Chapter 1

Introduction and Preliminaries

1.1 Historical Background

The classical theory of sets was introduced by Cantor towards the end of nineteenth century. This is still regarded as the foundation of mathematics in general. Based on this theory, vast amount of significant researches have been carried out by several distinguished mathematicians in different fields of Mathematics. To mention a few, subjects like Topology, Algebra, Functional Analysis, Measure Theory were immensely developed since the time of Cantor to the present day.

Even then, applications of this set theory had their own limitations. It is with this point of view as the background, Zadeh in 1965 enunciated a new branch of study known as fuzzy mathematics. He defined in his preliminary work [94], the concept of fuzzy sets as a function from an ordinary set X into the closed interval $[0, 1]$. In this contemporary period, Goguen [38] replaced the closed interval $[0, 1]$ by a complete distributive lattice and named it L-fuzzy set. Soon thereafter in 1971, Brown [7] modified the definition of Goguen by defining a fuzzy set as a function from a set to a Boolean lattice. He applied his definition to some of the results based on Zadeh's fuzzy sets and found them consistent in his own framework.

In this period, the theory of fuzzy sets had been developed considerably by several workers. Fuzzy set theory is applied in various branches of Mathematics, such as, Topology, Algebra, Graph Theory, Logic and so on. Apart from these, it has applications in subjects like Operation Research, Mathematical Programming and various diversified fields of applications as Pattern Recognition, Information Processing, Signal Processing, Production Management, Robotics, Automations, etc. Lucid presentation of the subject including some of its applications are given in the book of Zimmerman [95].

In 1968, C.L. Chang [10] became the pioneer to present the idea of fuzzy topology. Fundamentally, he replaced the classical notion of open sets by a notion called fuzzy open sets. However, later in 1976, Lowen [53] found that many of the well known results in general topology cannot be obtained if one follows Chang's definition of fuzzy topology. So, he redefined fuzzy topology by including constant fuzzy sets in it. This implies that in Chang's definition of fuzzy topology one more axiom was included. It should be mentioned that those who follow Chang's definition of fuzzy topology, call Lowen's definition of it as "fully stratified" fuzzy topology. In this thesis, we have also followed the same nomenclature.

To study in fuzzy topology, analogous local topological concepts, such as, local compactness, local connectedness, 1^{st} countability and continuity at a point, convergence of nets and filters to a point, the well known concept of fuzzy point was introduced by Wong in 1974 [93]. According to Wong's definition, a fuzzy point on a space X is a particular type of fuzzy set having value zero at all but one point of X, while the value at the later point x (say) of X is α (with $0 < \alpha < 1$). This was later modified by Pu and Liu [72] in 1980, to include $\alpha = 1$. It is only when $\alpha = 1$ is taken, that crisp singletons follow as a particular case of fuzzy points. In this thesis, we have adopted the above modified definition as given by Pu and Liu.

To Pu and Liu [72] is attributed the inception of the concepts of q-concidence and q-neighbourhoods, which in succeeding years led with great strides the study of fuzzy topology.

Among them, many investigators whose works involve the concept of fuzzy points, the names of Wong [92], Kotze [50], De Mitri and Pascali [17], Kandid and El-Etriby [47] need to be mentioned.

In 1976, Lowen [53] introduced two functors ω and i which allow one to see more clearly the fuzzy topological spaces and topological spaces. The functor ω carries over a topological space to a fuzzy topological space; whereas the functor i_α, for each $\alpha \in [0, 1)$, carries over a fuzzy topological space to a topological space. The topology $i_\alpha(\tau)$, where τ stands for a fuzzy topology, has been named by Kohli and Prasannan [49], strong α-level topology.

Pioneering researches on various separation axioms were done by Hutton [42], Hutton and Reilly [43], Rodabough [77] and many others, of whom the names of Ganguly and Saha [33], Sinha [83], [82] should be cited.

In the following sections, the gradual developments of the topics relevant to the thesis are enumerated in details.

1.2 Operators and functions on fuzzy topological spaces

In the last three or four decades, various mathematicians initiated a variety of closure-like operators in general topology. Significant among them are semi-closures [14], δ-closure [87], θ-closure [87], α-

closure [70], pre-closure [60] and δ-preclosure [76] operators. These in turn resulted in the introduction of their corresponding open-like sets. Several authors in this period applied these concepts in their studies of separation axioms, different compact-like covering properties, continuous-like functions, connectedness and so on.

