CHAPTER 5

Cosmological Models with Variable Gravitational and Cosmological Constants in R^2 Gravity

5.1 Introduction

In the last couple of decades there has been a growing interest to study cosmological models in the presence of a variable cosmological constant [126] including a time varying gravitational constant [127]. The interest stems from the observational analysis of type Ia supernova with redshift parameter $Z \leq 1$ [74]. Supernovae study provides evidence that we may be living in a low mass density universe of the order of $\Omega \sim 0.3$ [128]. However, the inflationary universe model, which is an essential ingredient in modern cosmology, predicts a density parameter of $\Omega \sim 1$. Consequently it comes out that a major part of the matter content in the universe remains unobserved, which is known as dark matter. Recent observations from type Ia supernovae predicts that the present universe is accelerating. To accommodate such a universe cosmological models are proposed with an assumption that there exists a small positive cosmological constant (Λ) even at the present epoch called ΛCDM model. In modern Cosmology, it is also believed that our universe might have originated from a phase of rapid expansion or inflation in the past. Such a rapid expansion or inflation may be obtained if one considers an Einstein-Hilbert action with a cosmological constant. To accommodate and to interpret the above facts, several physical models with variable cosmological and gravitational constants have been proposed in recent times. The effect of extra dimensions in the evolution of a homogeneous isotropic universe and the first quantitative study on the space and/ or time dependence of the structure and Newton's constants were considered by Forgacs and Horvath [129]. It was shown that the time and space variation of fundamental constants is a natural consequence of Kaluza-Klein theories. Bertolami [130] studied a cosmological model with a time dependent cosmological term. Later, Özer and Taha [131] and others [132] obtained models of the universe, that are found free from cosmological problems, by considering a varying cosmological constant in the case of a closed universe. Recently, a volume of literature [133]-[140] have appeared which addressed various cosmological issues, namely the age problem, observational constraints on Λ, structure formation, and gravitational lensing. Ratra and Peebles [79] studied the dark energy content of the universe making use of a varying cosmological constant permitted from a scalar field theory. Dolgov [109] analyzed the cosmological evolution of a free massless vector or tensor (not gauge) field minimally coupled to gravity and found that cosmological constant exactly cancels out. Sahni and Starobinsky [134] discussed the observational and theoretical aspects of a small positive cosmological constant term which may be relevant at the present epoch. The origin of such a small Λ has been discussed in the framework of field theoretic techniques. Such a model is allowed with a dynamical Λ term obtained from a scalar field [135]. Padmanabhan [136] also discussed several aspects of the cosmological constant both from cosmological and field theoretic perspectives in order to explain the cosmological constant problem in the framework of the string theory. Vishwakarma [137] explored different models to estimate the density parameter and/or deceleration parameter for a variable Λ and numerically analyzed the observational data in cosmology. It is found that the models with dynamical cosmological term (Λ) is in agreement with observations. Birkel and Sarkar [138] studied the nucleosynthesis and obtained a bound on a time varying Λ.

In the last few years, a number of literature in cosmology with variable cosmological and gravitational constants with a perfect fluid as a source in isotropic [139] as well as in anisotropic models [140] of the universe in the frame work of Einstein gravity has been considered. In this chapter isotropic cosmological model in a higher derivative theory (which is also known as a *'generalized theory'*) with variable cosmological and gravitational constants is considered to obtain cosmological solutions that are relevant to accommodate the recent observational evidences. A number of literature have appeared in the last few decades on the generalized theory of gravity with additional terms in R^2, $R_{\mu\nu\rho\delta}R^{\mu\nu\rho\delta}$, $C_{\mu\nu\rho\delta}C^{\mu\nu\rho\delta}$ in the Lagrangian also. However, in four dimensions, only the R^2 term is important and the combinations of terms e.g. Gauss-Bonnet combinations ($R_{abcd}R^{abcd}$ −

$4R_{ab}R^{ab} + R^2$) occurs as Euler number in the Einstein action. Thus GB term does not contribute in the dynamical evolution of the universe. The generalized theory of gravity has a number of useful features. It is a renormalizable, asymptotically free theory [107]. However, the unitarity problem of the theory is not settled [141].

The chapter is organized as follows. In sec. 5.2, the gravitational action and field equations are presented. Cosmological solutions are given in sec. 5.3, and finally in sec. 5.4 a brief discussion is given.

