CHAPTER 3

SOME RESULTS ON THE GENERALISED GROWTH ESTIMATES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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3.1 Introduction, Definitions and Notations.

Let f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$. It is well known that [**6**], $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty$ where $f\circ g$ stands for the composition of f and g. Singh [**42**] proved some comparative growth properties of $\log T(r,f\circ g)$ and $T(r,f)$. Some results on the comparative growth of $\log T(r,f\circ g)$ and $T(r,f)$ are also proved by Lahiri [**26**]. In this chapter we study some generalised comparative growth properties of composite entire and meromorphic functions in the direction of Singh [**42**] and Lahiri [**26**]. We also exetend our results for a special type of differential polynomials viz. wronskians generated by entire and meromorphic functions.In the sequel we use the following two notations:

$$\log^{[k]}x=\log\left(\log^{[k-1]}x\right)\text{ for }k=1,2,3,\cdots\text{and }\log^{[0]}x=x$$

The results of this chapter have been published in **News Bulletin of Calcutta Mathematical Society,** see [**13**].

and

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x) \text{ for } k = 1, 2, 3, \cdots \text{and } \exp^{[0]} x = x.$$

The following definitions are well known.

Definition 3.1.1 *The order ρ_f and the lower order λ_f of an entire function f are respectively defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

If f is meromorphic, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Extending this notion, Sato [**37**] defined the generalised order and generalised lower order of an entire function as follows :

Definition 3.1.2 *The generalised order $\rho_f^{[l]}$ and the generalised lower order $\lambda_f^{[l]}$ of an entire function f are defined as :*

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M(r, f)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M(r, f)}{\log r} \text{ where } l = 2, 3, 4...$$

If f is meromorphic, one can also verify that

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T(r, f)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} T(r, f)}{\log r} \text{ where } l = 2, 3, 4...$$

Definition 3.1.3 *A meromorphic function $a = a(z)$ is called small with respect to f if $T(r, a) = S(r, f)$.*

Definition 3.1.4 *Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k, f)$ the Wronskian determinant of $a_1, a_2, ..., a_k, f$ i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1^{(k)} & a_2^{(k)} & \cdots & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 3.1.5 *If $a \in \mathbb{C} \cup \{\infty\}$, the quantity*

$$\begin{aligned}\delta(a; f) &= 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} \\ &= \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}\end{aligned}$$

is called the Nevanlinna deficiency of the value $'a'$.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf.[[**21**], p.43]). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

3.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.2.1 *[1] If f is a meromorphic function and g is an entire function then for all sufficiently large values of r,*

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 3.2.2 *[6] If f and g are entire functions then for all sufficiently large values of r,*

$$M(r, f \circ g) \geq M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).$$

Lemma 3.2.3 *[2] Let f be a meromorphic function and g be an entire function and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f \circ g) \geq T(\exp(r^{\mu}), f).$$

Lemma 3.2.4 *[32] Let f be a transcendental meromorphic function having the maximum deficiency sum . Then*

$$\lim_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta(\infty; f).$$

Lemma 3.2.5 *If f be a transcendental meromorphic function with the maximum deficiency sum, then the generalised order and generalised lower order of $L(f)$ are same as those of f.*

Proof. By Lemma 3.2.4

$$\lim_{r\to\infty} \frac{\log^{[l-1]} T(r, L(f))}{\log^{[l-1]} T(r, f)}$$

exists and is equal to 1, where $l = 2, 3, 4.....$ So

$$\begin{aligned}
\rho_{L(f)}^{[l]} &= \limsup_{r\to\infty} \frac{\log^{[l-1]} T(r, L(f))}{\log r} \\
&= \limsup_{r\to\infty} \frac{\log^{[l-1]} T(r, f)}{\log r} . \lim_{r\to\infty} \frac{\log^{[l-1]} T(r, L(f))}{\log^{[l-1]} T(r, f)} \\
&= \rho_f^{[l]} . 1 \\
&= \rho_f^{[l]} .
\end{aligned}$$

In a similar manner, $\lambda_{L(f)}^{[l]} = \lambda_f^{[l]}$.
This proves the lemma. ■

3.3 Theorems.

In this section we present the main results of the chapter.

Theorem 3.3.1 *Let h be meromorphic, k, g be entire such that $\lambda_h^{[p]} > 0$, $\rho_g < \lambda_k \leq \rho_k$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for every entire function f with $0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty$,*

$$\limsup_{r\to\infty} \frac{\log^{[p+1-l]} T(r, h \circ k) \log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, f) \left\{ \log T(r, f \circ g) + \log^{[2]} M(r, g) \right\}} = \infty ,$$

where p and l are integers greater that unity.

Proof. Since $\rho_g < \rho_k$ we can choose $\varepsilon\,(> 0)$ and μ such that

$$\rho_g + \varepsilon < \mu < \rho_k. \tag{3.1}$$

By Lemma 3.2.3 we obtain for a sequence of values of r tending to infinity

$$\log^{[p-2]} T(r, h \circ k) > \exp\left[r^{\mu}\delta\right] \tag{3.2}$$

where $0 < \mu < \rho_k$ and $0 < \delta < \lambda_h^{[p]}$.

Again from Lemma 3.2.1 it follows for all sufficiently large values of r,

$$T(r, f \circ g) \log M(r, g) \leq \{1 + o(1)\} T(r, g) T(M(r, g), f)$$

$$\text{i.e., } T(r, f \circ g) \log M(r, g) \leq \{1 + o(1)\} r^{\rho_g + \varepsilon} \exp^{[l-2]} \{M(r, g)\}^{\rho_f^{[l]} + \varepsilon}$$

$$\text{i.e., } T(r, f \circ g) \log M(r, g) \leq \{1 + o(1)\} r^{\rho_g + \varepsilon} \exp^{[l-1]} \left\{r^{\rho_g + \varepsilon} \left(\rho_f^{[l]} + \varepsilon\right)\right\}$$

$$\text{i.e., } \log T(r, f \circ g) + \log^{[2]} M(r, g)$$
$$\leq (\rho_g + \varepsilon) \log r + \exp^{[l-2]} \left\{r^{\rho_g + \varepsilon} \left(\rho_f^{[l]} + \varepsilon\right)\right\} + O(1). \tag{3.3}$$

Now from (3.2) and (3.3) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p+1-l]} T(r, h \circ k)}{\log T(r, f \circ g) + \log^{[2]} M(r, g)}$$
$$> \frac{\exp^{[l-2]}\left[r^{\mu}\delta\right]}{(\rho_g + \varepsilon) \log r + \exp^{[l-2]} \left\{r^{\rho_g + \varepsilon} \left(\rho_f^{[l]} + \varepsilon\right)\right\} + O(1)}. \tag{3.4}$$

