Chapter 3

Pseudo fuzzy δ-continuous functions and Starplus near compactness

3.1 Introduction

As we have already mentioned in Chapter 1, Lowen [53] introduced two functors ω and i between the category of all fuzzy topological spaces and fuzzy continuous functions and the category of all topological spaces and continuous functions. It was also observed in the same work that for each α with $\alpha \in [0, 1)$, the functor i_α associates a topology $i_\alpha(\tau)$ with the fuzzy topology τ.

It is a natural question, whether the fuzzy topologies on a family of functions studied in Chapter 2, correspond to the known compact open topology and NR topology in the general setting, via the functor i_α, for each $\alpha \in [0, 1)$. In the process of their investigations, Kohli and Prasannan [49], introduced another form of fuzzy topology on function spaces, which they have called starplus compact open fuzzy topology and have shown that such fuzzy topology is the one that corresponds to the usual compact open topology.

In carrying out our research along the same line, we first notice that fuzzy regular open sets do not behave as fuzzy open sets under the above functorial correspondence.

In Section (3.2), we have observed that a strong α-level set $i_\alpha(\mu) = \mu^\alpha$ need not be regular open, if μ is fuzzy regular open. On the other hand, even the regular openness of μ^α, for all $\alpha \in [0, 1)$ may fail to imply the fuzzy regularity of μ. This observation leads to the definition of new type of fuzzy sets termed as pseudo regular open fuzzy sets. In a similar manner, pseudo regular closed, pseudo δ-open, pseudo δ-closed fuzzy sets are also introduced. It is quite interesting to notice that a pseudo regular closed fuzzy is not a complement of pseudo regular open fuzzy set, and a pseudo δ-closed fuzzy set is not a complement of pseudo δ-open fuzzy set. Also, surprisingly, a pseudo δ-open fuzzy set may not be expressible as a union of pseudo regular open fuzzy sets, though a union of pseudo regular open fuzzy sets is indeed a pseudo δ-open fuzzy set. This fact gave birth to two new fuzzy topologies, that we have called *ps*-δ and *ps*-*ro* fuzzy topologies on X. We have also discussed their interrelations with the original fuzzy topology τ, in the same section.

In Section (3.3), we have introduced a class of functions, called pseudo fuzzy δ-continuous functions whose functorial counterpart, via the functor i_α for each $\alpha \in [0,1)$ is precisely the family of all δ-continuous functions in general topology.

In Section (3.4), we have defined a new form of compact-like fuzzy sets and have called them starplus nearly compact fuzzy sets. We have discussed some fundamental properties of the same and also a couple of necessary conditions for starplus nearly compact fts, which play important role in determining when (X,τ) is not starplus nearly compact.

Through out this thesis, we denote $[0,1)$ by I_1.

3.2 New fuzzy topologies from old

We begin this section with an example showing that on a fts (X,τ), if μ is fuzzy regular open, then μ^α need not be regular open in the corresponding topological space $(X, i_\alpha(\tau)), \alpha \in I_1$ and also μ^α may be regular open in $(X, i_\alpha(\tau)), \alpha \in I_1$, inspite of μ being not fuzzy regular open in the fts (X, τ).

Example 3.2.1 Let X be a set with at least two elements. Fix an element $y \in X$. Clearly, $\tau = \{0, 1, A\}$ is a fuzzy topology on X, where A is defined as $A(x) = \begin{cases} 0.5, & \text{for } x = y \\ 0.3, & \textit{otherwise.} \end{cases}$

The fuzzy closed sets on (X, τ) are $0, 1$ and $1 - A$ where

$$(1 - A)(x) = \begin{cases} 0.5, & \text{for } x = y \\ 0.7, & \textit{otherwise.} \end{cases}$$

Clearly, $A \leq 1 - A$ and hence $int(clA) = A$. i.e., A is fuzzy regular open in (X, τ). Now, in the corresponding topological space $(X, i_\alpha(\tau))$, $\alpha \in I_1$, the open sets are Φ, X and A^α where $A^\alpha =$

$$\begin{cases} X, & \text{for } \alpha < 0.3 \\ \{y\}, & \text{for } 0.3 \leq \alpha < 0.5 \\ \Phi, & \text{for } \alpha \geq 0.5. \end{cases}$$

For, $0.3 \leq \alpha < 0.5$, the closed sets on $(X, i_\alpha(\tau))$ are Φ, X and $X - \{y\}$. It is clear that $int(clA^\alpha) = X$. Hence, A^α is not regular open in $(X, i_\alpha(\tau))$ for $0.3 \leq \alpha < 0.5$.

