CHAPTER 6

Brane-World Cosmology with Viscosity and Gauss-Bonnet term

6.1 Introduction

In the recent years, there has been considerable interest in the possibility that our observable four dimensional (4D) universe may be viewed as a brane hypersurface embedded in a higher dimensional bulk space [27]-[29], [143]. According to the brane world scenario the standard model fields are confined on a hypersurface (called brane) embedded in a higher dimensional space (called bulk). Only gravity and other exotic matter such as the dilaton can propagate in the bulk. The simplest phenomenological models describing such a scenario are the five dimensional Randall-Sundrum (RS)[144, 145] type brane world cosmologies. These models are motivated by ideas from string/ M theory. The observable universe is a four-dimensional brane surface embedded in a five-dimensional anti de Sitter (AdS_5) bulk spacetime , i.e. a spacetime with a negative cosmological constant. In Randall-Sundrum type II scenario (RS II) spatially isotropic and homogeneous brane propagates in a five dimensional Schwarzschild-anti-de Sitter space [145]. Although the fifth dimension is infinite, the curvature of the higher dimensional space acts to confine the graviton near the brane at low energies, so that general relativity is recovered. At high energies in the early universe, graviton localization fails and the Friedmann equation is modified i.e. extra-dimensional gravity dominating at high energies. This property can also be understood within the context of the AdS/ conformal field theory (CFT) correspondence [146] that the RS model is equivalent to four dimensional gravity coupled to a conformal field theory [147]. Using RS brane model some of the outstanding problems of particle physics get resolved, i.e.(i) the hierarchy problem [148], i.e. the large discrepancy between the Plank scale at 10^{19} GeV and the electroweak scale at 100 GeV, can be addressed, (ii) cosmological constant problem of standard cosmology may be addressed [149], (iii) at the early universe inflationary scenario may be addressed [143]. The brane-world may offers a right approach to our understanding of the evolution of the universe.

The Randall-Sundrum braneworld cosmology is based on the five dimensional Einstein-Hilbert action. At high energies, it is expected that this action needs quantum corrections. One approach to develop the brane-world scenario in a more string theoretic setting is to include higher order curvature invariants in the bulk action [150]. In general, resulting equations of motion of such terms contain more than second derivatives of metric and the theory is plagued by ghosts. However there exists a combination of quadratic terms, called Gauss-Bonnet (GB) term, whose equation of motion has no more than second derivatives of the metric and the theory is free from ghosts [151]. Another important property of GB term is that the Lagrangian is a pure divergence in four or less dimensions. Specifically, the Gauss-Bonnet (GB) term arises as the leading order for quantum correction in the heterotic string effective action [58, 152]. It has been shown that localization of the graviton zero mode on the brane is possible when such a term is included in the bulk action [153]. In higher dimensions (D> 4) Gauss-Bonnet terms ($GB = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ where R_{abcd} is the covariant form of Riemann Christoffel tensor, R_{ab} is the Ricci tensor and R is the Ricci scalar) plays a crucial role which appears as a low energy limit of string theory. It may be mentioned here that in the case of inflation drivèn by an inflaton field, presence of the Gauss-Bonnet term allows the spectral index of the scalar perturbation spectrum to take values in the range between 0.944 and 0.989 [154], thereby bringing the scenario in closer agreement with the most recent observations.

Several investigation considering perfect fluid as a source of matter on brane cosmological models have been studied in the literature [155] in order to obtain viable cosmological scenario of the early universe. In the early universe it is also known that a number of processes [18] (as mention earlier chapters) might have occurred leading to a deviation from perfect fluid assumption e.g. viscosity which is to be taken into account seriously. However, real fluids behave irreversibly, and some processes in cosmology and astrophysics cannot be understood except a dissipative processes. Further, the inclusion of dissipative terms in the energy-momentum tensor of the cosmological fluid seems to be best motivated generalization of the matter sector of the gravitational field equations on brane. It has been predicted from observations that a non negligible dissipative bulk stress on cosmological scales at the late universe phase might be important. The possible source of such viscosity may be due to (i) gaseous matter in the framework of relativistic gravity which may give rise to internal self-interaction leading to a negative cosmic bulk pressure [119], (ii) deviation of the non relativistic particle in the substratum from dust. For a non-relativistic substratum cosmic anti-friction may generate a negative fluid bulk pressure which has been noted [120] in the frame work of Einstein gravity.

