Chapter 4

Starplus nearly compact pseudo regular open fuzzy topology on function spaces

4.1 Introduction

With the help of pseudo regular open fuzzy sets and starplus nearly compact fuzzy sets studied in Chapter 3, we have constructed in this chapter, a new type of fuzzy topology on function spaces. We have termed it as Starplus nearly compact pseudo regular open fuzzy topology (τ_{*NC}) and found it finer than F_{NR}. On the other hand, if we start with the NR-topology [31], we are able to establish that $w(N_R)$ is coincident with τ_{*NC}.

In the following section, defining pseudo δ-admissibility, we have shown that any pseudo δ-admissible fuzzy topology is finer than starplus nearly compact pseudo regular open fuzzy topology. We have also obtained the result that the concept of pseudo δ-admissibility ensures δ-admissibility of the corresponding strong α-level topology.

4.2 Starplus nearly compact pseudo regular open fuzzy topology

Definition 4.2.1 Let (X,τ) and (Y,σ) be two *fts* and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each starplus nearly compact fuzzy set K on X and each pseudo regular open fuzzy set μ on Y, a fuzzy set K_μ on $\mathcal{F}$ is given by

$$K_\mu(g) = \inf_{x \in supp(K)} \mu(g(x))$$

The collection of all such K_μ forms a subbase for some fuzzy topology on $\mathcal{F}$, called starplus nearly compact pseudo regular open fuzzy topology and it is denoted by τ_{*NC}.

Definition 4.2.2 [48] Let (X,τ) and (Y,σ) be two *fts* and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each $x \in X$, define a map $e_x : \mathcal{F} \to Y$ by $e_x(g) = g(x)$. The map e_x is called the evaluation map at the point x. The initial fuzzy topology τ_p generated by the collection of maps $\{e_x : x \in X\}$ is called the pointwise fuzzy topology on $\mathcal{F}$.

Theorem 4.2.1 [48] If Y is a fuzzy Hausdorff fts, then the fts $(\mathcal{F}, \tau_p)$ is fuzzy Hausdorff.

Remark 4.2.1 $\forall\ x \in X$, $\forall\ g \in \mathcal{F}$ and any fuzzy set μ on Y, $(e_x{}^{-1}(\mu))(g) = \mu(e_x(g)) = \mu(g(x))$.

So, $K_\mu(g)$

$= inf\{\mu(g(x)) : x \in supp(K)\}$

$= inf\{(e_x{}^{-1}(\mu))(g) : x \in supp(K)\}$

$= (inf\{e_x{}^{-1}(\mu) : x \in supp(K)\})(g)$

Hence, K_μ

$= inf\{e_x^{-1}(\mu) : x \in supp(K)\}$

$= \bigwedge_{x\in supp(K)} e_x^{-1}(\mu)$.

Theorem 4.2.2 The starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} is finer than the pointwise fuzzy topology τ_p on $\mathcal{F}$.

Proof. As every fuzzy set with finite support is starplus compact, it is starplus nearly compact, and so the theorem follows.

Theorem 4.2.3 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be endowed with starplus nearly compact pseudo regular open fuzzy topology. Then $(\mathcal{F}, \tau_{*NC})$ is fuzzy Hausdorff when (Y, σ) is fuzzy Hausdorff.

Proof. By Theorem (4.2.2) and (4.2.1), the theorem follows.

Remark 4.2.2 If $\mathcal{F}$ is a collection of functions from X to Y, then we shall denote the set $\{f \in \mathcal{F} : f(T) \subset U\}$, by $[T, U]$, where $T \subset X$ and $U \subset Y$.

Theorem 4.2.4 If K_μ is a subbasic open fuzzy set on τ_{*NC}, then $K_\mu^\alpha = [supp(K), \mu^\alpha]$.

Proof. In view of Remark (4.2.1),

K_μ^α

$= (\bigwedge_{x \in supp(K)} e_x^{-1}(\mu))^\alpha$

$= \{f \in \mathcal{F} : \bigwedge_{x \in supp(K)} e_x^{-1}(\mu)(f) > \alpha\}$

$= \{f \in \mathcal{F} : \bigwedge_{x \in supp(K)} \mu(f(x)) > \alpha\}$

$= \{f \in \mathcal{F} : f(supp(K)) \subseteq \mu^\alpha\}$

$= [supp(K), \mu^\alpha]$

Remark 4.2.3 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. Let us denote by N_R^α the ordinary nearly compact regular open topology [31] when X is endowed with 0-level topology $i_0(\tau)$ and Y with $i_\alpha(\sigma), \forall\ \alpha \in I_1$.

