CHAPTER-8

ON THE GROWTH ESTIMATES OF ENTIRE FUNCTIONS SATISFYING SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

Chapter 8

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8.1 Introduction, Definitions and Notations.

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [10] proved that

$$\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty \quad \textit{and} \quad \lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty.$$

Singh [59] proved some comparative growth properties of $\log T(r,f\circ g)$ and $T(r,f)$. He [59] also raised the problem of investigating the comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$ which he was unable to solve. However, some results on the comparative growth of $\log T(r,f\circ g)$ and $T(r,g)$ are proved in [37].

Let f be an entire function defined in the open complex plane $\mathbb{C}$. Kwon [33] studied on the growth of an entire function f satisfying second order linear differential equation. Later Chen [12] proved some results on the growth of solutions of second order linear differential equations with meromorphic coefficents. Chen and Yang [13] established a few theorems on the zeros and growths of entire solutions of second order linear differential equations. The purpose of this chapter is to study on the growth of the solution

The results of this chapter have been published in International Mathematical Forum,see [23].

$f \not\equiv 0$ of the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0,$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions.
The following definitions are well known.

Definition 8.1.1 *The order ρ_f and lower order λ_f of an entire function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.
If f is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Definition 8.1.2 *The hyper order $\bar{\rho}_f$ and hyper lower order $\bar{\lambda}_f$ of an entire function f is defined as follows*

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

If f is meromorphic, then

$$\bar{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

Definition 8.1.3 *[43] Let f be an entire function of order zero. Then the quantities ρ_f^*, λ_f^* and $\bar{\rho}_f^{\,*}$, $\bar{\lambda}_f^{\,*}$ are defined in the following way :*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^{\,*} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^{\,*} = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

If f is meromorphic then clearly

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

and

$$\bar{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}, \quad \bar{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}.$$

Definition 8.1.4 *The type σ_f of an entire function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When f is meromorphic, then

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Definition 8.1.5 *Let $'a'$ be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of $'a'$ with respect to a meromorphic function f are defined as*

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.$$

8.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 8.2.1 *[1] If f is meromorphic and g is entire then for all sufficiently large values of r,*

$$T(r, f\circ g) \le \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 8.2.2 *[4] Let f be meromorphic and g be entire and suppose that $0 < \mu \le \rho_g \le \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f\circ g) \ge T(\exp(r^\mu), f).$$

Lemma 8.2.3 *[51]Let f and g be two transcendental entire functions with $\rho_g < \infty$, η be a constant satisfying $0 < \eta < 1$ and α be a positive number. Then*

$$\begin{aligned} T(r, f \circ g) + O(1) &\geq N(r, 0; f \circ g) \\ &\geq \log\left(\frac{1}{\eta}\right)\left[\frac{N(M((\eta r)^{\frac{1}{1+\alpha}}, g), 0, f)}{\log M((\eta r)^{\frac{1}{1+\alpha}}, g) - O(1)} - O(1)\right] \end{aligned}$$

as $r \to \infty$ through all values.

8.3 Theorems.

In this section we present the main results of the chapter.

Theorem 8.3.1 *Let f be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If (i) ρ_A, ρ_B are both finite, (ii) λ_A, λ_f are both positive, (iii) $\rho_B < \lambda_A$ and $\rho_B < \lambda_f$ i.e. $\rho_B < \min\{\lambda_A, \lambda_f\}$ and (iv) B be of regular growth i.e., $\lambda_B = \rho_B$ then*

$$\lim_{r\to\infty} \frac{\{\log T(r, A \circ B)\}^2}{T(r, f)T(r, A)} = 0.$$

Proof. It is well known that for an entire function B, $T(r, B) \leq \log^+ M(r, B)$. So in view of Lemma 8.2.1, we get for all sufficiently large values of r,

$$\begin{aligned} & T(r, A \circ B) \leq \{1 + o(1)\}T(M(r, B), A) \\ i.e., \quad & \log T(r, A \circ B) \leq \log\{1 + o(1)\} + \log T(M(r, B), A) \\ i.e., \quad & \log T(r, A \circ B) \leq o(1) + (\rho_A + \epsilon) \log M(r, B) \\ i.e., \quad & \log T(r, A \circ B) \leq o(1) + (\rho_A + \epsilon) r^{\rho_B + \epsilon}. \end{aligned} \tag{8.1}$$