Using the transport equations obtained from *EIT*, several works in Einstein gravity [19] and in the higher derivative gravity theory have been reported [121]. The motivation of the chapter is to explore cosmological solutions in the RS brane-world model with or without GB term in the presence with imperfect fluid described by *EIT*. We also study the stability of the equilibrium points of the dynamical system associated with the evolution of the viscous cosmological fluid in the RS scenario with or without a Gauss-Bonnet term. The chapter is organized as follows. In sec. 6.2, the gravitational action and the relevant field equations in brane world gravity are given. Cosmological solutions are presented in sec. 6.3. In sec. 6.4, stability analysis is presented. Finally, in sec. 6.5, results are summarized.

6.2 Field Equations in Brane-World Gravity

For a 5D bulk with Einstein-Gauss-Bonnet gravity, containing a 4D brane, the gravitational action is

$$S = \frac{1}{2k_5^2}\int d^5x\sqrt{-g_5}\left[-2\Lambda_5 + R + \alpha(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd})\right] - \int_{brane} d^4x\sqrt{-g}\,\sigma \tag{6.1}$$

where $x^a = (x^\mu, z)$, with z the extra-dimensional coordinate, $g_{ab} = g_{ab} - n_a n_b$ is the induced metric, with n^α is the unit normal to the brane, σ (> 0) is the brane tension, the Gauss-Bonnet coupling α has dimensions L^2 and Λ_5 (< 0) is the bulk cosmological constant. The fundamental energy scale of gravity is the 5D scale M_5, where $k_5^2 = \frac{8\pi}{M_5^3}$.

The Planck Scale $M_4 \sim 10^{16}$ TeV is an effective scale, describing gravity on the brane at low energies, and typically $M_4 >> M_5$. The Gauss-Bonnet (GB) term may be thought of as the lowest-order stringy correction to the 5D Einstein-Hilbert action, with coupling constant $\alpha > 0$. In this case $\alpha|R^2| << |R|$, so that

$$\alpha << l^2 \tag{6.2}$$

where l is the bulk curvature scale, $| R | \sim l^{-2}$. The RS type models are recovered for $\alpha = 0$. Paul and Sami [156] studied the tachyonic inflation in brane world cosmology with Gauss-Bonnet term in the bulk. An exact solution of slow roll equations in case of exponential potential is obtained and it was attempted to implement the proposal of Lidsey and Nunes [157] for the tachyon condensate rolling on the Gauss-Bonnet brane. They found that due to peculiar nature of tachyon field dynamics, the proposal of the Lidsey and Nunes does not seem to work in a natural way. A number of observational estimates using action (6.1) was taken up in the literature [158].

Imposing Z_2 symmetry across a Friedmann brane in an anti de Sitter bulk, the modified Friedmann equation on the (spatially flat) brane is [159]

$$k_5^2(\rho + \sigma) = 2\sqrt{H^2 + \mu^2}[3 - 4\alpha\mu^2 + 8\alpha H^2] \tag{6.3}$$

where H is the Hubble rate, μ ($\equiv l^{-1}$) is the energy scale associated with l. When $\rho = 0 = H$ in the eq. (6.3) we recover the expression for the critical brane tension which achieves zero cosmological constant on the brane,

$$k_5\sigma = 2\mu(3 - 4\alpha\mu^2) \tag{6.4}$$

The above equation may be rewritten in the useful form [157]

$$H^2 = \frac{1}{4\alpha}\left[(1 - 4\alpha\mu^2)\cosh\left(\frac{2\chi}{3} - 1\right)\right], \tag{6.5}$$

$$k_5^2(\rho + \sigma) = \left[\frac{2(1 - 4\alpha\mu^2)^3}{\alpha}\right]^{1/2} \sinh\chi \tag{6.6}$$

where χ is a dimensionless measure of the energy density. By eqs. (6.5) and (6.6) there is a characteristic GB energy scale,

$$m_\alpha = \left[\frac{2(1 - 4\alpha\mu^2)^3}{\alpha k_5^4}\right]^{\frac{1}{8}}, \tag{6.7}$$

such that the GB high energy regime ($\sinh\chi \gg 1$) corresponds to $\rho \gg m_\alpha^4$. Since the GB term is a correction to the RS action, we must have m_α^4 greater than the RS energy scale σ. Expanding eq.(6.5) in χ, one obtains two regimes for the dynamical history of the brane universe during the early period of evolution which are given below :

- In the GB regime,

$$\rho >> m_\alpha^4 \Rightarrow H^2 \approx \left[\frac{k_5^2}{16\alpha}\rho\right]^{2/3} \tag{6.8}$$

which gives, $\rho = \rho_0 H^3$, where $\rho_0 = \frac{16\alpha}{k_5^2}$.