Theorem 4.2.5 The strong α-level topology $i_\alpha(\tau_{*NC})$ on $\mathcal{F}$, where $\alpha \in I_1$, is coarser than N_R^α topology on $\mathcal{F}$.

Proof. Let $\beta = \bigwedge_{i=1}^n K_{\mu_i}^i$ be a basic fuzzy open set on τ_{*NC}. The strong α-level set β^α is given by

β^α

$= (\bigwedge_{i=1}^{n} K^i_{\mu_i})^\alpha$

$= \bigcap_{i=1}^{n} (K^i_{\mu_i})^\alpha$

$= \bigcap_{i=1}^{n} [supp(K^i), \mu_i^\alpha]$.

As each K^i is starplus nearly compact, $supp(K^i)$ is nearly compact in $i_0(\tau)$. Again $\forall\ \alpha \in I_1$, μ_i^α being regular open in $i_\alpha(\sigma)$, it follows that β^α is a basic open set in N_R^α. This proves the theorem.

We shall take the same example as discussed in [49] to show that in general, two topologies $i_\alpha(\tau_{*NC})$ and N_R^α on $\mathcal{F}$ are not same.

Example 4.2.1 Let X be an infinite set and τ be the fuzzy topology on X generated by the collection
$\{(\frac{1}{2}\chi_U) \vee \chi_{X-U} : U \in X$ and $(X-U)$ is finite$\}$. Then

$$i_\alpha(\tau) = \begin{cases} \textit{the discrete topology on } X, & for\ \alpha \geq \frac{1}{2} \\ \textit{the indiscrete topology on } X, & for\ \alpha < \frac{1}{2} \end{cases}$$

For an infinite subset T of X, $\chi_T = K$(say), is a fuzzy set on X. Now, $supp(K) = T$ is compact and hence nearly compact in $i_0(\tau)$ but K^α is not nearly compact in $i_\alpha(\tau)$, for $\alpha \geq \frac{1}{2}$. Hence, K^α is not starplus nearly compact on (X, τ) and hence the fuzzy set $K_\mu \notin \tau_{*NC}$, for any pseudo regular open fuzzy set on Y. In fact, $K_\mu^\alpha = [supp(K), \mu^\alpha] = [T, \mu^\alpha] \in N_R^\alpha$. Hence, $i_\alpha(\tau_{*NC}) \neq N_R^\alpha$.

Theorem 4.2.6 Let (X, T) and (Y, U) be topological spaces and let (X, T_f) and (Y, U_f) denote corresponding characteristic fts, respectively. Then for each $\alpha \in I_1$, $i_\alpha(\tau_{*NC}) = N_R$, where N_R is the ordinary nearly compact regular open topology on $\mathcal{F}$.

Proof. $\forall\ \alpha \in I_1$, $i_\alpha(T_f) = T$ and $i_\alpha(U_f) = U, N_R^\alpha = N_R$. By Theorem (4.2.5) $i_\alpha(\tau_{*NC}) \subseteq N_R$. Now, let $[K, V] \in N_R$, where K is nearly compact in X and V is regular open in Y. Then the fuzzy set $S = \chi_K$ is starplus nearly compact in X and $\mu = \chi_V$ is pseudo regular open fuzzy set on Y, and $S_\mu^\alpha = [K, V]$, for each $\alpha \in I_1$. So, $[K, V] \in i_\alpha(\tau_{*NC})$. Hence, for each $\alpha \in I_1$, $i_\alpha(\tau_{*NC}) = N_R$.

Theorem 4.2.7 Let (X, T_X) and (Y, T_Y) be topological spaces and let N_R denote nearly compact regular open topology on $\mathcal{F}$. Then $\tau_{*NC} = w(N_R)$, where X and Y are endowed with the fuzzy topologies $w(T_X)$ and $w(T_Y)$, respectively.

