CHAPTER-7

ON THE DEFINITION OF WEAK TYPE OF A MEROMORPHIC FUNCTION OF LOWER ORDER ZERO OR INFINITY AND SOME RELATED GROWTH PROPERTIES

Chapter 7

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7.1 Introduction, Definitions and Notations.

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. In the sequel we use the following two notations:

$\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$; and $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ for $k = 1, 2, 3, ...$and $\exp^{[0]} x = x$.

The lower order and weak type of a meromorphic function f are defined in the following way :

Definition 7.1.1 *The lower order λ_f of a meromorphic function f are defined as*

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} .$$

If f is entire, one can easily verify that

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} .$$

Some results of this chapter have been published in **International Journal of Mathematical Sciences and Engineering Applications**, see [24] and the remaining in **International Journal of Contemporary Mathematical Sciences**, see [25].

Definition 7.1.2 *[15]The weak type τ_f of a meromorphic function f is defined as follows :*

$$\tau_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty.$$

When f is entire, then

$$\tau_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty.$$

But when a meromorphic function f is of lower order zero or infinity, the weak type of f cannot be defined. In this chapter we introduce the definition of weak type of a meromorphic function of lower order zero or infinity and deduce its integral representation. In order to do this we just recall the definition of zero lower order of a meromorphic function. In this connection Liao and Yang [**43**] gave the following definition :

Definition 7.1.3 *[43] Let f be a meromorphic function of order zero. Then the quantity λ_f^* is defined as*

$$\lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r} .$$

If f is entire then clearly

$$\lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r} .$$

The following definition is also well known.

Definition 7.1.4 *The hyper lower order $\bar{\lambda}_f$ of a meromorphic function f is defined as follows :*

$$\bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} .$$

If f is entire,then

$$\bar{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} .$$

In this chapter we introduce the following definitions.

Definition 7.1.A The weak type τ_f^* of a meromorphic function of order zero is defined by

$$\tau_f^* = \liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^{\lambda_f^*}}, \quad 0 < \lambda_f^* < \infty.$$

Definition 7.1.B A meromorphic function f of order zero is said to be of weak type τ_f^* if the integral $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp(\log r)^{\lambda_f^*}\right]^{k+1}} dr$ $(r_0 > 0)$ is convergent for $k > \tau_f^*$ and divergent for $k < \tau_f^*$ where $0 < \lambda_f^* < \infty$.

Definition 7.1.C The weak type $\bar{\tau}_f$ of a meromorphic function of lower order infinity is defined as follows :

$$\bar{\tau}_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{r^{\bar{\lambda}_f}}, \quad 0 < \bar{\lambda}_f < \infty.$$

Definition 7.1.D A meromorphic function f of lower order infinity is said to be of weak type $\bar{\tau}_f$ if the integral $\int_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp\left(r^{\bar{\lambda}_f}\right)\right]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \bar{\tau}_f$ and diverges for $k < \bar{\tau}_f$.

In this chapter we establish the equivalence of Definition 7.1.A and Definition 7.1.C with Definition 7.1.B and Definition &7.1.D respectively. Further we duduce the relationship between the respective weak types of an entire function f and that of its kth derivative for $k = 0, 1, 2, 3, \ldots$ Some growth properties of Nevanlinna's characteristic function of composite meromorphic and entire functions with that of their left and right factors in terms of their weak types are also established in this chapter with examples. Also in the chapter, using the concept of weak type we establish some results related to the growth properties of composite entire functions.

7.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 7.2.1 *Let the integral* $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k+1}} dr \ (r_0 > 0)$ *converges for* $0 < k < \infty$. *Then*

$$\liminf_{r\to\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k}} = 0.$$

Proof. Since the integral $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k+1}} dr \ (r_0 > 0)$ is convergent for $0 < k < \infty$, given $\epsilon(> 0)$ there exists a number $R = R(\varepsilon)$ such that

$$\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k+1}} dr < \varepsilon \text{ for } r_0 > R$$

i.e., for $r_0 > R$,

$$\int_{r_0}^{r_0+\exp\{(\log r)^{\lambda_f^*}\}} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k+1}} dr < \varepsilon.$$

As $\exp\{T(r,f)\}$ is an increasing function of r, so

$$\begin{aligned} \int_{r_0}^{r_0+\exp\{(\log r)^{\lambda_f^*}\}} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k+1}} dr &\geq \frac{\exp\{T(r_0,f)\}}{\left[\exp\{(\log r_0)^{\lambda_f^*}\}\right]^{k+1}} . \left[\exp\{(\log r_0)^{\lambda_f^*}\}\right] \\ &= \frac{\exp\{T(r_0,f)\}}{\left[\exp\{(\log r_0)^{\lambda_f^*}\}\right]^{k}} \end{aligned}$$

$$i.e., \frac{\exp\{T(r_0,f)\}}{\left[\exp\{(\log r_0)^{\lambda_f^*}\}\right]^{k}} < \varepsilon \text{ for } r_0 > R,$$

from which it follows that

$$\liminf_{r\to\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k}} = 0.$$

This proves the lemma. ■

Lemma 7.2.2 *If the integral* $\int\limits_{r_0}^{\infty}\frac{T(r,f)}{\left[\exp\left(r^{\bar{\lambda}_f}\right)\right]^{k+1}}dr\ (r_0>0)$ *is convergent for* $0<k<\infty$*, then*

$$\liminf_{r\to\infty}\frac{T(r,f)}{\left[\exp\left(r^{\bar{\lambda}_f}\right)\right]^{k}}=0.$$