- In the RS regime,

$$m_\alpha^4 >> \rho >> \sigma \equiv m_\sigma^4 \Rightarrow H^2 \approx \frac{k_4^2}{6\sigma}\rho^2 \tag{6.9}$$

which gives $\rho = \rho_0 H$, with $\rho_0 = \left(\frac{6\sigma}{k_4^2}\right)^{1/2}$. The conservation equation for the matter is

$$\dot{\rho} = -3H(\rho + p), \tag{6.10}$$

where ρ and p are the energy density and pressure of the perfect fluid respectively. To include the effects of viscosity, the perfect fluid pressure should be replaced by an effective pressure p_{eff}, which is given by

$$p_{eff} = p + \Pi, \tag{6.11}$$

where p is the perfect fluid contribution and Π is the bulk viscous stress. The causal evolution equation for the bulk viscous pressure Π is given [160] by the equation

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{\epsilon}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right). \tag{6.12}$$

where ξ is the coefficient of bulk viscosity, τ is the relaxation coefficient for transient bulk viscous effect and T is the temperature of the universe. The parameter ϵ takes the value 0 or 1, the former represents the truncated Israel-Stewart theory (TIS) and the latter represents the full causal theory (FIS). One recovers the non causal Eckart theory for $\tau = 0$. We consider an equation of state for the perfect fluid pressure p and the energy density of the usual form

$$p = (\gamma - 1)\rho \tag{6.13}$$

where γ $(1 \leq \gamma \leq 2)$ is a constant. The conservation eq. (6.10) including viscous fluid now can be written as :

$$\dot{\rho} = -3H(\gamma\rho + \Pi). \tag{6.14}$$

An important observational quantity deceleration parameter (q) is related to H as

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1. \tag{6.15}$$

The positive sign of q corresponds to deceleration whereas the negative sign indicates acceleration of the universe.

6.3 Cosmological Solutions

The set of eqs. (6.8), (6.9), (6.12)-(6.14) will be used to obtain cosmological solutions. As the number of equations are more than the number of unknowns we use some *ad hoc* relations for the viscosity parameters. We assume the following simple phenomenological relation for the bulk viscosity coefficient and relaxation time respectively:

$$\xi = \eta \rho^s, \quad \tau = \frac{\xi}{\rho} = \eta \rho^{s-1} \tag{6.16}$$

where $s \geq 0$, $\eta \geq 0$ are constants. In GB region $\xi = \xi_0 H^{3s}$, $\tau = \frac{H^{3(s-1)}}{\tau_0}$ where $\xi_0 = \eta \rho_0^s$ and $\tau_o = \frac{\rho_0}{\xi_0}$ are constant. In RS region $\xi = \xi_0 H^s$ and $\tau = \frac{H^{(s-1)}}{\tau_0}$. In the next section we explore cosmologies solution in TIS and FIS theory respectively.

6.3.1 *Truncated Israel-Stewart (TIS) Theory* ($\epsilon = 0$) :

In the TIS theory the transport eq. (6.12) reduces to

$$\tau \dot{\Pi} + \Pi = -3\xi H. \tag{6.17}$$

I. In the GB Regime:

1. Power-law model : We choose a power-law expansion for the universe given by

$$a(t) = a_0 t^D \tag{6.18}$$

where a_0 and D are constants which are to be determined from the field equations. The energy density and bulk viscous stress are obtained from eqs. (6.8) and (6.14), which is given by are given by

$$\rho = \rho_0 t^{-3}, \qquad \Pi = -\Pi_0 t^{-3} \tag{6.19}$$

where $\rho_0 = \frac{D^3}{C^3}$, $C = \left(\frac{k_5^2}{16\alpha}\right)^{1/3}$ and $\Pi_0 = \left(\gamma\rho_0 - \frac{\rho_0^{2/3}}{C}\right) = \frac{D^2}{C^3}(\gamma D - 1)$. Using transport eq. (6.17) one obtains

$$A_1 t^{-3} + A_2 t^{-3s-1} = 0 \tag{6.20}$$

where $A_1 = \frac{D^2}{C^3}(1 - \gamma D)$ and $A_2 = 3\eta\frac{D^{3s}}{C^{3s}}\left(D - \frac{1}{D} + \gamma\right)$. We note the following :

(a) $A_1 = 0$ demands $D = \frac{1}{\gamma}$, i.e., the power law inflation may be obtained here for $\gamma < 1$;

(b) when $A_2 = 0$, it leads to either $\eta = 0$, *i.e.*, no viscosity or $\gamma = \frac{1-D^2}{D}$. Thus the universe in this case evolves as $a \sim t^{\frac{1}{\gamma}}$. If we set matter in the presence of viscosity then the universe during this period evolves as $a \sim t$. If we set $s = \frac{2}{3}$, it leads to $A_1 + A_2 = 0$ which yields viscosity term: $\eta = \frac{\gamma D - 1}{3C\left(D - \frac{1}{D} + \gamma\right)}$.

