Chapter - I

1 Introduction

1.1 Nonlinearity and partial differential equations :

The great physicist Richard Feynman [1] once commented that: "The next great era of awakening of human intellect may well produce a method of understanding the qualitative content of equation". The philosophy behind this can be appreciated from the observation of another great personality of physical science. Eugene Wigner pointed out that the chief role of mathematics in physics consists not in its being an instrument (i.e. computations) but in being the language of physics (details in [2]). This role of mathematics is being served for about last few hundred years chiefly by differential equations. The general practice is to formulate the laws of physics in the form of differential equations and then to solve the differential equations in different physical situations. Though the process of getting the solutions of differential equations is just the inverese of the process of the formation of those equations, it is for several reasons much more difficult to get the solutions. Thus, to get the solutions of differential equations has become a central problem of theoretical physics. The problem has become still more difficult with the fact that differential equations arising from physical situations are mostly nonlinear.

It is not far back when the nonlinear partial differential equations (nPDE's) were something of a closed chapter. The reason is that such equations are very difficult to study. Linear differential equations have the advantage that the principle of (linear) superposition holds in their cases, i.e. by adding two or more solutions, one can always get a new solution and the general solution can be expressed as a linear combination of the particular solutions.

The non-linear differential equations do not obey the principle of linear superposition. This is a severe loss on the part of non-linear differential equations and to obtain general exact solutions for the non-linear differential equations become more complicated. However, there exist classes of non-linear (and even linear) equations which possess non-linear superposition principles. Of course, there is no universal non-linear superposition (details in [3]).

Among numerous developments regarding the characteristics of nPDE's two are Solitary solutions and Painleve' analysis.

The boost for the study of nonlinear partial differential equations (nPDEs) and the mathematical study of solitary waves started with the work of Zabusky and Kruskal [4] in the year of 1965. Their work was stimulated by a physical problem and is also a classic example of how computational results may lead to the development of new mathematics, just as observational and experimental results have done since the time of Archimedes (for details see [5]).

1.2 Solitary solutions :

Examining the Fermi-Pasta-Ulm [6] model of phonons in an anharmonic lattice, Zabusky and Kruskal [4] were led to the work on Korteweg de Vries (KdV) equation. They considered the following initial-value problem in a periodic domain:

$$u_t + u\, u_x + \delta u_{xxx} = 0 \tag{1.1a}$$

where $u_t = \frac{\partial u}{\partial t}$, $\quad u_x = \frac{\partial u}{\partial x}$ etc.

$$u(2,t) = u(0,t)\ ,\ u_x(2,t) = u_x(0,t)\ ,\ u_{xx}(2,t) = u_{xx}(0,t)\ ,\ \text{for all } t. \tag{1.1b}$$

$$\text{and } u(x,0) = \cos \pi x \quad \text{for } 0 \leq x \leq 2 \tag{1.1c}$$

The motivation for choosing the boundary conditions stated above is that they suit numerical integration of the system. Zabusky and Kruskal found that the solution breaks up into a train of eight solitary waves, each like a *sech-squared* solution, that these waves move through one another as the faster ones catch up the slower ones, and that finally the initial state (or something very close to it) recurs. The word 'Soliton' was coined by Zabusky and Kruskal [4] after 'Photon', etc., to emphasize that a soliton is a localized entity which may keep its identity after an interaction. However, in the course

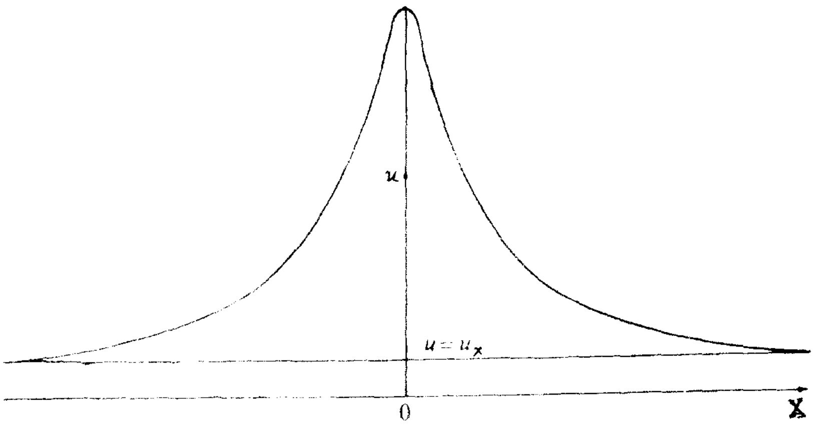

Figure 1: Profile of solitary wave joining constant states at $X = \pm\infty$ and localized in X according to equation (1.9).