Proof. Since the integral $\int\limits_{r_0}^{\infty}\frac{T(r,f)}{\left[\exp\left(r^{\bar{\lambda}_f}\right)\right]^{k+1}}dr$ converges for $0<k<\infty$, given $\epsilon(>0)$ there exists a number $R=R(\epsilon)$ such that

$$\int\limits_{r_0}^{\infty}\frac{T(r,f)}{\left[\exp\left(r^{\bar{\lambda}_f}\right)\right]^{k+1}}dr<\varepsilon \text{ for } r>R$$

$$i.e.,\ \int\limits_{r_0}^{r_0+\{\exp(r_0^{\bar{\lambda}_f})\}}\frac{T(r,f)}{\{\exp(r_0^{\bar{\lambda}_f})\}^{k+1}}dr\geq\frac{T(r_0,f)}{\{\exp(r_0^{\bar{\lambda}_f})\}^{k+1}}.\{\exp(r_0^{\bar{\lambda}_f})\}$$
$$=\frac{T(r_0,f)}{\{\exp(r_0^{\bar{\lambda}_f})\}^{k}}$$

$$i.e.,\ \frac{T(r_0,f)}{\{\exp(r_0^{\bar{\lambda}_f})\}^{k}}<\varepsilon \text{ for } r_0>R.$$

Now from above it follows that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\left[\exp\left(r^{\bar{\lambda}_f}\right)\right]^{k}}=0$$

from which the lemma follows. ■

Lemma 7.2.3 *[39] If f is a non-constant entire function, then*

$$T(r,f)\leq\log M(r,f)\leq\log T(2r,f)+o(1)$$

as $r\to\infty$.

Lemma 7.2.4 *[10]If f be and g are entire functions, for all sufficiently large values of r,*

$$M(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f)\leq M(r,f\circ g)\leq M\left(M\left(r,g\right),f\right).$$

Lemma 7.2.5 *[24]Let f be an entire function such that $0<\lambda_f<\infty$. If τ_f and $\tau_{f^{(k)}}$ be the respective weak types of f and $f^{(k)}$ then $\tau_{f^{(k)}}\leq(2^k)^{\lambda_f}\tau_f$ where $k=0,1,2,3,.....$*

Lemma 7.2.6 *[35] Let f be an entire function of finite lower order. If there exist entire functions a_i $(i=1,2,3,\cdots,n;n\leq\infty)$ satisfying $T(r,a_i)=o\{T(r,f)\}$ and $\sum\limits_{i=1}^{n}\delta(a_i,f)=1$, then $\lim\limits_{r\to\infty}\frac{T(r,f)}{\log M(r,f)}=\frac{1}{\pi}$.*

7.3 Theorems.

In this section we present the main results of the chapter.

Theorem 7.3.1 *Let f be meromorphic with lower order zero. Also let $0<\lambda_f^*<\infty$. Then Definition 7.1.A and Definition 7.1.B are equivalent.*

Proof. Case *I.* $\tau_f^*=\infty$.
Definition 7.1.A$\Rightarrow$ **Definition** 7.1.B
As $\tau_f^*=\infty$, from Definition 7.1.A we obtain for arbitrary positive G and for all sufficiently large values of r that

$$T(r,f)>G(\log r)^{\lambda_f^*}$$
$$i.e.,\ \exp[T(r,f)]>\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^G. \tag{7.1}$$

If possible, let the integral $\int\limits_{r_0}^{\infty}\frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{G+1}}dr$ $(r_0>0)$ be converge.
Then by Lemma 7.2.1,

$$\liminf_{r\to\infty}\frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^G}=0.$$

So for a sequence of values of r tending to infinity that

$$\exp\{T(r,f)\}<\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{G+1}. \tag{7.2}$$

Now from (7.1) and (7.2) we arrive at a contradiction.

Hence $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{G+1}} dr$ $(r_0 > 0)$ diverges whenever G is finite, which is Definition 7.1.B.

Definition 7.1.B$\Rightarrow$ **Definition** 7.1.A.

Let G be any positive number. Since $\tau_f^* = \infty$, from Definition 7.1.B the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{G+1}} dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for all sufficiently large values of r,

$$\exp\{T(r,f)\} > \left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{G-\varepsilon}$$
$$i.e.,\ T(r,f) > (G-\varepsilon)(\log r)^{\lambda_f^*}.$$

This gives that

$$\liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^{\lambda_f^*}} \geq (G-\varepsilon).$$

Since $G > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^{\lambda_f^*}} = \infty.$$

Thus Definition 7.1.A follows.

Case $II.$ $0 \leq \tau_f^* < \infty$.

Definition 7.1.A$\Rightarrow$ **Definition** 7.1.B.

Subcase (a).

