\Delta\left(\infty;f\right)\right\}-2\left\{1-\Delta_R^F\left(0;f\right)\right\}$$
$$+\left\{1-\delta\left(a;f\right)\right\}+\left\{1-\delta\left(b;f\right)\right\}$$

$$i.e.,\ 2\delta\left(a;f\right)+\delta\left(b;f\right)+2\Delta_R^F\left(\infty;f\right)\leq 2\Delta\left(\infty;f\right)+2\Delta_R^F\left(0;f\right).$$

Thus the theorem is established. ■

In this chapter we call the terms

$$\delta_A^F\left(a;f\right)=1-\limsup_{r\to\infty}\frac{N\left(r,a;F\right)}{T\left(r,F\right)}=\liminf_{r\to\infty}\frac{m\left(r,a;F\right)}{T\left(r,F\right)}$$

the usual Nevanlinna defect or the absolute Nevanlinna defect of the value $'a'$ with respect to F and

$$\Delta_A^F\left(a;f\right)=1-\liminf_{r\to\infty}\frac{N\left(r,a;F\right)}{T\left(r,F\right)}=\limsup_{r\to\infty}\frac{m\left(r,a;F\right)}{T\left(r,F\right)},$$

the usual Valiron defect or the absolute Valiron defect of the value $'a'$ with respect to F.

The following theorems are based on the relationship between absolute defect and relative defect of a certain value with respect to F.

Remark 9.3.3 *The condition that a and b are two distinct complex numbers in Theorem 9.3.2 is necessary which is evident from the following example.*

Example 9.3.4 *Let* $f = \exp z$,

$$n_0 = 1,\ n_1 = n_2 = ... = n_k = 0 \ \ \textit{and} \ \ n = 1.$$

Then

$$F = f^n Q[f] = (\exp z)(\exp z) = \exp^{[2]} z.$$

Also

$$\rho_f = 1, \quad \delta(0, f) = 1, \quad \Delta_R^F(\infty; f) = 1,$$
$$\Delta(\infty; f) = 1 \ \ \textit{and} \ \ \Delta_R^F(0; f) = 1.$$

Again let $a \neq 0, \infty,\ b \neq 0, \infty$ and $a = b$.
Since

$$\delta(0; f) = \delta(\infty; f) = 1,$$

we get that

$$\delta(a; f) = 0 \ \ for \ \ a \neq 0, \infty$$
$$\delta(b; f) = 0 \ \ for \ \ b \neq 0, \infty$$

Thus

$$2\delta(a; f) + \delta(b; f) + 2\Delta_R^F(\infty; f) = 2.0 + 0 + 2.1 = 2$$

and

$$2\Delta(\infty; f) + 2\Delta_R^F(0; f) = 2.1 + 2.1 = 4,$$

which contradicts Theorem 9.3.2.

Theorem 9.3.3 *If f be a transcendental meromorphic function of finite order ρ_f and $\delta(\infty; f) = 1$ then*

$$\Delta_R^F(\infty; f) + \delta(0; f) \leq \Delta_R^F(0; f) + \Delta_A^F(\infty; f).$$

Proof. Since $f = F\frac{f}{F}$ we get that

$$m(r, f) \leq m(r, F) + m\left(r, \frac{f}{F}\right). \tag{9.5}$$

Now by Nevanlinna's First Fundamental theorem and by Lemma 9.2.1 we obtain from (9.5) that

$$\begin{aligned} m(r,f) &\le m(r,F) + T\left(r,\frac{f}{F}\right) - N\left(r,\frac{f}{F}\right) \\ i.e.,\ m(r,f) &\le m(r,F) + T\left(r,\frac{F}{f}\right) - N\left(r,\frac{f}{F}\right) + O(1) \\ i.e.,\ m(r,f) &\le m(r,F) + N\left(r,\frac{F}{f}\right) + m\left(r,\frac{F}{f}\right) \\ &\quad - N\left(r,\frac{f}{F}\right) + O(1). \qquad (9.6) \end{aligned}$$

