CHAPTER 3

Viscous Cosmologies with Variable Λ in Higher Derivative Gravity

3.1 Introduction

The observed supernovae light curve data to Wilkinson Microwave Anisotropy Probe (WMAP) data [75] indicate that the present universe is passing through an accelerating phase. It is also known that the universe might have emerged to the present state from an inflationary phase in the past. It is known that inflation in the early universe is permitted with a cosmological constant. To accommodate the present accelerating phase of the universe one needs either to modify the Einstein's general theory of relativity (GR) or to include a time varying cosmological constant. It is also known that an accelerating universe may be obtained assuming exotic kind of fields in the gravitational theory. Such fields are not the normal fields available in the standard model of particle physics. To understand perhaps a new physics may permit exotic field which yet to come up. One of the attempts to modify GR is based on adding curvature squared terms in the Einstein-Hilbert action, which is known as generalized theory of gravity. Considerable work have been reported in the last decades in the framework of generalized theory of gravity which permits inflation to obtain early universe. Starobinsky [5] shown that higher order theories of gravity admits inflation, which was proposed before the seminal work of Guth. However, the efficacy of the theory is known only after the work of Guth [1], where a temperature dependent phase transition mechanism was used to realized inflation. Inflationary scenario of the early universe is attractive as it can solve some of the outstanding problems not understood both in particle physics and cosmology [93],[106]. It may be mentioned here that higher order theories of gravity have a number of good features. It is also known that with suitable counter terms viz., $C^{\mu\nu\rho\delta}C_{\mu\nu\rho\delta}$, R^2, and cosmological constant (Λ) added to the Einstein action, one gets a perturbation theory which is well behaved, formally renormalizable and asymptotically free [107].

In the literature [108] cosmological models with perfect fluid as a source of matter in the context of higher derivative theories are considered. It is also known that a higher derivative theory without matter admitting inflation / de Sitter solution is unstable [109, 110]. Although matter distribution in the observed universe is satisfactorily described by perfect fluid, a number of processes might have occurred in the early universe leading to viscosity. In the early universe, viscosity may arise due to many processes, e.g. the decoupling of neutrinos during the radiation era, the decoupling of matter from radiation during the recombination era, creation of superstrings during the quantum era, particle collisions involving gravitons, cosmological quantum particle creation, as well as during the formation of galaxies [18]. Therefore, it is important to consider dissipative processes and to study the role of these processes in the evolution of the universe. The effect of dissipative process in the matter distribution corresponds to deviation from perfect fluid distribution which is known as imperfect fluid. Cosmological models with imperfect fluid in the frame work of higher derivative theories of gravity are also studied [111] to explore early inflation. To accommodate the present cosmic acceleration a number of literature [112] appeared considering a modification to the Einstein-Hilbert action including a theory with a variable cosmological constant (Λ). The cosmological constant term Λ. which was originally introduced by Einstein to obtain a static model of the universe, once again has been considered by physicists in order to explain the observed universe motivated by recent observational predictions. The cosmological constant problem provoked a great interest in a variable Λ as it may be used to resolve many outstanding problems in a natural way. Recent applications of the apparent magnitude-redshift test, based on type Ia supernovae [113], strongly favours a significant and positive Λ term. Further, the age issue, structure formation, baryon excess is clusters, high redshift supernovae, and the cosmic microwave background are also in favor of a positive cosmological constant.

It is known that a dynamical cosmological constant in a gravitational theory permits an accelerating universe. A number of ansatze have been proposed for a dynamical Λ, which decays with time [114]. Berman and Som [115] pointed out that the relation $\Lambda \sim t^{-2}$ seems to play a major role in cosmology. To obtain a small value for the cosmological constant at the present epoch, a variable Λ coupled with Hubble parameter as $\Lambda = m\ H^2 + n\ \dot{H}$, where m and n are dimensionless constant, have also been considered in recent times. Cosmological models with viscosity in addition to a dynamical cosmological constant have been studied in GR [116]. In this chapter a higher order theory of gravity is taken up including a time varying cosmological parameter and viscosity, to explore the evolution of the universe. The objective of the work is to explore the effects of higher order terms in the evolution of the universe. We consider Eckart, truncated and full causal theories for describing the imperfect fluid to obtain cosmological solutions. Eckart theory [66] is the first theory in viscosity which is considered to describe viscous cosmology. But it is found that the theory suffers from serious shortcoming, viz., causality and stability [117]. It is now known that the problems may be resolved by including higher order deviation terms in the transport equation [118]. Israel and Stewart [67], and Pavon [68] developed a fully relativistic formulation of the theory taking into account second order deviation terms in the theory, which is termed extended irreversible thermodynamics (in short, *EIT*). Using the transport equation obtained from *EIT*, cosmological solutions have been obtained in Einstein gravity [19]. It may be pointed out here that the actual behavior of the various thermodynamic parameters in the theory is not fully realized as the number of unknowns is more than the number of field equations. A source of viscosity at the present time which gives rise to the observed acceleration could be the presence of dark matter [119] or a cosmic antifriction force giving rise to an effective negative pressure [120].

The chapter is organized as: in Sec. 3.2, the relevant field equations in the higher derivative theory is set up ; in Sec. 3.3, cosmological solutions are obtained with imperfect fluid described by Eckart, truncated Israel-Stewart (TIS) and full Israel-Stewart (FIS), respectively. Finally, in Sec. 3.4, a summary of the result is given.