In this contemporary period, generalizations of the above concepts have been made in fuzzy topological space by different authors. Among them, Azad [3] introduced what is known as fuzzy semi-closure operator. Following his concepts, fuzzy semi-separation axioms, semi-continuous and semi-closed mappings, fuzzy semi-connectedness and fuzzy S-closedness were developed by several authors and to name a few are Ganguly and Saha [32], Ghosh [37], Mukherjee and Ghosh [62] and Das [16].

The δ-closure operator as enunciated by Velicko [87] in general topology was extended in fuzzy settings by Ganguly and Saha [33], and was successfully applied in their study of fuzzy δ-continuity and fuzzy δ-connectedness. In topology, the δ-closure operator plays an important role to characterize near compactness. Later, these results were reviewed or generalized in fuzzy framework by Hayder [41], Mukherjee and Ghosh [63], [65], among others.

Apart from the operators described above, there are other significant operators on fuzzy topological spaces, of which α-closure operators for L-fuzzy topological spaces, as introduced by Rodabaugh [77] is one and fuzzy θ-closure operator, introduced by Mukherjee and Sinha [69] is one more among such operators.

A new pair of operators which we name as fuzzy ps-closure and fuzzy ps-interior has been used in our work in this thesis.

Different types of continuous-like functions in fuzzy topology were studied by many mathematicians. Chang [10] introduced fuzzy continuous maps between two fuzzy topological spaces, whereas Wong [92] initiated fuzzy open and closed maps. Mashhour and Ghanim [61] made a further generalization of fuzzy continuity in fuzzy closure spaces.

T. Noiri formulated various types of functions between topological spaces. His introduction of δ-continuity opened vast areas for further researches in general topology. The corresponding analogue in fuzzy version was fruitfully studied by Saha [78], Ganguly and Saha [33], El-Monsef et al. [28], Mukherjee and Ghosh [64] and others.

Apart from fuzzy δ-continuity, there are other forms of fuzzy continuity and their mutual relations.

In this thesis two distinct types of fuzzy continuous-like maps are defined and successfully applied in our investigations of certain compact-like fuzzy covering properties.

1.3 Fuzzy compactness, fuzzy near compactness and starplus compactness

Fuzzy compactness has been defined differently by different researchers. None of such definitions could singly construct many of the well known properties in fuzzy topology, in analogy to their corresponding properties in general topology.

C.L Chang [10] in 1968, showed that the fuzzy compactness is preserved under fuzzy continuous functions. In 1973, J.A. Goguen [39] pointed out a deficiency in Chang's definition of compactness, by showing that Tychonoff theorem does not hold for infinite product.

Later, C.K Wong [91], [92] introduced the notion of countable compactness, sequential compactness, semi-compactness and local compactness in fuzzy framework.

In the process of his investigations, Lowen [53] gave a successful definition of fuzzy compactness.

The concept of N-closed spaces was introduced by Carnahan [8] in general topology. Now, by replacing the topology of subspace by the topology of whole space is formed, what is known as nearly compact space. Singal and Mathur [81] are its profounders.

Hayder in 1987, defined near compactness in fuzzy topology and then after Mukherjee and Ghosh [63] made considerable studies of it. Fuzzy locally nearly compact space was introduced and characterized by Bakier [4]

In 2001 Kohli and Prasannan [49] introduced the notion of Starplus compactness by extending Lowen's notion of strong fuzzy cmpactness [53] to an arbitrary set. Earlier, Lowen had shown that for each $\alpha \in [0, 1)$, a functor i_α carries over a fuzzy topological space to a topological space. Kohli and Prasannan utilized this functor in defining Starplus compact fuzzy sets.

1.4 Fuzzy function spaces

Y. W. Peng [75], G. Jagar [45],[44], Kohli and Prasannan [49], [48] are among the early investigators on fuzzy function spaces. The introduction of fuzzy topologies on a given function space was first made by Peng [75]. He defined pointwise convergent topology and a version of compact open topology on a family of functions from a fuzzy topological space to another. Later, Alderton [1] considered such a problem from categorical point of view and made use of the theory of Cartesian closedness of monotopological categories to function spaces. Subsequently, Dang and Behera [15] induced a topology in lieu of a fuzzy topology, on a family of functions from a fuzzy topological space to another.

Later, Gunther Jagar [44] made significant contributions to the study of compactness in the category of fuzzy convergence spaces as defined by Lowen [53], Lowen and Wuyts [59]. The notions of splitting and conjoining structures on fuzzy subsets have been utilized by them in their works. As a special case, they studied a notion of fuzzy compact open topology.