of time mathematicians have coined a more general term than the so-called *sech-squared* solution of the KdV equation. They have talked about solitary wave. It is a solution of a nonlinear system, which represents a hump-shaped wave of permanent form, whether it is a soliton, or not.

A simple analytical approach is as follows [5]

$$u(x,t) = f(X)\ , \quad X = x - ct \tag{1.2}$$

Putting (1.2) in (1.1) and integrating once one gets

$$-cf + \frac{1}{2}f^2 + \delta f_{XX} - A = 0 \tag{1.3}$$

where A is an arbitrary constant of integration.

Integrating (1.3) once again we get

$$-3cf^2 + f^3 + 3\delta f_X^2 - 6Af - 6B = 0 \tag{1.4}$$

where B is another arbitrary constant of integration.

Since one seeks a solitary wave, one can add the boundary conditions that

$$f\ ,\ f_X\ ,\ f_{XX} \to 0 \quad \text{as} \quad X \to \pm\infty \tag{1.5}$$

Immediately we get $A\ ,\ B = 0$ and

$$3\delta f_X^2 = 3cf^2 - f^3 \tag{1.6}$$

Equation (1.6) can be rewritten as

$$X = \sqrt{3\delta} \int \frac{df}{f\sqrt{3c - f}} \tag{1.7}$$

The substitution

$$f = 3c\,\mathrm{sech}^2\,\theta f \tag{1.8}$$

then gives the solution

$$u = f(X) = 3c\,\mathrm{sech}^2 \left\{ \frac{1}{2}\sqrt{\frac{c}{\delta}}\,(X - X_0) \right\} \tag{1.9}$$

for any constants $c \geq 0$ and X_0 . Obviously, δ should also be greater than zero.

If one plots (1.9), one can get the solitary wave ($Fig.1$).

However, if one takes the equation (1) as

$$u_t - u\ u_x + \delta u_{xxx} = 0$$

we will still get a solitary wave but an inverted one.

1.2.1 *Balance between self-focussing and dispersion :*

The KdV equation can be thought of the superimposition of two waves given by:

$$u_t + u\ u_x = 0 \tag{1.10a}$$

$$u_t + \delta u_{xxx} = 0 \tag{1.10b}$$

Solution of (1.10b) is a dispersive one.

On the other hand, the solution of (1.10a) is a shock wave. The wave become self-focussed, i.e. the profiles of a particular wave become more and more sharpened.

In the KdV equation given by (1.1) these two types of solutions which are opposite in behavior seem to balance each other and to generate a single wave with permanent shape and size, i.e. the so-called solitary wave.

However, such kind of simple picture is not available for complicated cases like ours.

1.3 Painleve' analysis :

The Painleve' analysis for integrability has its origin in the contribution of great woman mathematician Sofya Kovalevskaya [7, 8] in relation to the equations in two groups defining the motion of a heavy rigid body about a fixed point which are given by

$$A\frac{dp}{dt}+(C-B)\,qr = Mg\,(y_0\gamma''-z_0\gamma')\ , \tag{1.11a}$$

$$B\frac{dq}{dt}+(A-C)\,rp = Mg\,(z_0\gamma-x_0\gamma'')\ . \tag{1.11b}$$

$$C\frac{dr}{dt}+(B-A)\,pq = Mg\,(x_0\gamma'-y_0\gamma) \tag{1.11c}$$

and

$$\frac{d\gamma}{dt} = r\gamma'-q\gamma''\ , \tag{1.12a}$$

$$\frac{d\gamma'}{dt} = p\gamma''-r\gamma\ , \tag{1.12b}$$

$$\frac{d\gamma''}{dt} = q\gamma-p\gamma'\ . \tag{1.12c}$$

Before Kovalevskaya two particular cases were known in which complete solution of the problem were possible.