Let f be of weak type τ_f^* where $0 < \tau_f^* < \infty$. Then for arbitrary positive $\varepsilon(> 0)$ and for a sequence of values of r tending to infinity that

$$\frac{T(r,f)}{(\log r)^{\lambda_f^*}} < \tau_f^* + \varepsilon$$
$$i.e.,\ T(r,f) < (\tau_f^* + \varepsilon)(\log r)^{\lambda_f^*}$$
$$i.e.,\ \exp\{T(r,f)\} < \exp\left\{(\tau_f^* + \varepsilon)(\log r)^{\lambda_f^*}\right\}$$
$$i.e.,\ \exp\{T(r,f)\} < [\exp\left\{(\log r)^{\lambda_f^*}\right\}]^{\tau_f^*+\varepsilon}$$
$$i.e.,\ \frac{\exp\{T(r,f)\}}{[\exp\left\{(\log r)^{\lambda_f^*}\right\}]^k} < \frac{[\exp\left\{(\log r)^{\lambda_f^*}\right\}]^{\tau_f^*+\varepsilon}}{[\exp\left\{(\log r)^{\lambda_f^*}\right\}]^k}$$
$$i.e.,\ \frac{\exp\{T(r,f)\}}{[\exp\left\{(\log r)^{\lambda_f^*}\right\}]^k} < \frac{1}{[\exp\left\{(\log r)^{\lambda_f^*}\right\}]^{k-(\tau_f^*+\varepsilon)}}.$$

Therefore $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{[\exp\{(\log r)^{\lambda_f^*}\}]^{k+1}} dr \; (r_0 > 0)$ converges for $k > \tau_f^*$ and diverges if $k < \tau_f^*$.

Subcase (b).

When f is of weak type $\tau_f^* (= 0)$, Definition 7.1.A gives for a sequence of values of r tending to infinity that

$$\frac{T(r,f)}{\{(\log r)^{\lambda_f^*}\}} < \varepsilon.$$

Then as before we obtain that $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{k+1}} dr \; (r_0 > 0)$ converges for $k > 0$ and diverges for $k < 0$.

Thus combining Subcase (a) and Subcase (b), Definition 7.1.B follows.

Definition 7.1.B$\Rightarrow$ **Definition** 7.1.A

Since f is of weak type τ_f^*, by Definition 7.1.B, for arbitrary positive $\varepsilon (> 0)$, the integral $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1+\epsilon)}} dr$ converges.

Then by Lemma 7.2.1,

$$\liminf_{r \to \infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+\epsilon)}} = 0.$$

i.e., for a sequence of values of r tending to infinity that

$$\frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+\epsilon)}} < \varepsilon$$

$$i.e., \; \exp[T(r,f)] < \varepsilon . \left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+\epsilon)}$$

$$i.e., \; T(r,f) < \log \varepsilon + \left(\tau_f^* + \varepsilon\right) (\log r)^{\lambda_f^*}$$

$$i.e., \; \frac{T(r,f)}{(\log r)^{\lambda_f^*}} < \frac{\log \varepsilon}{(\log r)^{\lambda_f^*}} + (\tau_f^* + \varepsilon)$$

$$i.e., \; \liminf_{r \to \infty} \frac{T(r,f)}{(\log r)^{\lambda_f^*}} \leq \tau_f^* + \varepsilon.$$

Since $\varepsilon\, (> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{T(r,f)}{(\log r)^{\lambda_f^*}} \leq \tau_f^*. \tag{7.3}$$

Again by Definition 7.1.B, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1-\epsilon)}} dr \; (r_0 > 0)$$

implies that for all sufficiently large values of r,

$$\frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1-\epsilon)}} > \frac{1}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(1+\epsilon)}}$$

$$i.e., \exp\{T(r,f)\} > \left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*-2\epsilon)}$$
$$i.e., \; T(r,f) > \left(\tau_f^* - 2\varepsilon\right)(\log r)^{\lambda_f^*}$$
$$i.e., \frac{T(r,f)}{(\log r)^{\lambda_f^*}} > \left(\tau_f^* - 2\varepsilon\right).$$

As $\varepsilon\,(>0)$ is arbitrary, we get that

$$\liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^{\lambda_f^*}} \geq \tau_f^*. \tag{7.4}$$

So from (7.3) and (7.4) it follows that

$$\liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^{\lambda_f^*}} = \tau_f^*.$$

Thus we obtain Definition 7.1.A.
Now combining Case I and Case II, the theorem follows. ■

Theorem 7.3.2 *The integral* $\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1)}} dr$ $(r_0 > 0)$ *converges if and only if the integral* $\int_{r_0}^{\infty} \frac{M(r,f)}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1)}} dr$ $(r_0 > 0)$ *converges.*

Proof. Let the integral $\int_{r_0}^{\infty} \frac{M(r,f)}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{\tau_f^*+1}} dr$ $(r_0 > 0)$ converges. Then by the first part of Lemma 7.2.3 we obtain that

$$\int_{r_0}^{\infty} \frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1)}} dr \leq \int_{r_0}^{\infty} \frac{M(r,f)}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1)}} dr$$

i.e., $\int\limits_{r_0}^{\infty}\frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{\tau_f^*+1}}dr$ converges.