Important developments of fuzzy topologies on function spaces were made by Kohli and Prasannan [48], [49]. In the first of these two papers, they introduced and studied three different fuzzy topologies on function spaces, which are analogues of the topology of pointwise convergence, compact open topology and topology of joint-continuity. Interrelations among them, analogous to their counter part in general topology, were also obtained. In their second paper, these authors extended Lowen's [53] notion of strong fuzzy compactness to an arbitrary fuzzy set; thereby they introduced the notion of starplus compact fuzzy set. They have shown that the category of starplus compact fuzzy topological space is productive and that starplus compactness is a good extension of the notion of compactness. Besides, they introduced the notion of starplus compact open fuzzy topology on a function space. Its interrelations with fuzzy topology of point-wise convergence and the fuzzy topology of joint fuzzy continuity were studied.

1.5 Left fuzzy topological ring

As is very well known, topological ring is the study of the topological behavior of a space, in presence of its ring structure. The properties of topological ring were extensively investigated by several renowned mathematicians, namely, Arnautov, Ursul, Warner and others. Lucid review of these works is given in the book "Topological rings satisfying compactness conditions" by Mihail Ursul [86] and also in the book "Introduction to the theory of topological rings and modules" by Arnautov, Glavatsky and Mikhalev [2].

Deb Ray [18] has recently framed a new form of fuzzy topological ring in the light of topological rings referred above. This author has named it a left fuzzy topological ring and comments that analogously, a right fuzzy topological ring can be studied. The basic aim of this paper is to characterize the fundamental system of fuzzy neighbourhoods of the fuzzy point 0_α $(0 < \alpha \leq 1)$ and to establish that any fuzzy neighbourhood of a fuzzy point x_α is of the form $x_\alpha + V$, where V is a fuzzy neighourhood of x_α.

In this section, many more properties of the above left fuzzy topological ring are obtained. Further, fuzzy continuous functions, having values in left fuzzy topological ring have been studied.

1.6 Basic definitions and results

The previously known definitions and results, which we have adopted in this thesis are enumerated as follows.

Definition 1.6.1 [94] Let X be a non empty set and I denote the closed interval $[0,1]$. A fuzzy set μ on X is a function from X into I. Let I^X denote the collection of all fuzzy sets on X. The support of a fuzzy set μ, denoted by $supp(\mu)$, is the crisp set $\{x \in X : \mu(x) > 0\}$. A fuzzy set with a singleton as its support is called a fuzzy point, denoted by x_α, where $0 < \alpha \leq 1$ and defined as

$$x_\alpha(z) = \begin{cases} \alpha, & for\ z = x \\ 0, & otherwise. \end{cases}$$

The fuzzy sets on X taking the constant values 0 and 1 respectively are denoted by 0_X and 1_X or simply by 0 and 1 respectively. A fuzzy set A on X is called non-null if $A \neq 0$. If a fuzzy set A takes only the values 0 and 1, then A is called a crisp subset of X, i.e., A becomes the characteristic function of an ordinary subset of X.

Definition 1.6.2 [94] Let f be a function from a set X into another set Y and A and B be fuzzy sets on X and Y respectively. Then $f(A)$ and $f^{-1}(B)$ are respective fuzzy sets on Y and X, given by,

$$f(A)(y) = \begin{cases} sup\{A(x) : x \in f^{-1}(y)\}, & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

and $f^{-1}(B)(x) = B(f(x))$

Definition 1.6.3 [94] Two fuzzy sets A and B on a set X are said to be equal, i.e., $A = B$, if $A(x) = B(x), \forall\, x \in X$. A fuzzy point x_α is said to belong or is contained in a fuzzy set A if $\alpha \leq A(x)$. For two fuzzy sets A and B on X, A is said to be subset of B, written as $A \leq B$ if $A(x) \leq B(x), \forall\, x \in X$. Clearly, $A = B$ iff $A \leq B$ and $B \leq A$.

Definition 1.6.4 [94] The complement of a fuzzy set A on X, denoted by $1 - A$ or A^c is given by $(1 - A)(x) = 1 - A(x), \forall\, x \in X$.