Example 3.2.2 Let $X = \{x, y, z\}$. Define fuzzy sets μ, γ and η as follows: $\mu(a) = 0.4$, $\gamma(a) = 0.55$ and $\eta(a) = 0.6$, $\forall\ a \in X$. If $\tau = \{0, 1, \mu, \gamma, \eta\}$ then (X, τ) is a fts. The closed fuzzy sets are $(1 - \mu) = \eta$, $(1 - \gamma)(a) = 0.45$, $\forall\ a \in X$ and $(1 - \eta) = \mu$. Here,

$cl(\gamma) = \eta$ and $int(cl\gamma) = \eta$. Hence, γ is not fuzzy regular open set. But, $\gamma^{\alpha} = \{x : \gamma(x) > \alpha\}$. For $\alpha \geq 0.55$, $\gamma^{\alpha} = \Phi$, which is regular open. For $\alpha < 0.55$, $\gamma^{\alpha} = \{x, y, z\} = X$, which is also regular open. Hence for all $\alpha \in I_1, \gamma^{\alpha}$ is regular open in $(X, i_{\alpha}(\tau))$.

In view of these examples we define the following:

Definition 3.2.1 A fuzzy open set (fuzzy closed set) μ on a *fts* (X, τ) is said to be pseudo regular open (respectively, pseudo regular closed) fuzzy set if the strong α-level set μ^{α} is regular open (respectively, regular closed) in $(X, i_{\alpha}(\tau))$, $\forall\ \alpha \in I_1$.

The following example establishes that pseudo regular closed and pseudo regular open fuzzy sets are not complements of each other.

Example 3.2.3 Let $X = \{x, y, z, w\}, \tau = \{0, 1, \mu\}$ where μ is defined as $\mu(x) = 0.1, \mu(y) = 0.2, \mu(z) = 0.3, \mu(w) = 0.4$.
Clearly, (X, τ) is a *fts*. If $\alpha = 0.3$, $i_{\alpha}(\tau) = \{\Phi, X, \mu^{\alpha}\}$ and $\mu^{\alpha} = \{x \in X : \alpha < \mu(x) \leq 1\} = \{w\}$. Closed sets on $(X, i_{\alpha}(\tau))$ are Φ, X and $\{x, y, z\}$. Here we shall show that μ is not pseudo regular open but its complement $(1 - \mu)$ is pseudo regular closed fuzzy set on (X, τ). As the smallest closed set containing μ^{α} is X, $cl(\mu^{\alpha}) = X$. So, $int(cl\mu^{\alpha}) = X \neq \mu^{\alpha}$. Hence, μ^{α} is not regular open in $(X, i_{\alpha}(\tau))$. This shows that μ is not pseudo regular open

in (X,τ). Here, $(1-\mu)(x) = 0.9, (1-\mu)(y) = 0.8, (1-\mu)(z) = 0.7, (1-\mu)(w) = 0.6$. $(1-\mu)^\alpha = \{x \in X : \alpha < (1-\mu)(x) \leq 1\}$. For, $\alpha \geq 0.6$, $i_\alpha(\tau) = \{\Phi, X\}$. So, $cl[int(1-\mu)^\alpha] = X$. Also, for $\alpha < 0.6, (1-\mu)^\alpha = X, cl[int(1-\mu)^\alpha] = X$, whatever be $i_\alpha(\tau)$. Hence, in any case $(1-\mu)^\alpha$ is regular closed in $(X, i_\alpha(\tau))$. This shows, $(1-\mu)$ is pseudo regular closed fuzzy set on (X,τ).

Definition 3.2.2 A fuzzy set μ on a fts (X,τ) is said to be pseudo δ-open (respectively, pseudo δ-closed) fuzzy set if the strong α-level set μ^α is δ-open (respectively, δ-closed) in $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$.

Theorem 3.2.1 The collection of all pseudo δ-open fuzzy sets on a fts (X,τ) forms a fuzzy topology on X.

Proof. Straightforward.

Definition 3.2.3 The fuzzy topology as obtained in the above theorem is called pseudo δ-fuzzy topology (in short *ps-δ* fuzzy topology) on X. The complements of the members of *ps-δ* fuzzy topology are known as *ps-δ* closed fuzzy sets.

Theorem 3.2.2 In a fts (X,τ), union of pseudo regular open fuzzy sets is pseudo δ-open.

