Chapter 2

Fuzzy topologies on Function spaces

2.1 Introduction

Function spaces play an important role in the study of various branches of mathematics, such as, functional analysis, topology, differential equations, differential geometry, and complex analysis among others. Considerable amount of work has been done since long past, by introducing different topologies on a given collection of functions. But not much researches have been done so far, by applying fuzzy topologies on function spaces. Some fruitful attempts in this direction which need be mentioned, were by Kohli and Prasannan [48], [49], Gunther Jagar [45] and Peng [75].

Section (2.2) begins with the definition of fuzzy compact open topology on function spaces, as given by Gunther Jagar in 1999 [45]. This author has slightly changed the definition of fuzzy compact open topology as due to Peng, by replacing Wang's [88] definition of N-compactness by Lowen's definition [53] of compactness. In [48], Kohli and Prasannan, following the definition of compactness due to Wang [88], made some studies on fuzzy compact open topology on function spaces and named it N-compact open topology. In this paper, the authors also left a problem open that if the range space is fuzzy regular, whether the space of all fuzzy continuous functions too is fuzzy regular or not. In this section, we have shown that a collection of functions $\mathcal{F}$ (from a fts X to another fts Y) endowed with fuzzy compact open topology becomes fuzzy T_2, when the range space Y is fuzzy T_2. Moreover, the fuzzy regularity of the range space induces somewhat regularity on the space of functions with fuzzy compact open topology.

Besides, we have introduced a new fuzzy topology called fuzzy F_{NR} topology on functions between two fts. The NR topology was introduced and studied in detail by Ganguly and Dutta [31] for functions between two topological spaces. Analogous to the definition of fuzzy compact open topology, given by Gunther Jagar in [45], we have introduced fuzzy nearly compact regular open (F_{NR}) topology and studied the function space under this fuzzy topology, imposing conditions on the respective range and domain fts. It is observed that the fuzzy GS-T_2-ness (fuzzy almost regularity) of the range fts Y induces fuzzy GS-T_2-ness (respectively, fuzzy somewhat almost regularity) in $\mathcal{F}$. We have also obtained that the evaluation map on $(\mathcal{F}, F_{NR})$ is fuzzy δ-continuous, if the range space is almost regular.

In Section (2.3), the notions of jointly fuzzy continuous on fuzzy compacta and jointly fuzzy δ-continuous on fuzzy near compacta are initiated, in connection with the fuzzy compact open topology and F_{NR} topology respectively. A special class of functions with fuzzy compact open topology is shown to be jointly fuzzy continuous on fuzzy compacta. The conditions for the F_{NR} topology to be jointly fuzzy δ-continuous are also formulated.

2.2 Fuzzy compact open and nearly compact regular open topology

In this section, we investigate fuzzy T_2-ness of function spaces under fuzzy compact open topology, as defined by Jagar and also by our own definition of fuzzy nearly compact regular open topology. Several forms of fuzzy T_2-ness are available in the literature, two of such being fuzzy T_2-ness [74] and GS-T_2-ness [33]. We find that

GS-T_2-ness of the codomain space is responsible for GS-T_2-ness of fuzzy compact open topology, Δ_{co} as well as fuzzy nearly compact regular open topology, F_{NR}. Moreover, we see that fuzzy T_2-ness of the codomain space also induces fuzzy T_2-ness on $(\mathcal{F}, \Delta_{co})$ and $(\mathcal{F}, F_{NR})$.

The last part of this section is mainly involved in finding out the behaviour of the function spaces, when the codomain space is fuzzy regular or fuzzy almost regular, under both the fuzzy topologies mentioned above.

Definition 2.2.1 [45] Let (X, τ) and (Y, σ) be two topological spaces and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each fuzzy compact set K on X and each fuzzy open set G on Y, a fuzzy set K_G on $\mathcal{F}$ is given by $K_G(g) =^{inf}_{x \in supp(K)} G(g(x))$. The collection of all such K_G forms a subbase for some fuzzy topology on $\mathcal{F}$, called fuzzy compact open topology and it is denoted by Δ_{co}.

Replacing fuzzy compact set K by fuzzy nearly compact set N on X and fuzzy open set G by fuzzy regular open set R on Y, we get a fuzzy set N_R on $\mathcal{F}$, given by $N_R(g) =^{inf}_{x \in supp(N)} R(g(x))$. The collection of all such fuzzy sets N_R, forms a subbase for some fuzzy topology on $\mathcal{F}$. We call this, fuzzy nearly compact regular open topology and denote it by F_{NR}.