Next let $\int\limits_{r_0}^{\infty}\frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{\tau_f^*+1}}dr$ $(r_0 > 0)$be convergent. Then by the second part of Lemma 7.2.3 we get that

$$\begin{aligned}
&\int\limits_{r_0}^{\infty}\frac{M(r,f)}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1)}}dr \\
&\leq \int\limits_{r_0}^{\infty}\frac{\exp\{T(2r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1)}}dr + \int\limits_{r_0}^{\infty}\frac{o(1)}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{(\tau_f^*+1)}}dr \\
&= \frac{1}{2[\exp(\frac{1}{2}\lambda_f^*)]}\int\limits_{r_0}^{\infty}\frac{\exp\{T(r,f)\}}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{\tau_f^*+1}}dr + o(1).
\end{aligned}$$

Thus,

$$\int\limits_{r_0}^{\infty}\frac{M(r,f)}{\left[\exp\left\{(\log)^{\lambda_f^*}\right\}\right]^{(\tau_f^*+1)}}dr \quad (r_0 > 0)$$

is convergent. This proves the theorem. ■

Now in veiw of Theorem 7.3.1 and Theorem7.3.2, we may give an alternative definition of weak type τ_f^* of an entire function f with lower order zero as follows:

An entire function f with lower order zero is said to be of weak type τ_f^* if the integral $\int\limits_{r_0}^{\infty}\frac{M(r,f)}{\left[\exp\{(\log r)^{\lambda_f^*}\}\right]^{\tau_f^*+1}}dr$ $(r_0 > 0)$ converges for $k > \tau_f^*$ and diverges for $k < \tau_f^*$

Theorem 7.3.3 *If f be a meromorphic function of infinite lower order and $0 < \overline{\lambda}_f < \infty$. Then Definition 7.1.C and Definition7.1.D are equivalent.*

Proof. Case *I.* $\overline{\tau}_f = \infty$.

Definition 7.1.C⇒ **Definition** 7.1.D..

As $\overline{\tau}_f = \infty$, from Definition 7.1.C we obtain for arbitrary positive G and for all sufficiently large values of r that

$$\log T(r,f) > G(r^{\bar{\lambda}_f})$$
$$i.e.,\ T(r,f) > \left[\exp(r^{\bar{\lambda}_f})\right]^G. \tag{7.5}$$

If possible, let the integral $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{G+1}} dr$ $(r_0 > 0)$ be converge.

Then by Lemma 7.2.2,

$$\liminf_{r\to\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^G} = 0.$$

So for a sequence of values of r,

$$T(r,f) < \left[\exp(r^{\bar{\lambda}_f})\right]^{G+1}. \tag{7.6}$$

Now from (7.5) and (7.6), we arrive at a contradiction.
Hence $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{G+1}} dr$ $(r_0 > 0)$ diverges whenever G is finite, which is Definition 7.1.D.

Definition 7.1.D$\Rightarrow$ **Definition** 7.1.C.

Let G be any positive number. Since $\bar{\tau}_f = \infty$, from Definition 7.1.D the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{G+1}} dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for all sufficiently large values of r that

$$T(r,f) > \left[\exp(r^{\bar{\lambda}_f})\right]^{G-\varepsilon}$$
$$i.e.,\ \log T(r,f) > (G-\varepsilon)\, r^{\bar{\lambda}_f}.$$

This gives that

$$\liminf_{r\to\infty} \frac{\log T(r,f)}{r^{\bar{\lambda}_f}} \geq (G-\varepsilon).$$

Since G is arbitrary, this shows that

$$\liminf_{r\to\infty}\frac{\log T(r,f)}{r^{\bar{\lambda}_f}}=\infty.$$

Thus Definition 7.1.C follows.

Case *II*. $0\le\bar{\tau}_f<\infty$.
Definition 7.1.C$\Rightarrow$ **Definition** 7.1.D.
Let G be any positive number. Since $\bar{\tau}_f=\infty$, from Definition 7.1.D the divergence of the integral $\int\limits_{r_0}^{\infty}\frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{G+1}}dr$ $(r_0>0)$ gives for arbitrary positive ε and for all large values of r tending to infinity,

$$T(r,f)>\left[\exp(r^{\bar{\lambda}_f})\right]^{G-\varepsilon}$$
$$i.e.,\ \log T(r,f)>(G-\varepsilon)\,r^{\bar{\lambda}_f}.$$

This gives that

$$\liminf_{r\to\infty}\frac{\log T(r,f)}{r^{\bar{\lambda}_f}}\ge(G-\varepsilon).$$

Since G is arbitrary, this shows that

$$\liminf_{r\to\infty}\frac{\log T(r,f)}{r^{\bar{\lambda}_f}}=\infty.$$

Thus Definition 7.1.C follows.
Case *II*. $0\le\bar{\tau}_f<\infty$.
Definition 7.1.C$\Rightarrow$ **Definition** 7.1.D.
Subcase (a).
Let f be of weak type $\bar{\tau}_f$ where $0\le\bar{\tau}_f<\infty$. Then for arbitrary positive ε

and for a sequence of values of r tending to infinity that

$$\log T(r,f) < (\bar{\tau}_f + \varepsilon) r^{\bar{\lambda}_f}$$

$$i.e.,\ T(r,f) < \exp[(\bar{\tau}_f + \varepsilon) r^{\bar{\lambda}_f}]$$

$$i.e.,\ T(r,f) < [\exp(r^{\bar{\lambda}_f})]^{(\bar{\tau}_f + \varepsilon)}$$

$$i.e., \frac{T(r,f)}{[\exp(r^{\bar{\lambda}_f})]^{k'}} < \frac{[\exp(r^{\bar{\lambda}_f})]^{\bar{\tau}_f + \varepsilon}}{[\exp(r^{\bar{\lambda}_f})]^{k'}}$$

$$i.e., \frac{T(r,f)}{[\exp(r^{\bar{\lambda}_f})]^{k'}} < \frac{1}{[\exp(r^{\bar{\lambda}_f})]^{k' - (\bar{\tau}_f + \varepsilon)}}.$$

Therefore, $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{[\exp(r^{\bar{\lambda}_f})]^{k'}} dr$ $(r_0 > 0)$ converges if $k' > \bar{\tau}_f$ and diverges if $k' < \bar{\tau}_f$.

i.e., $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{[\exp(r^{\bar{\lambda}_f})]^{k'+1}} dr$ $(r_0 > 0)$ converges if $k' > \bar{\tau}_f$ and diverges if $k' < \bar{\tau}_f$.