Now in view of $\{p.34, [\mathbf{29}]\}$ it follows from (9.6) that

$$\begin{aligned} m(r,f) &\le m(r,F) + N(r,F) + N\left(r,\frac{1}{f}\right) - N(r,f) \\ &\quad - N\left(r,\frac{1}{F}\right) + S(r,f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e.,\ \liminf_{r\to\infty}\frac{m(r,f)}{T(r,f)} &\le \liminf_{r\to\infty}\left\{\frac{N(r,F)}{T(r,f)} - \frac{N(r,f)}{T(r,f)} - \frac{N\left(r,\frac{1}{F}\right)}{T(r,f)}\right\} \\ &\quad + \limsup_{r\to\infty}\left\{\frac{N\left(r,\frac{1}{f}\right)}{T(r,f)} + \frac{m(r,F)}{T(r,F)}\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ \liminf_{r\to\infty}\frac{m(r,f)}{T(r,f)} &\le \liminf_{r\to\infty}\frac{N(r,F)}{T(r,f)} - \liminf_{r\to\infty}\frac{N(r,f)}{T(r,f)} - \liminf_{r\to\infty}\frac{N\left(r,\frac{1}{F}\right)}{T(r,f)} \\ &\quad + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f}\right)}{T(r,f)} + \limsup_{r\to\infty}\frac{m(r,F)}{T(r,f)}. \qquad (9.7) \end{aligned}$$

Since $\delta(\infty;f) = 1$ then $\Delta(\infty;f) = 1$. So by Lemma 9.2.2 we obtain from (9.7) that

$$\begin{aligned} \delta(\infty;f) &\le \left\{1 - \Delta_R^F(\infty;f)\right\} - \{1 - \Delta(\infty;f)\} - \left\{1 - \Delta_R^F(0;f)\right\} \\ &\quad + \{1 - \delta(0;f)\} + \limsup_{r\to\infty}\frac{m(r,F)}{T(r,F)} \cdot \lim_{r\to\infty}\frac{T(r,F)}{T(r,f)} \end{aligned}$$

$$i.e.,\ \delta(\infty;f)+\Delta_R^F(\infty;f)+\delta(0;f)\leq\Delta(\infty;f)+\Delta_R^F(0;f)+\Delta_A^F(\infty;f)$$
$$i.e.,\ 1+\Delta_R^F(\infty;f)+\delta(0;f)\leq 1+\Delta_R^F(0;f)+\Delta_A^F(\infty;f)$$
$$i.e.,\ \Delta_R^F(\infty;f)+\delta(0;f)\leq\Delta_R^F(0;f)+\Delta_A^F(\infty;f).$$

Thus the theorem is established. ■

Remark 9.3.4 *The sign* $'\leq'$ *in Theorem* 9.3.3 *cannot be replaced by* $'<'$ *only as we see in the following example.*

Example 9.3.5 *Let* $f=\exp z$,

$$n_0=1,\ n_1=n_2=...=n_k=0\ \ and\ \ n=1.$$

Then

$$F=f^nQ[f]=(\exp z)(\exp z)=\exp^{[2]}z.$$

Also

$$\rho_f=1,\ \ \delta(0,f)=\delta(\infty,f)=1,\ \ \Delta_A^F(\infty;f)=1,$$
$$and\ \ \Delta_R^F(0;f)=\Delta_R^F(\infty;f)=1.$$

Thus

$$\Delta_R^F(\infty;f)+\delta(0,f)=1+1=2$$

and

$$\Delta_R^F(0;f)+\Delta_A^F(\infty;f)=1+1=2.$$

Remark 9.3.5 *If we omit the condition* $\delta(\infty;f)=1$ *in Theorem* 9.3.3 *and the other conditions remaining the same, using the first part of Lemma* 9.2.3 *we may establish the next theorem.*