Thus from (3.1) and (3.4) it follows that

$$\limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k)}{\log T(r, f \circ g) + \log^{[2]} M(r, g)} = \infty. \tag{3.5}$$

Let us choose $0 < \varepsilon < \min\left\{\lambda_f^{[l]}, \lambda_g\right\}$. Now for all sufficiently large values of r we get from Lemma 3.2.2 that

$$M(r, f \circ g) \geq M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right)$$

so that for all sufficiently large values of r,

$$\log^{[l]} M(r, f \circ g) \geq \log^{[l]} M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right)$$

$$\text{i.e., } \log^{[l]} M(r, f \circ g) \geq \left(\lambda_f^{[l]} - \varepsilon\right) \log\left\{\frac{1}{16} M\left(\frac{r}{2}, g\right)\right\}$$

$$\text{i.e., } \log^{[l]} M(r, f \circ g) \geq \left(\lambda_f^{[l]} - \varepsilon\right) \log \frac{1}{16} + \left(\lambda_f^{[l]} - \varepsilon\right) \left(\frac{r}{2}\right)^{(\lambda_g - \varepsilon)}. \tag{3.6}$$

Again from the definition of $\rho_f^{[l]}$ it follows that for all large values of r,

$$\log^{[l]} M(\exp r, f) \leq \left(\rho_f^{[l]} + \varepsilon\right) \log(\exp r)$$
$$\text{i.e., } \log^{[l+1]} M(\exp r, f) \leq O(1) + \log r. \tag{3.7}$$

Thus from (3.6) and (3.7) we obtain for all sufficiently large values of r,

$$\frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, f)} \geq \frac{\left(\lambda_f^{[l]} - \varepsilon\right) \log \frac{1}{16} + \left(\lambda_f^{[l]} - \varepsilon\right) \left(\frac{r}{2}\right)^{(\lambda_g - \varepsilon)}}{O(1) + \log r}$$

$$\text{i.e., } \lim_{r \to \infty} \frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, f)} = \infty. \tag{3.8}$$

Now from (3.5) and (3.8) it follows that

$$\limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k) \log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, f) \left\{\log T(r, f \circ g) + \log^{[2]} M(r, g)\right\}}$$
$$= \limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k)}{\log T(r, f \circ g) + \log^{[2]} M(r, g)} \cdot \lim_{r \to \infty} \frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, f)}$$
$$= \infty \cdot \infty = \infty.$$

This proves the theorem. ■

Remark 3.3.1 *The condition $\lambda_h^{[p]} > 0$ in Theorem 3.3.1 is necessary as we see in the following example.*

Example 3.3.1 *Let $f = g = \exp z$, $h = z$, $k = \exp\left(z^2\right)$ and $p = l = 2$.*
Then $\rho_f = \lambda_f = 1 = \lambda_g = \rho_g$, $\lambda_h = 0$ and $\lambda_k = \rho_k = 2$.
Also

$$T(r, f \circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \, (r \to \infty).$$

Now

$$T(r, h \circ k) \leq \log^+ M(r, h \circ k)$$
$$i.e., \log T(r, h \circ k) \leq \log^{[2]} \exp\left(r^2\right)$$
$$i.e., \log T(r, h \circ k) \leq 2 \log r$$

and

$$M(r, f \circ g) = \exp^{[2]} r$$
$$i.e., \log^{[2]} M(r, f \circ g) = \log^{[2]} \exp^{[2]} r = r.$$

Again

$$M(\exp r, f) = \exp^{[2]} r$$
$$i.e., \log^{[3]} M(\exp r, f) = \log r$$

and

$$\log T(r, f \circ g) + \log^{[2]} M(r, g) \sim r - \frac{1}{2}\log r + O(1) + \log^{[2]} \exp r$$
$$= r + \frac{1}{2}\log r + O(1).$$

Therefore

$$\frac{\log T(r, h \circ k) \log^{[2]} M(r, f \circ g)}{\log^{[3]} M(\exp r, f)\left\{\log T(r, f \circ g) + \log^{[2]} M(r, g)\right\}}$$
$$\leq \frac{(2\log r) \cdot r}{(\log r)\left[r + \frac{1}{2}\log r + O(1)\right]} = \frac{2r}{r + \frac{1}{2}\log r + O(1)}$$

$$i.e., \limsup_{r\to\infty} \frac{\log T(r, h \circ k) \log^{[2]} M(r, f \circ g)}{\log^{[3]} M(\exp r, f)\left\{\log T(r, f \circ g) + \log^{[2]} M(r, g)\right\}} \leq 2,$$

which is a contrary to Theorem 3.3.1.

Theorem 3.3.2 *Let h be meromorphic, k, g be entire such that $\lambda_h^{[p]} > 0$, $\rho_g^{[l]} < \lambda_k \leq \rho_k$ and $0 < \lambda_g^{[l]} \leq \rho_g^{[l]} < \infty$. Then for every entire function f with $0 < \lambda_f \leq \rho_f < \infty$,*

$$\limsup_{r\to\infty} \frac{\log^{[p+1-l]} T(r, h \circ k) \log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, g)\left\{\log T(r, f \circ g) + \log^{[2]} M(r, g)\right\}} = \infty,$$

where p and l are integers greater that unity.

Proof. From Lemma 3.2.1 it follows for all sufficiently large values of r,

$$T(r, f \circ g) \log M(r, g) \leq \{1 + o(1)\} T(r, g) T(M(r, g), f)$$
$$\text{i.e., } T(r, f \circ g) \log M(r, g) \leq \{1 + o(1)\} \exp^{[l-2]} r^{\rho_g^{[l]}+\varepsilon} \{M(r, g)\}^{\rho_f+\varepsilon}$$
$$\text{i.e., } T(r, f \circ g) \log M(r, g) \leq \{1 + o(1)\} \exp^{[l-2]} r^{\rho_g^{[l]}+\varepsilon} \exp^{[l-1]} \left\{r^{\rho_g^{[l]}+\varepsilon} (\rho_f + \varepsilon)\right\}$$

$$\text{i.e., } \log T(r, f \circ g) + \log^{[2]} M(r, g)$$
$$\leq \exp^{[l-3]} r^{\rho_g^{[l]}+\varepsilon} + \exp^{[l-2]} \left\{(\rho_f + \varepsilon) r^{\rho_g^{[l]}+\varepsilon}\right\} + O(1). \tag{3.9}$$