Definition 1.6.5 [94] Let $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy sets on X, where Λ denotes an indexing set. The union $\cup\{A_\alpha : \alpha \in \Lambda\}$ and the intersection $\cap\{A_\alpha : \alpha \in \Lambda\}$ (also denoted by $\vee\{A_\alpha : \alpha \in \Lambda\}$ and $\wedge\{A_\alpha : \alpha \in \Lambda\}$, respectively) are given by

$(\vee\mathcal{A})(x) = sup\{A_\alpha(x) : \alpha \in \Lambda\}, \forall\, x \in X$

$(\wedge\mathcal{A})(x) = inf\{A_\alpha(x) : \alpha \in \Lambda\}, \forall\, x \in X$

De Morgan's laws were also established in [94]:

Theorem 1.6.1 [94] For any family $\{A_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on X, the following hold:

(i) $1 - \vee\{A_\alpha : \alpha \in \Lambda\} = \wedge\{1 - A_\alpha : \alpha \in \Lambda\}$

(ii) $1 - \wedge\{A_\alpha : \alpha \in \Lambda\} = \vee\{1 - A_\alpha : \alpha \in \Lambda\}$.

Definition 1.6.6 [46] Let X and Y be non empty sets and A, B be fuzzy sets on X and Y respectively. Then the product fuzzy set $A \times B$ on $X \times Y$ is defined by $(A \times B)(x, y) = inf\{A(x), B(y) : (x, y) \in X \times Y\}$

Theorem 1.6.2 [3], [10], [73] Let f be a function from a set X into a set Y. Then the following hold:

(i) $f^{-1}(1 - B) = 1 - f^{-1}(B)$, for any fuzzy set B on Y.

(ii) $B_1 \leq B_2 \Rightarrow f^{-1}(B_1) \leq f^{-1}(B_2)$, for any fuzzy sets B_1 and B_2 on Y.

(iii) $A_1 \leq A_2 \Rightarrow f(A_1) \leq f(A_2)$, for any fuzzy sets A_1 and A_2 on X.

(iv) $ff^{-1}(B) \leq B$, for any fuzzy set B on Y.

(v) $f^{-1}f(A) \geq A$, for any fuzzy set A on X.

(vi) $f^{-1}(B)^c = (f^{-1}(B))^c$, for any fuzzy set B on Y.

(vii) For any family $\{A_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on X,

1. $f(\vee\{A_\alpha : \alpha \in \Lambda\}) = \vee\{f(A_\alpha) : \alpha \in \Lambda\}$.

2. $f(\wedge\{A_\alpha : \alpha \in \Lambda\}) \leq \wedge\{f(A_\alpha) : \alpha \in \Lambda\}$.

(viii) For any family $\{B_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on Y,

1. $f^{-1}(\vee\{B_\alpha : \alpha \in \Lambda\}) = \vee\{f^{-1}(B_\alpha) : \alpha \in \Lambda\}$.

2. $f^{-1}(\wedge\{B_\alpha : \alpha \in \Lambda\}) = \wedge\{f^{-1}(B_\alpha) : \alpha \in \Lambda\}$.

Definition 1.6.7 [10] A collection $\tau \subseteq I^X$ is called a fuzzy topology on X if the following conditions are satisfied:

(i) $0, 1 \in \tau$

(ii) $\forall\ \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \wedge_{i=1}^{n} \mu_i \in \tau$

(iii) $\mu_\alpha \in \tau$, $\forall\ \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \vee\mu_\alpha \in \tau$.

Then (X, τ) is called a fuzzy topological space (fts, for short). The members of τ are called fuzzy open sets on X and their complements are called fuzzy closed sets on X.

The definition of fuzzy topology generalizes ordinary (classical) set topology. In what follow, we always mean a fuzzy topological space (X, τ) by fts.

Definition 1.6.8 [53] A collection $\tau \subseteq I^X$ is called a laminated [49] or stratified [48] or fully stratified [51] fuzzy topology on X if the following conditions are satisfied:

(i) $\forall\ c \in I, \overline{c} \in \tau$ where $\overline{c}(x) = c, \forall\ x \in X$.

(ii) $\forall\ \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \wedge_{i=1}^{n} \mu_i \in \tau$

(iii) $\mu_\alpha \in \tau$, $\forall\ \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \vee\mu_\alpha \in \tau$.

Definition 1.6.9 [92] Let (X, τ) be a fts. A subfamily $\mathcal{B}$ of τ is called a base for τ if each member of τ can be expressed as a union of some members of $\mathcal{B}$. A subfamily $\mathcal{F}$ of τ is called a subbase for τ if the collection of all finite intersections of members of $\mathcal{F}$ forms a base for τ. The members of $\mathcal{B}$ are called basic fuzzy open sets and that of $\mathcal{F}$ are called subbasic fuzzy open sets.