2. Exponential model : In this case de Sitter solution is obtained with energy density and bulk viscous stress are given by

$$\rho = \frac{H_0^3}{C^3} = const., \qquad \Pi = -\frac{\gamma}{C^3}H_0^3 = const. \tag{6.21}$$

The the coefficient of bulk viscosity is

$$\xi = \frac{\gamma}{3C^3}H_0^2 \tag{6.22}$$

with $\eta = \frac{\gamma C^{3(s-1)}}{3}H_0^{2-3s}$. For $s = \frac{2}{3}$, we find $\eta = \frac{\gamma}{3C}$. The scale factor of the universe evolves as

$$a(t) = a_0 \exp(H_0 t) \tag{6.23}$$

For sufficient inflation to solve the cosmological problems, one requires $H_0 t > 65$.

II. In the RS brane regime :

1. Power-law model : In this case we consider a power-law expansion of the universe given by eq. (6.18). The energy density and bulk viscous stress are given by

$$\rho = \rho_1 t^{-1}, \qquad \Pi = -\Pi_1 t^{-1} \tag{6.24}$$

where $\rho_1 = \frac{D}{C'}$, $C' = \left(\frac{k_4^2}{6\sigma}\right)^{1/2}$ and $\Pi_1 = \gamma\frac{D}{C'} - \frac{1}{3C'}$. From transport eq. (6.17) we obtain

$$A_1' t^{-1} + A_2' t^{-s-1} = 0 \tag{6.25}$$

where $A_1' = \frac{1}{3C'}(1-3\gamma D)$ and $A_2' = \eta \frac{D^s}{C'^s}\left(3D - \frac{1}{3D} + \gamma\right)$. We note the following :

(a) when $A_1' = 0$ one gets $D = \frac{1}{3\gamma}$, i.e., the power law inflation may be obtained here for $D > 1$ i.e., $\gamma < \frac{1}{3}$; (b) when $A_2' = 0$, it leads to either $\eta = 0$, *i.e.*, no viscosity or $\gamma = \frac{C'-9D^2}{3C'D}$. which is a new result. Thus the universe in this case evolves as $a \sim t^{\frac{1}{3\gamma}}$. If we set matter in the presence of viscosity one gets $D = \sqrt{\frac{C'}{9}}$, where C' can be chosen to take $D > 1$ i.e., the power law inflation may be obtained on brane. For $s = 0$, we get cosmological solutions when viscosity and matter coexist, and it leads to $A_1' + A_2' = 0$ which yields $\eta = \frac{D(3\gamma D-1)}{(9D^2+3\gamma D-1)C'}$.

2. Exponential model : In this case, one obtain de Sitter solution when energy density and bulk viscous stress become

$$\rho = \frac{H_0}{C'} \qquad \Pi = -\frac{\gamma}{C'}H_0. \tag{6.26}$$

The coefficient of bulk viscosity is determined as $\xi = \frac{\gamma}{3C'}$. However, one obtain $\eta = \frac{\gamma C'^{s-1}}{3}H_0^{-s}$ and for $s = 0$, we find $\eta = \frac{\gamma}{3C'}$. The evolution of scale factor of the universe is given by eq. (6.23).

6.3.2 *Full Israel-Stewart (FIS) Theory* ($\epsilon = 1$)

In the full causal theory eq. (6.12) becomes

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{1}{2}\tau\Pi\left[3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right]. \tag{6.27}$$