Now from (3.2) and (3.9) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p+1-l]} T(r, h \circ k)}{\log T(r, f \circ g) + \log^{[2]} M(r, g)}$$
$$> \frac{\exp^{[l-2]} [r^{\mu} \delta]}{\exp^{[l-3]} r^{\rho_g^{[l]}+\varepsilon} + \exp^{[l-2]} \left\{(\rho_f + \varepsilon) r^{\rho_g^{[l]}+\varepsilon}\right\} + O(1)}. \tag{3.10}$$

Now if we choose $\varepsilon\,(> 0)$ and μ such that $\rho_g + \varepsilon < \mu < \rho_k$ then from above it follows that

$$\limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k)}{\log T(r, f \circ g) + \log^{[2]} M(r, g)} = \infty. \tag{3.11}$$

Let us choose $0 < \varepsilon < \min\left\{\lambda_f, \lambda_g^{[l]}\right\}$. Now for all sufficiently large values of r we get from Lemma 3.2.2 that

$$\log^{[2]} M(r, f \circ g) \geq \log^{[2]} M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right)$$
$$\text{i.e., } \log^{[2]} M(r, f \circ g) \geq (\lambda_f - \varepsilon) \log\left\{\frac{1}{16} M\left(\frac{r}{2}, g\right)\right\}$$
$$\text{i.e., } \log^{[l]} M(r, f \circ g) \geq \log^{[l-1]} M\left(\frac{r}{2}, g\right) + O(1)$$
$$\text{i.e., } \log^{[l]} M(r, f \circ g) \geq \left(\frac{r}{2}\right)^{\left(\lambda_g^{[l]} - \varepsilon\right)} + O(1). \tag{3.12}$$

Again from the definition of $\rho_g^{[l]}$ it follows that for all large values of r,

$$\log^{[l]} M(\exp r, g) \leq \left(\rho_g^{[l]} + \varepsilon\right) \log(\exp r)$$

$$\text{i.e., } \log^{[l+1]} M(\exp r, g) \leq O(1) + \log r. \tag{3.13}$$

Thus from (3.12) and (3.13) we obtain for all sufficiently large values of r,

$$\frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, g)} \geq \frac{\left(\frac{r}{2}\right)^{\left(\lambda_g^{[l]} - \varepsilon\right)} + O(1)}{O(1) + \log r}$$

$$\text{i.e., } \lim_{r \to \infty} \frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, g)} = \infty. \tag{3.14}$$

Now from (3.11) and (3.14) it follows that

$$\begin{aligned}
&\limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k) \log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, g) \left\{\log T(r, f \circ g) + \log^{[2]} M(r, g)\right\}} \\
&= \limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k)}{\log T(r, f \circ g) + \log^{[2]} M(r, g)} \cdot \lim_{r \to \infty} \frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, g)} \\
&= \infty \cdot \infty = \infty.
\end{aligned}$$

This proves the theorem. ■

Remark 3.3.2 *Considering* $f = g = \exp z$, $h = z$, $k = \exp\left(z^2\right)$ *and* $p = l = 2$ *one can easily prove that the condition* $\lambda_h^{[p]} > 0$ *in Theorem* 3.3.2 *is necessary* .

The next two theorems are just the improvement of Theorem 3.3.1 and Theorem 3.3.2 respectively.

Theorem 3.3.3 *Let* h *be meromorphic,* k, g *be entire such that* $\lambda_h^{[p]} > 0$, $\rho_g < \lambda_k \leq \rho_k$ *and* $0 < \lambda_g \leq \rho_g < \infty$. *Then for every transcendental entire function* f *having the maximum deficiency sum with* $0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty$,

$$\limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k) \log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, L(f)) \left\{\log T(r, f \circ g) + \log^{[2]} M(r, g)\right\}} = \infty ,$$

where p *and* l *are integers greater that unity.*

Proof. For all sufficiently large values of r; we obtain in view of Lemma 3.2.5

$$\log^{[l]} M(\exp r, L(f)) \leq \left(\rho_{L(f)}^{[l]} + \varepsilon\right) \log(\exp r)$$
$$\text{i.e., } \log^{[l]} M(\exp r, L(f)) \leq \left(\rho_{f}^{[l]} + \varepsilon\right) \log(\exp r)$$
$$\text{i.e., } \log^{[l+1]} M(\exp r, L(f)) \leq O(1) + \log r. \tag{3.15}$$

Thus from (3.6) and (3.15) we obtain for all sufficiently large values of r,

$$\frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, L(f))} \geq \frac{\left(\lambda_f^{[l]} - \varepsilon\right) \log \frac{1}{16} + \left(\lambda_f^{[l]} - \varepsilon\right) \left(\frac{r}{2}\right)^{(\lambda_g - \varepsilon)}}{O(1) + \log r}$$

$$\text{i.e., } \lim_{r \to \infty} \frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, L(f))} = \infty. \tag{3.16}$$

Now from (3.5) and (3.16) it follows that

$$\limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k) \log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, L(f)) \left\{\log T(r, f \circ g) + \log^{[2]} M(r, g)\right\}}$$
$$= \limsup_{r \to \infty} \frac{\log^{[p+1-l]} T(r, h \circ k)}{\log T(r, f \circ g) + \log^{[2]} M(r, g)} \cdot \lim_{r \to \infty} \frac{\log^{[l]} M(r, f \circ g)}{\log^{[l+1]} M(\exp r, L(f))}$$
$$= \infty \cdot \infty = \infty.$$

This proves the theorem. ■

Remark 3.3.3 *The condition* $\lambda_h^{[p]} > 0$ *in Theorem* 3.3.3 *is necessary as we see in the following example.*

Example 3.3.2 *Let* $f = g = \exp z$, $h = z$, $k = \exp\left(z^2\right)$ *and* $p = l = 2$.

Then $\rho_f = \lambda_f = 1 = \lambda_g = \rho_g$, $\lambda_h = 0$, $\lambda_k = \rho_k = 2$ *and* $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$.

Taking $a_1 = 1$, $a_2 = \ldots = a_k = 0$ *and* $k = 1$ *in Definition* 3.1.4 *we get that*

$$L(f) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now one can easily verify that

$$\limsup_{r\to\infty}\frac{\log T(r,h\circ k)\log^{[2]}M(r,f\circ g)}{\log^{[3]}M(\exp r,L(f))\left\{\log T(r,f\circ g)+\log^{[2]}M(r,g)\right\}}\leq 2,$$

which is a contrary to Theorem 3.3.3.