5.2 Gravitational Action and Field Equations

Let us consider a gravitational action with higher order terms in the scalar curvature (R) including a time varying cosmological constant ($\Lambda(t)$) and gravitational constant ($G(t)$) which is given by

$$I = -\int \left[\frac{f(R) - 2\Lambda}{16\pi G(t)} + L_m\right] \sqrt{-g}\, d^4x, \tag{5.1}$$

where $f(R)$ is a function of scalar curvature (R) and its higher power and L_m represents the matter lagrangian.

Variation of action (5.1) with respect to $g_{\mu\nu}$ yields

$$\begin{aligned} f'(R)\, R_{\mu\nu} - \frac{1}{2}\ (f(R) - 2\Lambda)\ g_{\mu\nu} + f''(R)\,(\nabla_\mu\nabla_\nu R - g_{\mu\nu}\nabla^\mu\nabla^\nu R g_{\mu\nu}) + \\ f'''(R)\,(\nabla_\mu R\nabla_\nu R - \nabla^\sigma R\nabla_\sigma R\, g_{\mu\nu}) = -\, 8\pi G\, T_{\mu\nu}, \end{aligned} \tag{5.2}$$

where ∇_μ is the covariant differential operator, and a prime represents the derivative with respect to R, $T_{\mu\nu}$ is the energy momentum tensor for matter fields determined by L_m .

We consider here an isotropic universe given by Robertson-Walker space-time in eq. (2.4). The trace and (0,0) components of eq.(5.2) are given by

$$Rf'(R) - 2\,(f(R) - 2\Lambda) + 3f''(R)\left(\ddot{R} + 3\frac{\dot{a}}{a}\dot{R}\right) + 3f'''(R)\dot{R} + 8\pi G\ \ T = 0, \tag{5.3}$$

$$f'(R)R_{00} + \frac{1}{2}\,(f(R) - 2\Lambda) - 3f''(R)\frac{\dot{a}}{a}\dot{R} + 8\pi G\ \ T_{00} = 0, \tag{5.4}$$

where the dot represents the derivative with respect to comoving time (t).

Here a polynomial in R upto second order is considered which is $f(R) = R + \alpha R^2$ in the above. Using the eq. (2.5) for the scalar curvature (R) in eqs. (5.3) and (5.4), which yield

$$H^2 + \frac{k}{a^2} - 6\alpha\left[2H\ddot{H} - \dot{H}^2 - \frac{2kH^2}{a^2} + 6\dot{H}H^2 + \frac{k^2}{a^4}\right] = \frac{8\pi G(t)\rho}{3} + \frac{\Lambda(t)}{3}, \tag{5.5}$$

$$\frac{d\rho}{dt} + 3(\rho+p)H = -\left(\frac{\dot{G}}{G}\,\rho + \frac{\dot{\Lambda}}{8\pi G}\right), \tag{5.6}$$

where ρ and p represent the cosmic fluid energy density and the pressure respectively. The equation of state is given by

$$p = (\gamma - 1)\rho \tag{5.7}$$

where γ is a constant ($1 \leq \gamma \leq 2$) and $\gamma = 1$ corresponds to matter, $\gamma = \frac{4}{3}$ represents radiation respectively. The deceleration parameter (q) is related to H as

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1.$$

Thus $q < 0$ represents acceleration, $q > 0$ represents deceleration of the universe.

5.3 Cosmological Solutions

The dynamical eqs. (5.5)-(5.7) are considered to obtain cosmological solutions. For simplicity we consider a flat universe ($k = 0$). It may be pointed out here that a flat universe is supported by recent cosmological predictions. Now, the set of equations contain six unknowns: to solve them, it is necessary to consider time variation of two of them *a priori* for a given equation of state.

Let us consider power law behavior of the universe which is given by

$$a(t) = a_0\, t^D \tag{5.8}$$

where a_0 and D are constants to be determined from the field equations. For two different phases of the universe namely, (i) the radiation dominated and (ii) the matter dominated, the cosmological solutions are discussed in the next section.