Remark 2.2.1 Every crisp fuzzy point x_1 on a fts is fuzzy compact.

Proof. If x_1 is a crisp fuzzy point and $\mathcal{U}$ is any collection of fuzzy open sets on X with $x_1 \leq sup\{U : U \in \mathcal{U}\}$ then $1 \leq sup\{U : U \in \mathcal{U}\}(x)$ and for $y \neq x$, $0 \leq sup\{U : U \in \mathcal{U}\}(y)$. By definition of supremum, for any $\epsilon > 0$, $\exists\, U_k \in \mathcal{U}$ such that $U_k(x) > 1 - \epsilon$. Again for $y \neq x, U_k(y) \geq 0$, so that $(U_k + \epsilon)(y) > 0$. Hence, $\{U_k\}$ forms a finite subcollection of $\mathcal{U}$ such that $x_1 < U_k + \epsilon$. This proves that x_1 is fuzzy compact.

Remark 2.2.2 As x_1 is fuzzy compact, it is fuzzy nearly compact.

Remark 2.2.3 Here, in particular if we take $N = x_1$, we get $(x_1)_G(f) = Gf(x)$. Then the fuzzy topology generated by all $(x_1)_G$ is called fuzzy point regular open topology, denoted by F_{PR}.

Lemma 2.2.1 In a fts X,

(1) if x_α and y_β are any two fuzzy points with $x \neq y$ and A_1, B_1 are fuzzy open sets with $x_\alpha \in A_1$, $y_\beta \in B_1$ and $A_1 \not{q} B_1$ then there exist fuzzy regular open sets A and B such that $x_\alpha \in A$, $y_\beta \in B$ and $A \not{q} B$.

(2) if x_α and x_β $(\alpha < \beta)$ are any two fuzzy points and A_2, B_2 are fuzzy open sets with $x_\alpha \in A_2$, $x_\beta q B_2$ and $A_2 \not{q} B_2$ then there exist fuzzy regular open sets A and B such that $x_\alpha \in A$, $x_\beta q B$ and $A \not{q} B$.

Proof. **(1)** Consider $A = \overline{A_1}^0$ and $B = \overline{B_1}^0$. $x_\alpha \in A_1 \Rightarrow \alpha \leq A_1(x)$.

So, $\alpha \leq \overline{A_1}^0(x)$. Hence, $x_\alpha \in \overline{A_1}^0 = A$. Similarly, $y_\beta \in B_1$

$\Rightarrow y_\beta \in \overline{B_1}^0 = B$. $A_1 \not{q} B_1 \Rightarrow \forall\, x \in X,\; A_1(x) + B_1(x) \leq 1$.

$\Rightarrow A_1(x) \leq 1 - B_1(x)$.

$\Rightarrow \overline{A_1}(x) \leq (1 - B_1)(x)$.

$\Rightarrow \overline{A_1}^0(x) \leq \overline{A_1}(x) \leq (1 - B_1)(x)$.

$\Rightarrow \overline{A_1}^0(x) + B_1(x) \leq 1$.

$\Rightarrow B_1(x) \leq 1 - \overline{A_1}^0(x), \forall\, x \in X$

As $\overline{A_1}^0$ is a fuzzy open set, by similar argument as above,

$\overline{B_1}^0(x) \leq \overline{B_1}(x) \leq 1 - \overline{A_1}^0(x), \forall\, x \in X$.

Hence, $\overline{A_1}^0(x) + \overline{B_1}^0(x) \leq 1, \forall\, x \in X \Rightarrow \overline{A_1}^0 \not{q} \overline{B_1}^0 \Rightarrow A \not{q} B$. Hence, the result.

(2) x_α and x_β $(\alpha < \beta)$ are any two fuzzy points and A_2, B_2 are fuzzy open sets with $x_\alpha \in A_2$, $x_\beta q B_2$ and $A_2 \not{q} B_2$. $\alpha \leq A_2$, $\beta + B_2(x) > 1$ and $A_2(z) + B_2(z) \leq 1, \forall\, z \in X$. Choose $A = \overline{A_2}^0$ and $B = \overline{B_2}^0$. Then $\alpha \leq A_2(x) \leq \overline{A_2}^0(x) = A(x)$. So, $x_\alpha \in A$.

$\beta + B(x) = \beta + \overline{B_2}^0(x) \geq \beta + B_2(x) > 1$. So, $x_\beta q B$ and as in (1), $A \not{q} B$.

Theorem 2.2.1 Let (X, τ) and (Y, σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with F_{NR} topology is fuzzy GS-T_2 when (Y, σ) is so.