Also we obtain for all sufficiently large values of r,

$$T(r, A) \geq r^{\lambda_A - \epsilon}. \tag{8.2}$$

Now combining (8.1) and (8.2) it follows for all sufficiently large values of r,

$$\begin{aligned} & \frac{\log T(r, A \circ B)}{T(r, A)} \leq \frac{o(1) + (\rho_A + \epsilon) r^{(\rho_B + \epsilon)}}{r^{\lambda_A - \epsilon}} \\ i.e., \quad & \limsup_{r\to\infty} \frac{\log T(r, A \circ B)}{T(r, A)} \leq \limsup_{r\to\infty} \frac{o(1) + (\rho_A + \epsilon) r^{(\rho_B + \epsilon)}}{r^{\lambda_A - \epsilon}}. \end{aligned}$$

Since $\rho_B < \lambda_A$, we can choose ϵ (> 0) in such a way that $\rho_B + \epsilon < \lambda_A - \epsilon$ and so it follows from above that

$$\limsup_{r\to\infty} \frac{\log T(r, A \circ B)}{T(r, A)} = 0. \tag{8.3}$$

Again we get for all sufficiently large values of r,

$$\begin{aligned} \log T(r,f) &\geq (\lambda_f - \epsilon)\log r \\ i.e., \quad T(r,f) &\geq r^{\lambda_f - \epsilon}. \end{aligned} \tag{8.4}$$

Since $\rho_B < \lambda_f$, we can choose ϵ (> 0) in such a way that

$$\rho_B + \epsilon < \lambda_f - \epsilon. \tag{8.5}$$

Now combining (8.1), (8.4) and (8.5) it follows for all sufficiently large values of r,

$$\frac{\log T(r, A \circ B)}{T(r,f)} \leq \frac{o(1) + (\rho_A + \epsilon) r^{(\rho_B + \epsilon)}}{r^{\lambda_f - \epsilon}}$$

$$\begin{aligned} i.e., \quad \limsup_{r\to\infty} \frac{\log T(r, A \circ B)}{T(r,f)} &= 0 \\ i.e., \quad \lim_{r\to\infty} \frac{\log T(r, A \circ B)}{T(r,f)} &= 0. \end{aligned} \tag{8.6}$$

Therefore in view of (8.3) and (8.6), we obtain that

$$\begin{aligned} &\lim_{r\to\infty} \frac{\{\log T(r, A \circ B)\}^2}{T(r,f)T(r,A)} \\ &= \lim_{r\to\infty} \frac{\log T(r, A \circ B)}{T(r,f)} . \lim_{r\to\infty} \frac{\log T(r, A \circ B)}{T(r,A)} \\ &= 0. \end{aligned}$$

$$i.e., \quad \lim_{r\to\infty} \frac{\{\log T(r, A \circ B)\}^2}{T(r,f)T(r,A)} = 0.$$

This proves the theorem. ■

Remark 8.3.1 *The condition* $\rho_B < \min\{\lambda_A, \lambda_f\}$ *in Theorem* 8.3.1 *is essential as we see in the following example.*

Example 8.3.1 *Let* $f(z) = \exp z,\ A(z) = \exp z$ *and* $B(z) = \exp z$ *with* $1 + 2e^z = 0$.
Then $\rho_A = \lambda_A = 1,\ \rho_B = \lambda_B = 1$ *and* $\lambda_f = 1$.
Also $T(r, f) = T(r, \exp z) = \frac{r}{\pi}$,
$T(r, A) = T(r, \exp z) = \frac{r}{\pi}$ *and*

$$T(r, A \circ B) = T(r, \exp^{[2]} z) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \quad (r \to \infty)$$

$$i.e., \quad \log T(r, A \circ B) \sim r - \frac{1}{2}\log r + O(1).$$

Therefore,

$$\begin{aligned}
&\lim_{r\to\infty} \frac{\{\log T(r, A \circ B)\}^2}{T(r.f)T(r, A)} \\
&= \lim_{r\to\infty} \frac{\left\{r - \frac{1}{2}\log r + O(1)\right\}^2}{\frac{r}{\pi}.\frac{r}{\pi}} \\
&\geq \lim_{r\to\infty} \frac{\pi^2 \left\{\frac{1}{2}r + O(1)\right\}^2}{r^2} \\
&= \lim_{r\to\infty} \pi^2 \left\{\frac{1}{2} + \frac{O(1)}{r}\right\}^2 \\
&= \frac{\pi^2}{4},
\end{aligned}$$

which contradicts Theorem 8.3.1.