Proof. Let $\mu = \vee\{\mu_i : i \in \Lambda\}$, where μ_i is pseudo regular open fuzzy set on a fts (X,τ), for each $i \in \Lambda$. Here, μ_i^α is regular open in

$(X, i_\alpha(\tau)), \forall\, i \in \Lambda$. As, $(\vee \mu_i)^\alpha = \cup \mu_i^\alpha$ is δ-open in $(X, i_\alpha(\tau)), \forall\, \alpha \in I_1$, μ is pseudo δ-open fuzzy set on (X, τ).

The following example shows that the converse of Theorem (3.2.2) is not true in general. i.e., Any pseudo δ-open fuzzy set on a *fts* (X, τ) need not be expressible as union of pseudo regular open fuzzy sets.

Example 3.2.4 Let $X = \{x, y, z\}$ and the fuzzy topology τ, generated by μ , γ , η where $\mu(x) = 0.4, \mu(y) = 0.4, \mu(z) = 0.5$, $\gamma(x) = 0.4$,$\gamma(y) = 0.6$, $\gamma(z) = 0.4$ and $\eta(x) = 0.5$, $\eta(y) = 0.5$, $\eta(z) = 0.6$. Consider $i_\alpha(\tau)$, for each α as follows:

Case 1: For $\alpha < 0.4$, $\mu^\alpha = \gamma^\alpha = \eta^\alpha = X$ and hence $i_\alpha(\tau) = \{X, \Phi\}$. Consequently, $\mu^\alpha, \gamma^\alpha$ and η^α are all regular open.

Case 2: For $\alpha \geq 0.6$, $\mu^\alpha = \gamma^\alpha = \eta^\alpha = \Phi$ and hence $i_\alpha(\tau) = \{X, \Phi\}$. Consequently, $\mu^\alpha, \gamma^\alpha$ and η^α are all regular open.

Case 3: For $0.4 \leq \alpha < 0.5$, $\mu^\alpha = \{z\}$, $\gamma^\alpha = \{y\}$, $\eta^\alpha = X$ and hence $i_\alpha(\tau) = \{X, \Phi, \{y\}, \{z\}, \{y, z\}\}$. We observe that $int(cl\mu^\alpha) = \mu^\alpha$, $int(cl\gamma^\alpha) = \gamma^\alpha$ and $int(cl\eta^\alpha) = \eta^\alpha$, proving all of them to be regular open. But $(\mu \vee \gamma)^\alpha = \{y, z\}$ is not regular open as $int(cl\{y, z\}) = X \neq \{y, z\}$.

Case 4: For $0.5 \leq \alpha < 0.6$, $\mu^\alpha = \Phi$, $\gamma^\alpha = \{y\}$, $\eta^\alpha = \{z\}$ and hence $i_\alpha(\tau) = \{X, \Phi, \{y\}, \{z\}, \{y, z\}\}$. In this case too all μ^α,γ^α and η^α are regular open but $(\gamma \vee \eta)^\alpha = \{y, z\}$ is not so.

Now, we consider a fuzzy set K on X as follows: $K(x) = 0.4$, $K(y) = 0.6$ and $K(z) = 0.6$. Clearly,

$$K^\alpha = \begin{cases} X, & \text{for } \alpha < 0.4 \\ \{y, z\}, & \text{for } 0.4 \leq \alpha < 0.6 \\ \Phi, & \text{for } \alpha \geq 0.6. \end{cases}$$

Hence, $K^\alpha = \begin{cases} X, & \text{for } \alpha < 0.4 \\ \mu^\alpha \cup \gamma^\alpha, & \text{for } 0.4 \leq \alpha < 0.5 \\ \gamma^\alpha \cup \eta^\alpha, & \text{for } 0.5 \leq \alpha < 0.6 \\ \Phi, & \text{for } \alpha \geq 0.6. \end{cases}$

and so, K^α is δ-open in $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$. Therefore, K is pseudo δ-open fuzzy set in (X, τ). It can be shown easily that K is neither a pseudo regular open fuzzy set in (X, τ) nor is expressible as union of pseudo regular open fuzzy sets on (X, τ).

It is clear from the above example that $A^\alpha = B^\alpha$, for some $\alpha \in I_1$ does not imply $A = B$. However we have the following Theorem:

Theorem 3.2.3 If A and B are two fuzzy sets on a fts (X, τ) such that $A^\alpha = B^\alpha$, $\forall\ \alpha \in I_1$ then $A = B$.