Theorem 3.3.4 *Let h be meromorphic, k be entire and g be transcendental entire function having the maximum deficiency sum such that $\lambda_h^{[p]}>0$, $\rho_g^{[l]}<\lambda_k\leq\rho_k$ and $0<\lambda_g^{[l]}\leq\rho_g^{[l]}<\infty$. Then for every entire function f with $0<\lambda_f\leq\rho_f<\infty$,*

$$\limsup_{r\to\infty}\frac{\log^{[p+1-l]}T(r,h\circ k)\log^{[l]}M(r,f\circ g)}{\log^{[l+1]}M(\exp r,L(g))\left\{\log T(r,f\circ g)+\log^{[2]}M(r,g)\right\}}=\infty\ ,$$

where p and l are integers greater that unity.

We omit the proof of the theorem because it can be carried out in the line of Theorem 3.3.3 and with the help of Lemma 3.2.5.

Remark 3.3.4 *The condition $\lambda_h^{[p]}>0$ in Theorem* 3.3.4 *is necessary as we see in the following example.*

Example 3.3.3 *Let $f=g=\exp z$, $h=z$, $k=\exp\left(z^2\right)$ and $p=l=2$.*

Then $\rho_f=\lambda_f=1=\lambda_g=\rho_g$, $\lambda_h=0$, $\lambda_k=\rho_k=2$ and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$.

Taking $a_1=1$, $a_2=...=a_k=0$ and $k=1$ in Definition 3.2.4 *we get that*

$$L(g)=\begin{vmatrix}1 & \exp z\\ 0 & \exp z\end{vmatrix}=\exp z.$$

Now one can easily verify that

$$\limsup_{r\to\infty}\frac{\log T(r,h\circ k)\log^{[2]}M(r,f\circ g)}{\log^{[3]}M(\exp r,L(g))\left\{\log T(r,f\circ g)+\log^{[2]}M(r,g)\right\}}\leq 2,$$

which is a contrary to Theorem 3.3.4.

Theorem 3.3.5 *Let f be entire with $0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty$ and g, h, k be three transcendental entire functions such that $0 < \lambda_g \leq \rho_g < \infty$, $0 < \lambda_h^{[p]} \leq \rho_h^{[p]} < \infty$ and $\rho_g < \lambda_k$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g) \log^{[p-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, h) \log^{[l-1]} T(r, f\circ g)} = \infty.$$

where $l = 2, 3, 4 \cdots$ and $p = 2, 3, 4 \cdots$.

Proof. We know that for $r > 0$ $\{cf.[\mathbf{36}]\}$

$$T(r, f\circ g) \geq \frac{1}{3} \log M\left\{\frac{1}{8} M\left(\frac{r}{4}, g\right) + o(1), f\right\}. \tag{3.17}$$

Since $\lambda_f^{[l]}$ and λ_g are the generalised lower order and lower order of f and g respectively, for given ε and for all large values of r we get that

$$\log^{[l-1]} M(r, f) \geq r^{\lambda_f^{[l]} - \varepsilon} \text{ and } \log M(r, g) \geq r^{\lambda_g - \varepsilon}$$

where $0 < \varepsilon < \min\left\{\lambda_f^{[l]}, \lambda_g\right\}$.

So from (3.17) we get for all large values of r,

$$\log^{[l-1]} T(r, f\circ g) \geq \left(\lambda_f^{[l]} - \varepsilon\right) \log\left\{\frac{1}{8} M\left(\frac{r}{4}, g\right) + o(1)\right\} + O(1)$$

$$\text{i.e., } \log^{[l-1]} T(r, f\circ g) \geq \left(\lambda_f^{[l]} - \varepsilon\right) \log\left\{\frac{1}{9} M\left(\frac{r}{4}, g\right)\right\} + O(1)$$

$$\text{i.e., } \log^{[l-1]} T(r, f\circ g) \geq \left(\lambda_f^{[l]} - \varepsilon\right) \left(\frac{r}{4}\right)^{\lambda_g - \varepsilon} + O(1)$$

$$\text{i.e., } \log^{[l]} T(r, f\circ g) \geq (\lambda_g - \varepsilon) \log r + O(1). \tag{3.18}$$

Also for a sequence of values of r tending to infinity

$$\log^{[p-1]} T(r, h) \leq \left(\lambda_h^{[p]} + \varepsilon\right) \log r. \tag{3.19}$$

So from (3.18) and (3.19) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r, h)} \geq \frac{(\lambda_g - \varepsilon) \log r + O(1)}{\left(\lambda_h^{[p]} + \varepsilon\right) \log r}. \tag{3.20}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (3.20) that

$$\limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)}{\log^{[p-1]}T(r,h)}\geq\frac{\lambda_g}{\lambda_h^{[p]}}. \tag{3.21}$$

As $\rho_g<\lambda_k$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho_g+\varepsilon<\lambda_k-\varepsilon. \tag{3.22}$$

Using the inequality $T(r,h\circ k)\geq\frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},k\right)+o(1),h\right\}$ $\{cf.[\mathbf{36}]\}$, we obtain for all large values of r that

$$\log^{[p-1]}T(r,h\circ k)\geq O(1)+\left(\lambda_h^{[p]}-\varepsilon\right)\log M\left(\frac{r}{4},k\right)$$

$$\text{i.e., }\log^{[p-1]}T(r,h\circ k)\geq O(1)+\left(\lambda_h^{[p]}-\varepsilon\right)\left(\frac{r}{4}\right)^{\lambda_k-\varepsilon}. \tag{3.23}$$

Again by Lemma 3.2.1 and the inequality $T(r,g)\leq\log^{+}M(r,g)$ we obtain for all sufficiently large values of r that

$$T(r,f\circ g)\leq\{1+o(1)\}T(M(r,g),f)$$

$$\text{i.e., }\log^{[l-1]}T(r,f\circ g)\leq O(1)+\left(\rho_f^{[l]}+\varepsilon\right)\log M(r,g)$$

$$\text{i.e., }\log^{[l-1]}T(r,f\circ g)\leq O(1)+\left(\rho_f^{[l]}+\varepsilon\right)r^{\rho_g+\varepsilon}. \tag{3.24}$$

Combining (3.23) and (3.24) we get for all large values of r that

$$\frac{\log^{[p-1]}T(r,h\circ k)}{\log^{[l-1]}T(r,f\circ g)}\geq\frac{O(1)+\left(\lambda_h^{[p]}-\varepsilon\right)\left(\frac{r}{4}\right)^{\lambda_k-\varepsilon}}{O(1)+\left(\rho_f^{[l]}+\varepsilon\right)r^{\rho_g+\varepsilon}}. \tag{3.25}$$