Theorem 8.3.2 *Let f be an entire function satisfying the second order linear differential equation* $f'' + A(z)f' + B(z)f = 0$ *where* $A(z)$ *and* $B(z) \not\equiv 0$ *are entire functions. If* $\rho_B = 0$ *then* $\rho_{A\circ B} \geq \lambda_A^*.\mu$ *where* $0 < \mu < \rho_B$.

Proof. In view of Lemma 8.2.2 and for $0 < \mu < \rho_B$ we get that

$$\begin{aligned}
\rho_{A\circ B} &= \limsup_{r\to\infty} \frac{\log T(r, A\circ B)}{\log r} \\
&\geq \liminf_{r\to\infty} \frac{\log T(\exp(r^{\mu}), A)}{\log r} \\
&\geq \liminf_{r\to\infty} \frac{\log T(\exp(r^{\mu}), A)}{\log^{[2]}(\exp(r^{\mu}))} . \liminf_{r\to\infty} \frac{\log^{[2]}(\exp(r^{\mu}))}{\log r} \\
&= \lambda_A^{*} . \liminf_{r\to\infty} \frac{\log r^{\mu}}{\log r} \\
&= \lambda_A^{*} . \mu.
\end{aligned}$$

Thus the theorem is established. ■

Remark 8.3.2 *The condition $\mu < \rho_B$ in Theorem* 8.3.2 *is necessary which is evident from the following example.*

Example 8.3.2 *Let $f = \exp z,\ A(z) = z$ and*

$B(z) = \exp z$ with $1 + z + e^{z} = 0$. Also let $\mu = 2$.
Then $\rho_A = \lambda_A = 0,\ \lambda_A^{*} = 1$ and $\rho_{A\circ B} = 1$.
Thus $\rho_{A\circ B} = 1 < 2 = 1.2 = \lambda_A^{*}.\mu$, which is contrary to Theorem 8.3.2.

Theorem 8.3.3 *Let f be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If ρ_A, ρ_B are both finite and λ_f is positive then for any $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{[\log\{T(r, A\circ B)\log M(r, B)\}]^{1+\alpha}}{T(\exp r, f)} = 0.$$

Proof. If $1 + \alpha \leq 0$, the theorem is obvious. So we suppose that $1 + \alpha > 0$. In view of Lemma 8.2.1, we have for all sufficiently large values of r,

$$\begin{aligned}
&\log\{T(r, A\circ B)\log M(r, B)\} \\
&\leq \log T(r, B) + \log T(M(r, B), A) + \log\{1 + o(1)\} \\
&\leq (\rho_B + \epsilon)\log r + (\rho_A + \epsilon)r^{\rho_B+\epsilon} + o(1) \\
&\leq r^{\rho_B+\epsilon}\left\{(\rho_A + \epsilon) + \frac{(\rho_B + \epsilon)\log r + o(1)}{r^{\rho_B+\epsilon}}\right\}. \qquad (8.7)
\end{aligned}$$

Again we get for all sufficiently large values of r,

$$\log T(\exp r, f) \geq (\lambda_f - \epsilon) \log \{\exp r\}$$
$$i.e., \quad T(\exp r, f) \geq \exp \{(\lambda_f - \epsilon) r\}. \tag{8.8}$$

Now combining (8.7) and (8.8) it follows for all sufficiently large values of r,

$$\frac{[\log\{T(r, A \circ B) \log M(r, B)\}]^{1+\alpha}}{T(\exp r, f)}$$

$$\leq \frac{r^{(\rho_B+\epsilon)(1+\alpha)\left\{(\rho_A+\epsilon)+\frac{(\rho_B+\epsilon)\log r+o(1)}{r^{\rho_B+\epsilon}}\right\}^{1+\alpha}}}{\exp\{(\lambda_f-\epsilon)r\}}$$

$$i.e., \quad \limsup_{r\to\infty} \frac{[\log\{T(r, A \circ B) \log M(r, B)\}]^{1+\alpha}}{T(\exp r, f)} = 0,$$

from which the theorem follows. ■

Remark 8.3.3 *The condition $\lambda_f > 0$ in the Theorem* 8.3.3 *is essential as we see in the following example.*