Proof. For $\alpha = 0, A^0 = B^0 \Rightarrow A(x) > 0$ iff $B(x) > 0$. Hence, $A(x) = 0$ iff $B(x) = 0$. Suppose, $y \in Y$ is such that $A(y) > 0, B(y) > 0$ and

$A(y) \neq B(y)$. Let $A(y) = \alpha_1$ and $B(y) = \alpha_2$. Without any loss of generality, let us take $\alpha_1 > \alpha_2$. Since $A(y) = \alpha_1 > \alpha_2$, $y \in A^{\alpha_2}$, but $y \notin B^{\alpha_2}$ i.e., $A^{\alpha_2} \neq B^{\alpha_2}$, which is a contradiction. Hence, $\forall\ y \in Y$, $A(y) = B(y)$ i.e., $A = B$.

Theorem 3.2.4 If $\{\mu_i\}$ be a collection of all pseudo regular open fuzzy sets on a *fts* (X, τ), then

(i) $0, 1 \in \{\mu_i\}$.

(ii) $\forall\ \mu_1, \mu_2 \in \{\mu_i\} \Rightarrow \mu_1 \wedge \mu_2 \in \{\mu_i\}$.

Proof. As 0 and 1 are pseudo regular open fuzzy sets on a *fts* (X, τ), $0, 1 \in \{\mu_i\}$. Let $\mu_1, \mu_2 \in \{\mu_i\}$. Then $\mu_1^\alpha, \mu_2^\alpha$ are regular open in $(X, i_\alpha(\tau))$. Now, $\mu_1^\alpha \cap \mu_2^\alpha$ is regular open in $(X, i_\alpha(\tau))$. i.e., $(\mu_1 \wedge \mu_2)^\alpha = \mu_1^\alpha \cap \mu_2^\alpha$ is regular open in $(X, i_\alpha(\tau))$. Which shows $\mu_1 \wedge \mu_2$ is pseudo regular open fuzzy set on (X, τ).

Remark 3.2.1 In view of Theorem (3.2.4) the collection of all pseudo regular open fuzzy sets on (X, τ) generates a fuzzy topology, which we call pseudo regular open fuzzy topology (in short, *ps-ro* fuzzy topology) on X. The members of this topology are termed as *ps-ro* open fuzzy sets and their complements as *ps-ro* closed fuzzy sets.

Theorem 3.2.5 In a *fts* (X, τ), *ps-ro* fuzzy topology is coarser than τ.

Proof. Straightforward.

Theorem 3.2.6 In a fts (X, τ), *ps-ro* fuzzy topology is coarser than *ps-*δ fuzzy topology.

Proof. Let $\mu \in$ *ps-ro* fuzzy topology. So, $\mu = \bigvee_i \gamma_i$ where γ_i's are pseudo regular open fuzzy sets on (X, τ). Hence, $\mu^\alpha = (\bigvee_i \gamma_i)^\alpha = \bigcup_i \gamma_i^\alpha$ is δ-open in $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$. This shows that $\mu \in$ *ps-*δ fuzzy topology on X.

Remark 3.2.2 In view of Example (3.2.4), in general *ps-ro* fuzzy topology is strictly coarser than *ps-*δ fuzzy topology in a fts (X, τ).

3.3 Pseudo fuzzy δ-continuous functions

As proposed in the introduction of this Chapter, defining pseudo fuzzy δ-continuous functions, we establish that such functions correspond to the well known δ-continuous functions in general topology, under the functorial correspondence i_α, for each $\alpha \in [0, 1)$. Moreover, we characterize such pseudo fuzzy δ-continuous functions in terms of *ps-*δ closed fuzzy sets as well as fuzzy points.

Definition 3.3.1 A function f from a fts X to a fts Y is pseudo fuzzy δ-continuous if $f^{-1}(U)$ is pseudo δ-open fuzzy set on X, for each pseudo δ-open fuzzy set U on Y.

Theorem 3.3.1 If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is pseudo fuzzy δ-continuous then $f : (X, i_\alpha(\tau)) \rightarrow (Y, i_\alpha(\sigma))$, is δ-continuous for each $\alpha \in I_1$, where $(X, \tau), (Y, \sigma)$ are *fts*.

Proof. Let v be a δ-open set in $i_\alpha(\sigma)$. As every δ-open set is open, $v \in i_\alpha(\sigma)$ and so there exist, $\mu \in \sigma$ such that $v = \mu^\alpha$. Now, $f^{-1}(\mu^\alpha)$

$= \{x \in X : f(x) \in \mu^\alpha\}$

$= \{x \in X : \mu(f(x)) > \alpha\}$

$= \{x \in X : (\mu f)(x) > \alpha\}$

$= \{x \in X : (f^{-1}(\mu))(x) > \alpha\}$

$= \{x \in X : x \in (f^{-1}(\mu))^\alpha\}$

$= (f^{-1}(\mu))^\alpha$.