Proof. Let f_λ and g_μ be two fuzzy points on $\mathcal{F}$.

Case (i): Suppose, $f \neq g$. Then $f(x) \neq g(x)$ for some $x \in X$. Now,

$(f(x))_\lambda$ and $(g(x))_\mu$ be two fuzzy points on Y with $f(x) \neq g(x)$ and Y is fuzzy GS-T_2, there exist fuzzy open sets U_1, V_1 with $(f(x))_\lambda \in U_1$, $(g(x))_\mu \in V_1$ and $U_1 \not{q} V_1$. By Lemma (2.2.1(1)), there exist fuzzy regular open sets U and V such that $(f(x))_\lambda \in U$, $(g(x))_\mu \in V$ and $U \not{q} V$. Which gives, $\lambda \leq U(f(x))$, $\mu \leq V(g(x))$ and $U \not{q} V$.

$\Rightarrow \lambda \leq (x_1)_U(f)$, $\mu \leq (x_1)_V(g)$ and $U \not{q} V$.

$\Rightarrow f_\lambda \in (x_1)_U$, $g_\mu \in (x_1)_V$ and $U \not{q} V$.

To show $U \not{q} V \Rightarrow (x_1)_U \not{q} (x_1)_V$.

$U \not{q} V \Rightarrow \forall\, z \in X, U(z) + V(z) \leq 1$. Now,

$\forall\, h \in \mathcal{F}, (x_1)_U(h) + (x_1)_V(h) = U(h(x)) + V(h(x)) \leq 1, \forall\, x \in X$.

Hence, $(x_1)_U \not{q} (x_1)_V$.

Case (ii): Suppose $f = g$. without loss of generality we assume $\lambda < \mu$. Then $f(x) = g(x), \forall\, x \in X$. $(f(x))_\lambda$ and $(g(x))_\mu$ with $\lambda < \mu$ are fuzzy points on Y with $f(x) = g(x)$. By fuzzy GS-T_2 ness of Y, $\exists$ fuzzy open sets A_2, B_2 on Y such that $(f(x))_\lambda \in A_2$, $(g(x))_\mu q B_2$ and $A_2 \not{q} B_2$. By Lemma (2.2.1(2)), there exist fuzzy regular open sets A and B on Y such that $(f(x))_\lambda \in A$, $(g(x))_\mu q B$ and $A \not{q} B$

$\Rightarrow \lambda \leq A(f(x))$, $\mu + B(g(x)) > 1$ and $A(z) + B(z) \leq 1, \forall\, z \in X$.

$\Rightarrow f_\lambda \in (x_1)_A$, $g_\mu(g) + (x_1)_B(g) > 1$ and $A(z) + B(z) \leq 1, \forall\, z \in X$.

Thus, $f_\lambda \in (x_1)_A$, $g_\mu q (x_1)_B$ and $A(z) + B(z) \leq 1, \forall\, z \in X$. Now, $\forall\, \psi \in \mathcal{F}$, $(x_1)_A(\psi) + (x_1)_B(\psi) = A(\psi(x)) + B(\psi(x)) \leq 1$.

So, $(x_1)_A \not q\ (x_1)_B$. Hence, $\mathcal{F}$ endowed with F_{NR} topology is fuzzy GS-T_2.

As F_{PR} is a special case of F_{NR} topology, we have,

Corollary 2.2.1 Let (X,τ) and (Y,σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with F_{PR} topology is fuzzy GS-T_2 when (Y,σ) is fuzzy GS-T_2.

We append here, an analogous result for fuzzy compact open topology. As the method of proving this result is similar to that of Theorem (2.2.1), the proof is not given.

Theorem 2.2.2 Let (X,τ) and (Y,σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with fuzzy compact open topology Δ_{co}. Then $(\mathcal{F}, \Delta_{co})$ is fuzzy GS-T_2 if (Y,σ) is so.

We also find that fuzzy T_2-ness on the codomain space induces fuzzy T_2-ness on functions with fuzzy compact open topology.

Theorem 2.2.3 Let (X,τ) and (Y,σ) be two fuzzy topological spaces and $\mathcal{F}$ be a nonempty collection of functions from X to Y, endowed with the fuzzy compact open topology Δ_{co}. Then $(\mathcal{F}, \Delta_{co})$ is fuzzy T_2 when (Y,σ) is fuzzy T_2.

Proof. Let f_λ and g_μ be two fuzzy points in $\mathcal{F}$ having supp$(f_\lambda)\neq$ supp(g_μ); i.e., $f \neq g$. Hence, there exists $x \in X$ such that $f(x) \neq g(x)$. Now, let us consider two fuzzy points $f(x)_\lambda$ and $g(x)_\mu$ on Y.