Example 8.3.3 *Let $f = z$, $A(z) = B(z) = \exp z$ and $\alpha = 0$ with $z + 1 = 0$. Then $\rho_A = \rho_B = 1$ and $\lambda_f = 0$.*

Also

$$T(r, A \circ B) = T(r, \exp^{[2]} z) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \quad (r \to \infty),$$

$$M(r, B) = M(r, \exp z) = \exp r$$

and

$$T(\exp r, z) \leq \log^+ M(\exp r, z) = \log(\exp r) = r.$$

Therefore,

$$\begin{aligned}
&\lim_{r\to\infty}\frac{[\log\{T(r,A\circ B)\log M(r,B)\}]^{1+\alpha}}{T(\exp r,f)}\\
&=\lim_{r\to\infty}\frac{\log T(r,A\circ B)+\log^{[2]}M(r,B)}{T(\exp r,f)}\\
&=\frac{r-\frac{1}{2}\log r+O(1)+\log^{[2]}\exp r}{r}\\
&=\frac{r-\frac{1}{2}\log r+O(1)+\log r}{r}\\
&=\frac{r+\frac{1}{2}\log r+O(1)}{r}\\
&=1,
\end{aligned}$$

which contradicts Theorem 8.3.3.

Theorem 8.3.4 *Let f be an entire function satisfying the second order linear differential equation $f''+A(z)f'+B(z)f=0$ where $A(z)$ and $B(z)\not\equiv 0$ are entire functions. If ρ_A,ρ_B are both finite and λ_f is positive then for any $\alpha\,\epsilon\,(-\infty,\infty)$,*

$$\lim_{r\to\infty}\frac{[\log\{T(r,A\circ B)\log M(r,B)\}]^{1+\alpha}}{T(\exp r,f)}=0\quad if\quad 0<1+\alpha<\frac{1}{\rho_B}.$$

Proof. If $1+\alpha\le 0$, the theorem is obvious. So we take $1+\alpha>0$. We obtain for all sufficiently large values of r,

$$\begin{aligned}
&\log T(\exp r,f)\ge(\lambda_f-\epsilon)\log\{\exp r\}\\
&i.e.,\quad T(\exp r,f)\ge\exp\{(\lambda_f-\epsilon)r\}. \qquad (8.9)
\end{aligned}$$

Now combining (8.7) and (8.9) it follows for all sufficiently large values of r,

$$\begin{aligned}
&\frac{[\log\{T(r,A\circ B)\log M(r,B)\}]^{1+\alpha}}{T(\exp r,f)}\\
&\le\frac{r^{(\rho_B+\epsilon)(1+\alpha)\left\{(\rho_A+\epsilon)+\frac{(\rho_B+\epsilon)\log r+o(1)}{r^{\rho_B+\epsilon}}\right\}^{1+\alpha}}}{(\lambda_f-\epsilon)r}. \qquad (8.10)
\end{aligned}$$

Since $1+\alpha<\frac{1}{\rho_B}$, we can choose $\epsilon\ (>0)$ in such a way that

$$(\rho_B+\epsilon)(1+\alpha)<1. \qquad (8.11)$$

Thus the theorem follows from (8.10) and (8.11). ■

Theorem 8.3.5 *If f be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If $0 < \bar{\lambda}_{A \circ B} \leq \bar{\rho}_{A \circ B} < \infty$ and $0 < \bar{\rho}_f < \infty$ then for any positive number α,*

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, A \circ B)}{\log^{[2]} T(r^\alpha, f)} \leq \frac{\bar{\rho}_{A \circ B}}{\alpha \bar{\rho}_f} \leq \limsup_{r \to \infty} \frac{\log^{[2]} T(r, A \circ B)}{\log^{[2]} T(r^\alpha, f)}.$$

Proof. From the definition of hyper order we get for all sufficiently large values of r,

$$\log^{[2]} T(r, A \circ B) \leq (\bar{\rho}_{A \circ B} + \epsilon) \log r. \tag{8.12}$$

Again we have for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r^\alpha, f) \geq (\bar{\rho}_f - \epsilon) \log r^\alpha$$
$$i.e., \quad \log^{[2]} T(r^\alpha, f) \geq \alpha(\bar{\rho}_f - \epsilon) \log r. \tag{8.13}$$