Subcase (b).

When f is of weak type $\bar{\tau}_f = 0$, Definition 7.1.C gives for a sequence of values of r tending to infinity,

$$\frac{\log T(r,f)}{r^{\bar{\lambda}_f}} < \varepsilon.$$

Then as before, we obtain that $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{k'+1}} dr$ $(r_0 > 0)$ converges for $k' > 0$ and diverges for $k' < 0$.

Thus combining Subcase (a) and Subcase (b), Definition 7.1.D follows.

Definition 7.1.D$\Rightarrow$ **Definition** 7.1.C.

Since f is of weak type $\bar{\tau}_f$, by Definition 7.1.D for arbitrary positive $\varepsilon(> 0)$, the integral $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f + 1 + \epsilon}} dr$ converges

Then by Lemma 7.2.2, we obtain that

$$\liminf_{r\to\infty}\frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau_f}+\epsilon)}}=0,$$

i.e., for a sequence of values of r tending to infinity that

$$\frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau_f}+\epsilon)}}<\varepsilon$$

$$\begin{aligned}
i.e.,\ T(r,f) &< \varepsilon.\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau_f}+\epsilon)} \\
i.e., \log\ T(r,f) &< \log\varepsilon+\left(\bar{\tau_f}+\varepsilon\right)r^{\bar{\lambda}_f} \\
i.e.,\ \frac{\log T(r,f)}{r^{\bar{\lambda}_f}} &< \frac{\log\varepsilon}{r^{\bar{\lambda}_f}}+(\bar{\tau_f}+\varepsilon) \\
i.e.,\ \liminf_{r\to\infty}\frac{\log T(r,f)}{r^{\bar{\lambda}_f}} &\leq (\bar{\tau_f}+\varepsilon).
\end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log T(r,f)}{r^{\bar{\lambda}_f}}\leq\bar{\tau_f}. \tag{7.7}$$

Again by Definition 7.1.D, for arbitrary positive ε the divergence of the integral $\int_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f+1-\epsilon)}} dr$ implies that for all sufficiently large values of r,

$$\frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f+1-\epsilon)}} > \frac{1}{\left[\exp(r^{\bar{\lambda}_f})\right]^{1+\epsilon}}$$

$$i.e., T(r,f) > \left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f-2\epsilon)}$$

$$i.e., \log T(r,f) > \left(\bar{\tau}_f - 2\varepsilon\right) r^{\bar{\lambda}_f}$$

$$i.e., \frac{\log T(r,f)}{r^{\bar{\lambda}_f}} > \left(\bar{\tau}_f - 2\varepsilon\right)$$

$$i.e., \liminf_{r\to\infty} \frac{\log T(r,f)}{r^{\bar{\lambda}_f}} \geq \left(\bar{\tau}_f - 2\varepsilon\right).$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty} \frac{\log T(r,f)}{r^{\bar{\lambda}_f}} \geq \bar{\tau}_f. \tag{7.8}$$

Now from (7.7) and (7.8) it follows that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\bar{\lambda}_f}} = \bar{\tau}_f.$$

Thus we get Definition 7.1.C.

Hence combining Case I and Case II, the theorem follows. ■

Theorem 7.3.4 *The integral* $\int_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f+1}} dr$ $(r_0 > 0)$ *converges if and only if the integral* $\int_{r_0}^{\infty} \frac{\log M(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f+1)}} dr$ $(r_0 > 0)$ *converges.*

Proof. Let $\int\limits_{r_0}^{\infty} \frac{\log M(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f+1)}} dr$ $(r_0 > 0)$ be convergent. Then by the first part of Lemma 7.2.3 we obtain that

$$\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f+1}} dr \leq \int\limits_{r_0}^{\infty} \frac{\log M(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f+1)}} dr$$

i.e., $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f+1}} dr$ $(r_0 > 0)$ converges.

Next, let $\int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f+1}} dr$ $(r_0 > 0)$ convergent. Then by the second part of Lemma 7.2.3 we get that

$$\int\limits_{r_0}^{\infty} \frac{\log M(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f+1)}} dr$$

$$\leq \int\limits_{r_0}^{\infty} \frac{T(2r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f+1}} dr + \int\limits_{r_0}^{\infty} \frac{o(1)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f+1}} dr$$

$$= \frac{1}{2[\exp(\frac{1}{2}\bar{\lambda}_f)]} \int\limits_{r_0}^{\infty} \frac{T(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{\bar{\tau}_f+1}} dr + o(1).$$

Thus $\int\limits_{r_0}^{\infty} \frac{\log M(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{(\bar{\tau}_f+1)}} dr$ $(r_0 > 0)$ is convergent.

This proves the theorem. ■

Now in veiw of Theorem 7.3.3 and Theorem 7.3.4, we may give an alternative definition of the weak type $\bar{\tau}_f$ of an entire function f with infinite order as follows :

An entire function f with infinite order is said to be of weak type $\bar{\tau}_f$ if the integral $\int_{r_0}^{\infty} \frac{\log M(r,f)}{\left[\exp(r^{\bar{\lambda}_f})\right]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \bar{\tau}_f$ and diverges for $k < \bar{\tau}_f$.