(i) Euler case, when $x_0 = y_0 = z_0 = 0$.

(ii) Lagrange case, for which $A = B$, $x_0 = y_0 = 0$.

Kovaleskaya adopted a new approach to the problem. She considered time t to be a complex variable. This enabled her to apply the theory of functions of a complex variable. She sought a solution by assuming that the functions $p, q, r, \gamma, \gamma', \gamma''$ have poles in the complex plane of variable t. If one of the poles is $t = t_1$, then we can seek a solution in the form of series

$$p = \tau^{-n_1}\left(p_0 + p_1\tau + p_2\tau^2 + ...\right), \tag{1.13a}$$

$$q = \tau^{-n_2}\left(q_0 + q_1\tau + q_2\tau^2 + ...\right), \tag{1.13b}$$

$$r = \tau^{-n_3}\left(r_0 + r_1\tau + r_2\tau^2 + ...\right), \tag{1.13c}$$

$$\gamma = \tau^{-m_1}\left(f_0 + f_1\tau + f_2\tau^2 + ...\right), \tag{1.13d}$$

$$\gamma' = \tau^{-m_2}\left(g_0 + g_1\tau + g_2\tau^2 + ...\right), \tag{1.13e}$$

$$\gamma'' = \tau^{-m_3}\left(h_0 + h_1\tau + h_2\tau^2 + ...\right). \tag{1.13f}$$

Substituting these series into equations (1.11) and (1.12), Kovalevskaya determined the order of the possible poles:

$$m_1 = m_2 = m_3 = 2, \qquad n_1 = n_2 = n_3 = 1$$

and the existence conditions for solutions in the from (1.13). It turned out that they are possible both for the known two cases ((i) Euler and (ii) Lagrange) mentioned above and for another case. This new case is a discovery of Kovalevskaya and is given by

$$A = B = 2C, \qquad z_0 = 0.$$

After that the technique was examined by several mathematicians. But all of them were meant for nonlinear Ordinary differential equations (ODE's) only.

The first application of this approach was extended to nonlinear PDE's by Ablowitz, Ramani and Segur [9]. They conjectured that every nonlinear ODE obtained by an exact reduction (i.e. through similarity transformation) of an integrable nonlinear PDE has Painleve' property.

They also suggested an algorithm for inspecting whether a particualr nonlinear ODE has the Painleve property or not.

The so called ARS-conjecture [i.e. conjecture proposed by Ablowitz, Ramani and Segur [9]] is incomplete in the sense that there is no way to know exactly how many similarity reductions a particular nonlinear PDE permits.

Weiss, Tabor and Carnevale [11] generalized the approach of Ablowitz, Ramani and Segur [9] and applied the basic philosophy of the ARS conjecture directly to a particualr nonlinear PDE.

The approach of Weiss, Tabor and Carnevale [11] is again a conjecture which has many sucesses in its credit. For almost all celebrated nPDE's with single dependent variable the Painleve' analysis according to Weiss *et. al.* itself could provide Lax-pair, the basic condition for integrability of nPDE's. This success could not be achieved for coupled nPDE's. Still, the approach of Weiss *et. al.* could provide interesting exact solution in addition to the establishment of an association of the existence of Painleve' property and integrability.

Success of Painleve' analysis extends even beyond the above. The results of the analysis have been found to have a strong correlation with the existence or absence of chaos [11]. Normally, for ODE's by chaos we mean sensitive dependence on initial conditions. For, then the approximately known initial conditions do not give distant states with comparable approximation.

For some of the cases of the nPDE presented here one gets solutions that are initially solitary waves with oscillatory profile and become highly irregular with space and time. These may be stated as the examples for spatio-temporal chaos [12, 13]. Differential equations that do not pass the Painleve' test are found, in most of the cases, to have chaotic properties as well. In this thesis the present author reports the result of their investigations regarding solitary solutions and Painleve' properties for some nPDE's from physical orgin. The equations are highly non-linear and coupled. For reasons stated above simple variations in the value of the parameters present in the equations lead to severe deviations in the nature of solutions and Painleve' properties. This thesis elaborates such deviations as well.