Clearly, $supp(f(x)_\lambda) \neq supp(g(x)_\mu)$. Hence, by fuzzy T_2-ness of Y, there exist *q-nbd.s* B and C of $f(x)_\lambda$ and $g(x)_\mu$ respectively such that $B \wedge C = 0$. As B is a *q-nbd.* of $f(x)_\lambda$, there exist a fuzzy open set A in Y such that $f(x)_\lambda \, qA$ and $A \leq B$. i.e., $f(x)_\lambda(f(x)) + A(f(x)) > 1$ and putting $K = x_1$ we get $K_A \leq K_B$ where K is fuzzy compact in X. i.e., $\lambda + A(f(x)) > 1$ and $K_A \leq K_B$.

i.e., $f_\lambda + inf[A(f(x) : x \in \{x\}] > 1$ and $K_A \leq K_B$.

i.e., $f_\lambda(f) + K_A(f) > 1$ and $K_A \leq K_B$.

i.e., $f_\lambda q K_A$ and $K_A \leq K_B$.

Hence, K_B is a *q-nbd.* of f_λ. Similarly, we can prove that K_C is a *q-nbd.* of g_μ. Now,

$(K_B \wedge K_C)(h)$

$= inf(K_B(h), K_C(h))$

$= inf[inf\{B(h(x)) : x \in supp(K)\}, inf\{C(h(x)) : x \in supp(K)\}]$

$= inf[B(h(x)), C(h(x))]$

$= (B \wedge C)(h(x))$

$= 0$

Corollary 2.2.2 Let (X, τ) and (Y, σ) be two fts, $\mathcal{F} \subseteq Y^X$, endowed with F_{PR} topology is fuzzy T_2 when (Y, σ) is fuzzy T_2.

Proof. Follows from the above Theorem, as F_{PR} is a special case of fuzzy compact open topology.

Lemma 2.2.2 Let (X,τ) and (Y,σ) be two fts, $\mathcal{F} \subset Y^X$ and $N = x_1$ be a crisp fuzzy point on X.

(i) If $\mathcal{F}$ is endowed with fuzzy nearly compact regular open topology, F_{NR} and F is fuzzy regular open on Y then $N_{\overline{F}} \geq \overline{N_F} \geq N_F$.

(ii) If $\mathcal{F}$ is endowed with fuzzy compact open topology, Δ_{co} and F is fuzzy open on Y then $N_{\overline{F}} \geq \overline{N_F} \geq N_F$

Proof. **(i)** Suppose, $N = x_1$, for some $x \in X$. Then,

$(1 - (x_1)_F)(g)$

$= 1 - (x_1)_F(g)$

$= 1 - F(g(x))$

$= inf\{(1 - F)(g(x)) : x \in supp(x_1)\}$

$= N_{1-F}(g)$. Hence, $1 - N_F = N_{1-F}$. Since $(x_1)_F$ is a member of F_{NR}, $1-(x_1)_F$ is fuzzy closed in $\mathcal{F}$. So, N_{1-F} is closed in $\mathcal{F}$. In particular if V is a fuzzy regular open set, $\overline{V}$ is fuzzy regular closed in Y. Hence, $N_{\overline{V}} = N_{1-(1-\overline{V})} = 1 - N_{1-\overline{V}}$. In other words, $N_{\overline{V}}$ is a fuzzy closed set on $\mathcal{F}$ such that $N_V \leq N_{\overline{V}}$, as $V(f(x)) \leq \overline{V}(f(x))$. Consequently, $\overline{N_V} \leq N_{\overline{V}}$, as $\overline{N_V}$ is the smallest fuzzy closed set containing N_V.

(ii) Similar to (i).

Theorem 2.2.4 Let $\mathcal{F}$ be a collection of functions from a fts (X,τ) to a fts (Y,σ). Consider the evaluation map $e_x : \mathcal{F} \to Y$ defined by $e_x(f) = f(x)$ for each $x \in X$.

(i) If $\mathcal{F}$ is endowed with F_{NR} topology and Y is fuzzy almost regular, then each e_x is fuzzy δ-continuous.

(ii) If $\mathcal{F}$ is endowed with Δ_{co} topology and Y is fuzzy regular, then each e_x is fuzzy continuous.