3.2 The Field Equations in Higher Derivative Gravity

Let us consider a gravitational action which is given by

$$I = -\int \left[\frac{1}{2}f(R) + L_m\right]\sqrt{-g}\, d^4x \tag{3.1}$$

where $f(R)$ is a polynomial function of scalar curvature (R) and its higher power including a cosmological constant, g is the determinant of the four dimensional metric and L_m represents the matter Lagrangian, choosing a unit with $8\pi G = c = 1$.
Variation of the action (3.1) with respect to $g_{\mu\nu}$ yields

$$f_R(R)\, R_{\mu\nu} - \frac{1}{2}\, f(R)\, g_{\mu\nu} + f_{RR}(R)\,(\nabla_\mu\nabla_\nu R - g_{\mu\nu}\nabla^\mu\nabla^\nu R g_{\mu\nu}) +$$

$$f_{RRR}(R)\,(\nabla_\mu R\nabla_\nu R - \nabla^\sigma R\nabla_\sigma R\, g_{\mu\nu}) = -\ T_{\mu\nu} \tag{3.2}$$

where ∇_μ is covariant differential operator, $f_R(R)$ represents the derivative of $f(R)$ with respect to R and $T_{\mu\nu}$ is the energy momentum tensor for matter determined by matter Lagrangian L_m . In a flat Robertson-Walker metric given by eq. (2.4) the eq. (3.2) and its trace can be written as

$$Rf_R(R) - 2f(R)\ +\ 3f_{RR}(R)\left(\ddot{R} + 3\frac{\dot{a}}{a}\dot{R}\right) + 3f_{RRR}(R)\dot{R} + T = 0, \tag{3.3}$$

$$f_R(R)\, R_{00} + \frac{1}{2}f(R) - 3f_{RR}(R)\frac{\dot{a}}{a}\dot{R} + T_{00} = 0. \tag{3.4}$$

Now we consider a higher derivative theory of gravity described by $f(R) = R+\alpha R^2 - 2\Lambda(t)$. Using eq. (2.5) in eqs. (3.3) and (3.4), we get

$$H^2 - 6\alpha\left[2H\ddot{H} - \dot{H}^2 + 6\dot{H}H^2\right] = \frac{\rho}{3} + \frac{\Lambda(t)}{3}, \tag{3.5}$$

the conservation equation is given by

$$\dot{\rho} + 3(\rho + p)H = -\dot{\Lambda}, \tag{3.6}$$

where ρ is the energy density and p is the pressure of the perfect fluid. It may be pointed out here that eqs. (3.5) and (3.6) are the set of equations used to study cosmological models with a perfect fluid using a time varying Λ. Now the effect of viscosity, is to be included for which the perfect fluid pressure in eq. (3.6) is replaced by an effective pressure p_{eff}, which is given by

$$p_{eff} = p + \Pi, \tag{3.7}$$

where Π is the bulk viscous stress. In *EIT*, the transport equation for Π is given by

$$\Pi + \tau\dot{\Pi} = -3\zeta H - \frac{\epsilon}{2}\tau\Pi\left[3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T}\right] \tag{3.8}$$

where ζ is the coefficient of bulk viscosity, τ is the relaxation coefficient for transient bulk viscous effects and T is the absolute temperature of the universe. The parameter ϵ takes the value 0 or 1, the former represents the truncated Israel-Stewart causal theory and the latter represents full causal theory. One recovers the non-causal Eckart theory for $\tau = 0$. Incorporating viscosity the conservation eq. (3.6) can be written as :

$$\dot{\rho} + 3(\rho + p + \Pi)H = -\dot{\Lambda}. \tag{3.9}$$

For a constant Λ and $\Pi = 0$, eq. (3.9) leads to a continuity equation that corresponds to a perfect fluid. The above equation shows that, in order to satisfy the energy conservation, a decaying vacuum term and viscosity necessarily lead to matter production. We consider an equation of state for the isotropic fluid, which is given by

$$p = (\gamma - 1)\rho \tag{3.10}$$

where γ $(1 \leq \gamma \leq 2)$ is a constant. Equation (3.9) is employed to determine the bulk viscous stress.

3.3 Cosmological Solutions in the presence of Viscosity

The system of eqs. (3.5) and (3.8)-(3.10) is employed to obtain cosmological solutions. The system of equations is not closed as it has seven unknowns ρ, γ, τ, ζ, Λ, $a(t)$, T to be determined from the equations. We assume the following widely accepted *ad hoc* relations:

$$\zeta = \beta\rho^s, \quad \tau = \beta\rho^{s-1} \tag{3.11}$$

where $\zeta \geq 0$, $\tau \geq 0$, $\beta \geq 0$ and $s \geq 0$. We assume further a dynamical cosmological constant given by $\Lambda = mH^2 + n\dot{H}$ where m and n are dimensionless constants to solve the equations. We determine time variation of Λ in the next sections, admitting a viable cosmological scenario.

3.3.1 *Eckart theory ($\tau = 0$)*

In this case, the transport eq. (3.8) reduces to

$$\Pi = -3\,\zeta\,H. \tag{3.12}$$

We now use eq. (3.12) to obtain cosmological solutions from eqs. (3.5), (3.9) and (3.10). For simplicity we first consider power-law expansion.

1. Power-law model: We choose a power-law expansion of the universe,

$$a(t) = a_0\,t^D, \tag{3.13}$$

where a_0 and D are constants which are to be determined from the field equations. The energy density is given by

$$\rho = \rho_{01}\,t^{-2} + \rho_{02}\,t^{-4} \tag{3.14}$$

where $\rho_{01} = ((3-m)D+n)D$ and $\rho_{02} = 54\alpha D^2(2D-1)$. The coefficient of bulk viscosity is determined from eq. (3.12), which is

$$\zeta = \zeta_{01}\,t^{-1} + \zeta_{02}\,t^{-3} \tag{3.15}$$

where $\zeta_{01} = \frac{1}{3}(n\gamma - 2 + D\gamma(3-m))$ and $\zeta_{02} = 6\alpha(2D-1)(3\gamma D-4)$.