I. In the GB regime:

1. Power-law model : In this case we consider power-law expansion of the universe given by eq. (6.18). The energy density and bulk viscous stress are given by

$$\rho = \rho_0 t^{-3}, \qquad \Pi = -\Pi_0 t^{-3}, \tag{6.28}$$

where $\rho_0 = \frac{D^3}{C^3}$ and $\Pi_0 = \frac{D^2}{C^3}(\gamma D - 1)$. Using eq. (6.27) and (6.28) we obtain a differential equation which determines the temperature of the universe,

$$\frac{\dot{T}}{T} = B_1 t^{3s-3} - B_2 t^{-1} \tag{6.29}$$

where $B_1 = 2\tau_0 D^{3(1-s)} = \frac{2}{\eta}\left(\frac{C}{D}\right)^{3(s-1)}$ and $B_2 = \left(3 + \frac{6D^2}{\gamma D-1} - 3D\right)$. For $s = \frac{2}{3}$, eq. (6.29) yields $T \sim t^{B_1 - B_2}$. The solutions admits a decreasing mode of temperature when $B_2 > B_1$

i.e., for a range of values of η satisfying a lower limit $\eta > \frac{2D}{C}\frac{\gamma D-1}{6D^2+(3-3D)(\gamma D-1)}$. If $s \neq \frac{2}{3}$ eq. (6.29) yields the evolution of the temperature which is given by $T = T_0 t^{-B_2} e^{\frac{B_1}{3s-2}t^{3s-2}}$. It is evident that the temperature decreases when $s < \frac{2}{3}$.

2. Exponential model : In this case de Sitter solution is obtained with constant energy density and the bulk viscous stress as given in eq. (6.21). For constant values of ρ and Π the evolution of temperature is obtained from eq. (6.27), which yields $T = T_{01}e^{\alpha_1 t}$, where $\alpha_1 = H_0(3 - \frac{6}{\gamma} + 2\tau_0 {H_0}^{2-3s})$. The decreasing mode of temperature is obtained for $\alpha_1 < 0$. For $\alpha_1 = 0$ the temperature of the universe remain constant. For $s = \frac{2}{3}$ and $\tau_0 = \frac{6-3\gamma}{2\gamma}$ the value of temperature becomes $T = const.$ The scale factor of the universe evolves as

$$a(t) = a_o \exp(H_0 t) \tag{6.30}$$

For sufficient inflation to solve the cosmological problems, one requires $H_0 t > 65$.

II. In the RS brane regime :

1. Power-law model : We consider power-law expansion of the universe given by eq. (6.18). The energy density and bulk viscous stress are given by

$$\rho = \rho_1 t^{-1}, \qquad \Pi = -\Pi_1 t^{-1} \tag{6.31}$$

where $\rho_1 = \frac{D}{C'}$ and $\Pi_1 = \frac{1}{3C'}(3\gamma D - 1)$. Using eqs. (6.27) and (6.31) we obtain a differential equation for the temperature of the universe,

$$\frac{\dot{T}}{T} = B_1' t^{s-1} - B_2' t^{-1} \tag{6.32}$$

where $B_1' = \frac{2}{\eta}\frac{C'}{D}^{s-1}$ and $B_2' = (1 + \frac{18D^2}{1-3\gamma D} - 3D)$. For $s = 0$ eq. (6.32) yields $T \sim t^{B_1' - B_2'}$. The solutions admits a decreasing mode of temperature when $B_2' > B_1'$ i.e., for a range of values of η satisfying a lower limit $\eta > \frac{2D}{C'}\frac{1-3\gamma D}{18D^2+(1-3D)(1-3\gamma D)}$. If $s \neq 0$ eq. (6.32) yield the evolution of the temperature which is given by $T = T_0 t^{-B_2'} e^{\frac{B_1'}{s}t^s}$.

2. Exponential model : In this case de Sitter solution is obtained with constant energy density and the bulk viscous stress as given in eq. (6.26). For constant values of ρ and Π the evolution of temperature is obtained from eq. (6.27), which yields $T = T_{02}e^{\alpha_2 t}$, where $\alpha_2 = H_0(3 - \frac{6}{\gamma} + 2\tau_0 H^{-s})$. The decreasing mode of temperature is obtained for $\alpha_2 < 0$. For $s = 0$ and $\alpha_2 = 0$ i.e., $T = const.$ the value of τ_0 becomes $\tau_0 = \frac{6-3\gamma}{2\gamma}$. The evolution of scale factor of the universe is shown in eq. (6.30).