Proof. **(i)** Let f_λ be a fuzzy point on $\mathcal{F}$. As $e_x(f_\lambda)$ is a fuzzy set on Y, for any $y \in Y$,we have $e_x(f_\lambda) = (f(x))_\lambda$. Let $(f(x))_\lambda$ be a fuzzy point on Y and U be a fuzzy regular open q-*nbd.* of $(f(x))_\lambda$. As Y is fuzzy almost regular, there exist fuzzy regular open set V such that $(f(x))_\lambda qV$ and $\overline{V} \leq U$. Hence, $\lambda + V(f(x)) > 1$ and $V \leq \overline{V} \leq U$
$\Rightarrow 1 - \lambda < V(f(x)) \leq \overline{V}(f(x)) \leq U(f(x))$.

So, $(1 - f_\lambda)(f) < (x_1)_V(f) \leq (x_1)_{\overline{V}}(f) \leq (x_1)_U(f)$.

Using Lemma (2.2.2), we have $\overline{(x_1)_V}^0 \leq \overline{(x_1)_V} \leq (x_1)_{\overline{V}} \leq (x_1)_U$.

Hence, $f_\lambda q\overline{(x_1)_V}^0$. Now, for each $y \in Y$,

$e_x[\overline{(x_1)_V}^0](y)$

$= \sup_{e_x(g)=y}[\overline{(x_1)_V}^0(g)]$

$\leq \sup_{e_x(g)=y}[(x_1)_{\overline{V}}(g)]$

$= \sup_{e_x(g)=y}[\overline{V}(g)(x)]$

$\leq U(y)$. Hence, $e_x[\overline{(x_1)_V}^0] \leq U$. So, $\overline{(x_1)_V}^0$ is fuzzy regular q-*nbd.* of f_λ, as desired.

(ii) Similar to (i).

Theorem 2.2.5 (i) If K_μ is a fuzzy open set on $\mathcal{F}$ endowed with F_{NR}

topology, then $K_{\overline{\mu}}$ is a fuzzy δ-closed set on $\mathcal{F}$ and $K_\mu \leq \overline{K_\mu} \leq \delta$-$cl(K_\mu) \leq K_{\overline{\mu}}$.

(ii) If K_μ is a fuzzy open set on $\mathcal{F}$ endowed with Δ_{co} topology, then $K_{\overline{\mu}}$ is a fuzzy closed set on $\mathcal{F}$ and $K_\mu \leq \overline{K_\mu} \leq K_{\overline{\mu}}$.

Proof. (i) Since μ is a fuzzy regular open set on Y, $\overline{\mu}$ is fuzzy regular closed and hence by fuzzy δ-continuity of $e_x : \mathcal{F} \to Y$ given by $e_x(f) = f(x), \forall\, x \in X$, $e_x^{-1}(\overline{\mu})$ is δ-closed in $\mathcal{F}$. Now, $\forall f \in \mathcal{F}$,

$K_{\overline{\mu}}(f)$

$= inf\{\overline{\mu}(f(x)) : x \in supp(K)\}$

$= inf\{e_x^{-1}(\overline{\mu})(f) : x \in supp(K)\}$.

So, $K_{\overline{\mu}} = inf\{e_x^{-1}(\overline{\mu}) : x \in supp(K)\}$. Hence, $K_{\overline{\mu}}$ is fuzzy δ-closed set on $\mathcal{F}$. Again, $\forall\, x \in X$, $e_x^{-1}(\mu) \leq e_x^{-1}(\overline{\mu})$. Hence, $K_\mu \leq \overline{K_\mu} \leq \delta$-$cl(K_\mu) \leq K_{\overline{\mu}}$.

(ii) Since μ is a fuzzy open set on Y, $\overline{\mu}$ is fuzzy closed and hence by fuzzy continuity of $e_x : \mathcal{F} \to Y$ given by $e_x(f) = f(x), \forall\, x \in X$, $e_x^{-1}(\overline{\mu})$ is fuzzy closed in $\mathcal{F}$. Now, $\forall\, f \in \mathcal{F}$,

$K_{\overline{\mu}}(f)$

$= inf\{\overline{\mu}(f(x)) : x \in supp(K)\}$

$= inf\{e_x^{-1}(\overline{\mu})(f) : x \in supp(K)\}$.

So, $K_{\overline{\mu}} = inf\{e_x^{-1}(\overline{\mu}) : x \in supp(K)\}$.

Hence, $K_{\overline{\mu}}$ is fuzzy closed set on $\mathcal{F}$. Again, $\forall\, x \in X$, $e_x^{-1}(\mu) \leq e_x^{-1}(\overline{\mu})$.