We note the following:

(i) $D = \frac{1}{2}$, the bulk viscous coefficient and the cosmological constant become

$$\zeta = \frac{(3\gamma - 4 - \lambda_0\gamma)}{6t}, \quad \Lambda = \frac{\lambda_0}{t^2} \tag{3.16}$$

where $\lambda_0 = \frac{m-2n}{4}$. A viscous universe evolves as $a(t) \sim \sqrt{t}$ when $1 \leq \gamma \leq 2$ with $-1 \leq (m-2n) \leq 1$. However, a universe with both radiation and viscosity admits the solution for $m < 2n$.

(ii) $D = \frac{4}{3\gamma}$, the bulk viscous coefficient and the cosmological constant are

$$\zeta = \frac{1}{3t}\left[2 + \frac{3\lambda_0\gamma^2}{4}\right], \quad \Lambda = \frac{\lambda_0}{t^2}. \tag{3.17}$$

where $\lambda_0 = \frac{4(4m-3\gamma n)}{9\gamma^2}$. In this case the following point are noted :

• for stiff fluid $\gamma = 2$, $m \leq \frac{3n+3}{2}$ yielding $a(t) \sim t^{2/3}$,

• for radiation $\gamma = \frac{4}{3}$, $m \leq n + \frac{3}{2}$ yielding $a(t) \sim t$ admitting expansion faster than the usual expansion in GR,

• for matter dominated epoch $\gamma = 1$ and in the presence of viscosity, the scale factor of the universe evolves as $a(t) \sim t^{\frac{4}{3}}$ with $m \leq \frac{6+3n}{4}$ which admits an accelerating universe. The latter solution obtained in the case of a matter dominated universe is new, and interesting as it accommodates present *accelerating phase* of the universe.

(iii) $D = \frac{n-2}{m-3}$, the bulk viscous coefficient and the cosmological constant evolve as

$$\zeta = \frac{6\alpha(m-2n+1)(4m-3n-6)}{(3-m)^2 t^3}, \quad \Lambda = \frac{(2-n)(2m-3n)}{(3-m)^2 t^2} \tag{3.18}$$

during matter domination stage. The scale factor of the universe evolves as $a(t) \sim t^{\frac{n-2}{m-3}}$, which admits an accelerating universe if $m-n < 1$, with a positive α. It may be mentioned here that cosmological solution with positive cosmological constant may be obtained for $n > 2$.

(iv) The forms of Λ in eqs. (3.16)-(3.18) are compatible with the Landau-Lifshitz theory of non equilibrium fluctuations and hence with cosmic observations [68]. In addition, as pointed out in the introduction, such a variation is required to solve the cosmological constant problem.

2. **Exponential model** : The field eqs (3.5), (3.9), (3.10) and (3.12) now admit an exponential solution where Hubble parameter satisfies a first order differential equation given by

$$\dot{H} = \frac{1}{\sqrt{6\alpha}} H - \frac{3}{2} H^2. \tag{3.19}$$

On integrating eq. (3.19), which is given by

$$a(t)_{\pm} = \left[a_1 \pm a_2 \, e^{\left(\frac{t}{\sqrt{6\alpha}}\right)} \right]^{\frac{2}{3}}. \tag{3.20}$$

The solution has a number of good features. It may be employed to construct an emergent universe scenario from a static state. Near $t \sim 0$ the universe is matter dominated, which thereafter leads to an accelerated phase. It has no singularity for a_+ and the universe originated from a static state in the past.

3.3.2 *Truncated Israel-Stewart theory* ($\epsilon = 0$)

In this case, the transport eq. (3.8) becomes

$$\Pi + \tau\dot{\Pi} = -3\zeta H. \tag{3.21}$$

1. Power- law model: We consider a power law model given by eq. (3.13), i.e., $a = a_0\, t^D$. In this case the energy density and bulk viscous stress become

$$\rho = \rho_1 t^{-2}, \qquad \Pi = \Pi_1 t^{-2} \tag{3.22}$$

where $\rho_1 = \frac{3-m+2n}{4}$ and $\Pi_1 = \frac{4-3\gamma+\gamma(m-2n)}{4}$ with $D = \frac{1}{2}$. We note that the cosmological constant takes the form $\Lambda = \frac{\lambda_0}{4}\frac{1}{t^2}$, where $\lambda_0 = (m-2n)$. Using the eqs. (3.11) and (3.22) in the eq. (3.21), one obtains

$$A_1\, t^{-2} + A_2\, t^{-2s-1} = 0 \tag{3.23}$$

where $A_1 = \frac{4-(3-\lambda_0)\gamma}{4}$, $A_2 = (\frac{3-\lambda_0}{4})^{(s-1)}\left(\frac{4\gamma(3-\lambda_0)-7-3\lambda_0}{8}\right)\beta$.

We note the following:

(i) If $s \neq \frac{1}{2}$ both A_1 and A_2 are zero. $A_2 = 0$ leads to $\beta = 0$, no viscosity and $A_1 = 0$ yields, $\gamma = \frac{4}{3-\lambda_0}$, with $1 \geq \lambda_0 \geq -1$ since $2 \geq \gamma > 1$. We note that value of the constant (λ_0) may pick up 1, 0, -1, for stiff fluid ($\gamma = 2$), radiation ($\gamma = \frac{4}{3}$) and matter ($\gamma = 1$) dominated era respectively.

(ii) If $s = \frac{1}{2}$ we get $A_1 + A_2 = 0$ which yields $\beta = \frac{(4-\gamma(3-\lambda_0))\sqrt{3-\lambda_0}}{7+3\lambda_0-4\gamma(3-\lambda_0)}$. We note the following bounds on λ_0 that determines the cosmological constant:

(a) stiff fluid, one requires $\frac{17}{11} \leq \lambda_0 \leq 3$ (b) radiation, one requires $\frac{27}{25} \leq \lambda_0 \leq 3$. (c) matter, one requires $\frac{5}{7} \leq \lambda_0 \leq 3$. However, in the absence of cosmological constant we get $\beta = \frac{\sqrt{3}(3\gamma-4)}{12\gamma-7}$ i.e. the power-law solutions are physically relevant for $\frac{4}{3} < \gamma \leq 2$.