Now in view of (3.22) it follows from (3.25) that

$$\lim_{r\to\infty}\frac{\log^{[p-1]}T(r,h\circ k)}{\log^{[l-1]}T(r,f\circ g)}=\infty. \tag{3.26}$$

From (3.21) and (3.26) we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)\log^{[p-1]}T(r,h\circ k)}{\log^{[p-1]}T(r,h)\log^{[l-1]}T(r,f\circ g)}$$

$$=\limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)}{\log^{[p-1]}T(r,h)}\cdot\lim_{r\to\infty}\frac{\log^{[p-1]}T(r,h\circ k)}{\log^{[l-1]}T(r,f\circ g)}\geq\frac{\lambda_g}{\lambda_h^{[p]}}\cdot\infty=\infty.$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[l]} T(r, f\circ g)\log^{[p-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, h)\log^{[l-1]} T(r, f\circ g)} = \infty.$$

Thus the theorem is established. ■

Remark 3.3.5 *The condition $\rho_g < \lambda_k$ in Theorem 3.3.5 is essential which is evident from the following two examples.*

Example 3.3.4 *Let $f = \exp z$, $g = \exp\left(z^2\right)$, $h = k = \exp z$ and $p = l = 2$. Then $\lambda_f = \rho_f = 1$, $\lambda_g = \rho_g = 2$, $\lambda_h = \rho_h = 1$ and $\lambda_k = 1$.*
Now

$$\begin{aligned} T(r, f\circ g) &= T\left(r, \exp^{[2]}\left(z^2\right)\right) \\ &\le \log M\left(r, \exp^{[2]}\left(z^2\right)\right) = \log\left(\exp^{[2]}\left(r^2\right)\right) = \exp\left(r^2\right) \end{aligned}$$

i.e., $\log T(r, f\circ g) \le r^2$.

Also

$$T(r, h\circ k) = T\left(r, \exp^{[2]}(z)\right) \sim \frac{\exp r}{\left(2\pi^3 r\right)^{\frac{1}{2}}}\ (r\to\infty)$$

$$\text{i.e., } \log T(r, h\circ k) \sim r - \frac{1}{2}\log r + O(1).$$

Again in view of the inequality

$$T(r, f) \le \log^+ M(r, f) \le \frac{R+r}{R-r}T(R, f)\ \{cf.\ [4]\}$$

we get that

$$\begin{aligned} T(r, f\circ g) &\ge \frac{1}{3}\log M\left(\frac{r}{2}, f\circ g\right) \\ \text{i.e., } T\left(r, \exp^{[2]}\left(z^2\right)\right) &\ge \frac{1}{3}\log M\left(\frac{r}{2}, \exp^{[2]}\left(z^2\right)\right) \\ \text{i.e., } T\left(r, \exp^{[2]}\left(z^2\right)\right) &\ge \frac{1}{3}\log\left(\exp^{[2]}\left(\frac{r^2}{4}\right)\right) = \frac{1}{3}\exp\left(\frac{r^2}{4}\right) \\ \text{i.e., } \log T\left(r, \exp^{[2]}\left(z^2\right)\right) &\ge \frac{r^2}{4} + O(1). \end{aligned}$$

Also

$$\begin{aligned} T(r, h) &= T(r, \exp z) = \frac{r}{\pi} \\ \text{i.e., } \log T(r, h) &= \log r + O(1). \end{aligned}$$

Therefore

$$\frac{\log^{[2]} T(r, f\circ g)\log T(r, h\circ k)}{\log T(r,h)\log T(r, f\circ g)} \leq \frac{(2\log r)\left(r-\frac{1}{2}\log r+O(1)\right)}{(\log r+O(1))\left(\frac{r^2}{4}+O(1)\right)}$$

$$= \frac{2}{1+\dfrac{O(1)}{\log r}}\cdot\frac{1-\dfrac{1}{2}\dfrac{\log r}{r}+\dfrac{O(1)}{r}}{\dfrac{r}{4}+\dfrac{O(1)}{r}}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)\log T(r, h\circ k)}{\log T(r,h)\log T(r, f\circ g)} = 0,$$

which is contrary to Theorem 3.3.5.

Example 3.3.5 *Let* $f = g = h = k = \exp z$ *and* $p = l = 2$.
Then $\lambda_f = \rho_f = 1$, $\lambda_g = \rho_g = 1$, $\lambda_h = \rho_h = 1$ *and* $\lambda_k = 1$.
Now

$$T(r, f\circ g) = T(r, h\circ k) = T\left(r, \exp^{[2]}(z)\right) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\ (r\to\infty)$$

So

$$\log T(r, f\circ g) = \log T(r, h\circ k) \sim r - \frac{1}{2}\log r + O(1)\ .$$

Also

$$T(r, f\circ g) = T\left(r, \exp^{[2]}(z)\right)$$
$$\leq \log M\left(r, \exp^{[2]}(z)\right) = \log\exp^{[2]} r = \exp r$$
$$i.e., \log T(r, f\circ g) \leq \log\exp r = r$$
$$i.e., \log^{[2]} T(r, f\circ g) \leq \log r.$$

And

$$T(r,h) = T(r, \exp z) = \frac{r}{\pi}$$
$$i.e., \log T(r, \exp z) = \log r + O(1)\,.$$

Therefore

$$\limsup_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)\log T(r, h\circ k)}{\log T(r,h)\log T(r, f\circ g)}$$
$$\leq \limsup_{r\to\infty}\frac{(\log r)\left(r-\frac{1}{2}\log r+O(1)\right)}{(\log r+O(1))\left(r-\frac{1}{2}\log r+O(1)\right)} = 1,$$

which contradicts Theorem 3.3.5.

Remark 3.3.6 *The conclusion of Theorem* 3.3.5 *can also be drawn under some different conditions as we see in the following theorem.*

Theorem 3.3.6 *Let f, g, h and k be any four transcendental entire functions such that $\lambda_f^{[l]} > 0$, $\rho_g > 0$ and $\rho_h^{[p]} < \infty$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[l]} T(r, f\circ g)\log^{[p-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, h)\log^{[l-1]} T(r, f\circ g)} = \infty,$$

where $l = 2, 3, 4\cdots$ and $p = 2, 3, 4\cdots$.