Consequently, $K_\mu \leq \overline{K_\mu} \leq K_{\overline{\mu}}$.

Definition 2.2.2 A *fts* X is said to be fuzzy somewhat regular if for each fuzzy point x_α $(0 < \alpha < 1)$ and any fuzzy open set A with $x_\alpha qA$, there exists a fuzzy open set B and γ with $0 < \alpha < \gamma < 1$ such that $x_\gamma qB$ and $\overline{B} \leq A$.

Definition 2.2.3 A *fts* X is said to be fuzzy somewhat almost regular if for each fuzzy point x_α $(0 < \alpha < 1)$ and any fuzzy regular open set A with $x_\alpha qA$, there exists a fuzzy regular open set B and γ with $0 < \alpha < \gamma < 1$ such that $x_\gamma qB$ and $\bar{B} \leq A$.

Remark 2.2.4 It is clear that a fuzzy regular space is fuzzy somewhat regular and a fuzzy almost regular space is fuzzy somewhat almost regular.

Finally, we observe in the following two results that fuzzy regularity (fuzzy almost regularity) of the range space induces fuzzy somewhat regularity (respectively, fuzzy somewhat almost regularity) on the function space, equipped with fuzzy compact open topology (respectively, fuzzy nearly compact regular open topology).

Theorem 2.2.6 Let (X, τ) and (Y, σ) be two *fts* and $\mathcal{F} \subset Y^X$ be endowed with Δ_{co} topology. Then $\mathcal{F}$ is fuzzy somewhat regular if Y is fuzzy regular.

Proof. Let (Y, σ) be fuzzy regular. Let g_λ be any fuzzy point on $\mathcal{F}$ and K_G be a subbasic fuzzy open set on $\mathcal{F}$ such that $g_\lambda q K_G$, where K is fuzzy compact on X and G is fuzzy open on Y. So, $g_\lambda(g) + K_G(g) > 1$. i.e., $1 - \lambda < \inf\{G(g(x)) : x \in supp(K)\}$. Hence, for all $x \in supp(K)$ and $(g(x))_\lambda$ on Y, we get $(g(x))_\lambda q G$. By fuzzy regularity of Y there exist a fuzzy open set F on Y, such that $(g(x))_\lambda q F$ and $\overline{F} \leq G$. i.e., $1 - \lambda < F(g(x)) \leq \overline{F}(g(x)) \leq G(g(x))$

So, $1 - \lambda \leq K_F(g) \leq K_{\overline{F}}(g) \leq K_G(g)$. Choose any β such that $0 < \lambda < \beta < 1$. Then $1-\beta < 1-\gamma$ and $1-g_\beta(g) \leq K_F(g) \leq K_{\overline{F}}(g) \leq K_G(g)$. Hence, using Theorem (2.2.5) $1-g_\beta(g) \leq K_F(g) \leq \overline{K_F}(g) \leq K_{\overline{F}}(g) \leq K_G(g)$. So, $g_\beta q K_F$ and $\overline{K_F} \leq K_G$. Hence, $(\mathcal{F}, \Delta_{co})$ is fuzzy somewhat regular.

Theorem 2.2.7 Let (X, τ) and (Y, σ) be two fts and $\mathcal{F} \subset Y^X$ be endowed with F_{NR} topology. Then $\mathcal{F}$ is fuzzy somewhat almost regular if Y is fuzzy almost regular.

Proof. Let (Y, σ) be fuzzy almost regular. Let g_λ be any fuzzy point on $\mathcal{F}$ and N_R be a subbasic fuzzy open set on $\mathcal{F}$ such that $g_\lambda q N_R$, where N is fuzzy nearly compact on X and R is fuzzy regular open in Y. So, $g_\lambda(g) + N_R(g) > 1$.

i.e., $1 - \lambda < \inf\{R(g(x)) : x \in supp(N)\}$. Hence, $(g(x))_\lambda q R$, $\forall\ x \in supp(N)$. By fuzzy almost regularity of Y there exist a fuzzy regular open set F in Y such that $(g(x))_\lambda qF$ and $\bar{F} \leq R$. i.e., $1-\lambda < F(g(x)) \leq \overline{F}(g(x)) \leq R(g(x))$

$\Rightarrow 1-\lambda \leq N_F(g) \leq N_{\overline{F}}(g) \leq N_R(g)$. Choose any β such that $0 < \lambda < \beta < 1$. Then $1-\beta < 1-\gamma$ and $1 - g_\beta(g) \leq N_F(g) \leq N_{\overline{F}}(g) \leq N_R(g)$. Hence, using Theorem (2.2.5) $1 - g_\beta(g) \leq N_F(g) \leq \overline{N_F}^0(g) \leq \overline{N_F}(g) \leq N_{\overline{F}}(g) \leq N_R(g)$. So, $g_\beta q \overline{N_F}^0$ and $\overline{N_F}^0 \leq N_R$. Hence, $(\mathcal{F}, F_{NR})$ is fuzzy somewhat almost regular.