However for $D \neq \frac{1}{2}$, it is not simple to find analytic solutions. So, we adopt a numerical technique to study cosmological dynamics. For simplicity, we choose $s = 1$. The deceleration parameter q is given by $q = -\frac{\dot{H}}{H^2} - 1$ which is used to derive the following differential equation from the field equation :

$$q''' + \frac{4q'q''}{q+1} + \frac{q'^3}{(q+1)^2} + \left[\frac{14(q+1)H - c_1}{(q+1)H^2}\right] q'' + \left[\frac{22(q+1)H - c_1}{(q+1)^2H^2}\right] q'^2$$

$$+\left[\frac{58(q+1)^2H^2-8c_1(q+1)H-15(q+1)H^2+c_2}{(q+1)^2H^4}\right]q'$$

$$\left[\frac{6(4q-1)}{H^3}-\frac{6c_1}{H^4}+\frac{2c_2(q+1)^2H+c_4(q+1)+c_3(q+1)^2H^2+c_5}{(q+1)^3H^6}\right]=0, \qquad (3.24)$$

where the co-efficients are

$c_1=(3H+\frac{1}{\beta}+3\gamma H)$,

$c_2=(9\gamma H^2-\frac{1}{6\alpha}+\frac{n\gamma}{12\alpha}+\frac{3H}{\beta}-\frac{3H(3\beta H-\gamma)}{\beta})$,

$c_3=(\frac{6}{\beta}+\frac{3(3\beta H-\gamma)}{2\beta}+18\gamma H)$,

$c_4=(\frac{1}{6\alpha\beta}+\frac{\gamma(3-m)}{6\alpha}H+(\frac{n}{12\alpha}+9H^2)\frac{3\beta H-\gamma}{\beta})$,

$c_5=\frac{(3\beta H-\gamma)(3-m)}{12\alpha\beta}$,

a prime represents the derivative with respect to the Hubble parameter H. Numerical solutions may be obtained choosing three initial conditions for q, q' and q'' for a given H as it is a differential equation of third order. We choose units so that H_0, the present value of H, is unity and picks up the set of values for q, q' and q'' for $H=1$ (i.e., the present value) from observationally consistent region [103] and study variation of q with H by plotting them numerically [101, 105]. As the inverse of H has the dimension of time, it gives an estimation for the cosmic age, future is given by $H<1$ and the past by $H>1$. One obtains desired feature of a negative q which describes an accelerating universe at $H=1$ and found that it attains the negative phase only in the recent past. Furthermore, in the near future, q may pass through another sign flip in the opposite direction, forcing to enter into a decelerated phase in the future once again. In the next section some special cases are considered to understand cosmological evolution.

Special Cases :

(i) Case I : For different values of $q''[1]$, the variation of q versus H with one set of other conditions is shown in fig. (3.1). The plot shows how the nature of the curve depends on the initial conditions and that there will be a phase change of q, in the near future leading to a decelerated phase. For a high value of $q''[1]$, the deceleration attained in the past was higher and the accelerating phase is shorter. For $q''[1]=1.3$, we get three phases of sign flips. In the past it happened twice and in future there may be another sign change for q. The first cosmic acceleration may be interpreted as the early inflation followed by a deceleration phase where radiation or matter dominated universe results and at the

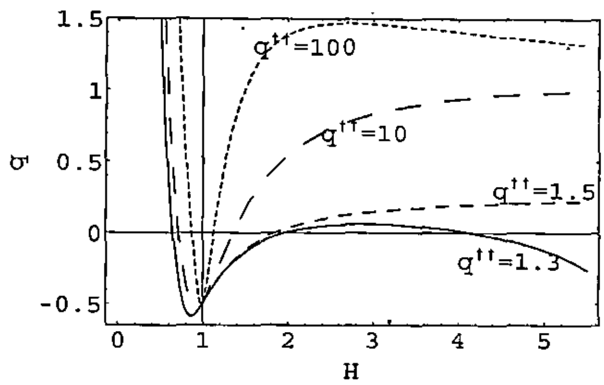

Figure 3.1: shows the plot of q vs H for different values of $q''(1)$, we choose the initial conditions $q'(1) = q^{\dagger}(1) = 1$, $q(1) = -0.5$, $\beta = 1$, $n = -2$, $m = 2$, $\gamma = 1$ and $\alpha = \frac{1}{6}$.

present moment, we obtain acceleration once again. This picture of the evolution of the universe is very interesting and now supported by experiment.

(ii) Case II : The variation of q with H for different value of γ, i.e. the equation of state, is shown in fig. (3.2). The plot shows that the deceleration in the past was higher for larger value of γ (i.e., EOS parameter).

(iii) Case III : The variation of q with H for different values of the coupling constant α is shown in fig. (3.3). The fig. (3.3) shows that the rate of deceleration in the past was higher for larger values of α.

(iv) Case IV : The variation of q with H for different values of β is shown in fig. (3.4). The plot shows that if β is taken small, say $\beta = 0.1$ the duration of present accelerating phase is found to increase and transition from deceleration to acceleration is quick with a possibility of an early inflation in the infinite past epoch.