In the subsequent theorems we establish some growth properties of composite entire and meromorphic functions on the basis of their weak types.

Theorem 7.3.5 *Let f be an entire function such that $0 < \lambda_f < \infty$. If τ_f and $\tau_{f^{(k)}}$ be the respective weak types of f and $f^{(k)}$ then $\tau_{f^{(k)}} \le (2^k)^{\lambda_f} \tau_f$ where $k = 0, 1, 2, 3, \ldots..$*

Proof. It is known from G.Valiron {[**64**], p.35} that

$$\frac{1}{r}\{M(r,f) - |f(0)|\} \le M(r,f) \le \frac{1}{r} M(2r,f).$$

Noting that $\lambda_{f^{(k)}} = \lambda_f$ we get from the second part of the inequality for $r \ge 1$,

$$M(r, f^{(k)}) \le M(2^k r, f)$$

$$i.e., \frac{\log M(r, f^{(k)})}{r^{\lambda_{f^{(k)}}}} \le \frac{\log M(2^k r, f)}{(2^k r)^{\lambda_f}} . (2^k)^{\lambda_f}$$

$$i.e., \liminf_{r\to\infty} \frac{\log M(r, f^{(k)})}{r^{\lambda_{f^{(k)}}}} \le (2^k)^{\lambda_f} \liminf_{r\to\infty} \frac{\log M(2^k r, f)}{(2^k r)^{\lambda_f}}$$

$$i.e., \tau_{f^{(k)}} \le (2^k)^{\lambda_f} \tau_f,$$

which proves the theorem. ■

Theorem 7.3.6 *Let f be meromorphic and g be entire such that (i) $0 < \lambda_g < \infty$, (ii) $0 < \tau_g < \infty$, (iii) $\tau_{fog} = \tau_g$ and (iv) $0 < \lambda_{fog} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{T(r, f \circ g)}{T(r,g)} \le \frac{\tau_{fog}}{\tau_g} \le \limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r,g)} .$$

Proof. From the definition of weak type of a composite meromorphic function we have for arbitrary positive ϵ and for a sequence of values of r tending to infinity that

$$T(r, f \circ g) \le (\tau_{fog} + \varepsilon) r^{\lambda_{fog}} . \tag{7.9}$$

Also for all large values of r,

$$T(r,g) \ge (\tau_g - \varepsilon) r^{\lambda_g} . \tag{7.10}$$

As $\lambda_{fog} = \lambda_g$ from (7.9) and (7.10) it follows for a sequence of values of r tending to infinity,

$$\frac{T(r, f \circ g)}{T(r, g)} \le \frac{(\tau_{fog} + \varepsilon)}{(\tau_g - \varepsilon)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{T(r, f \circ g)}{T(r, g)} \le \frac{\tau_{fog}}{\tau_g}\,. \tag{7.11}$$

Again for all sufficiently large values of r,

$$T(r, f \circ g) \ge (\tau_{fog} - \varepsilon)\, r^{\lambda_{fog}} \tag{7.12}$$

and for a sequence of values of r tending to infinity that

$$T(r, g) \le (\tau_g + \varepsilon)\, r^{\lambda_g}\,. \tag{7.13}$$

By condition *(iii)* we obtain from (7.12) and (7.13) for a sequence of values of r tending to infinity that

$$\frac{T(r, f \circ g)}{T(r, g)} \ge \frac{(\tau_{fog} - \varepsilon)}{(\tau_g + \varepsilon)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get that

$$\limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r, g)} \ge \frac{\tau_{fog}}{\tau_g}\,. \tag{7.14}$$

Thus the theorem follows from (7.11) and (7.14). ■

Remark 7.3.1 *The sign '$\le$' in Theorem* 7.3.6 *cannot be replaced by '$<$' only as we see in the following example.*

Example 7.3.1 *Let $f = z$ and $g = \exp z$. Then $\lambda_g = \lambda_{fog} = \tau_g = \tau_{fog} = 1$. Also $T(r, f \circ g) = T(r, g) = \frac{r}{\pi}$.*
Therefore

$$\liminf_{r\to\infty} \frac{T(r, f \circ g)}{T(r, g)} = \liminf_{r\to\infty} \frac{\frac{r}{\pi}}{\frac{r}{\pi}} = 1 = \frac{\tau_{fog}}{\tau_g}$$
$$= \limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r, g)}.$$

Remark 7.3.2 *If f and g be both entire and the other conditions remain the same then the conclusion of Theorem* 7.3.6 *is still valid with $T(r, f \circ g)$ and $T(r, g)$ respectively replaced by* $\log M(r, f \circ g)$ *and* $\log M(r, g)$ *as we see in the following theorem.*

Theorem 7.3.7 *Let f and g be two entire functions such that* *(i)* $0 < \lambda_g < \infty$, *(ii)* $0 < \tau_g < \infty$, *(iii)* $\tau_{fog} = \tau_g$ *and (iv)* $0 < \lambda_{fog} < \infty$. *Then*

$$\liminf_{r\to\infty} \frac{\log M(r, f \circ g)}{\log M(r, g)} \leq \frac{\tau_{fog}}{\tau_g} \leq \limsup_{r\to\infty} \frac{\log M(r, f \circ g)}{\log M(r, g)}.$$