2.3 Jointly fuzzy continuous and δ-continuous fuzzy topologies

The notion of joint continuity and joint δ-continuity have significant roles in general topology. In this section, defining the analogues of such topologies in fuzzy setting, we observe their interrelations with fuzzy compact open topology and fuzzy nearly compact regular open topology. We have also found a family of functions on which Δ_{co} (F_{NR}) becomes jointly fuzzy continuous on fuzzy compacta (respectively, jointly fuzzy δ-continuous on fuzzy near compacta).

Definition 2.3.1 Let (X, τ) and (Y, σ) be two fuzzy topological spaces and $\mathcal{F}$ be a nonempty collection of functions from X to Y. A fuzzy topology on $\mathcal{F}$ is said to be

(i) jointly fuzzy continuous (jointly fuzzy continuous on fuzzy compacta) if a function $P : \mathcal{F} \times X \to Y$ given by $P(f,x) = f(x)$ is fuzzy continuous (respectively, $P\mid_{\mathcal{F}\times supp(K)}$ is fuzzy continuous for each fuzzy compact set K on X).

(ii) jointly fuzzy δ-continuous (jointly fuzzy δ-continuous on fuzzy near compacta) if a function $P : \mathcal{F} \times X \to Y$ given by $P(f,x) = f(x)$ is fuzzy δ-continuous (respectively, $P\mid_{\mathcal{F}\times supp(N)}$ is fuzzy δ-continuous for each fuzzy nearly compact set N on X).

Definition 2.3.2 Let (X,τ) and (Y,σ) be two fuzzy topological spaces. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be

(i) fuzzy continuous on a fuzzy compact set K on X if for any fuzzy open set μ, $f\mid^{-1}_{supp(K)}(\mu)$ is fuzzy open on $supp(K)$. (ii) fuzzy δ-continuous on a fuzzy nearly compact set N on X if for any fuzzy regular open set μ, $f\mid^{-1}_{supp(N)}(\mu)$ is fuzzy regular open on $supp(N)$.

The following two results are immediate from the definitions and hence the proofs are omitted.

Theorem 2.3.1 (i) Each fuzzy topology on $\mathcal{F}$ which is jointly fuzzy continuous on fuzzy compacta is larger than the fuzzy compact open topology on $\mathcal{F}$.

(ii) Each fuzzy topology on $\mathcal{F}$ which is jointly fuzzy δ-continuous on fuzzy near compacta is larger than the fuzzy nearly compact regular open topology on $\mathcal{F}$.

We observe that each member of the collection of functions $\mathcal{F}$ is necessarily fuzzy continuous (fuzzy δ-continuous) if $\mathcal{F}$ is endowed with any jointly fuzzy continuous topology (respectively, jointly fuzzy δ-continuous topology). Since the methods of developments for both the cases are similar, we prove only one case in the following theorem and state the other as its subsequent theorem.

Theorem 2.3.2 If $\mathcal{F}$ is endowed with the fuzzy topology which is jointly fuzzy δ-continuous then each $f \in \mathcal{F}$ is fuzzy δ-continuous.

Proof. Let Δ be a fuzzy topology on $\mathcal{F}$ which is jointly fuzzy δ-continuous. Then P is fuzzy δ-continuous. Let x_β be any fuzzy point on X and V any fuzzy regular open *nbd.* of $(f(x))_\beta$ on Y. Now, $(f_1 \times x_\beta)(h,t)$

$= f_1(h) \wedge x_\beta(t)$

$$= \begin{cases} \beta, & if h = f \text{ and } t = x \\ 0, & otherwise. \end{cases}$$

Hence, $(f_1 \times x_\beta) = (f,x)_\beta$. So, $P(f_1 \times x_\beta) = P((f,x)_\beta) = (f(x))_\beta$. Using the fuzzy δ continuity of P, there exist fuzzy regular open *nbd.s* U_1 of f_1 in $\mathcal{F}$ and U_2 of x_β on X such that $P(U_1 \times U_2) \leq V$. Now, $P(U_1 \times U_2)(y)$

$= \sup\{(U_1 \times U_2)(h,t) : (h,t) \in\ P^{-1}(y)\}$, *where* $(h,t) \in \mathcal{F} \times X$.