Now combining (8.12) and (8.13) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, A \circ B)}{\log^{[2]} T(r^\alpha, f)} \leq \frac{(\bar{\rho}_{A \circ B} + \epsilon) \log r}{\alpha(\bar{\rho}_f - \epsilon) \log r}.$$

Since ϵ (> 0) is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, A \circ B)}{\log^{[2]} T(r^\alpha, f)} \leq \frac{\bar{\rho}_{A \circ B}}{\alpha \bar{\rho}_f}. \tag{8.14}$$

Also for arbitrary positive ϵ and for all sufficiently large values of r,

$$\log^{[2]} T(r^\alpha, f) \leq (\bar{\rho}_f + \epsilon) \log r^\alpha$$
$$i.e., \quad \log^{[2]} T(r^\alpha, f) \leq \alpha(\bar{\rho}_f + \epsilon) \log r. \tag{8.15}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, A \circ B) \geq (\bar{\rho}_{A \circ B} - \epsilon) \log r. \tag{8.16}$$

Now from (8.15) and (8.16) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, A \circ B)}{\log^{[2]} T(r^{\alpha}, f)} \geq \frac{(\bar{\rho}_{A \circ B} - \epsilon) \log r}{\alpha(\bar{\rho}_f + \epsilon) \log r}.$$

As ϵ (> 0) is arbitrary, we have from above that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, A \circ B)}{\log^{[2]} T(r^{\alpha}, f)} \geq \frac{\bar{\rho}_{A \circ B}}{\alpha \bar{\rho}_f}. \quad (8.17)$$

Thus the theorem follows from (8.14) and (8.17). ■

Remark 8.3.4 *The sign '* $\leq$*' in Theorem* 8.3.5 *cannot be replaced by '* $<$*' only as we see in the following example.*

Example 8.3.4 *Let* $f = \exp^{[2]} z$, $A(z) = B(z) = \exp z$ *and* $\alpha = 1$, *with* $1 + e^z = 0$.

Then $\bar{\lambda}_{A \circ B} = \bar{\rho}_{A \circ B} = 1$ and $\bar{\rho}_f = 1$.
Also

$$T(r, A \circ B) = T(r, \exp^{[2]} z) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \quad (r \to \infty)$$

$$\text{i.e.,} \quad \log T(r, A \circ B) \sim r - \frac{1}{2} \log r + O(1)$$
$$\text{i.e.,} \quad \log^{[2]} T(r, A \circ B) \sim \log(r - \frac{1}{2} \log r + O(1)).$$

Again

$$T(r^{\alpha}, f) = T(r, \exp^{[2]} z) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}$$
$$\text{i.e.,} \quad \log T(r^{\alpha}, f) \sim r - \frac{1}{2} \log r + O(1)$$
$$\text{i.e.,} \quad \log^{[2]} T(r^{\alpha}, f) \sim \log(r - \frac{1}{2} \log r + O(1)).$$

Therefore,

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\log^{[2]}T(r,A\circ B)}{\log^{[2]}T(r^{\alpha},f)}\\&=\liminf_{r\to\infty}\frac{\log(r-\frac{1}{2}\log r+O(1))}{\log(r-\frac{1}{2}\log r+O(1))}\\&=1\end{aligned}$$

and

$$\begin{aligned}&\limsup_{r\to\infty}\frac{\log^{[2]}T(r,A\circ B)}{\log^{[2]}T(r^{\alpha},f)}\\&=\limsup_{r\to\infty}\frac{\log(r-\frac{1}{2}\log r+O(1))}{\log(r-\frac{1}{2}\log r+O(1))}\\&=1.\end{aligned}$$

Also

$$\frac{\bar{\rho}_{A\circ B}}{\alpha\bar{\rho}_f}=\frac{1}{1.1}=1.$$

Theorem 8.3.6 *Let f be an entire function satisfying the second order linear differential equation $f''+A(z)f'+B(z)f=0$ where $A(z)$ and $B(z)\not\equiv 0$ are entire functions. If (i) $0<\rho_f<\infty$, (ii) $\sigma_f<\infty$, (iii) $\rho_{A\circ B}=\rho_f$ and (iv) $0<\sigma_{A\circ B}<\infty$ then*

$$\liminf_{r\to\infty}\frac{T(r,A\circ B)}{T(r,f)}\le\frac{\sigma_{A\circ B}}{\sigma_f}\le\limsup_{r\to\infty}\frac{T(r,A\circ B)}{T(r,f)}.$$