(v) Case V : The variation of q with respect to H for different values of n is shown in fig. (3.5). In this case, we choose $m = 2$ and consider n satisfying the bound ($-16.95 < n < 5.75$) an interesting aspect is noted ; i.e., where more than one change of sign of q is found to exist. However for a large value of n, say ($n > 5.75$) one obtains early inflation followed by a decelerating phase, thereafter an accelerating phase once again, and then a decelerating phase in future. However, for smaller values of n (~ -16.95), as we go to

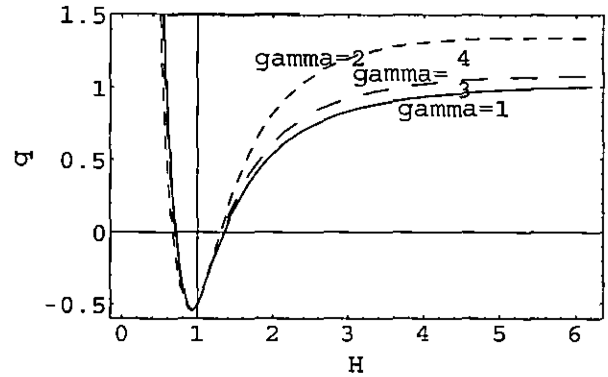

Figure 3.2: shows the plot of q vs H for different values of γ with $q''(1) = 10$, $q'(1) = 1$, $q(1) = -0.5$, $\beta = 1$, $n = -2$, $m = 2$ and $\alpha = \frac{1}{6}$.

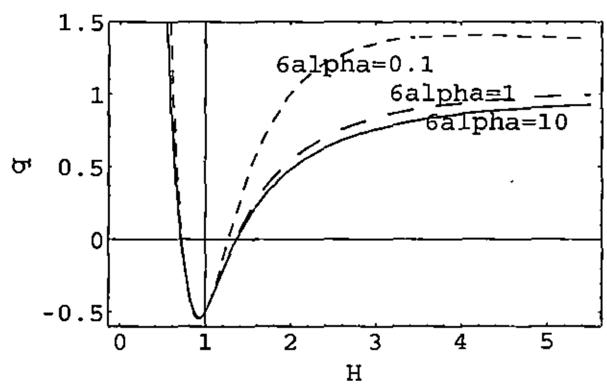

Figure 3.3: shows the plot of q vs H for different values of α with $q''(1) = 10$, $q'(1) = 1$, $q(1) = -0.5$, $\beta = 1$, $n = -2$, $m = 2$ and $\gamma = 1$.

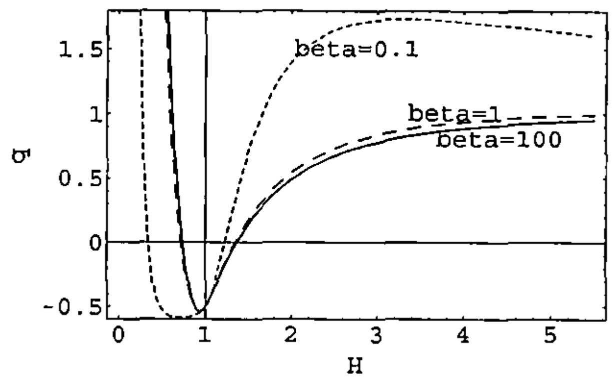

Figure 3.4: shows the plot of q vs H for different values of β, we choose the initial conditions $q''(1) = 10$, $q'(1) = 1$, $q(1) = -0.5$, $\alpha = \frac{1}{6}$, $n = -2$, $m = 2$ and $\gamma = 1$.

the future, the accelerating phase changes to a decelerating phase, followed by another accelerating phase. The period of the accelerating phase is found to depend on n, i.e., on the cosmological constant (Λ), which is shown in fig. (3.5) . It is noted that a small value of n results an increase in deceleration.

(vi) Case VI : The variation of q with H for different values of m is shown in fig. (3.6).

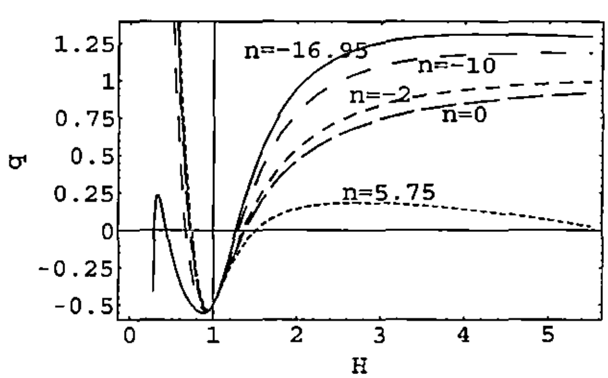

Figure 3.5: shows the plot of q vs H for different values of n, we choose the initial conditions $q''(1) = 10$, $q'(1) = 1$, $q(1) = -0.5$, $\beta = 1$, $m = 2$, $\gamma = 1$ and $\alpha = \frac{1}{6}$.

In this case evolutionary behaviour of the universe is found to have same characteristics as that noted in the case V.

(vii) Case VII : In the absence of a cosmological constant the variation of q with H for

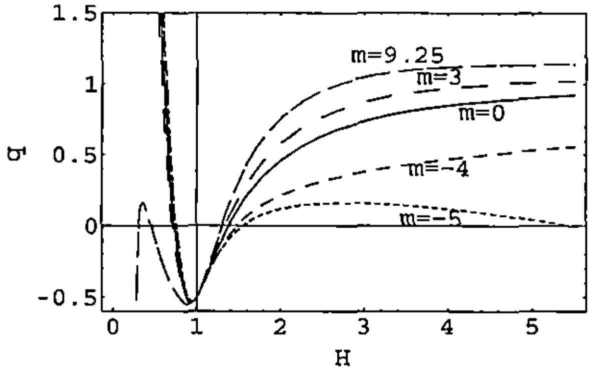

Figure 3.6: shows the plot of q vs H for a different values of m, we choose the initial conditions $q''(1) = 10$, $q'(1) = 1$, $q(1) = -0.5$, $\beta = 1$, $n = -2$, $\gamma = 1$ and $\alpha = \frac{1}{6}$.

different values of γ is plotted in fig. (3.7). It is evident that once the equation of state is known, the result is not very much dependent on Λ.