Definition 1.6.10 [10] For any fuzzy set A on a fts (X, τ), the fuzzy closure of A, denoted by clA or $\bar{A}$, and the fuzzy interior, denoted by $intA$ or A^0, are defined respectively as follows:

$clA = inf\{B \in I^X : A \leq B, (1 - B) \in \tau\}$

$intA = sup\{B \in I^X : B \leq A, B \in \tau\}$.

It is clear that clA is the smallest fuzzy closed set containing A and $intA$ is the largest fuzzy open set contained in A.

Theorem 1.6.3 [3] For any fuzzy set A on a fts (X, τ),

(i) $1 - intA = cl(1 - A)$

(ii) $1 - clA = int(1 - A)$

(iii) A is fuzzy open (closed) iff $A = intA$ (respectively, $A = clA$)

Definition 1.6.11 [3] For any family $\{A_\alpha : \alpha \in \Lambda\}$ of fuzzy sets on a fts (X, τ),

(i) $\vee\{clA_\alpha : \alpha \in \Lambda\} \leq cl(\vee\{A_\alpha : \alpha \in \Lambda\})$, the equality holds if Λ is finite.

(ii) $\vee\{intA_\alpha : \alpha \in \Lambda\} \leq int(\vee\{A_\alpha : \alpha \in \Lambda\})$.

Definition 1.6.12 [72] In a fts (X, τ), a fuzzy set A is said to be a neighborhood (*nbd.* for short) of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha \in B \leq A$. In addition, if A is fuzzy open, the *nbd.* is called fuzzy open *nbd.*

Definition 1.6.13 [72] In a fts (X, τ), a fuzzy set A is said to be a quasi neighborhood or simply *q-nbd.* of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha q B \leq A$. In addition, if A is fuzzy open, the *q-nbd.* is called fuzzy open *q-nbd.*

Definition 1.6.14 [72] In a fts (X, τ), a fuzzy set A is said to be quasi coincident (q-coincident, for short) with another fuzzy set B (written as AqB) if there exists $x \in X$ such that $A(x) + B(x) > 1$. If A and B are not q-coincident, we write $A \not{q} B$.

A fuzzy point x_α is q-coincident with a fuzzy set B, denoted by $x_\alpha q B$, if $\alpha + B(x) > 1$.

Theorem 1.6.4 [72] A fuzzy set is fuzzy open iff it is a fuzzy *nbd.* of every point contained in it.

Theorem 1.6.5 [72] In a fts (X, τ), a family $\mathcal{B}$ of τ is a base for τ iff for each fuzzy point x_α in (X, τ) and for each fuzzy open *q-nbd.* U of x_α, there exists a member $B \in \mathcal{B}$ such that $x_\alpha q B \leq A$.

Definition 1.6.15 [72] In a fts (X, τ), a fuzzy point x_α is called a fuzzy cluster point of a fuzzy set A if every q-$nbd.$ (or equivalently, every fuzzy open q-$nbd.$) of x_α is q-coincident with A.

Theorem 1.6.6 [72] For a fuzzy point x_α and a fuzzy set A on a fts (X, τ), $x_\alpha \in clA$ iff x_α is a fuzzy cluster point of A.

Definition 1.6.16 [3] A fuzzy set A on a fts (X, τ) is called fuzzy regular open in X if $int(clA) = A$. A is called fuzzy regular closed in X if $cl(intA) = A$.

Theorem 1.6.7 [3] A fuzzy set A on a fts (X, τ) is fuzzy regular open iff $(1 - A)$ is fuzzy regular closed in X.

Definition 1.6.17 [32] A fuzzy point x_α is said to be a fuzzy δ-cluster point of a fuzzy set A on a fts (X, τ), if every fuzzy regular open q-$nbd.$ U of x_α is q-coincident with A. The union of fuzzy δ-cluster points of A is called the fuzzy δ-closure of A and is denoted by δ-clA. A fuzzy set A is called fuzzy δ-closed if $A = \delta$-clA and the complement of such fuzzy set is called fuzzy δ-open. Another operator δ-interior of a fuzzy set A, denoted by δ-$intA$ is defined as δ-$intA = 1 - \delta$-$cl(1 - A)$. It can be proved that A is fuzzy δ-open iff $A = \delta$-$intA$.

Remark 1.6.1 [3] A fuzzy δ-open set is the union of some fuzzy regular open sets. The complement of a fuzzy δ-open set is fuzzy δ-closed set. Hence, every fuzzy regular open set is fuzzy δ-open.