Proof. By the definition of type, we have for arbitrary positive ϵ and for all sufficiently large values of r,

$$T(r,A\circ B)\le(\sigma_{A\circ B}+\epsilon)r^{\rho_{A\circ B}}. \tag{8.18}$$

Again we get for a sequence of values of r tending to infinity,

$$T(r,f)\ge(\sigma_f-\epsilon)r^{\rho_f}. \tag{8.19}$$

Since $\rho_{A\circ B}=\rho_f$ from (8.18) and (8.19) it follows for a sequence of values of r tending to infinity,

$$\frac{T(r,A\circ B)}{T(r,f)}\le\frac{(\sigma_{A\circ B}+\epsilon)}{(\sigma_f-\epsilon)}.$$

As ϵ (> 0) is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{T(r, A\circ B)}{T(r,f)} \leq \frac{\sigma_{A\circ B}}{\sigma_f}. \tag{8.20}$$

Again for a sequence of values of r tending to infinity,

$$T(r, A\circ B) \geq (\sigma_{A\circ B} - \epsilon) r^{\rho_{A\circ B}}. \tag{8.21}$$

Also for all sufficiently large values of r,

$$T(r,f) \leq (\sigma_f + \epsilon) r^{\rho_f}. \tag{8.22}$$

Now in view of condition (*iii*) we get from (8.21) and (8.22) for a sequence of values of r tending to infinity,

$$\frac{T(r, A\circ B)}{T(r,f)} \geq \frac{(\sigma_{A\circ B} - \epsilon)}{(\sigma_f + \epsilon)}.$$

Since ϵ (> 0) is arbitrary, we obtain from above that

$$\limsup_{r\to\infty}\frac{T(r, A\circ B)}{T(r,f)} \geq \frac{\sigma_{A\circ B}}{\sigma_f}. \tag{8.23}$$

Thus the theorem follows from (8.20) and (8.23). ■

Remark 8.3.5 *The sign* $'\leq'$ *in Theorem* 8.3.6 *cannot be replaced by* $'<'$ *only as we see in the following example.*

Example 8.3.5 *Let* $f = \exp z$, $A(z) = \exp z$, $B(z) = z$ *and*

$\alpha = 1$ with $1 + z + e^z = 0$.
Then $\rho_f = 1$, $\rho_{A\circ B} = 1$, $\sigma_f = 1$ and $\sigma_{A\circ B} = 1$.
Also

$$T(r, A\circ B) = T(r, \exp z) = \frac{r}{\pi}$$

and

$$T(r,f) = T(r, \exp z) = \frac{r}{\pi}.$$

Therefore,

$$\liminf_{r\to\infty}\frac{T(r, A\circ B)}{T(r,f)} = \liminf_{r\to\infty}\frac{\frac{r}{\pi}}{\frac{r}{\pi}} = 1$$

and

$$\limsup_{r\to\infty}\frac{T(r,A\circ B)}{T(r,f)}=\limsup_{r\to\infty}\frac{\frac{r}{\pi}}{\frac{r}{\pi}}=1.$$

Also

$$\frac{\sigma_{A\circ B}}{\sigma_f}=\frac{1}{1}=1.$$

Theorem 8.3.7 *Let f be an entire function satisfying the second order linear differential equation $f''+A(z)f'+B(z)f=0$ where $A(z)$ and $B(z)\not\equiv 0$ are entire functions. If (i) $0<\lambda_B\le\rho_B<\infty$, (ii) $\lambda_A>0$, (iii) $\rho_f<\infty$ and (iv) $\Delta(0;A)<1$ then*

$$\lim_{r\to\infty}\frac{\log T(r,A\circ B)}{\log T(r^{\beta},f)}=\infty,$$

where β is a real constant.

Proof. We suppose that $\beta>0$ because otherwise the theorem is obvious.

For given ϵ $(0<\epsilon<1-\Delta(0;A))$,

$$N(r,0;A)>(1-\Delta(0;A)-\epsilon)T(r,A)$$

for all sufficiently large values of r.