The above $(q - H)$ figures have been drawn using MATHEMATICA, the plot provides a

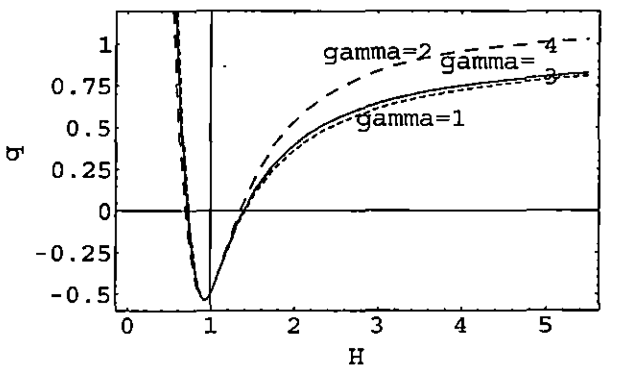

Figure 3.7: shows the plot of q vs H for a different values of γ, we choose the initial conditions $q''(1) = 10$, $q'(1) = 1$, $q(1) = -0.5$, $\beta = 1$, $n = 0$, $m = 0$ and $\alpha = \frac{1}{6}$.

sufficient data set. These numerical values may be used to determine a closed analytical mathematical forms for q and H by assuming a polynomial function for q as $q = \Sigma_0^n a_i H^i$. A functional form to explore $q(H)$ may be determined corresponding to fig. (3.1) with $q''(1) = 10$. In this case the corresponding analytical function can be expressed as

$$q = -1.059H^6 + 8.344H^5 - 20.163H^4 + 6.373H^3 + 40.1H^2 - 53.183H + 19.057.$$

In fig. (3.8), a plot of q vs H is shown for both numerical and analytical expression ob-

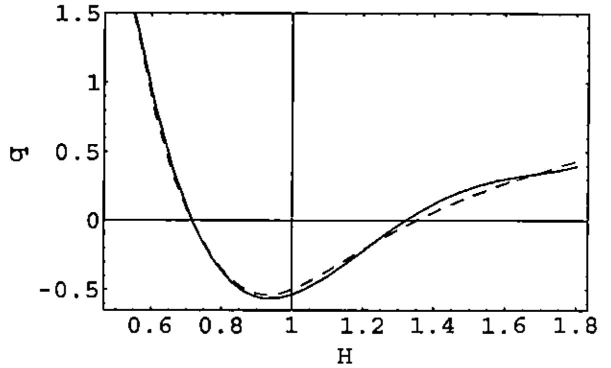

Figure 3.8: shows dashed curve due to polynomial expression and the solid curve due to from fig. (3.1) with $q''(1) = 10$.

tained above. The comparison between the numerical and analytical expression is shown in fig. (3.8). This comparison holds well only when H is reasonably close to 1.

2. Exponential model: In this case, we do not get exponential solution of the form (3.20). However, the field equation admits a de Sitter solution ($H = H_0$) with energy density

$$\rho = \frac{81\alpha}{2}H_0^4 - (m - \frac{3}{2}n)H_0^2 - \frac{n}{\sqrt{6\alpha}}H_0. \tag{3.25}$$

The bulk viscous stress is given by

$$\Pi = -9\sqrt{6\alpha}H_0^3 + 81\alpha(1 - \frac{\gamma}{2})H_0^4 + (m - \frac{3n}{2})\gamma H_0^2 + \frac{n\gamma}{\sqrt{6\alpha}}H_0. \tag{3.26}$$

From eq. (3.21) for $s = \frac{1}{2}$, the bulk viscous constant $\beta = \frac{9\sqrt{6\alpha}H_0^2 - 81\alpha(1-\frac{\gamma}{2})H_0^3 - (m - \frac{3n}{2})\gamma H_0 + \frac{n\gamma}{\sqrt{6\alpha}}}{3(\frac{81\alpha}{2}H_0^4 - (m - \frac{3}{2}n)H_0^2 - \frac{n}{\sqrt{6\alpha}}H_0)}$. For $m = n = 0$, i.e., without a cosmological constant and $\gamma = 2$, we get $\beta = \frac{2\sqrt{3}}{3}$. From eq. (3.19) we get, $a \sim e^{H_0 t}$ where $H_0 = \frac{2}{3\sqrt{6\alpha}}$. For sufficient inflation to solve the cosmological problems, one must have $H_0 t > 65$. It is not possible to predict temperature in this case. However, in the next section we describe a FIS theory where temperature of the universe can be determined.

3.3.3 *Full Israel-Stewart theory ($\epsilon = 1$)*

In the full causal theory the transport eq. (3.8) becomes,

$$\Pi + \tau\dot{\Pi} = -3\zeta H - \frac{1}{2}\tau\Pi\left[3H + (\frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T})\right]. \tag{3.27}$$

1. Power-law model: In this case we consider a power law expansion of the universe given by $a(t) = a_0 t^D$ with $D = \frac{1}{2}$. The energy density and the bulk viscous stress are given by

$$\rho = \rho_1 t^{-2}, \qquad \Pi = \Pi_1 t^{-2} \tag{3.28}$$

where $\rho_1 = \frac{3-m+2n}{4}$ and $\Pi_1 = \frac{4-3\gamma+\gamma(m-2n)}{4}$. The cosmological constant under this condition become $\Lambda = \frac{\lambda_0}{4}$, where $\lambda_0 = (m - 2n)$. Using eqs. (3.11), (3.27) and (3.28) we obtain a differential equation for temperature,