Definition 1.6.18 Let (X, τ) and (Y, σ) be two fuzzy topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) fuzzy continuous [10], if $f^{-1}(\mu)$ is fuzzy open on X, for all fuzzy open set μ on Y.

(ii) fuzzy open (fuzzy closed) [93], if for each fuzzy open (fuzzy closed) set μ on X , $f(\mu)$ is fuzzy open (respectively fuzzy closed) on Y.

(iii) fuzzy homeomorphism [10], if f is bijective and both f and f^{-1} are fuzzy continuous.

A property $\mathcal{P}$ of a fts (X, τ) is called a fuzzy topological property if it remains invariant under fuzzy homeomorphism.

Theorem 1.6.8 [73] Let (X, τ) and (Y, σ) be two fuzzy topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function, then the following are equivalent:

(i) f is fuzzy continuous.

(ii) $f^{-1}(\mu)$ is fuzzy closed on X, for all fuzzy closed set μ on Y.

(iii) For each member V of a subbase $\mathcal{B}$ for σ, $f^{-1}(V)$ is τ open.

(iv) For each fuzzy point x_β on X and any fuzzy *nbd.* V of $(f(x))_\beta$ on Y, there exists fuzzy *nbd.* U of x_β on X such that $f(U) \leq V$.

(v) For each x_β on X and any fuzzy *q-nbd.* V of $(f(x))_\beta$ on Y, there exists fuzzy *q-nbd.* U of x_β on X such that $f(U) \leq V$.

(vi) For any fuzzy set A on X, $f(clA) \leq clf(A)$.

(vii) For any fuzzy set B on Y, $cl(f^{-1}(B)) \leq f^{-1}(clB)$.

Theorem 1.6.9 [73] Let f be a function from a fts X to another fts Y. Then the following hold:

(i) If $A_1, A_2 \in I^X$ and $A_1 \wedge A_2 \neq 0_X$, then $f(A_1 \wedge A_2) \neq 0_Y$.

(ii) If $A_1, A_2 \in I^X$, $f(A_1 \wedge A_2) \leq f(A_1) \wedge f(A_2)$.

(iii) If $A \in I^X$ and $B \in I^Y$, then $f(A) \leq B \Rightarrow A \leq f^{-1}(B)$.

(iv) If $\dot{A}, B \in I^X$ such that AqB, then $f(A)qf(B)$.

(v) If $A, B \in I^Y$ such that $A \not{q} B$, then $f^{-1}(A) \not{q} f^{-1}(B)$.

Definition 1.6.19 [92] Let (X, τ) and (Y, σ) be two fts. The fuzzy product space $X \times Y$ of X and Y is a fts which is endowed with the product fuzzy topology ρ generated by the family $\{(p_1^{-1}(A), p_2^{-1}(B)) : A \in \tau, B \in \sigma\}$ as a base, where p_1 and p_2 are the usual projection mappings of $X \times Y$ onto X and Y respectively.

According to [74], the definition of fuzzy T_2 space is

Definition 1.6.20 A fts (X, τ) is called fuzzy T_2 if for any two fuzzy points x_α and y_β with $x \neq y$, there exists *q-nbd.* B and C of x_α and y_β respectively, such that $B \wedge C = 0$

According to [49], the definition of fuzzy Hausdorff space is

Definition 1.6.21 A fts (X, τ) is called fuzzy Hausdorff if for every pair of fuzzy points x_α, y_α on X with distinct supports there exist fuzzy open sets U and V on X such that $x_\alpha \in U$, $y_\alpha \in V$ and $U \wedge V = 0$

Definition 1.6.22 [33] A fts (X, τ) is called fuzzy GS-T_2 if for any two distinct fuzzy points x_α and y_β,

(i) when $x \neq y$, x_α and y_β have fuzzy open *nbds*. which are not q-coincident,

(ii) when $x = y$ and $\alpha < \beta$(say), x_α has a fuzzy open *nbd.* U and y_β has a fuzzy open *q-nbd.* V such that $U \not{q} V$.

Definition 1.6.23 [67] A fts (X, τ) is called fuzzy regular if each fuzzy point x_α on X and each fuzzy open *q-nbd.* U of x_α there exists a fuzzy open *q-nbd.* V such that $\overline{V} \leq U$,where $\overline{V}$ stands for the closure of V.