So from Lemma 8.2.3 we get for all large values of r,

$$\begin{aligned}&T(r,A\circ B)+O(1)\\&\ge(\log\frac{1}{\eta})\left[\frac{(1-\Delta(0;A)-\epsilon)T\left\{M(\eta r)^{\frac{1}{1+\alpha}},B),A\right\}}{\log M((\eta r)^{\frac{1}{1+\alpha}},B)-O(1)}-O(1)\right].\end{aligned}\tag{8.24}$$

Since for all large values of r, $\log M(r,B)<r^{\rho_B+\epsilon}$, it follows from (8.23) that for all sufficiently large values of r,

$$\begin{aligned}T(r,A\circ B)+O(1)\ge{}&O(\log r)+\log T\left\{M((\eta r)^{\frac{1}{1+\alpha}},B),A\right\}\\&+\log\left[1-\frac{\log M((\eta r)^{\frac{1}{1+\alpha}},B)O(1)}{(1-\Delta(0;A)-\epsilon)T\left\{M(\eta r)^{\frac{1}{1+\alpha}},B),A\right\}}\right].\end{aligned}$$

Since f is transcendental, it follows that

$$\lim_{r\to\infty}\frac{\log M((\eta r)^{\frac{1}{1+\alpha}},B)}{T\left\{M((\eta r)^{\frac{1}{1+\alpha}},B),A\right\}}=0.$$

So from above we get for all large values of r,

$$\log T(r, A \circ B) \geq O(\log r) + \log T\left\{M((\eta r)^{\frac{1}{1+\alpha}}, B), A\right\} + o(1). \tag{8.25}$$

Also we see that for all large values of r,

$$M(r, B) > \exp\left\{(r)^{(1/2)\lambda_B}\right\},$$
$$\log T(r, A) > \frac{1}{2}\lambda_A \log r$$
$$and \quad T(r, f) < r^{\rho_f + 1}.$$

So from (8.25) we obtain for all sufficiently large values of r,

$$\frac{\log T(r, A \circ B)}{\log T(r^\beta, f)} > \frac{O(\log r)}{\beta(1 + \rho_B)\log r} + \frac{\lambda_A}{2}.\frac{(\eta r)^{\frac{\lambda_B}{2(1+\alpha)}}}{\beta(1 + \rho_B)\log r} + o(1),$$

which implies that

$$\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{\log T(r^\beta, f)} = \infty.$$

This proves the theorem. ■

Remark 8.3.6 *The condition $\lambda_A > 0$ in Theorem 8.3.7 is necessary as we see in the following example.*

Example 8.3.6 *Let $f = \exp z$, $A(z) = z$, $B(z) = \exp z$ and*

$\beta = 1$ with $1 + z + e^z = 0$.
Then $\rho_f = 1$, $\lambda_A = 0$, $\lambda_B = \rho_B = 1$ and $\Delta(0; A) < 1$.
Also

$$T(r, A \circ B) = T(r, \exp z) = \frac{r}{\pi}$$

and

$$T(r^\beta, f) = T(r, \exp z) = \frac{r}{\pi}.$$

Then

$$\lim_{r \to \infty} \frac{\log T(r, A \circ B)}{\log T(r^\beta, f)} = \lim_{r \to \infty} \frac{\log \frac{r}{\pi}}{\log \frac{r}{\pi}} = \lim_{r \to \infty} \frac{\log r + O(1)}{\log r + O(1)} = 1,$$

which is contrary to Theorem 8.3.7.

Remark 8.3.7 *If we consider $\rho_A > 0$ instead of $\lambda_A > 0$, the theorem remains true with 'limit' replaced by 'limit superior' as we see in the following theorem.*

Theorem 8.3.8 *Let f be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If (i) $0 < \lambda_B \leq \rho_B < \infty$, (ii) $\rho_A > 0$, (iii) $\rho_f < \infty$ and (iv) $\Delta(0; A) < 1$ then*

$$\limsup_{r\to\infty} \frac{\log T(r, A \circ B)}{\log T(r^{\beta'}, f)} = \infty,$$

where β' is a real constant.