5.3.1 *Radiation Dominated Universe ($p = \frac{1}{3}\rho$)*

Radiation dominated phase of the universe corresponds to $\gamma = \frac{4}{3}$ i.e. $p = \frac{1}{3}\rho$ and in the absence of particle creation eq. (5.6) leads to two different decoupled equations which are given by

$$\frac{d\rho}{dt} + 4H\rho = 0, \tag{5.9}$$

$$\dot{\Lambda}(t) + 8\pi\rho \dot{G(t)} = 0. \tag{5.10}$$

On integration eq. (5.9) yields a relation between the energy density and the scale factor of the universe, which is

$$\rho = \frac{\rho_0}{a^4}, \tag{5.11}$$

where ρ_0 is a constant of integration. For a power law expansion given by eq. (5.8), time variation of cosmological and gravitational constants [128] are determined using eqs. (5.5), (5.10) and (5.11), which are as follows :

$$\Lambda(t) = \frac{3D}{2t^2}(2D-1) + \frac{54\alpha D(D-1)(2D-1)}{t^4}, \tag{5.12}$$

$$G(t) = \frac{3a_0^4}{16\pi\rho_0}\left[\frac{D}{t^{2-4D}} + \frac{36\alpha D(2D-1)}{t^{4-4D}}\right], \tag{5.13}$$

where α is the coupling parameter of R^2- term in the gravitational action. The deceleration parameter becomes: $q = \frac{1-D}{D}$, which is positive for $D < 1$ and negative for $D > 1$.

For simplicity we consider the following cosmological solutions :

(i) For $D = \frac{1}{2}$, the scale factor of the universe grows as $a(t) \sim \sqrt{t}$. In this case one obtains

$$\Lambda = 0 \;\; ; \qquad\qquad q > 0,$$

$$G = \frac{3a_0^4}{32\pi\rho_0} = const.$$

Thus it is evident that the presence of higher derivative terms, does not modify the behaviour of the early universe evolution in a radiation dominated universe when $\Lambda = 0$ and $G = const.$ It has a particle horizon. The cosmological evolution in this case is similar to that obtained for a radiation dominated era in the Einstein gravity.

(ii) For $D = 1$, the scale factor of the universe grows as $a(t) \sim t$. In this case we get

$$\Lambda(t) = \frac{3}{2t^2}, \qquad\qquad q = 0,$$

$$G(t) = \frac{3a_0^4}{16\pi\rho_0}\left(t^2 + 36\alpha\right).$$

It is evident that $\Lambda(t)$ decreases with time where as $G(t)$ grows. To begin with, at the early epoch, G shows constant behaviour determined by the coupling constant (α), initial size and matter density. For $\alpha < 0$, the solution is physically relevant for the epoch $t \geq 6\sqrt{|\alpha|}$. In this case the universe has no particle horizon.

In the presence of particle creation during radiation dominated era the field equation permits some cosmological solutions which are presented here. In a radiation dominated era eq. (5.6) takes the form:

$$\frac{d\rho}{dt} + 4\rho H = -\left(\frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G}\right). \tag{5.14}$$

The field eqs. (5.5) and (5.14), determine the evolution of the energy density and the cosmological constant, which are given by

$$\rho = \frac{3D}{16\pi G}\left[\frac{1}{t^2} + \frac{36\alpha(2D-1)}{t^4}\right], \tag{5.15}$$

$$\Lambda = \frac{3D}{2t^2}(2D-1) + \frac{54\alpha D(2D-1)(D-1)}{t^4}. \tag{5.16}$$

Special cases :

(i) When $D = \frac{1}{2}$, we get

$$\Lambda = 0, \quad \rho = \frac{3}{32\pi G}\frac{1}{t^2}.$$

Here, the solution is similar to that one obtains during usual radiation dominated universe. The solution is also similar to that obtained with $\alpha \neq 0$ in the absence of a cosmological constant. Thus R^2 terms does not effect the dynamics during radiation dominated era.

(ii) When $D = 1$, we get

$$\Lambda = \frac{3}{2t^2}, \quad \rho = \frac{3}{16\pi G}\left[\frac{1}{t^2} + \frac{36\alpha}{t^4}\right].$$

In this case the cosmological constant is decreasing with time but it is independent of the higher derivative term. The energy density is determined by the coupling parameter α. For $\alpha < 0$, physically relevant solutions are obtained for $t \geq 6\sqrt{|\alpha|}$. In the above cosmological solutions are permitted for a constant gravitational parameter.