Proof. From (3.17) we obtain for a sequence of values of r tending to infinity that

$$\log^{[l-1]} T(r, f\circ g) \geq O(1) + \left(\lambda_f^{[l]} - \varepsilon\right)\left(\frac{r}{4}\right)^{\rho_g - \varepsilon}$$

$$\text{i.e., } \log^{[l]} T(r, f\circ g) \geq O(1) + (\rho_g - \varepsilon)\log r. \tag{3.27}$$

Also for all large values of r we get that

$$\log^{[p-1]} T(r, h) \leq \left(\rho_h^{[p]} + \varepsilon\right)\log r. \tag{3.28}$$

Now from (3.27) and (3.28) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r, h)} \geq \frac{O(1) + (\rho_g - \varepsilon)\log r}{\left(\rho_h^{[p]} + \varepsilon\right)\log r}. \tag{3.29}$$

As $\varepsilon\,(0 < \varepsilon < \rho_g)$ is arbitrary, we obtain from (3.29) that

$$\limsup_{r\to\infty}\frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r, h)} \geq \frac{\rho_g}{\rho_h^{[p]}}. \tag{3.30}$$

So from (3.26) and (3.30) we get that

$$\limsup_{r\to\infty}\frac{\log^{[l]} T(r, f\circ g)\log^{[p-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, h)\log^{[l-1]} T(r, f\circ g)}$$

$$= \limsup_{r\to\infty}\frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r, h)} \cdot \lim_{r\to\infty}\frac{\log^{[p-1]} T(r, h\circ k)}{\log^{[l-1]} T(r, f\circ g)} \geq \frac{\rho_g}{\rho_h^{[p]}}\cdot\infty = \infty.$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)\log^{[p-1]}T(r,h\circ k)}{\log^{[p-1]}T(r,h)\log^{[l-1]}T(r,f\circ g)}=\infty.$$

This proves the theorem. ■

In the following two theorems we improve Theorem 3.3.5 and Theorem 3.3.6 respectively for a wronskian generated by a transcendental entire function .

Theorem 3.3.7 *Let f be entire with $0<\lambda_f^{[l]}\le\rho_f^{[l]}<\infty$, h be transcendental entire function having the maximum deficiency sum such that $0<\lambda_h^{[p]}\le\rho_h^{[p]}<\infty$ and g,k be two transcendental entire functions satisfying $0<\lambda_g\le\rho_g<\infty$ and $\rho_g<\lambda_k$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)\log^{[p-1]}T(r,h\circ k)}{\log^{[p-1]}T(r,L(h))\log^{[l-1]}T(r,f\circ g)}=\infty,$$

where $l=2,3,4\cdots$ and $p=2,3,4\cdots$.

Proof. In view of Lemma 3.3.5 we obtain for a sequence of values of r tending to infinity that

$$\log^{[p-1]}T(r,L(h))\le\left(\lambda_{L(h)}^{[p]}+\varepsilon\right)\log r$$

$$i.e.,\ \log^{[p-1]}T(r,L(h))\le\left(\lambda_h^{[p]}+\varepsilon\right)\log r. \tag{3.31}$$

So from (3.18) and (3.31) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]}T(r,f\circ g)}{\log^{[p-1]}T(r,h)}\ge\frac{(\lambda_g-\varepsilon)\log r+O(1)}{\left(\lambda_h^{[p]}+\varepsilon\right)\log r}. \tag{3.32}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (3.32) that

$$\limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)}{\log^{[p-1]}T(r,h)}\ge\frac{\lambda_g}{\lambda_h^{[p]}}. \tag{3.33}$$

From (3.21) and (3.33) we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)\log^{[p-1]}T(r,h\circ k)}{\log^{[p-1]}T(r,L(h))\log^{[l-1]}T(r,f\circ g)}$$

$$=\limsup_{r\to\infty}\frac{\log^{[l]}T(r,f\circ g)}{\log^{[p-1]}T(r,L(h))}\cdot\lim_{r\to\infty}\frac{\log^{[p-1]}T(r,h\circ k)}{\log^{[l-1]}T(r,f\circ g)}\ge\frac{\lambda_g}{\lambda_h^{[p]}}\cdot\infty=\infty.$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[l]} T(r, f\circ g)\log^{[p-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, L(h))\log^{[l-1]} T(r, f\circ g)} = \infty.$$

This completes the proof. ■

Remark 3.3.7 *The condition $\rho_g < \lambda_k$ in Theorem 3.3.7 is essential which is evident from the following two examples.*

Example 3.3.6 *Let $f = \exp z$, $g = \exp\left(z^2\right)$, $h = k = \exp z$ and $p = l = 2$.*

Then $\lambda_f = \rho_f = 1$, $\lambda_g = \rho_g = 2$, $\lambda_h = \rho_h = 1$, $\lambda_k = 1$ and $\sum\limits_{a\neq\infty}\delta(a;h) + \delta(\infty;h) = 2$.

Taking $a_1 = 1$, $a_2 = \ldots = a_k = 0$ and $k = 1$ in Definition 3.1.4 we get that

$$L(h) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now one can easily verify that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)\log T(r, h\circ k)}{\log T(r, L(h))\log T(r, f\circ g)} = 0,$$

which is contrary to Theorem 3.3.7.

Example 3.3.7 *Let $f = g = h = k = \exp z$ and $p = l = 2$.*

Then $\lambda_f = \rho_f = 1$, $\lambda_g = \rho_g = 1$, $\lambda_h = \rho_h = 1$, $\lambda_k = 1$ and $\sum\limits_{a\neq\infty}\delta(a;h) + \delta(\infty;h) = 2$.

Taking $a_1 = 1$, $a_2 = \ldots = a_k = 0$ and $k = 1$ in Definition 3.1.4 we get that

$$L(h) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now in the line of Example 3.3.5 it can be shown that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T(r, f\circ g)\log T(r, h\circ k)}{\log T(r, L(h))\log T(r, f\circ g)} \leq 1,$$

which contradicts Theorem 3.3.7.

Theorem 3.3.8 *Let h be transcendental entire function having the maximum deficiency sum and f, g,k be any three transcendental entire functions such that $\lambda_f^{[l]} > 0$, $\rho_g > 0$ and $\rho_h^{[p]} < \infty$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[l]} T(r, f\circ g)\log^{[p-1]} T(r, h\circ k)}{\log^{[p-1]} T(r, L(h))\log^{[l-1]} T(r, f\circ g)} = \infty,$$

where $l = 2, 3, 4\cdots$, and $p = 2, 3, 4\cdots$.

We omit the proof of the theorem because it can be carried out in the line of Theorem 3.3.6 and with the help of Lemma 3.2.5.

Theorem 3.3.9 *Let f be a meromorphic function and g, h be two entire functions such that $0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty$, $\rho_g < \infty$ and $\lambda_h^{[p]} > 0$. Then*

$$\liminf_{r\to\infty} \frac{\left\{\log^{[l]} T(r, f\circ g)\right\}^2}{\log^{[p-1]} T(r,h) \log^{[l]} T(\exp r, f)} = 0$$

where $l = 2, 3, 4 \cdots$. and $p = 2, 3, 4 \cdots$.