6.4 Stability Analysis of the Equilibrium Points

6.4.1 *RS brane regime with GB terms*

The general evolution equation of the bulk viscous cosmological fluid on the brane in *GB* regime can be obtained using eqs. (6.8), (6.12), (6.14), (6.16). The dynamical equation is a second order non-linear differential equation of the form

$$\ddot{H} + R(H, \dot{H}) = 0 \tag{6.33}$$

where $R(H, X) = \left(1 - \frac{3\epsilon}{2}(r+1)\right)\frac{\dot{H}^2}{H} + \left(3\gamma H + \tau_0 H^{-3(s-1)} + \frac{3\epsilon H}{2}(1 - \gamma(r+1))\right)\dot{H} + \tau_0\gamma H^{-3s+5} + 3H^3(\frac{\gamma\epsilon}{2} - 1)$. Here $\epsilon = 0$ for TIS theory, $\epsilon = 1$ for FIS theory and we consider barotropic behaviour of temperature ($T \sim \rho^r$, where $r = \frac{\gamma-1}{\gamma}$). In order to study the stability of the equilibrium points of the evolution equation of the viscous cosmological fluid on the brane with *GB* term, we shall rewrite eq. (6.33) in the form of an autonomous dynamical system, by introducing a new variable $X = \dot{H}$. We get

$$P(H, X) = \frac{\partial H}{\partial t} = X, \tag{6.34}$$

$$Q(H, X) = \frac{\partial X}{\partial t} = -\left(1 - \frac{3\epsilon}{2}(r+1)\right)\frac{X^2}{H} -$$

$$\left(3\gamma H + \tau_0 H^{-3(s-1)} + \frac{3\epsilon H}{2}(1 - \gamma(1+r)\right)X - \tau_0\gamma H^{-3s+5} - \frac{3}{2}H^3(\gamma\epsilon - 2). \tag{6.35}$$

The critical points are $P_0(0,0)$ and $P_0(h_0, 0)$. The critical points correspond to a Minkowskian space time ($a(t) = const.$) and to a de Sitter inflationary phase, with $a(t) = \exp(h_0 t)$. The Point $P_0(h_0, 0)$ will be stable if $p = \frac{\partial Q(H,X)}{\partial X} = -\frac{\partial R(H,\dot{H}=X)}{\partial X} < 0$ and $q_1 = -\frac{\partial Q(H,X)}{\partial H} = \frac{\partial R(H,\dot{H}=X)}{\partial H} > 0$. The stability of critical point for different type of fluid characterized by γ for TIS and FIS theory is given in table (6.1).

According to the standard theory for this type of differential equation [161] the equilibrium state ($H = 0, X = 0$) is stable if it satisfies the following conditions

$$HR(H, 0) = \tau_0\gamma H^{5-3s} + 3H^3\left(\frac{\gamma\epsilon}{2} - 1\right) > 0; \quad H \neq 0,$$

$$\frac{\partial R(H, X)}{\partial X} = \left(1 - \frac{3\epsilon}{2}(r+1)\right)\frac{2X}{H} + \left(3\gamma H + \tau_0 H^{3-3s} + \frac{3\epsilon}{2}(1 - \gamma(r+1))H\right)$$

$$= H\left(-(1 - \frac{3\epsilon}{2}(r+1))2(q+1) + (3\gamma + \tau_0 H^{2-3s} + \frac{3\epsilon}{2}(1 - \gamma(r+1))\right) \geq 0;$$

γ	TIS	FIS
2	$h_0 > (\frac{-\tau_0}{6})^{\frac{1}{3s-2}}$, $s \leq \frac{23}{12}$	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{3s-2}}, s < \frac{5}{3}$
$\frac{4}{3}$	$h_0 > (\frac{-\tau_0}{4})^{\frac{1}{3s-2}}, s \leq \frac{107}{48}$	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{3s-2}}$, $s \leq \frac{23}{12}$
1	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{3s-2}}, s \leq \frac{8}{3}$	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{3s-2}}, s \leq \frac{13}{6}$

Table 6.1: shows the variation of the stability condition of critical point in RS Brane regime without GB terms for different value of γ.

$$\frac{\partial R(H,X)}{\partial H} = -\left(1 - \frac{3\epsilon}{2}(r+1)\right)\frac{X^2}{H^2} + \left(3\gamma + \tau_0(3-3s)H^{2-3s} + \frac{3\epsilon}{2}(1-\gamma(r+1))\right)X + \gamma\tau_0(5-3s)H^{4-3s} + 9(\frac{\gamma\epsilon}{2}-1)H^2 = X\left(3\gamma + \tau_0(3-3s)H^{2-3s} + \frac{3\epsilon}{2}(1-\gamma(r+1))\right) - \left((1-\frac{3\epsilon}{2}(r+1))(q+1)\frac{X}{H^2} + \gamma\tau_0(5-3s)H^{2-3s}\frac{1}{(q+1)} + 9(\frac{\gamma\epsilon}{2}-1)\frac{1}{(q+1)}\right)X > 0.$$

The stability criteria of the critical points are also noted in terms of the deceleration parameter. For a large value of time, i.e., $H \to 0$, the term $H^{-(3s-2)}$ dominates for $(s > \frac{2}{3})$ in the expression of $\frac{\partial R(H,X)}{\partial X}$ and $\frac{\partial R(H,X)}{\partial H}$ making it obviously non-negative. For the small value of time, i.e., $H \to \infty$, the term $H^{-(3s-2)} \to 0$ for $(s > \frac{2}{3})$, the possibility of getting stable critical point for different fluid medium is determined in terms of q, which are given in table (6.2).