$$\frac{\dot{T}}{T} = T_0\, t^{2s-2} - T_{01}\, t^{-1} \tag{3.29}$$

where $T_0 = \frac{2}{\beta}\rho_1^{1-s}$ and $T_{01} = (\frac{1}{2} - \frac{3\rho_1}{\Pi_1})$, which may be integrated easily. For $s = \frac{1}{2}$, eq. (3.29) yields, $T \sim t^{T_0 - T_{01}}$. The solution admits a decreasing mode of temperature when $T_{01} > T_0$, i.e. for a range of values of β satisfying a lower limit $\beta > \frac{2(4-3\gamma+\gamma(m-2n))\sqrt{3-m+2n}}{(-14-3\gamma+(6+\gamma)(m-2n))}$.
In the above we note the following:
(a) $\lambda_0 \leq 1$ for stiff fluid $\gamma = 2$; (b) $\lambda_0 \leq 0$ for radiation $\gamma = \frac{4}{3}$; (c) $\lambda_0 \leq -1$ for matter $\gamma = 1$. In all the cases a small dynamical cosmological constant is required to obtain viable cosmological scenario. However in the absence of a cosmological constant (Λ), i.e., $m = n = 0$ or $m = 2n$, a decreasing mode of temperature is admitted for $\beta > \frac{2\sqrt{3}(3\gamma-4)}{3\gamma+14}$.
A physically realistic solution, i.e., a positive coefficient of bulk viscosity, is obtained for $\frac{4}{3} \leq \gamma \leq 2$.

If $s \neq \frac{1}{2}$, on integrating eq. (3.29) we get the evolution of temperature at that epoch, which follows a power law and exponential behaviour given by $T = T_{02}\, t^{-T_{01}}\, e^{\left(\frac{T_0}{2s-1}t^{(2s-1)}\right)}$. It is evident that the temperature decreases for $s \leq \frac{1}{2}$. We note the following:

(a) $\lambda_0 \leq 1$ for stiff fluid $\gamma = 2$; (b) $\lambda_0 \leq 0$ for radiation $\gamma = \frac{4}{3}$; (c) $\lambda_0 \leq -1$ for matter $\gamma = 1$ with decreasing mode of temperature. However in the absence of a cosmological constant a decreasing mode of temperature is obtained for $s \leq \frac{1}{2}$ in the regime $\frac{4}{3} \leq \gamma \leq 2$.

It is not simple to find analytic solution for FIS theory with $D \neq \frac{1}{2}$. We adopt a numerical technique to study FIS theory with $\alpha = 0$. Let us consider $s = 1$, and assume a power law of the temperature: $T = T_0\rho^r$, where T_0 is a dimensionless constant and $r = \frac{\gamma-1}{\gamma}$. With the helps of eqs. (3.11) and (3.27) we get the following:

$$q' + \left[\frac{2(q+1)}{H} + \frac{c_3\,(q+1)^2H^2 - c_4(q+1) + c_5}{(c_2 - c_1(q+1))(q+1)H^2}\right] = 0, \tag{3.30}$$

where the coefficients are

$c_1 = \frac{1}{2}n\beta\pi_1(r-\beta)$,

$c_2 = \frac{1}{2}\rho_1\beta(n\gamma(\beta+r) + 2\pi_1)$,

$c_3 = \frac{\beta}{2}(3n\pi_1 + 2\pi_1\rho_1(\beta+r) + 4\gamma\rho_1 n) - n(3nH + \pi_1)$,

$c_4 = \frac{\beta H\rho_1}{2}(3\pi_1 - 3n\gamma - 2\rho_1\gamma(\beta+r) + 4\gamma\rho_1) + \rho_1(6nH + \pi_1 - n\gamma)$,

$c_5 = \frac{1}{2}\rho_1^2(3\beta\gamma H - 6H + 2\gamma)$,

a prime represents the time derivatives by the derivative with respect to H.

Special Cases :

(i) Case I: The variation of q with H for the different values of n with $m = 3$ is shown in the fig. (3.9). It is evident that, for a larger value of n, the present accelerating state begins at a later epoch compared to a smaller value of n. However, the early deceleration rate is large for large n with a shorter period of accelerating phase compared to smaller n.

(ii) Case II: The variation of q with H for different values β is shown in fig. (3.10). It is evident that, for a large value of β, the present accelerating state might have occurred at any early epoch and the period of acceleration is found to enhance compared to a smaller value of β. However, the early deceleration rate is found to be large for small β.

(iii) Case III: The variation of q with H for a different values of γ is shown in fig. (3.11).

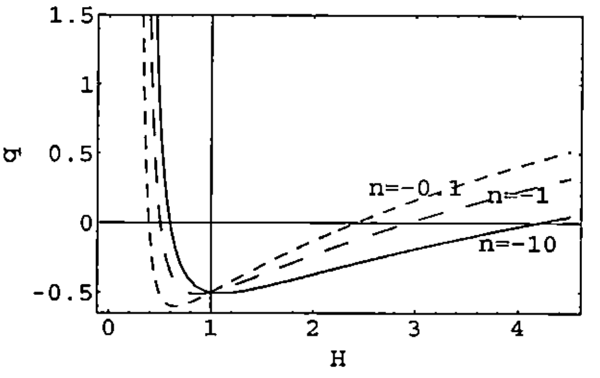

Figure 3.9: shows the plot of q vs H for a different values of n with $q(1) = -.5$, $\beta = 2$, $\gamma = 1$, and $m = 3$.

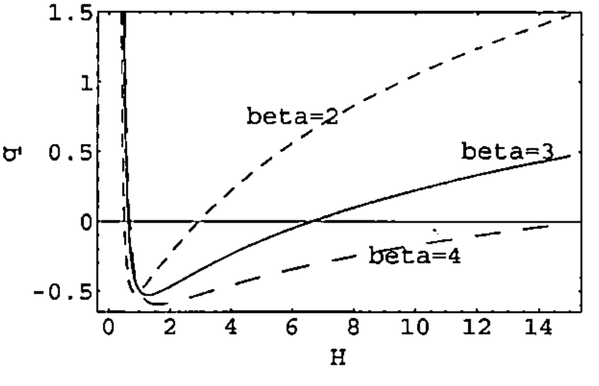

Figure 3.10: shows the plot of q vs H for a different values of β with $q(1) = -0.5$, $\gamma = 2$, $n = -2$, and $m = 3$.