Proof. For all large values of r,

$$\log^{[l]} T(\exp r, f) \geq \left(\lambda_f^{[l]} - \varepsilon\right) \log(\exp r) = \left(\lambda_f^{[l]} - \varepsilon\right) r \tag{3.34}$$

where $0 < \varepsilon < \lambda_f^{[l]}$.

Again from (3.24) we get for all sufficiently large values of r that

$$\log^{[l]} T(r, f\circ g) \leq O(1) + (\rho_g + \varepsilon) \log r. \tag{3.35}$$

Now from (3.34) and (3.35) it follows for all sufficiently large values of r,

$$\frac{\log^{[l]} T(r, f\circ g)}{\log^{[l]} T(\exp r, f)} \leq \frac{O(1) + (\rho_g + \varepsilon)\log r}{\left(\lambda_f^{[l]} - \varepsilon\right) r}$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[l]} T(\exp r, f)} = 0$$

$$\text{i.e., } \lim_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[l]} T(\exp r, f)} = 0. \tag{3.36}$$

In view of Lemma 3.2.1 and the inequality $T(r, g) \leq \log^+ M(r, g)$ we obtain for all sufficiently large values of r that

$$T(r, f\circ g) \leq \{1 + o(1)\} T(M(r, g), f)$$

$$\text{i.e., } \log^{[l-1]} T(r, f\circ g) \leq O(1) + \left(\rho_f^{[l]} + \varepsilon\right) \log M(r, g). \tag{3.37}$$

Also for a sequence of values of r tending to infinity,

$$\log M(r, g) \leq r^{\lambda_g + \varepsilon}. \tag{3.38}$$

So from (3.37) and (3.38) it follows for a sequence of values of r tending to infinity,

$$\log^{[l-1]} T(r, f \circ g) \leq O(1) + \left(\rho_f^{[l]} + \varepsilon\right) r^{\lambda_g + \varepsilon}$$

$$\text{i.e., } \log^{[l]} T(r, f \circ g) \leq O(1) + (\lambda_g + \varepsilon) \log r. \tag{3.39}$$

For all large values of r,

$$\log^{[p-1]} T(r, h) \geq \left(\lambda_h^{[p]} - \varepsilon\right) \log r \tag{3.40}$$

where $0 < \varepsilon < \lambda_h^{[p]}$.

Now combining (3.39) and (3.40) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T(r, f \circ g)}{\log^{[p-1]} T(r, h)} \leq \frac{O(1) + (\lambda_g + \varepsilon) \log r}{\left(\lambda_h^{[p]} - \varepsilon\right) \log r}$$

$$\text{i.e., } \liminf_{r \to \infty} \frac{\log^{[l]} T(r, f \circ g)}{\log^{[p-1]} T(r, h)} \leq \frac{\lambda_g + \varepsilon}{\lambda_h^{[p]} - \varepsilon}. \tag{3.41}$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from (3.41) that

$$\liminf_{r \to \infty} \frac{\log^{[l]} T(r, f \circ g)}{\log^{[p-1]} T(r, h)} \leq \frac{\lambda_g}{\lambda_h^{[p]}}. \tag{3.42}$$

From (3.36) and (3.42) we obtain that

$$\liminf_{r \to \infty} \frac{\left\{\log^{[l]} T(r, f \circ g)\right\}^2}{\log^{[p-1]} T(r, h) \log^{[l]} T(\exp r, f)}$$

$$= \liminf_{r \to \infty} \frac{\log^{[l]} T(r, f \circ g)}{\log^{[p-1]} T(r, h)} \cdot \lim_{r \to \infty} \frac{\log^{[l]} T(r, f \circ g)}{\log^{[l]} T(\exp r, f)} \leq \frac{\lambda_g}{\lambda_h^{[p]}} \cdot 0 = 0.$$

$$\text{i.e., } \liminf_{r \to \infty} \frac{\left\{\log^{[l]} T(r, f \circ g)\right\}^2}{\log^{[p-1]} T(r, h) \log^{[l]} T(\exp r, f)} = 0.$$

Thus the theorem is established. ■

Remark 3.3.8 *The conclusion of Theorem* 3.3.9 *can also be deduced under some different conditions as we see in the following theorem.*

Theorem 3.3.10 *Let f be a meromorphic function and g, h be two entire functions with $\rho_f^{[l]} < \infty$, $\rho_g < \infty$ and $\rho_h^{[p]} > 0$. Then*

$$\liminf_{r\to\infty} \frac{\left\{\log^{[l]} T(r, f\circ g)\right\}^2}{\log^{[p-1]} T(r,h) \log^{[l]} T(\exp r, f)} = 0$$

where $l = 2, 3, 4 \cdots$ and $p = 2, 3, 4 \cdots$.

Proof. For a sequence of values of r tending to infinity and for $0 < \varepsilon < \rho_h^{[p]}$,

$$\log^{[p-1]} T(r,h) \geq \left(\rho_h^{[p]} - \varepsilon\right) \log r. \tag{3.43}$$

So from (3.35) and (3.43) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r,h)} \leq \frac{O(1) + (\rho_g + \varepsilon)\log r}{\left(\rho_h^{[p]} - \varepsilon\right)\log r}. \tag{3.44}$$

Since $\varepsilon\,(>0)$ is arbitrary we obtain from (3.44) that

$$\liminf_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r,h)} \leq \frac{\rho_g}{\rho_h^{[p]}}. \tag{3.45}$$

Now from (3.36) and (3.45) we obtain that

$$\begin{aligned}
&\liminf_{r\to\infty} \frac{\left\{\log^{[l]} T(r, f\circ g)\right\}^2}{\log^{[p-1]} T(r,h) \log^{[l]} T(\exp r, f)} \\
&= \liminf_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r,h)} \cdot \lim_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[l]} T\left(\exp r, f^{(k)}\right)} \leq \frac{\rho_g}{\rho_h^{[p]}} \cdot 0 = 0.
\end{aligned}$$

$$\text{i.e., } \liminf_{r\to\infty} \frac{\left\{\log^{[l]} T(r, f\circ g)\right\}^2}{\log^{[p-1]} T(r,h) \log^{[l]} T(\exp r, f)} = 0.$$

This proves the theorem. ■

In the following two theorems we extend Theorem 3.3.9 and Theorem 3.3.10 respectively for special type of linear differential polynomials viz. the Wronskians.