Consider a fuzzy set ζ on Y given by

$$\zeta(z) = \begin{cases} 1 & if\ \mu(z) > \alpha \\ \alpha & otherwise. \end{cases}$$

Then $$\zeta^\beta = \begin{cases} \mu^\alpha & if\ \beta \geq \alpha \\ Y & \beta < \alpha. \end{cases}$$

Consequently, ζ^β is δ-open for all $\beta \in I_1$. Hence, ζ is a pseudo δ-open on Y. Since, f is pseudo fuzzy δ-continuous, $f^{-1}(\zeta)$ is pseudo δ-open fuzzy set on X. Now, $(f^{-1}(\zeta))^\alpha = f^{-1}(\zeta^\alpha) = f^{-1}(\mu^\alpha) = f^{-1}(v)$. Hence, $f^{-1}(v)$ is δ-open set whenever v is so. This proves that f is δ-continuous, for each $\alpha \in I_1$.

Theorem 3.3.2 A function $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$, is δ-continuous for each $\alpha \in I_1$, where $(X, \tau), (Y, \sigma)$ are fts then $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous.

Proof. Let μ be any fuzzy pseudo δ-open set in (Y, σ). μ^α is δ-open in $(Y, i_\alpha(\sigma))$. By the δ-continuity of $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$, $f^{-1}(\mu^\alpha) = (f^{-1}(\mu))^\alpha$ is δ-open in $(X, i_\alpha(\tau))$. Hence, $f^{-1}(\mu)$ is pseudo δ-open fuzzy set on (X, τ), proving f to be pseudo fuzzy δ-continuous.

A necessary condition for pseudo fuzzy δ-continuous functions follows next.

Theorem 3.3.3 If a function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous then $f^{-1}(\mu)$ is pseudo δ-closed fuzzy set on (X, τ), for all pseudo δ-closed fuzzy set μ in (Y, σ).

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous.

So, $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous for each $\alpha \in I_1$. Now, μ is pseudo δ-closed fuzzy set on (Y, σ). Hence, $\forall\ \alpha \in I_1$, μ^α is δ-closed in $(Y, i_\alpha(\sigma))$, that is $(Y - \mu^\alpha)$ is δ-open in $(Y, i_\alpha(\sigma))$. Now,

$f^{-1}(Y - \mu^\alpha)$

$= \{x \in X : f(x) \notin \mu^\alpha\}$

$= \{x \in X : \mu(f(x)) \leq \alpha\}$

$= \{x \in X : (\mu f)(x)) \leq \alpha\}$

$= X - \{x \in X : (\mu f)(x)) > \alpha\}$

$= X - \{x \in X : (f^{-1}(\mu))(x) > \alpha\}$

$= X - (f^{-1}(\mu))^{\alpha}$. As, $f^{-1}(Y - \mu^{\alpha})$ is δ-open, $(f^{-1}(\mu))^{\alpha}$ is δ-closed in $(X, i_{\alpha}(\tau))$, $\forall\ \alpha \in I_1$. Hence, $f^{-1}(\mu)$ is pseudo δ-closed fuzzy set on (Y, σ).

As a pseudo δ-closed set need not be the complement of a pseudo δ-open set, the converse of the Theorem (3.3.3) may not hold true. However, the following theorem characterizes pseudo fuzzy δ-continuous functions in terms of *ps*-δ closed fuzzy sets.

Theorem 3.3.4 A function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous iff $f^{-1}(\mu)$ is *ps*-δ closed fuzzy set on a *fts* (X, τ), where μ is *ps*-δ closed fuzzy set on a *fts* (Y, σ).

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous.
Then $f : (X, i_{\alpha}(\tau)) \to (Y, i_{\alpha}(\sigma))$ is δ-continuous, for each $\alpha \in I_1$.
Let μ be *ps*-δ closed fuzzy set on *fts* (Y, σ). Then $1 - \mu$ is pseudo δ-open fuzzy set on (Y, σ). So $(1 - \mu)^{\alpha}$ is δ-open and $f^{-1}((1 - \mu)^{\alpha}) = (f^{-1}(1 - \mu))^{\alpha}$ is δ-open fuzzy set on $(X, i_{\alpha}(\tau)), \forall\ \alpha \in I_1$. This shows that $f^{-1}(1 - \mu)$ is pseudo δ-open fuzzy set on (X, τ). Now,

$(1 - f^{-1}(1 - \mu))(x)$

$= 1 - f^{-1}(1 - \mu)(x)$

$= 1 - (1 - \mu)(f(x))$

$= \mu(f(x))$

$= f^{-1}(\mu)(x)$.