γ	TIS	FIS
2	$-1 < q \leq 2$	$-\frac{11}{5} < q \leq \frac{7}{5}$
$\frac{4}{3}$	$-1 < q \leq 1$	$-1 < q \leq 4.35$
1	$-1 < q \leq \frac{1}{2}$	$-1 < q \leq 5.4$

Table 6.2: shows the variation of stability condition of critical point in terms of q in RS Brane model with GB terms for different value of γ.

6.4.2 *RS brane regime without GB terms*

The general evolution equation of the bulk viscous cosmological fluid on the brane in GB regime can be obtained using eqs. (6.9), (6.12), (6.14), (6.16). The dynamical equation is a second order non-linear differential equation of the form

$$\ddot{H} + R(H, \dot{H}) = 0 \tag{6.36}$$

where $R(H,\dot{H}) = -\left(1+(r+1)\frac{\epsilon}{2}\right)\frac{\dot{H}^2}{H} + H\left(3\gamma + \tau_0 H^{-s} + \frac{3\epsilon}{2}(1-(r+1)\gamma)\right)\dot{H}$ $+3\gamma\tau_0 H^{3-s} + 9(\frac{\gamma\epsilon}{2} - 1)H^3$. Here $\epsilon = 0$ for TIS theory, $\epsilon = 1$ for FIS theory and we consider barotropic behaviour of temperature ($T \sim \rho^r$, where $r = \frac{\gamma-1}{\gamma}$). In order to study the stability of the equilibrium points of the evolution equation of the viscous cosmological fluid on the brane, we shall rewrite eq. (6.36) in the form of an autonomous dynamical system, by introducing a new variable $X = \dot{H}$, we get

$$P(H,X) = \frac{\partial H}{\partial t} = X, \tag{6.37}$$

$$Q(H,X) = \left(1+(r+1)\frac{\epsilon}{2}\right)\frac{X^2}{H} -$$

$$H\left(3\gamma + \tau_0 H^{-s} + \frac{3\epsilon}{2}(1-(r+1)\gamma)\right)X - 3\gamma\tau_0 H^{3-s} - 9(\frac{\gamma\epsilon}{2} - 1)H^3. \tag{6.38}$$

The critical points are $P_0(0,0)$ and $P_0(h_0,0)$. They corresponds to a Minkowskian space time ($a(t) = const.$) and to a de Sitter inflationary phase, with $a(t) = exp(h_0 t)$. The Point $P_0(h_0,0)$ will be stable if $p = \frac{\partial Q(H,X)}{\partial X} = -\frac{\partial R(H,\dot{H}=X)}{\partial X} < 0$ and $q_1 = -\frac{\partial Q(H,X)}{\partial H} = \frac{\partial R(H,\dot{H}=X)}{\partial H} > 0$. We note that the critical point for different equation of state are stable for the following cases as given in table (6.3).

γ	TIS	FIS
2	$h_0 > (\frac{-\tau_0}{6})^{\frac{1}{s}}$, $s \le \frac{15}{4}$	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{s}}, s < 3$
$\frac{4}{3}$	$h_0 > (\frac{-\tau_0}{4})^{\frac{1}{s}}$, $s \le \frac{75}{16}$	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{s}}$, $s \le \frac{15}{4}$
1	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{s}}$, $s \le 6$	$h_0 > (\frac{-\tau_0}{3})^{\frac{1}{s}}$, $s \le \frac{9}{2}$

Table 6.3: shows variation of the stability condition of critical point in RS Brane regime without GB terms for different value of γ.