Proof. Let N_R be the nearly compact regular open topology on $\mathcal{F}$, where X and Y are endowed with the fuzzy topologies $w(T_X)$ and $w(T_Y)$, respectively. Let $K_\mu \in \tau_{*NC}$, where K is starplus nearly compact in $w(T_X)$ and μ is pseudo regular open fuzzy set on $w(T_Y)$. By Theorem (4.2.4), $K_\mu^\alpha = [supp(K), \mu^\alpha]$. Using Theorem (3.4.9), $supp(K)$ is nearly compact in T_X. As μ^α is regular open in T_Y, $[supp(K), \mu^\alpha]$ is a subbasic open set in N_R and so K_μ is lower semicontinuous. Hence, $K_\mu \in w(N_R)$. Consequently, $\tau_{*NC} \subseteq w(N_R)$.

Conversely, let $v \in w(N_R)$. Then $\forall\ \alpha \in I_1, v^{-1}(\alpha, 1] \in N_R$. We will show that v is a τ_{*NC} neighborhood of each of its points. Let $g_\lambda \in v$. Then for $\alpha < \lambda$, $g \in v^{-1}(\alpha, 1]$. Since $v^{-1}(\alpha, 1] \in N_R$, there exist nearly compact sets $K_1, K_2, ...K_n$ in X and regular open sets $U_1, U_2, ...U_n$ in Y such that $g \in \cap_{i=1}^{n}[K_i, U_i] \subset v^{-1}(\alpha, 1]$. Now, the fuzzy set $S^i = \chi_{K_i}$ is starplus nearly compact in $w(T_X)$ with support K_i. Since for each $i = 1, 2..., n$, $U_i \in T_Y$ and $g(K_i) \subset U_i$, then the fuzzy set $\mu_i = (\chi_{U_i} \wedge \lambda) \in w(T_Y)$ such that $\mu_i(g(x)) = \lambda$, for each $x \in K_i$ and $\mu_i^\alpha = U_i$ for $\alpha < \lambda$. Hence, $g \in \cap_{i=1}^{n}[supp(S^i), \mu_i^\alpha] \subset v^{-1}(\alpha, 1]$. Now, using Theorem (4.2.5), $g_\lambda \in \wedge_{i=1}^{n} S_{\mu_i}^i \leq v$. Hence, v is a τ_{*NC} neighborhood of each of its points.

4.3 Pseudo δ-admissible fuzzy topology

Definition 4.3.1 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. A fuzzy topology T on $\mathcal{F}$ is said to be pseudo δ-admissible (pseudo δ-admissible on starplus near compacta) if a function $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is pseudo fuzzy δ-continuous (respectively, $P\ |_{\mathcal{F} \times supp(K)}$ is pseudo fuzzy δ-continuous for each starplus nearly compact set K on X), where $\mathcal{F} \times X$ is endowed with the product fuzzy topology.

Theorem 4.3.1 If T is a pseudo δ-admissible fuzzy topology on $\mathcal{F}$, then for each $\alpha \in I_1$, the strong α-level topology $i_\alpha(T)$ is jointly δ-continuous.

Proof. Let T be a pseudo δ-admissible fuzzy topology on $\mathcal{F}$. So, $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is pseudo fuzzy δ-continuous. Hence, $P : (\mathcal{F}, i_\alpha(T)) \times (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma)$ is δ-continuous, for all $\alpha \in I_1$. This shows that $i_\alpha(T)$ is jointly δ-continuous for all $\alpha \in I_1$.

Definition 4.3.2 Let μ be a fuzzy set on a fts (X, τ). The subspace fuzzy topology on μ is given by $\{v \mid_{supp(\mu)} : v \in \tau\}$ and is denoted by τ_μ. The pair $(supp(\mu), \tau_\mu)$ is called the subspace fts of μ.

Definition 4.3.3 Let (X, τ) and (Y, σ) be two fts. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be pseudo fuzzy δ-continuous on a fuzzy set μ on X, if $f \mid_{supp(\mu)}$ is pseudo fuzzy δ-continuous, where $supp(\mu)$ is endowed with the subspace fuzzy topology τ_μ.

Theorem 4.3.2 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F} \subset Y^X$. Then every fuzzy topology on $\mathcal{F}$ which is pseudo δ-admissible on starplus near compacta is finer than the starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} on $\mathcal{F}$.