5.3.2 *Matter Dominated Universe ($p = 0$)*

A matter dominated universe, is characterized by equation of state $p = 0$ i.e. $\gamma = 1$. In the absence of particle creation and $\gamma = 1$, eq. (5.6) can be written as two decoupled equations:

$$\frac{d\rho}{dt} + 3H\rho = 0, \tag{5.17}$$

$$\dot{\Lambda}(t) + 8\pi\rho \dot{G}(t) = 0. \tag{5.18}$$

On integration eq. (5.17) yields

$$\rho = \frac{\rho_0}{a^3}, \tag{5.19}$$

where ρ_0 is an integration constant. Now variation of cosmological constant and gravitational constant are obtained as follows from the field eqs.(5.5), (5.18) and (5.19), which yield :

$$\Lambda(t) = \frac{D}{t^2}(3D-2) + \frac{18\alpha D(2D-1)(3D-4)}{t^4}, \tag{5.20}$$

$$G(t) = \frac{a_0^3}{4\pi\rho_0}\left[\frac{D}{t^{2-3D}} + \frac{36\alpha D(2D-1)}{t^{4-3D}}\right]. \tag{5.21}$$

The following cosmological solutions are noted:

(i) For $D = \frac{1}{2}$, the scale factor of the universe grows as $a(t) \sim \sqrt{t}$, we get the variation of the cosmological constant

$$\Lambda(t) = -\frac{1}{4t^2}, \qquad q > 0$$

and the corresponding variation of Newton's gravitational constant is given by

$$G(t) = \frac{a_0^3}{8\pi\rho_0\sqrt{t}}.$$

Thus cosmological solutions are obtained here in the presence of a negative cosmological constant. The gravitational constant remains positive to begin with which however decreases with time. Thus a comparatively smaller rate of expansion in the matter dominated regime is obtained.

(ii) For $D = \frac{2}{3}$, the scale factor of the universe grows as $a(t) \sim t^{\frac{2}{3}}$, we get

$$\Lambda(t) = -\frac{8\alpha}{t^4}, \qquad q > 0,$$

$$G(t) = \frac{a_0^3}{4\pi\rho_0}\left[\frac{2}{3} + \frac{8\alpha}{t^2}\right].$$

Thus in the matter dominated era, the field equation permits cosmological solution with (i) $\Lambda < 0$ if $\alpha > 0$ or (ii) $\Lambda > 0$ if $\alpha < 0$. It is interesting to note that Newton's gravitational constant G attains a constant value at a later epoch, i.e. at $t \to \infty$. This cosmology has a particle horizon. The solution is new and interesting in model building for describing a late universe. The cosmological constant becomes small depending upon time and the coupling parameter at a late epoch. Here, one recovers the matter dominated solution of Einstein gravity for $\alpha = 0$ (as $\Lambda = 0$ and $G = G_0$). Similar solution in higher derivative gravity is permitted for $\Lambda < 0$ with a varying G as discussed above.

(iii) For $D = \frac{4}{3}$, the scale factor of the universe evolves as $a(t) \sim t^{\frac{4}{3}}$, which is a rapid power law expansion accommodating present accelerating phase. In this epoch, one obtains

$$\Lambda(t) = \frac{8}{3t^2}, \qquad q < 0,$$
$$G(t) = \frac{a_0^3}{4\pi\rho_0}\left[80\alpha + \frac{4}{3}t^2\right].$$

Thus, in a matter dominated era, an accelerating universe is permitted in the higher order gravity for a decaying cosmological constant with an increasing gravitational constant $G(t)$ for $\alpha > 0$. This is a new and interesting cosmological solution relevant for a viable cosmological scenario. It admits a late accelerating universe which is supported by cosmological observation. For $\alpha < 0$, at $t \to 0$, one gets solution with $G(t)$ negative which later attains positive values for $t > \sqrt{|60\alpha|}$, and thereafter $G(t)$ grows. The solution does not admit a particle horizon.