The plot shows that with the higher value of γ the duration of the present accelerating phase become shorter.

2. Exponential model: The field equations (3.5) and (3.9) lead to a differential

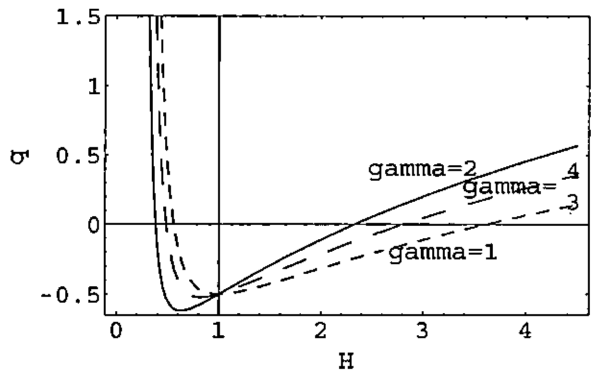

Figure 3.11: shows the plot of q vs H for a different values of γ with $q(1) = -0.5$, $\beta = 2$, $n = -2$, and $m = 3$.

equation for Hubble parameter which is given by

$$\dot{H} = \frac{1}{\sqrt{6\alpha}}H - \frac{3}{2}H^2 \tag{3.31}$$

as was obtained in eq. (3.19). The corresponding energy density is given by $\rho = \frac{81\alpha}{2}H^4 - (m - \frac{3}{2}n)H^2 - \frac{1}{\sqrt{6\alpha}}nH$.

We note the following : (i) for $\Lambda = \frac{n}{\sqrt{6\alpha}}\ H$, one obtains a lower bound on Hubble parameter $H > \left(\frac{2n}{81\sqrt{6\alpha^3}}\right)^{1/3}$ (ii) for $\Lambda = mH^2$, it becomes $H > \left(\frac{2m}{81\alpha}\right)^{1/2}$ for a realistic solution.

The viscous stress becomes

$$\Pi = -9\sqrt{6\alpha}H^3 + 81\alpha\left(1 - \frac{\gamma}{2}\right)H^4 + \left(m - \frac{3n}{2}\right)\gamma H^2 + \frac{n\gamma}{\sqrt{6\alpha}}H. \tag{3.32}$$

The bulk viscous coefficient is given by

$$\zeta = 27\alpha H^3\left(\frac{\gamma}{2} - 1\right) + \frac{18\alpha H^2}{\sqrt{6\alpha}} - \frac{\gamma}{6}(2m - 3n)H - \frac{n\gamma}{3\sqrt{6\alpha}}.$$

Eq.(3.27) yields a differential equation determining the temperature of the universe, which is given by

$$\frac{\dot{T}}{T} = 3H - \frac{\dot{\rho}}{\rho} + \frac{6H\rho}{\Pi} + \rho^{1-s}\frac{2}{\beta} + \frac{2\dot{\Pi}}{\Pi}. \tag{3.33}$$

In the absence of cosmological constant with $s = \frac{1}{2}$ and $\gamma = 2$, one obtains $\beta = \frac{2\sqrt{3}}{3}$ and the temperature of the universe evolves as

$$T = T_0 \frac{T}{\zeta} a^{-6} e^{\left(\frac{6}{\sqrt{6\alpha}} t\right)}. \tag{3.34}$$

The scale factor is obtained from eq. (3.31), which yields

$$a(t) = \left[a_1 + a_2 e^{(t/\sqrt{6\alpha})}\right]^{2/3}. \tag{3.35}$$

In this case, we note that the solution represents a universe which begins with a finite size in the past and grows exponentially thereafter. The increasing temperature mode obtained here in the FIS theory is relevant for describing fluid of the later case. It is now possible to determine the evolution of temperature in an emergent universe scenario.

3.4 Discussion

In this chapter both the power law and exponential expansion of the universe are considered separately in higher derivative theories of gravity with a temporal variation of cosmological constant and an imperfect fluid to explore cosmological models. In the Eckart theory we note a viscous universe which is evolving as $a(t) \sim \sqrt{t}$, during the radiation dominated epoch admitting Λ ranging from negative to positive values. Considering isotropic matter in the theory with a positive Λ, we obtain an interesting solution which admits an accelerating universe where the scale factor of the universe evolves as $a(t) \sim t^{\frac{4}{3}}$ which is a new solution. A viscous universe described by truncated and full Israel-Stewart theories are considered as Eckart theory is not suitable for describing causal fluid distribution of the universe. For obtaining a power law solution, it is found that the relevant field equations are highly nonlinear which however is solved numerically. We adopt here a technique similar to that used in *Ref.* [101]. The relevant equations are rewritten in terms of the Hubble parameter (H) and the deceleration parameter (q) and its derivatives. We plot q against H for different parameters that are used in the higher derivative action, namely α, β, Λ and the equation of state parameter γ. In all the theories it is found that the present acceleration phase of expansion of the universe is followed by a decelerating phase in the past. The period of the present acceleration depends on the parameters that are plotted in fig (3.1)-(3.11) which are discussed. It is found that there exist phases with a number of accelerating and decelerating phases which are important to build a cosmological scenario of the universe. In the latter case the present universe emerged from an early accelerating phase followed by a deceleration, thereafter the present acceleration phase. The early acceleration is vacuum dominated and the present phase is matter dominated. In the plot for q vs H we note that one can predict that there may be another decelerating phase followed by rapid acceleration in the future. It is noted that early deceleration rate was higher for smaller values of α or β. For simplicity in the FIS theory a power law solution is obtained with $\alpha = 0$ using numerical analysis. In the FIS theory decreasing mode of temperature is taken up to study the evolution of the universe.