Definition 1.6.24 [67] A fts (X, τ) is called fuzzy almost regular if for each fuzzy point x_α and any fuzzy regular open set A with $x_\alpha q A$, there exists a fuzzy regular open set B such that $x_\alpha q B$ and $\overline{B} \leq A$.

Definition 1.6.25 [53] A fuzzy set A on a fts (X, τ) is called fuzzy compact if for every family $\mathcal{U}$ of fuzzy open sets on X such that $A \leq sup\{U : U \in \mathcal{U}\}$ and for every $\epsilon > 0$, there exist, a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $A \leq sup\{U : U \in \mathcal{U}'\} + \epsilon$. Extending this notion to X, the definition of fuzzy compact space is obtained.

Similarly, fuzzy nearly compact set is defined as follows:

Definition 1.6.26 A fuzzy set A is called fuzzy nearly compact if for every family $\mathcal{U}$ of fuzzy open sets on X such that $A \leq sup\{U : U \in \mathcal{U}\}$ and for every $\epsilon > 0$, there exist, a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $A \leq sup\{\overline{U}^0 : U \in \mathcal{U}'\} + \epsilon$, where $\overline{U}^0$ denotes $int(clU)$.

Definition 1.6.27 [53], [54] For each $i \in \Lambda$, if $f_i : X \to (Y_i, \tau_i)$ are the functions from a set X into fts (Y_i, τ_i), then the smallest fuzzy topology on X for which the functions $f_i, i \in \Lambda$ are fuzzy continuous is called initial fuzzy topology on X generated by the collection of functions $\{f_i : i \in \Lambda\}$. For a fuzzy set μ on X, the set $\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-level set of X. For a topological space (X, τ), $w(\tau)$ denotes the collection of all lower semi continuous functions from X into I, i.e., $w(\tau) = \{\mu : \mu^{-1}(a, 1] \in \tau\}, \forall\ a \in [0, 1]$. For a topolog τ on X, $w(\tau)$ is a fully stratified fuzzy topology on X. In a fuzzy topological space (X, τ), for each $\alpha \in I_1 = [0, 1)$, the collection $i_\alpha(\tau) = \{\mu^{-1}(\alpha, 1] : \mu \in \tau\}$ is a topology on X and is called strong α-level topology.

We have also used the following definitions and results from general topology in our work.

Definition 1.6.28 [87] A set S in a topological space in (X, τ) is called regular open if $S = int(clS)$. Complement of a regular open set is regular closed. A δ-open set in X is the union of some regular open sets in X . Every regular open set is δ-open set.

Definition 1.6.29 [81] A set S is called nearly compact if every regular open cover of S has a finite subcover.

Theorem 1.6.10 [81] A topological space (X, τ) is nearly compact iff every family $\mathcal{F}$ of regular closed sets with $\bigcap\{f : f \in \mathcal{F}\} = \Phi$, there exist a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\bigcap\{f : f \in \mathcal{F}_0\} = \Phi$.

Theorem 1.6.11 [81] (i) Every regular closed set in nearly compact space is nearly compact.

(ii) In a Hausdorff space, every nearly compact set is δ-closed.

Definition 1.6.30 [71] Let X and Y be two topological spaces. A function $f : X \to Y$ is δ-continuous if $f^{-1}(V)$ is δ-open in X for each δ-open set V in Y.

Definition 1.6.31 [31] Let X, Y be two topological spaces. For any nearly compact set C in X and regular open set U in Y, $T(C, U) = \{f \in Y^X : f(C) \subset U\}$. Then the collection $\{T(C, U)\}$ forms a subbase for some topology on Y^X called nearly compact regular open topology.

We have denoted this topology by N_R topology in our work.

Definition 1.6.32 [31] Let X, Y be two topological spaces and $Z \subset Y^X$. A topology on Z is said to be jointly δ-continuous or δ-admissible if the evaluation mapping $P : Z \times X \to Y$ given by $P(f,x) = f(x)$, where $f \in Z$, $x \in X$ is δ-continuous and $Z \times X$ is endowed with the product topology.

Definition 1.6.33 [79] A category $\mathcal{C}$ consists of three items:

(a) a class of objects, to be denoted by letters A, B, C,..., etc.,

(b) for each pair (A,B) of objects in $\mathcal{C}$, a set $Mor(A,B)$, called the set of morphisms or arrows $f : A \to B$, and

(c) a product $Mor(A,B) \times Mor(B,C) \to Mor(A,C)$ for each triple (A,B,C) of objects in $\mathcal{C}$, called composition and taking the pair (f,g) to $g \circ f \in Mor(A,C)$.