However in the presence of particle creation and the matter domination era, eq. (5.6) becomes

$$\frac{d\rho}{dt} + 3H\rho = -\left(\frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G}\right). \tag{5.22}$$

In the case of power law expansion of the universe, the energy density and the cosmological constant are obtained from eq. (5.5) and (5.22) which are given by

$$\rho = \frac{D}{4\pi G}\left[\frac{1}{t^2} + \frac{36\alpha(2D-1)}{t^4}\right], \tag{5.23}$$

$$\Lambda = \frac{D}{t^2}(3D-2) + \frac{18\alpha D(2D-1)(3D-4)}{t^4}. \tag{5.24}$$

Special cases :

(i) For $D = \frac{1}{2}$, the scale factor of the universe grows as $a(t) \sim \sqrt{t}$, the cosmological parameter and the energy density varies as,

$$\Lambda = -\frac{1}{4t^2}, \quad \rho = \frac{1}{8\pi G t^2}.$$

(ii) For $D = \frac{2}{3}$, the scale factor of the universe evolves as $a(t) \sim t^{\frac{2}{3}}$, we get

$$\Lambda = -\frac{8\alpha}{t^4}, \quad \rho = \frac{1}{6\pi G}\left[\frac{1}{t^2} + \frac{12\alpha}{t^4}\right].$$

Thus, the usual matter dominated solution is recovered with $\Lambda < 0$, in the framework of the higher derivative gravity.

(iii) For $D = \frac{4}{3}$, the scale factor of the universe evolves as $a(t) \sim t^{\frac{4}{3}}$, which is a rapid power law expansion. In this epoch, the cosmological constant and the energy density evolves as

$$\Lambda = \frac{8}{3t^2}, \quad \rho = \frac{1}{3\pi G}\left[\frac{1}{t^2} + \frac{60\alpha}{t^4}\right].$$

For $D = \frac{1}{2}$ and $D = \frac{4}{3}$, the cosmological constant is independent of the higher derivative term. Moreover, a positive cosmological constant is required for $D = \frac{4}{3}$ and a negative cosmological constant is required for $D = \frac{1}{2}$. However, in the former case, energy density depends on the coupling parameter associated with the higher derivative term. In the former case, it has no particle horizon. It is interesting to note that in the case of $D = \frac{2}{3}$, both the cosmological constant and energy density are determined by α. It may be pointed out here that the gravitational constant G is required to be a constant in this cases. The solution obtained here describes a universe which is accelerating even in the matter dominated phase, this is supported by present cosmological observations.

5.4 Discussion

In this chapter, isotropic cosmological models with varying G and Λ in R^2 theory is studied. Both the radiation dominated and matter dominated phase of evolution of the universe with varying G and Λ are taken up to obtain cosmological solutions. In this case some new cosmological solutions in the presence of matter dominated era are obtained which are not admissible in Einstein gravity. These solutions are relevant for accommodating a late accelerating universe. As the field equation is highly non-linear some special cases are taken up here to compare the cosmological solutions in the absence of higher derivative terms. It is found that a Λ and G varying parameters admits new solution where G is increasing similar to that found by Abdel Rahaman [127] and Arbab [142] in an Einstein gravity. The cosmological constant Λ, however, found to decrease with time here. In the frame work of an R^2 theory, we note an interesting solution in the matter dominated regime. The gravitational constant G decreases initially and ultimately attains a constant at a later stage of evolution of the universe. The universe evolves as $a \sim t^{\frac{2}{3}}$, which is similar to that which one obtains in an Einstein gravity during matter domination. In this case a realistic variation of the parameter G is obtained where to begin with G is large in the early universe, which, However, at the present epoch, attains a constant. The above cosmological solutions are permitted either with (i) $\Lambda > 0$ if $\alpha < 0$ or (ii) with $\Lambda < 0$ if $\alpha > 0$. In the matter dominated era a new solution exists in R^2 theory which admits an accelerating universe (supported by supernova observation), with a decaying cosmological constant and a constant G. It may be pointed out here that an inflationary solution ($a(t) \sim e^{H_0 t}$) is also permitted in R^2 theory if one considers a non-zero Λ even with $\rho = p = 0$. A power law inflation is also admitted in the matter dominated universe (MDU) or in the radiation dominated universe (RDU) with varying Λ and G in the presence of particle creation. In MDU, a power law expansion of the universe is permissible if one allows G to increase at a later stage. However, when $\Lambda \to 0$, the solution is no longer valid and the usual matter dominated universe solution may be recovered with a constant G. In the presence of particle creation a number of interesting solutions are obtained which accommodates the present accelerating phase accommodating other observable features of the universe.