According to the standard theory for this type of differential equation [161] the equilibrium state ($H = 0, X = 0$) is stable, when

$$HR(H,0) = 3\gamma\tau_0 H^{3-s} + 9H^3(\frac{\gamma\epsilon}{2} - 1) > 0; \quad H \neq 0,$$

$$\frac{\partial R(H,X)}{\partial X} = -\left(1+(r+1)\frac{\epsilon}{2}\right)\frac{2X}{H} + H\left(3\gamma + \tau_0 H^{-s} + \frac{3\epsilon}{2}(1-(r+1)\gamma)\right)$$
$$= H\left((1+(r+1)\frac{\epsilon}{2})2(q+1) + 3\gamma + \tau_0 H^{-s} + \frac{3\epsilon}{2}(1-(r+1)\gamma)\right) \geq 0$$

$\frac{\partial R(H,X)}{\partial H} = \left(1+(r+1)\frac{\epsilon}{2}\right)\frac{X^2}{H^2} + \left(3\gamma + \tau_0(1-s)H^{-s} + \frac{3\epsilon}{2}(1-(r+1)\gamma)\right)X +$
$3\gamma\tau_0(3-s)H^{2-s} + 27(\frac{\gamma\epsilon}{2}-1)H^2 = X\left(3\gamma + \tau_0(1-s)H^{-s} + \frac{3\epsilon}{2}(1-\gamma(r+1))\right) +$
$\left((1+\frac{3\epsilon}{2}(r+1))(q+1)\frac{X}{H^2} - 3\gamma\,\tau_0(3-s)H^{-s}\frac{1}{(q+1)} - 27(\frac{\gamma\epsilon}{2}-1)\frac{1}{(q+1)}\right)X > 0.$

The stability of the critical point can also be determined in terms of the deceleration parameter. For large time limit, $H \to 0$, the term H^{-s} dominates ($s > 0$) in the expression of $\frac{\partial R(H,X)}{\partial X}$ and $\frac{\partial R(H,X)}{\partial H}$ making it obviously non-negative. For the small time limit, $H \to \infty$ the term $H^{-s} \to 0$ for ($s > 0$), the condition of the stability of the critical point in terms of q is given in table (6.4).

γ	TIS	FIS
2	$-1 < q \leq 8$	$-\frac{13}{7} < q \leq \frac{5}{7}$
$\frac{4}{3}$	$-1 < q \leq 6.56$	$-1 < q \leq 2.45$
1	$-1 < q \leq 5.9$	$-1 < q < 3.15$

Table 6.4: shows the variation of stability condition of critical point in terms of q in RS Brane model without GB terms for different value of γ.

The special case $\gamma = 2$ in FIS theory is studied in *Ref* [162], a condition on q are obtained which satisfy the lower limit $\frac{-13}{7} \leq q$ for a stable critical point for RS model. In our case (FIS theory), we found that stable critical points are permitted for $\gamma = 2$ with ($\frac{-13}{7} \leq q < \frac{5}{7}$).

6.5 Discussion

In this chapter both power law and exponential solutions are studied in the Randall-Sundrum type II (RS) brane-world scenario in the GB regime with viscous fluid. The truncated and Full Israel-Stewart theory are employed here to describe the viscous fluid. It is found that the effect of viscosity in general is to increase the rate of expansion of the universe. We do not get a viscous universe with power law behaviour; (a) in the GB- regime for $s = \frac{2}{3}$ and $D = \frac{1}{\gamma}$, (b) in the RS regime for $s = 0$ and $D = \frac{1}{3\gamma}$ in TIS theory. In TIS theory for $\gamma = 1$ we note $a(t) \sim t$ in GB but $a(t) \sim t^{1/3}$ without GB. Thus the rate of expansion is more in the presence of GB term than that when GB-term is absent in RS-brane. The coefficient of viscosity η is determined in terms of γ. The evolution of characteristic temperature of the viscous universe is determined in FIS theory. Exponential de Sitter solution ($H = H_0$) is explored in FIS theory both in the GB regime and in the RS regime. In this chapter we also study the stability of the equilibrium points of the dynamical system associated with the evolution of the viscous cosmological fluid both in the RS regime and in the GB regime. The stability condition of critical point in the GB and in the RS regime are given in tables (6.1)-(6.4). The tables shows the upper limit of deceleration parameter (q) for stability of critical point which decreases with higher values of γ in FIS theory, but in TIS that decreases for lower values of γ both in GB and RS regime. For a stable critical point in the GB regime the value of (q) lies between $-\frac{13}{7} > q > \frac{5}{7}$ for stiff fluid in FIS theory, that for RS brane regime the value of (q) lies between $-\frac{11}{5} > q > \frac{7}{5}$. In the GB regime the equilibrium point is stable if $h_0 > \left(-\frac{\tau_0}{3}\right)^{\frac{1}{3s-2}}$ for $s \leq \frac{5}{3}$ and in the RS regime the equilibrium point is stable if $h_0 > \left(-\frac{\tau_0}{s}\right)^{\frac{1}{3}}$ for $s \leq 3$ for stiff fluid.