Proof. From the definition of weak type of a composite meromorphic function we have for arbitrary positive ϵ and for a sequence of values of r tending to infinity that

$$\log M(r, f \circ g) \leq (\tau_{fog} + \varepsilon) r^{\lambda_{fog}} . \tag{7.15}$$

Also for all large values of r,

$$\log M(r, g) \geq (\tau_g - \varepsilon) r^{\lambda_g} . \tag{7.16}$$

As $\lambda_{fog} = \lambda_g$ from (7.15) and (7.16) it follows for a sequence of values of r tending to infinity,

$$\frac{\log M(r, f \circ g)}{\log M(r, g)} \leq \frac{(\tau_{fog} + \varepsilon)}{(\tau_g - \varepsilon)}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log M(r, f \circ g)}{\log M(r, g)} \leq \frac{\tau_{fog}}{\tau_g} . \tag{7.17}$$

Again for all sufficiently large values of r,

$$\log M(r, f \circ g) \geq (\tau_{fog} - \varepsilon) r^{\lambda_{fog}} \tag{7.18}$$

and for a sequence of values of r tending to infinity that

$$\log M(r, g) \leq (\tau_g + \varepsilon) r^{\lambda_g} . \tag{7.19}$$

By condition (*iii*) we obtain from (7.18) and (7.19) for a sequence of values of r tending to infinity that

$$\frac{\log M(r, f \circ g)}{\log M(r, g)} \geq \frac{(\tau_{fog} - \varepsilon)}{(\tau_g + \varepsilon)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we get that

$$\limsup_{r\to\infty}\frac{\log M\,(r,f\circ g)}{\log M\,(r,g)}\geq\frac{\tau_{fog}}{\tau_g}\,. \tag{7.20}$$

Thus the theorem follows from (7.17) and (7.20). ∎

Remark 7.3.3 *The sign '$\leq$' in Theorem 7.3.7 cannot be replaced by '$<$' only which is evident from the following example.*

Example 7.3.2 *Let $f=z$ and $g=\exp z$. Then $\lambda_g=\lambda_{fog}=\tau_g=\tau_{fog}=1$. Also $\log M\,(r,f\circ g)=\log M\,(r,g)=\log(\exp r)=r$.*
Therefore

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log M\,(r,f\circ g)}{\log M\,(r,g)}&=\liminf_{r\to\infty}\frac{r}{r}=1=\frac{\tau_{fog}}{\tau_g}\\&=\limsup_{r\to\infty}\frac{\log M\,(r,f\circ g)}{\log M\,(r,g)}.\end{aligned}$$

Theorem 7.3.8 *Let f and g be two non-constant entire functions satisfying (i) $0<\lambda_f\leq\rho_f<\infty$, (ii) $0<\lambda_g<\infty$, (iii) $\lambda_f=\lambda_g$, (iv) $\tau_f>0$ and (v) $\tau_g<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log T\,(r,f\circ g)}{T\,(r,f)}\leq\rho_f.\frac{\tau_g}{\tau_f}=\rho_g.\frac{\tau_g}{\tau_f}.$$

Proof. It is well known that [29] for an entire function f and for $r>0$,

$$T(r,f)\leq\log^{+}M(r,f). \tag{7.21}$$

Also by the second part of Lemma 7.2.4

$$\log M(r.f\circ g)\leq\log M(M(r,g),f). \tag{7.22}$$

Now from (7.21) and (7.22) we get for all sufficiently large values of r that

$$\begin{aligned}&T\,(r,f\circ g)\leq\log M(M(r,g),f)\leq\{M(r,g)\}^{\rho_f+\epsilon}\\&i.e.,\ \frac{\log T\,(r,f\circ g)}{T\,(r,f)}\leq(\rho_f+\epsilon)\frac{\log M(r,g)}{T\,(r,f)}\\&i.e.,\ \liminf_{r\to\infty}\frac{\log T\,(r,f\circ g)}{T\,(r,f)}\leq(\rho_f+\epsilon)\liminf_{r\to\infty}\frac{\log M(r,g)}{T\,(r,f)}.\end{aligned} \tag{7.23}$$

Also for a sequence of values of r tending to infinity we have

$$\log M(r,g) < (\tau_g + \varepsilon) r^{\lambda_g} \tag{7.24}$$

and for all large values of r,

$$T(r,f) > (\tau_f - \varepsilon) r^{\lambda_f}. \tag{7.25}$$

Since $\lambda_f = \lambda_g$, therefore from (7.24) and (7.25) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log M(r,g)}{T(r,f)} < \frac{(\tau_g + \varepsilon)}{(\tau_f - \varepsilon)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,f)} \leq \frac{\tau_g}{\tau_f}. \tag{7.26}$$

Therefore from (7.23) and (7.26) we get that

$$\liminf_{r\to\infty} \frac{\log T\,(r, f\circ g)}{T\,(r,f)} \leq (\rho_f + \epsilon)\frac{\tau_g}{\tau_f}.$$

Since $\varepsilon\,(> 0)$ is arbitrary and $\lambda_f = \lambda_g$, it follows that

$$\liminf_{r\to\infty} \frac{\log T\,(r, f\circ g)}{T\,(r,f)} \leq \rho_f . \frac{\tau_g}{\tau_f} = \rho_g . \frac{\tau_g}{\tau_f}.$$