Hence, $f^{-1}(\mu)$ is ps-δ closed fuzzy set on (X, τ).

Conversely, Let μ be any pseudo δ-open and so, $(1-\mu)$ is ps-δ closed fuzzy set on (Y, σ). As, $f^{-1}(1-\mu)$ is ps-δ closed, $1 - f^{-1}(1-\mu)$ is pseudo δ-open fuzzy set on $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. Again, $f^{-1}(\mu) = 1 - f^{-1}(1-\mu), f^{-1}(\mu)$ is pseudo δ-open fuzzy set on (X, τ). Hence, f is pseudo fuzzy δ-continuous.

We see in the following theorem that pseudo fuzzy δ-continuous functions may also be characterized by means of fuzzy points.

Theorem 3.3.5 A function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous iff for any pseudo δ-open fuzzy set μ on Y with $(f(x))_\alpha q \mu$ there exists a pseudo δ-open fuzzy set ν on X with $x_\alpha q \nu$ and $f(\nu) \leq \mu$.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous and μ be any pseudo δ-open fuzzy set on Y with $(f(x))_\alpha q \mu$. Then $\mu(f(x)) + \alpha > 1$. i.e., $(f^{-1}(\mu))(x) + \alpha > 1$. So, $x_\alpha q f^{-1}(\mu)$. Since f pseudo fuzzy δ-continuous, $f^{-1}(\mu)$ is pseudo δ-open in X. Now, $f(f^{-1}(\mu)) \leq \mu$ is always true, which proves the result.

Conversely, let the condition hold. We shall prove $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha \in I_1$, which is sufficent to prove $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous. Let μ^α be δ-open in Y

with $f(x) \in \mu^{\alpha}$. i.e., $\mu f(x) > \alpha$. Let us consider a fuzzy set ζ on Y by $\zeta(z) = \begin{cases} 1, & \text{if } \mu(t) > \alpha \\ \alpha, & \text{otherwise.} \end{cases}$

For $\beta > \alpha$, $y \in \zeta^{\beta} \Rightarrow \zeta(y) > \beta$

$\Rightarrow \zeta(y) > \alpha \Rightarrow \zeta(y) = 1 \Rightarrow \mu(y) > \alpha$. So, $(\zeta)^{\beta} \subseteq (\mu)^{\alpha}$. Similarly, we have $(\mu)^{\alpha} \subseteq (\zeta)^{\beta}$. So, $(\mu)^{\alpha} = (\zeta)^{\beta}$. For $\alpha > \beta, \forall\, y \in Y$, by definition of ζ, $\zeta(y) > \beta \Rightarrow y \in \zeta^{\beta}$. i.e, $Y \subseteq \zeta^{\beta}$. So, $Y = \zeta^{\beta}$. Also, for $\alpha = \beta$, $(\mu)^{\alpha} = (\zeta)^{\beta}$. So, $\zeta^{\beta} = \begin{cases} \mu^{\alpha}, & \text{if } \beta \geq \alpha \\ Y, & \text{if } \beta < \alpha. \end{cases}$

Hence, ζ^{β} is δ-open, $\forall\ \beta \in I_1$. Thus, ζ is pseudo δ-open fuzzy set on Y. As $\mu f(x) > \alpha$, $\zeta(f(x)) = 1 > \alpha \Rightarrow \zeta(f(x)) + (1 - \alpha) > 1 \Rightarrow (f(x))_{1-\alpha} q \zeta$. By the given condition there exist a pseudo δ-open fuzzy set ν on X with $x_{1-\alpha} q \nu$ and $f(\nu) \leq \zeta$. i.e., with $1 - \alpha + \nu(x) > 1 \Rightarrow \nu(x) > \alpha \Rightarrow x \in \nu^{\alpha}$ and $(f(\nu))^{\alpha} \subseteq \zeta^{\alpha}$, as $f(\nu) \leq \zeta \Rightarrow (f(\nu))^{\alpha} \subseteq \zeta^{\alpha}$. Hence, $f : (X, i_{\alpha}(\tau)) \to (Y, i_{\alpha}(\sigma))$ is δ-continuous, $\forall\ \alpha \in I_1$ and so, $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous.

Combining all the results proved earlier, we get:

Theorem 3.3.6 For a function f from a *fts* (X, τ) to another *fts* (Y, σ), the following are equivalent:

(a) f is pseudo fuzzy δ-continuous.