Proof. For all sufficiently large values of r,

$$M(r, B) > \exp\left\{(r)^{(1/2)\lambda_B}\right\}$$
$$\text{and} \quad T(r, f) < r^{\rho_f + 1}.$$

Also for a sequence of values of r tending to infinity,

$$\log T(r, A) > \frac{1}{2}\rho_A \log r.$$

So from (8.25) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r, A \circ B)}{\log T(r^{\beta'}, f)} > \frac{O(\log r)}{\beta'(1 + \rho_B)\log r} + \frac{\rho_A}{2} . \frac{(\eta r)^{\frac{\lambda_B}{2(1+\alpha)}}}{\beta'(1 + \rho_B)\log r} + o(1),$$

which implies that

$$\limsup_{r\to\infty} \frac{\log T(r, A \circ B)}{\log T(r^{\beta'}, f)} = \infty.$$

Thus the theorem is established. ∎

Remark 8.3.8 *The conclusion of Theorem 8.3.8 can also be drawn under the condition $\delta(0; A) < 1$ instead of $\Delta(0; A) < 1$ and the other conditions remaining the same as we see in the next theorem.*

Theorem 8.3.9 *Let f be an entire function satisfying the second order linear differential equation $f'' + A(z)f' + B(z)f = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are entire functions. If (i) $0 < \lambda_B \leq \rho_B < \infty$, (ii) $\lambda_A > 0$, (iii) $\rho_f < \infty$ and (iv) $\delta(0; A) < 1$ then*

$$\limsup_{r\to\infty} \frac{\log T(r, A \circ B)}{\log T(r^{\beta_0}, f)} = \infty,$$

where β_0 is a real constant.

Proof. We suppose that $\beta_0 > 0$ because otherwise the theorem is obvious.

For given ϵ $(0 < \epsilon < 1 - \delta(0; A))$,

$$N(r, 0; A) > (1 - \delta(0; A) - \epsilon)T(r, A)$$

for a sequence of values of r tending to infinity,

So from Lemma 8.2.3 we get for a sequence of values of r tending to infinity,

$$\begin{aligned} &T(r, A \circ B) + O(1) \\ &\geq (\log \frac{1}{\eta}) \left[\frac{(1 - \delta(0; A) - \epsilon)T\left\{M(\eta r)^{\frac{1}{1+\alpha}}, B), A\right\}}{\log M((\eta r)^{\frac{1}{1+\alpha}}, B) - O(1)} - O(1) \right]. \end{aligned} \tag{8.26}$$

Since for all large values of r,

$$\log M(r, B) < r^{\rho_B + \epsilon},$$

it follows from (8.26)that for a sequence of values of r tending to infinity,

$$\begin{aligned} T(r, A \circ B) + O(1) &\geq O(\log r) + \log T\left\{M((\eta r)^{\frac{1}{1+\alpha}}, B), A\right\} \\ &+ \log \left[1 - \frac{\log M((\eta r)^{\frac{1}{1+\alpha}}, B) O(1)}{(1 - \delta(0; A) - \epsilon)T\left\{M(\eta r)^{\frac{1}{1+\alpha}}, B), A\right\}}\right]. \end{aligned}$$

Since f is transcendental, it follows that

$$\limsup_{r \to \infty} \frac{\log M((\eta r)^{\frac{1}{1+\alpha}}, B)}{T\left\{M((\eta r)^{\frac{1}{1+\alpha}}, B), A\right\}} = 0.$$

So from above we get for a sequence of values of r tending to infinity,

$$\log T(r, A \circ B) \geq O(\log r) + \log T\left\{M((\eta r)^{\frac{1}{1+\alpha}}, B), A\right\} + o(1). \tag{8.27}$$

Also we see that for all large values of r,

$$\begin{aligned} M(r, B) &> \exp\left\{(r)^{(1/2)\lambda_B}\right\}, \\ \log T(r, A) &> \frac{1}{2}\lambda_A \log r \\ \textit{and} \quad T(r, f) &< r^{\rho_f + 1}. \end{aligned}$$

So from (8.27) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log T(r, A \circ B)}{\log T(r^{\beta_0}, f)} > \frac{O(\log r)}{\beta_0(1+\rho_B)\log r} + \frac{\lambda_A}{2}.\frac{(\eta r)^{\frac{\lambda_B}{2(1+\alpha)}}}{\beta_0(1+\rho_B)\log r} + o(1),$$

which implies that

$$\limsup_{r\to\infty} \frac{\log T(r, A \circ B)}{\log T(r^{\beta_0}, f)} = \infty.$$

This proves the theorem. ■

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