Theorem 3.3.11 *If f be a meromorphic function having the maximum deficiency sum and g be an entire function such that $0 < \lambda_f^{[l]} \leq \rho_f^{[l]} < \infty$, $\rho_g < \infty$ then for any entire function h having the maximum deficiency sum with $\lambda_h^{[p]} > 0$,*

$$\liminf_{r\to\infty} \frac{\left\{\log^{[l]} T(r, f\circ g)\right\}^2}{\log^{[p-1]} T(r, L(h)) \log^{[l]} T(\exp r, L(f))} = 0$$

where $l = 2, 3, 4 \cdots$ and $p = 2, 3, 4 \cdots$.

Proof. In view of Lemma 3.2.5 we obtain for all large values of r,

$$\log^{[l]} T(\exp r, L(f)) \geq \left(\lambda_{L(f)}^{[l]} - \varepsilon\right) \log(\exp r) = \left(\lambda_f^{[l]} - \varepsilon\right) r \tag{3.46}$$

where $0 < \varepsilon < \lambda_f^{[l]}$.

Now from (3.35) and (3.46) it follows for all sufficiently large values of r,

$$\frac{\log^{[l]} T(r, f\circ g)}{\log^{[l]} T(\exp r, L(f))} \leq \frac{O(1) + (\rho_g + \varepsilon)\log r}{\left(\lambda_f^{[l]} - \varepsilon\right) r}$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[l]} T(\exp r, L(f))} = 0$$

$$\text{i.e., } \lim_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[l]} T(\exp r, L(f))} = 0. \tag{3.47}$$

Again in view of Lemma 3.2.5 it follows for all large values of r,

$$\log^{[p-1]} T(r, L(h)) \geq \left(\lambda_{L(h)}^{[p]} - \varepsilon\right) \log r$$

$$i.e., \ \log^{[p-1]} T(r, L(h)) \geq \left(\lambda_h^{[p]} - \varepsilon\right) \log r \tag{3.48}$$

where $0 < \varepsilon < \lambda_h^{[p]}$.

Now combining (3.39) and (3.48) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r, L(h))} \leq \frac{O(1) + (\lambda_g + \varepsilon)\log r}{\left(\lambda_h^{[p]} - \varepsilon\right) \log r}$$

$$\text{i.e., } \liminf_{r\to\infty} \frac{\log^{[l]} T(r, f\circ g)}{\log^{[p-1]} T(r, L(h))} \leq \frac{\lambda_g + \varepsilon}{\lambda_h^{[p]} - \varepsilon}. \tag{3.49}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (3.49) that

$$\liminf_{r\to\infty}\frac{\log^{[l]}T\left(r,f\circ g\right)}{\log^{[p-1]}\left(r,L(h)\right)}\leq\frac{\lambda_g}{\lambda_h^{[p]}}. \tag{3.50}$$

From (3.47) and (3.50) we obtain that

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\left\{\log^{[l]}T\left(r,f\circ g\right)\right\}^2}{\log^{[p-1]}T\left(r,L(h)\right)\log^{[l]}T\left(\exp r,L(f)\right)}\\&=\liminf_{r\to\infty}\frac{\log^{[l]}T\left(r,f\circ g\right)}{\log^{[p-1]}T\left(r,L(h)\right)}\cdot\lim_{r\to\infty}\frac{\log^{[l]}T\left(r,f\circ g\right)}{\log^{[l]}T\left(\exp r,L(f)\right)}\leq\frac{\lambda_g}{\lambda_h^{[p]}}\cdot 0=0.\end{aligned}$$

$$\text{i.e., }\liminf_{r\to\infty}\frac{\left\{\log^{[l]}T\left(r,f\circ g\right)\right\}^2}{\log^{[p-1]}T\left(r,L(h)\right)\log^{[l]}T\left(\exp r,L(f)\right)}=0.$$

Thus the theorem is established. ■

Theorem 3.3.12 *If f be a meromorphic function having the maximum deficiency sum and g be an entire function such that $\rho_f^{[l]}<\infty$, $\rho_g<\infty$ then for any entire function h having the maximum deficiency sum with $\rho_h^{[p]}>0$,*

$$\liminf_{r\to\infty}\frac{\left\{\log^{[l]}T\left(r,f\circ g\right)\right\}^2}{\log^{[p-1]}T\left(r,L(h)\right)\log^{[l]}T\left(\exp r,L(f)\right)}=0$$

where $l=2,3,4\cdots$ and $p=2,3,4\cdots$.

Proof. In view of Lemma 3.2.5 it follows for a sequence of values of r tending to infinity and for $0<\varepsilon<\rho_h^{[p]}$,

$$\log^{[p-1]}T\left(r,L(h)\right)\geq\left(\rho_{L(h)}^{[p]}-\varepsilon\right)\log r$$

$$i.e.,\ \log^{[p-1]}T\left(r,L(h)\right)\geq\left(\rho_h^{[p]}-\varepsilon\right)\log r. \tag{3.51}$$

So from (3.35) and (3.51) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]}T\left(r,f\circ g\right)}{\log^{[p-1]}T\left(r,L(h)\right)}\leq\frac{O\left(1\right)+\left(\rho_g+\varepsilon\right)\log r}{\left(\rho_h^{[p]}-\varepsilon\right)\log r}. \tag{3.52}$$

Since $\varepsilon\,(>0)$ is arbitrary we obtain from (3.52) that

$$\liminf_{r\to\infty}\frac{\log^{[l]}T\left(r,f\circ g\right)}{\log^{[p-1]}T\left(r,L(h)\right)}\leq\frac{\rho_g}{\rho_h^{[p]}}. \tag{3.53}$$

Now from (3.47) and (3.53) we obtain that

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\left\{\log^{[l]}T\left(r,f\circ g\right)\right\}^2}{\log^{[p-1]}T\left(r,L(h)\right)\log^{[l]}T\left(\exp r,L(f)\right)}\\&=\liminf_{r\to\infty}\frac{\log^{[l]}T\left(r,f\circ g\right)}{\log^{[p-1]}T\left(r,L(h)\right)}\cdot\lim_{r\to\infty}\frac{\log^{[l]}T\left(r,f\circ g\right)}{\log^{[l]}T\left(\exp r,L(f)\right)}\leq\frac{\rho_g}{\rho_h^{[p]}}\cdot 0=0.\end{aligned}$$

$$\text{i.e., }\liminf_{r\to\infty}\frac{\left\{\log^{[l]}T\left(r,f\circ g\right)\right\}^2}{\log^{[p-1]}T\left(r,L(h)\right)\log^{[l]}T\left(\exp r,L(f)\right)}=0.$$

This proves the theorem. ■

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