(b) $f : (X, i_{\alpha}(\tau)) \to (Y, i_{\alpha}(\sigma))$ is δ-continuous for each $\alpha \in I_1$.

(c) $f^{-1}(\mu)$ is *ps*-δ closed fuzzy set on (X, τ), where μ is *ps*-δ closed fuzzy set on (Y, σ).

(d) for any pseudo δ-open fuzzy set μ on Y with $(f(x))_\alpha q\mu$ there exists a pseudo δ-open fuzzy set ν on X with $x_\alpha q\nu$ and $f(\nu) \leq \mu$.

3.4 Starplus nearly compact fuzzy sets

In this section, we define compact-like fuzzy sets called starplus nearly compact fuzzy sets, and observe that it generalizes the existing notion of starplus compact fuzzy sets [49]. Some properties of starplus nearly compact fuzzy sets are discussed here, along with a couple of necessary conditions for a fuzzy set to be starplus nearly compact.

Definition 3.4.1 A fuzzy set μ on a fts (X, τ) is said to be starplus nearly compactif μ^α is nearly compact on $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. A fts (X, τ) is said to be starplus nearly compact fts if $(X, i_\alpha(\tau))$ is nearly compact, $\forall\ \alpha \in I_1$.

It is clear from the definition that starplus compact [49] implies starplus nearly compact.

Theorem 3.4.1 The pseudo fuzzy δ-continuous image of a starplus nearly compact fuzzy set is also so.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous and μ, a starplus nearly compact fuzzy set on X. By Theorem (3.3.6), $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous for each $\alpha \in I_1$. As μ^α is nearly compact on $(X, i_\alpha(\tau))$, $f(\mu^\alpha)$ is nearly compact on $(Y, i_\alpha(\sigma))$. Again since $(f(\mu))^\alpha = f(\mu^\alpha), f(\mu)$ is starplus nearly compact fuzzy set in X.

Similarly, we have

Theorem 3.4.2 Every pseudo regular closed fuzzy set is starplus nearly compact on a starplus nearly compact fts.

Proof. Let (X, τ) be starplus nearly compact fts and μ be a pseudo regular closed fuzzy set on (X, τ). Hence, μ^α is regular closed in the nearly compact topological space $(X,\ i_\alpha(\tau))$ for each $\alpha \in I_1$. As every regular closed set on nearly compact space is nearly compact, μ^α is nearly compact in $(X,\ i_\alpha(\tau))$. Hence, μ is starplus nearly compact on (X, τ).

Theorem 3.4.3 Every pseudo δ-closed fuzzy set is starplus nearly compact fuzzy set on starplus nearly compact fts.

Theorem 3.4.4 The union of a finite number of starplus nearly compact fuzzy sets is starplus nearly compact.

Proof. Let μ, γ be two starplus nearly compact fuzzy sets on a fts

(X, τ). For all $\alpha \in I_1$, μ^α, γ^α and hence $\mu^\alpha \cup \gamma^\alpha$ is nearly compact on $(X, i_\alpha(\tau))$. As $(\mu \vee \gamma)^\alpha = \mu^\alpha \cup \gamma^\alpha$, $(\mu \vee \gamma)^\alpha$ is nearly compact on $(X, i_\alpha(\tau))$. Hence, $(\mu \vee \gamma)$ is starplus nearly compact fuzzy set on (X, τ).

Theorem 3.4.5 If μ is starplus nearly compact fuzzy set on a fts (X, τ), for any pseudo regular closed fuzzy set ϑ in $(X, \tau), \mu \wedge \vartheta$ is starplus nearly compact.

Proof. Considering that every regular closed set on nearly compact space is nearly compact and $(\mu \wedge \vartheta)^\alpha = \mu^\alpha \cap \vartheta^\alpha$, the theorem follows.

Theorem 3.4.6 Every starplus nearly compact fuzzy set on Hausdorff fts is pseudo δ-closed fuzzy set.

Proof. Let μ be starplus nearly compact fuzzy set on Hausdorff fts (X, τ). Then, $(X, i_\alpha(\tau))$ is Hausdorff and μ^α is nearly compact in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. So, μ^α is δ-closed in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. Hence, μ is pseudo δ-closed in (X, τ).

Theorem 3.4.7 A fuzzy set μ on a fts (X, τ) with a finite support is starplus nearly compact.

Proof. Straightforward and hence omitted.

Theorem 3.4.8 A fuzzy set μ on a fts (X, τ) with a finite fuzzy topology is starplus nearly compact.