Proof. Let $(\mathcal{F}, T)$ be pseudo δ-admissible on starplus near compacta. Let K_μ be any subbasic fuzzy open set on τ_{*NC} where K is starplus near compact on X and μ is pseudo regular open fuzzy set on Y. The function $P\mid_{\mathcal{F}\times supp(K)}$: $\mathcal{F}\times supp(K)\rightarrow Y$ given by $P(f,x)=f(x)$ is pseudo fuzzy δ-continuous. So, $P\mid^{-1}_{\mathcal{F}\times supp(K)}(\mu)$ is pseudo δ-open fuzzy set on $\mathcal{F}\times supp(K)$. For simplicity of notation , instead of $P\mid_{\mathcal{F}\times supp(K)}$ we shall use the symbol P only. Let f_α be any fuzzy point in K_μ. i.e., $K_\mu(f)\geq\alpha\Rightarrow inf\{\mu(f(x)):x\in supp(K)\}\geq\alpha$. We now prove, $f_\alpha\times\chi_{supp(K)}\leq P^{-1}(\mu)$. Now,

$(f_\alpha\times\chi_{supp(K)})(g,t)$

$=f_\alpha(g)\wedge\chi_{supp(K)}(t)$

$$=\begin{cases}\alpha, & iff=g\ and\ t\in supp(K)\\ 0, & otherwise.\end{cases}$$

Again, $P^{-1}(\mu)(g,t)=\mu(P(g,t))=\mu(g(t))\geq\alpha$.

Hence, $f_\alpha\times\chi_{supp(K)}\leq P^{-1}(\mu)$.

Consider the first projection $\Pi_1:(\mathcal{F},i_\alpha(T))\times(X,i_\alpha(\tau))\rightarrow(\mathcal{F},i_\alpha(T))$.

Now, $(\Pi_1(P^{-1}(\mu)))^\alpha$

$=\{f:\Pi_1(P^{-1}(\mu))(f)>\alpha\}$

$=\{f:^{sup}_{\Pi_1(g,t)=f}[P^{-1}(\mu)(g,t)>\alpha]\}$

$=\{f:^{sup}_{g=f}P^{-1}(\mu)(g,t)>\alpha\}$

$=\{f:P^{-1}(\mu)(f,t)>\alpha\}$

$=\{\Pi_1(f,t):(f,t)\in(P^{-1}(\mu))^\alpha\}$

$=\Pi_1(P^{-1}(\mu))^\alpha$

So, $(\Pi_1(P^{-1}(\mu)))^\alpha=\Pi_1(P^{-1}(\mu))^\alpha$. As $P^{-1}(\mu)$ is pseudo δ-open fuzzy set, $(P^{-1}(\mu))^\alpha$ is δ-open. Π_1 being projection mapping, $(\Pi_1(P^{-1}(\mu)))^\alpha = \Pi_1(P^{-1}(\mu))^\alpha$ is δ-open set. So, $\Pi_1(P^{-1}(\mu))$ is pseudo δ-open fuzzy set. As, $f_\alpha \in K_\mu$ we have,

$K_\mu(f) \geq \alpha$

$\Rightarrow inf\{\mu(f(x)) : x \in supp(K)\} \geq \alpha$. So, $\mu(f(s)) \geq \alpha, \forall\ s \in supp(K)$. $\Pi_1(P^{-1}(\mu))(g, s)$

$=^{sup}_{\Pi_1(g,s)=f} [P^{-1}(\mu)(g, s)]$

$= P^{-1}(\mu)(f, s)$

$= \mu(f(s))$

$\geq \alpha$. i.e., $f_\alpha \leq \Pi_1(P^{-1}(\mu))$.

Now, $\Pi_1(P^{-1}(\mu)) \times \chi_{supp(K)})(g, t)$

$= \Pi_1(P^{-1}(\mu))(g) \wedge \chi_{supp(K)}(t)$

$=^{sup}_{\Pi_1(f,t)=g} [P^{-1}(\mu)(g, t)] \wedge \chi_{supp(K)}(t)$

$= P^{-1}(\mu)(g, t) \wedge \chi_{supp(K)}(t)$

$$= \begin{cases} \mu(g(t)), & if\ t \in supp(K) \\ 0, & otherwise. \end{cases}$$

$\leq P^{-1}(\mu)(g, t)$.