Thus the theorem is established. ■

Theorem 7.3.9 *Let f and g be two entire functions such that $0 < \lambda_f < \infty$ and $0 < \lambda_g < \infty$. Also let $0 < \tau_g < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} M\,(r, f\circ g)}{\log M\left(r, g^{(k)}\right)} \geq \frac{\lambda_f}{2^{(k+1)}\lambda_g} \text{ for } k = 1,2,3,...$$

Proof. Let $0 < \epsilon < \min\{\lambda_f, \tau_g\}$. Then for all sufficiently large values of r we obtain that

$$\log M(\frac{r}{2}, g) > (\tau_g - \epsilon)(\frac{r}{2})^{\lambda_g}. \tag{7.27}$$

Again from the first part of Lemma 7.2.5, we get for all sufficiently large values of r that

$$\log^{[2]} M\,(r, f\circ g) > (\lambda_f - \epsilon)\log\frac{1}{16} + (\lambda_f - \epsilon)\log M(\frac{r}{2}, g). \tag{7.28}$$

Now for all sufficiently large values of r it follows from (7.27) and (7.28) that

$$\log^{[2]} M\left(r, f \circ g\right) > (\lambda_f - \epsilon) \log \frac{1}{16} + (\lambda_f - \epsilon)(\tau_g - \epsilon)(\frac{r}{2})^{\lambda_g}. \qquad (7.29)$$

Again by Lemma 7.2.6, we get for a sequence of values of r tending to infinity that

$$\log M\left(r, g^{(k)}\right) < \left(\tau_g(k) + \varepsilon\right) r^{\lambda_g(k)} \leq \left((2^k)^{\lambda g}\tau_g + \varepsilon\right) r^{\lambda_g}. \qquad (7.30)$$

So from (7.29)and (7.30) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} M\left(r, f \circ g\right)}{\log M\left(r, g^{(k)}\right)} > \frac{(\lambda_f - \epsilon) \log \frac{1}{16} + (\lambda_f - \epsilon)(\tau_g - \epsilon)(\frac{r}{2})^{\lambda_g}}{\left(2^{k\lambda_g}\tau_g + \varepsilon\right) r^{\lambda_g}}. \qquad (7.31)$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get from (7.31) that

$$\limsup_{r \to \infty} \frac{\log^{[2]} M\left(r, f \circ g\right)}{\log M\left(r, g^{(k)}\right)} \geq \frac{\lambda_f}{2^{(k+1)}\lambda_g}.$$

This proves the theorem. ■

Theorem 7.3.10 *Let f and g be two entire functions with $(i)\, \rho_f = \rho_g$ and $(ii)\, 0 < \lambda_g \leq \rho_g < \infty$. Also, let there exist entire functions a_i ($i = 1, 2, 3, \cdots, n$; $n \leq \infty$) satisfying $T\left(r, a_i\right) = o\left\{T\left(r, g\right)\right\}$ and $\sum_{i=1}^{n} \delta\left(a_i, g\right) = 1$. Then*

$$\min\{\limsup_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(\exp(r^{\lambda_g}), f^{(k)})}, \limsup_{r \to \infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(\exp(r^{\lambda_g}), g^{(k)})}\} \geq (\frac{1}{2})^{\lambda_g}.\pi\tau_g$$

for $k = 0, 1, 2, 3, \ldots$.

Proof. In view of Lemma 7.2.6 and by the first part of Lemma 7.2.5, we obtain that

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(\exp(r^{\lambda_g}),f^{(k)})}\\
&\geq \limsup_{r\to\infty}\frac{\log^{[2]} M\left(\frac{1}{16}M\left(\frac{r}{2},g\right),f\right)}{\log^{[2]} M(\exp(r^{\lambda_g}),f^{(k)})}\\
&= \limsup_{r\to\infty}\frac{\log^{[2]} M\left(\frac{1}{16}M\left(\frac{r}{2},g\right),f\right)}{\log\{\frac{1}{16}M\left(\frac{r}{2},g\right)\}}\\
&\quad \lim_{r\to\infty}\frac{\log M\left(\frac{r}{2},g\right)}{T\left(\frac{r}{2},g\right)}.\liminf_{r\to\infty}\frac{T\left(\frac{r}{2},g\right)}{(\frac{r}{2})^{\lambda_g}}.\\
&\quad \liminf_{r\to\infty}\frac{(\frac{1}{2})^{\lambda_g}\log\{\exp(r^{\lambda_g})\}}{\log^{[2]} M(\exp(r^{\lambda_g}),f^{(k)})}\\
&= \frac{\rho_f.\pi\tau_g(\frac{1}{2})^{\lambda_g}}{\rho_f}=\pi\tau_g(\frac{1}{2})^{\lambda_g}. \qquad (7.32)
\end{aligned}$$

In a similar way exactly proceeding as above and in view of condition (i), we get that

$$\begin{aligned}
&\limsup_{r\to\infty}\frac{\log^{[2]} M(r,fog)}{\log^{[2]} M(\exp(r^{\lambda_g}),g^{(k)})}\\
&\geq \frac{\rho_f.\pi\tau_g(\frac{1}{2})^{\lambda_g}}{\rho_g}=\pi\tau_g(\frac{1}{2})^{\lambda_g}. \qquad (7.33)
\end{aligned}$$

Thus the theorem follows from (7.32) and (7.33). ■

$* * * * * * X * * * * * *$