Theorem 9.3.4 *Let* f *be a transcendental meromorphic function of finite order. Then*

$$\delta(\infty;f)+\delta(0;f)\leq\Delta(\infty;f)+\Delta_R^F(0;f).$$

Proof. Using the first part of Lemma 9.2.3 and the inequality (9.7) it follows that

$$\delta(\infty;f)\leq\left\{1-\Delta_R^F(\infty;f)\right\}-\{1-\Delta(\infty;f)\}-\left\{1-\Delta_R^F(0;f)\right\}$$
$$+\{1-\delta(0;f)\}+\Delta_R^F(\infty;f)$$
$$i.e.,\ \delta(\infty;f)+\delta(0;f)\leq\Delta(\infty;f)+\Delta_R^F(0;f).$$

Thus the theorem is proved. ■

Theorem 9.3.5 *Let 'a' be a finite complex number and b, c be two distinct non-zero complex numbers. Then for any transcendental meromorphic function f with finite order*

$$\delta(a;f)+\delta_A^F(b;f)+\delta_A^F(c;f)\le 2.$$

Proof. Since $\frac{1}{f-a}=\frac{F}{f-a}.\frac{1}{F}$, by Lemma 9.2.1 we obtain that

$$m\left(r,\frac{1}{f-a}\right)\le m\left(r,\frac{1}{F}\right)+m\left(r,\frac{F}{f-a}\right)$$

$$i.e.,\ m\left(r,\frac{1}{f-a}\right)\le m\left(r,\frac{1}{F}\right)+S(r,f). \tag{9.8}$$

Applying Nevanlinna's First Fundamental theorem we get from (9.8) that

$$m\left(r,\frac{1}{f-a}\right)\le T\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{F}\right)+S(r,f). \tag{9.9}$$

Now by Nevanlinna's second fundamental theorem it follows from (9.9) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right)\le \bar{N}\left(r,\frac{1}{F}\right)+\bar{N}\left(r,\frac{1}{F-b}\right)+\bar{N}\left(r,\frac{1}{F-c}\right)\\ -N\left(r,\frac{1}{F}\right)+S(r,f). \end{aligned} \tag{9.10}$$

Since $\bar{N}\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{F}\right)\le 0$, we obtain from (9.10) in view of Lemma 9.2.2 that

$$m\left(r,\frac{1}{f-a}\right)\le \bar{N}\left(r,\frac{1}{F-b}\right)+\bar{N}\left(r,\frac{1}{F-c}\right)+S(r,f)$$

$$i.e.,\ m\left(r,\frac{1}{f-a}\right)\le N\left(r,\frac{1}{F-b}\right)+N\left(r,\frac{1}{F-c}\right)+S(r,f)$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right)\le T\left(r,\frac{1}{F-b}\right)+T\left(r,\frac{1}{F-c}\right)-m\left(r,\frac{1}{F-b}\right)\\ -m\left(r,\frac{1}{F-c}\right)+S(r,f) \end{aligned}$$

$$i.e., \ m\left(r,\frac{1}{f-a}\right) \leq 2T\left(r,F\right) - m\left(r,\frac{1}{F-b}\right) - m\left(r,\frac{1}{F-c}\right) + S\left(r,f\right)$$

$$i.e., \ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)} \leq 2\liminf_{r\to\infty}\frac{T\left(r,F\right)}{T\left(r,f\right)} - \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-b}\right)}{T\left(r,f\right)} - \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-c}\right)}{T\left(r,f\right)}$$

$$i.e., \ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)} \leq 2\liminf_{r\to\infty}\frac{T\left(r,F\right)}{T\left(r,f\right)} - \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-b}\right)}{T\left(r,F\right)}.\lim_{r\to\infty}\frac{T\left(r,F\right)}{T\left(r,f\right)} - \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-c}\right)}{T\left(r,F\right)}.\lim_{r\to\infty}\frac{T\left(r,F\right)}{T\left(r,f\right)}$$

$$i.e., \ \delta\left(a;f\right) \leq 2 - \delta_A^F\left(b;f\right) - \delta_A^F\left(c;f\right)$$

$$i.e., \ \delta\left(a;f\right) + \delta_A^F\left(b;f\right) + \delta_A^F\left(c;f\right) \leq 2.$$