Hence, $\Pi_1(P^{-1}(\mu)) \times \chi_{supp(K)} \leq P^{-1}(\mu)$.

Now, $\Pi_1(P^{-1}(\mu))(g)$

$=^{sup}_{\Pi_1(h,s)=g} [P^{-1}(\mu)(h, s)\ : s \in supp(K)]$

$= P^{-1}(\mu)(g, s)$

$= \mu(g(s)), \forall\ s \in supp(K)$

$=^{inf}_{t \in supp(K)} \mu(g(t))$

$= \ K_\mu(g).$

So, for any fuzzy point g_λ on $\Pi_1(P^{-1}(\mu)) \in T$, $g_\lambda \in K_\mu$, which proves the theorem.

We now state a lemma to prove the final result of this section. It is to mention here that this lemma has already been established by us in Theorem(3.3.5).

Lemma 4.3.1 A function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous iff for any pseudo δ-open fuzzy set μ on Y with $(f(x))_\alpha q\mu$ there exists a pseudo δ-open fuzzy set ν on X with $x_\alpha q\nu$ and $f(\nu) \leq \mu$.

Theorem 4.3.3 In a fully stratified Hausdorff fts (X, τ), if each member of $\mathcal{F} \subset Y^X$ is pseudo fuzzy δ-continuous on every starplus nearly compact fuzzy set of X, then the starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} on $\mathcal{F}$ is pseudo δ-admissible on starplus near compacta.

Proof. Let (X, τ) be a fully stratified Hausdorff fts. Let $f \in \mathcal{F}$ and $(f(x))_\alpha q\mu$, for any pseudo δ-open fuzzy set on Y. Since K is starplus nearly compact in $(supp(K), \tau_K)$, where τ_K is the subspace fuzzy topology on $supp(K)$, K^α is nearly compact on $(supp(K), i_\alpha(\tau_K)), \forall\ \alpha \in$

I_1. Again $f:(supp(K),\tau_K)\to(Y,\sigma)$ is pseudo fuzzy δ-continuous, $f:(supp(K),i_\alpha(\tau_K))\to(Y,i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha\in I_1$. (X,τ) being Hausdorff fts, $(supp(K),\tau_K)$ is Hausdorff $\forall\ \alpha\in I_1$. Hence, there exist a nearly compact *nbd.* M_α of x in $(supp(K),i_\alpha(\tau_K))$ and $f(M_\alpha)\subset\mu^\alpha$, as μ^α is open in $i_\alpha(\tau_K),\forall\ \alpha\in I_1$. Now, we choose $\beta>1-\lambda$. Let $K^*=(\chi_{M_\beta}\wedge\beta)$. As,

$$(K^*)^\alpha$$
$$=(\chi_{M_\beta}\wedge\beta)^\alpha$$
$$=\{x:(\chi_{M_\beta}\wedge\beta)(x)>\alpha\}$$
$$=\{x:\beta>\alpha\ and\ x\in M_\beta\}$$
$$=\begin{cases}M_\beta, & if\ \beta>\alpha\\ \Phi, & if\ \beta\leq\alpha\end{cases}$$

K^* is starplus nearly compact in the subspace fts $(supp(K),\tau_K)$ such that $f(K^*)\leq\mu$. Now,

$$(K^*_\mu\times\chi_{supp(K)})(f,x)$$
$$=K^*_\mu(f)\wedge\chi_{supp(K)}(x)$$
$$=\begin{cases}K^*_\mu(f), & if\ x\in supp(K)\\ 0, & otherwise\end{cases}$$
$$=inf\{\mu f(z):z\in supp(K^*)\},x\in supp(K)$$
$$>1-\lambda,\ as\ (f(x))_\lambda q\mu\ and\ z\in supp(K^*)\Rightarrow\mu f(z)>\beta.$$

Hence, $(K^*_\mu\times\chi_{supp(K)})\mid_{(\mathcal{F}\times supp(K))}$ is a q-*nbd.* of the fuzzy point $(f,x)_\lambda$

on $(\mathcal{F}\times supp(K))$. Also, it can be seen that $(K^*_\mu \times \chi_{supp(K)})\mid_{(\mathcal{F}\times supp(K))} \leq P^{-1}(\mu)$. Hence the theorem.