Also, $P(U_1 \times U_2)(y)$

$\geq \sup\{(f_1 \times U_2)(h,t) : P(h,t) = y\}$

$= \sup\{(f_1 \times U_2)(h,t) : h(t) = y\}$

$= \sup\{(f_1(h) \wedge U_2(t) : h(t) = y\}$

$= \sup\{U_2(t) : f(t) = y\}$

$= f(U_2)(y)$

Hence, $f(U_2) \leq P(U_1 \times U_2) \leq V$, as desired.

Theorem 2.3.3 If $\mathcal{F}$ is endowed with the fuzzy topology which is jointly fuzzy continuous then each $f \in \mathcal{F}$ is fuzzy continuous.

Definition 2.3.3 A family $\mathcal{F}$ of functions from a fts X to a fts Y is said to be

(i) fuzzy equicontinuous on fuzzy compacta, if for each fuzzy compact set K on X and any fuzzy open set V on Y with $(f(x))_\alpha qV$, for some $f \in \mathcal{F}, x \in supp(K)$, then there exist a fuzzy open set U on $supp(K)$ such that $\forall\ h \in \mathcal{F}$, $h(U) < V$ and $t_\alpha qU, \forall\ t \in supp(K)$.

(ii) fuzzy δ-equicontinuous on fuzzy near compacta, if for each fuzzy nearly compact set N on X and any fuzzy regular open set V on Y with $(f(x))_\alpha qV$, for some $f \in \mathcal{F}, x \in supp(N)$, then there exist a fuzzy regular open set U on $supp(N)$ such that $\forall\ h \in \mathcal{F}$, $h(U) < V$ and $t_\alpha qU, \forall\ t \in supp(N)$.

Theorem 2.3.4 Let (X, τ) and (Y, σ) be two fts. If $\mathcal{F}$ is fuzzy δ-

equicontinuous on fuzzy near compacta then $\mathcal{F}$ endowed with F_{NR} topology is jointly fuzzy δ-continuous on fuzzy near compacta.

Proof. Let $(f,x)_\alpha$ be a fuzzy point on $\mathcal{F} \times supp(N)$ and V be any fuzzy regular open set on Y with $(P(f,x))_\alpha qV$ i.e., $(f(x))_\alpha qV$, $x \in supp(N)$. Since $\mathcal{F}$ is fuzzy δ-equicontinuous, there exist a fuzzy regular open set U on $supp(N)$ such that $\forall\ h \in \mathcal{F},\ h(U) < V$ and $t_\alpha qU, \forall\ t \in supp(N)$. Now, it is easy to see that $h(U) < V \Rightarrow U(z) < V(h(z)), \forall\ z \in X$. since $(N_{\overline{V}})^0$ and U are respectively fuzzy regular open in $\mathcal{F}$ and $supp(N)$, we need to show that $(f,x)_\alpha q\ ((N_{\overline{V}})^0 \times U)$ and $P((N_{\overline{V}})^0 \times U) \leq V$. Now,

$((N_{\overline{V}})^0 \times U)(f,x) + \alpha$

$\geq (N_V(f) \wedge U(x)) + \alpha$

$= [\inf_{t \in supp(N)} V(f(t)) \wedge U(x)] + \alpha$

$> [\inf_{t \in supp(N)} U(t) \wedge U(x)] + \alpha$

$\geq (1-\alpha) + \alpha.$

Hence, $(f,x)_\alpha q\ ((N_{\overline{V}})^0 \times U)$.

Again, $P((N_{\overline{V}})^0 \times U)(y)$

$= \sup_{P(h,t)=y} [((N_{\overline{V}})^0 \times U)(h,t)]$

$\leq \sup_{P(h,t)=y} [(N_{\overline{V}} \times U)(h,t)]$

$= \sup_{h(t)=y} [\inf_{s \in supp(N)} \overline{V}(h(s)) \wedge U(t)]$

$< \sup_{h(t)=y} [\inf_{s \in supp(N)} \overline{V}(h(s)) \wedge V(h(t))]$

$\leq_{h(t)=y}^{sup} [V(h(t))]$

$= V(y)$

Hence, $P((N_{\overline{V}})^0 \times U) \leq V$.

In a similar way it can be shown that:

Theorem 2.3.5 Let (X, τ) and (Y, σ) be two fts. If $\mathcal{F}$ is fuzzy equicontinuous on fuzzy compacta then $\mathcal{F}$ endowed with Δ_{co} is jointly fuzzy continuous on fuzzy compacta.