Thus the theorem is established. ■

Remark 9.3.6 *The condition that b and c are two distinct non-zero complex numbers in Theorem* 9.3.5 *is essential as we see in the following examples.*

Example 9.3.6 *Let* $f = \exp z$,

$$n_0 = 1, \ n_1 = n_2 = ... = n_k = 0 \ \ and \ \ n = 1.$$

Then

$$F = f^n Q[f] = (\exp z)(\exp z) = \exp^{[2]} z.$$

Also

$$\rho_f = 1, \quad a = 0, \quad b = \infty \ \ and \ \ c = \infty.$$

Thus

$$\delta\left(a,f\right) = \delta\left(0,f\right) = 1,$$

$$\delta_A^F\left(b;f\right) = \delta_A^F\left(\infty;f\right) = 1$$

$$and \ \ \delta_A^F\left(c,f\right) = \delta_A^F\left(\infty,f\right) = 1.$$

So

$$\delta(a,f)+\delta_A^F(b;f)+\delta_A^F(c,f)=1+1+1=3,$$

which contradicts Theorem 9.3.5.

Example 9.3.7 *Let* $f=\exp z$,

$$n_0=1,\ n_1=n_2=...=n_k=0 \quad and \quad n=1.$$

Then

$$F=f^nQ[f]=(\exp z)(\exp z)=\exp^{[2]} z.$$

Also

$$\rho_f=1,\quad a=0,\quad b=0 \quad and \quad c=\infty.$$

Thus

$$\begin{aligned} \delta(a,f)&=\delta(0,f)=1,\\ \delta_A^F(b;f)&=\delta_A^F(0;f)=1\\ and\quad \delta_A^F(c,f)&=\delta_A^F(\infty,f)=1. \end{aligned}$$

So

$$\delta(a,f)+\Delta_A^F(b;f)+\delta_A^F(c,f)=1+1+1=3,$$

which is contrary to Theorem 9.3.5.

Example 9.3.8 *Let* $f=\exp z$,

$$n_0=1,\ n_1=n_2=...=n_k=0 \quad and \quad n=1.$$

Then

$$F=f^nQ[f]=(\exp z)(\exp z)=\exp^{[2]} z.$$

Also

$$\rho_f=1,\quad a=0,\quad b=\infty \quad and \quad c=0.$$

Thus

$$\begin{aligned} \delta(a,f)&=\delta(0,f)=1,\\ \delta_A^F(b;f)&=\delta_A^F(\infty;f)=1\\ and\quad \delta_A^F(c,f)&=\delta_A^F(0,f)=1. \end{aligned}$$

So

$$\delta(a,f)+\delta_A^F(b;f)+\delta_A^F(c,f)=1+1+1=3,$$

which contradicts Theorem 9.3.5.

Remark 9.3.7 *The sign* $' \leq'$ *in Theorem* 9.3.5 *cannot be replaced by* $' <'$ *only which is evident from the following examples.*

Example 9.3.9 *Let*

$$f = \exp z,$$

$$n_0 = 1,\ n_1 = n_2 = \ldots = n_k = 0 \quad and \quad n = 1.$$

Then

$$F = f^n Q[f] = (\exp z)(\exp z) = \exp^{[2]} z.$$

Also

$$\rho_f = 1, \quad a = 0, \quad b = \infty \quad and \quad c \neq 0, \infty.$$

Thus

$$\begin{aligned} \delta(a, f) &= \delta(0, f) = 1, \\ \delta_A^F(b; f) &= \delta_A^F(\infty; f) = 1 \\ and \quad \delta_A^F(c, f) &= 0 \quad for\ c \neq 0, \infty. \end{aligned}$$

So

$$\delta(a, f) + \delta_A^F(b; f) + \delta_A^F(c, f) = 1 + 1 + 0 = 2.$$

$$****** X ******$$