Proof. Straightforward and hence omitted.

Lemma 3.4.1 [54] Let (X, τ) be a topological space and let $(X, w(\tau))$ be the fully stratified fts. Then for each $\alpha \in I_1, i_\alpha(w(\tau)) = \tau$.

Theorem 3.4.9 Let (X, τ) be a topological space and let $(X, w(\tau))$ be the corresponding fully stratified fts. Then for any starplus nearly compact fuzzy set μ in fts $(X, w(\tau))$, $supp(\mu)$ is nearly compact in (X, τ).

Proof. As μ is starplus nearly compact fuzzy set on fts $(X, w(\tau)), \mu^\alpha$ is nearly compact in $i_\alpha(w(\tau)), \forall\ \alpha \in I_1$. By Lemma (3.4.1), μ^α is nearly compact in $\tau, \forall\ \alpha \in I_1$. In particular, $supp(\mu) = \mu^0$ is nearly compact in τ.

We conclude this section with a couple of necessary conditions for starplus nearly compact fts, which play important role in determining when (X, τ) is not starplus nearly compact.

Theorem 3.4.10 If a fts (X, τ) is starplus nearly compact then
(i) every collection of pseudo regular open fuzzy sets $\{\mu_i\}$ with $\vee\mu_i = 1$, implies, there exist a finite subcollection $\{\mu_i : i = 1, 2, .., n\}$ such that, $\bigvee_{i=1}^{n} \mu_i = 1$.
(ii) every family $\mathcal{F}$ of pseudo regular closed fuzzy sets with $\wedge\{\mu_i : \mu_i \in \mathcal{F}\} = 0$ implies for each $\alpha \in I_1 - \{0\}$ there exist a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\wedge\{\mu_i : \mu_i \in \mathcal{F}_0\} \leq \alpha$.

Proof. (i) Let (X, τ) be a starplus nearly compact fts and $\{\mu_i\}$ be a collection of pseudo regular open fuzzy sets with $\vee\mu_i = 1$. $(X, i_\alpha(\tau))$ is nearly compact $\forall\ \alpha \in I_1$. Now, $X = 1^\alpha = (\vee\mu_i)^\alpha = \cup\mu_i^\alpha$, which shows that $\{\mu_i^\alpha\}$ is a regular open cover of X. Since X is nearly compact, $\{\mu_i^\alpha\}$ has a finite subcover. i.e., there exist $i = 1, 2, .., n$ such that $X = \bigcup_{i=1}^n {\mu_i}^\alpha = (\bigvee_{i=1}^n \mu_i)^\alpha$. Hence, for each $\forall\ \alpha \in I_1$, $1^\alpha = (\bigvee_{i=1}^n \mu_i)^\alpha \Rightarrow 1 = \bigvee_{i=1}^n \mu_i$.

(ii) Let (X, τ) be a starplus nearly compact fts and $\mathcal{F}$ be a family of pseudo regular closed fuzzy sets with $\wedge\{\mu_i : \mu_i \in \mathcal{F}\} = 0$.

Hence, $\{{\mu_i}^\alpha\}$ is a collection of regular closed sets in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1 - \{0\}$. We claim that $\cap{\mu_i}^\alpha = \Phi$. If not, let $x \in \cap{\mu_i}^\alpha$

$\Rightarrow \forall\ i,\ x \in {\mu_i}^\alpha$

$\Rightarrow \mu_i(x) > \alpha$

$\Rightarrow \alpha < \mu_i(x) < 1 \Rightarrow \wedge\{\mu_i\} \neq 0$, which is a contradiction. Hence, $\cap{\mu_i}^\alpha = \Phi$. As (X, τ) is starplus nearly compact fts, $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1 - \{0\}$ is nearly compact. So, for each $\alpha \in I_1 - \{0\}$ there is a finite sub family $\mathcal{F}_0 = \{{\mu_i}^\alpha : i = 1, 2, ..., n\}$ of $\mathcal{F}$ such that $\cap\{{\mu_i}^\alpha : {\mu_i}^\alpha \in \mathcal{F}_0\} = \Phi$. We claim that $\wedge\{\mu_i : i = 1, 2, ..., n\} \leq \alpha$. If possible let for $i = 1, 2, ..., n$, $inf\{\mu_i(x)\} > \alpha$. So, $\mu_i(x) > \alpha$, for all $i = 1, 2, ..., n$. $\Rightarrow \cap{\mu_i}^\alpha \neq \Phi$, which is a contradiction. Hence, $\wedge\{\mu_i : i = 1, 2, ..., n\} \leq \alpha$