These are required to satisfy the following two axioms:

(i) For any three morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$, i.e., the composition of morphisms is associative.

(ii) for each object A in $\mathcal{C}$, there exist a morphism $I_A : A \to A$, called identity morphism, which has the property that for all $f : A \to B$ and all $g : C \to A$, we have $f \circ I_A = f$, $I_A \circ g = g$.

A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a collection of some of the objects and some of the arrows of $\mathcal{C}$ such that $\mathcal{S}$ itself is a category.

Definition 1.6.34 [79] A covariant functor F from a catrgory $\mathcal{C}$ to a category $\mathcal{D}$ is a pair of functions (both defined by the same letter F) which maps objects of $\mathcal{C}$ to the objects of $\mathcal{D}$ and, for any pair (A, B) of objects of $\mathcal{C}$, it maps the set $Mor(A, B)$ to the set $Mor(F(A), F(B))$ and is required to satisfy the folowing two conditions:

(i) $F(I_A) = I_{F(A)}$, for every $A \in \mathcal{C}$

(ii) $F(g \circ f) = F(g) \circ F(f) : F(A) \to F(C)$ for any two morphisms $f : A \to B$ and $g : B \to C$ in the category $\mathcal{C}$.

A contravariant functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is similarly defined. The only difference is that the contravariant functor maps a morphism $f : A \to B$ to a morphism $F(f) : F(B) \to F(A)$ in the opposite direction to that of f. Thus, the two conditions in this case will be

(i) $F(I_A) = I_{F(A)}$ for every $A \in \mathcal{C}$

(ii) $F(g \circ f) = F(f) \circ F(g) : F(C) \to F(A)$

A functor $T : \mathcal{C} \to \mathcal{D}$ is full when to every pair c, c' of objects of $\mathcal{C}$ and to every morphism $g : Tc \to Tc'$ of $\mathcal{D}$, there is a morphism $f : c \to c'$ of $\mathcal{C}$ such that $g = Tf$.

Proposition 1.6.1 [79] If $\mathcal{S}$ is a subcategory of a category $\mathcal{C}$ and $i : \mathcal{S} \to \mathcal{C}$ sends each object and each arrow of $\mathcal{S}$ to itself in $\mathcal{C}$, then i is a functor, called inclusion functor.

Definition 1.6.35 [79] A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is called full subcategory if the inclusion functor from $\mathcal{S}$ to $\mathcal{C}$ is a full functor.

1.7 A brief survey of this thesis

As follows, contents of the various Chapters of this thesis are reviewed in short.

In chapter 2, two fuzzy topologies on function spaces have been studied, one known as fuzzy compact open topology, as given by Gunther Jagar and the other defined by us as fuzzy nearly compact regular open topology.

Under both the topologies, we find that a collection of functions $\mathcal{F}$ from a fts X to another fts Y become fuzzy GS-T_2, if the range space Y is fuzzy GS-T_2. Moreover, the fuzzy regularity (fuzzy almost regularity) of the range space Y induces somewhat regularity (somewhat almost regularity) on the space of functions with fuzzy compact open topology (respectively, fuzzy nearly compact regular open topology).

The concepts of jointly fuzzy continuous on fuzzy compacta and jointly fuzzy δ-continuous on fuzzy near compacta have been introduced and studied.

In Chapter 3, by suitably defining pseudo fuzzy δ-continuous functions, we find that such functions correspond to the well known δ-continuous functions in general topology, under the functorial correspondence i_α, for each $\alpha \in [0, 1)$ and some of its characterizations are obtained.

By introducing a compact-like notion of fuzzy sets, which we call starplus near compactness, we see that it generalizes the existing notion of starplus compactness, and some of its properties are studied.

In Chapter 4, with the help of pseudo regular open fuzzy sets and starplus nearly compact fuzzy sets, discussed in Chapter 3, we construct a fuzzy topology on function spaces. Also, defining pseudo admissible fuzzy topology we have investigated the interrelation between this and the fuzzy topology stated above.

In Chapter 5, the notion of pseudo near compactness is introduced and it has been studied via fuzzy nets, fuzzy filterbase and so on.

Besides, two operators, named by us, fuzzy ps-closure and fuzzy ps-interior are introduced and studied.

Further, a new type of continuous like functions, which we have called pseudo fuzzy ro-continuous functions, has been introduced and its characterizations are obtained. We have also shown that this type of functions preserve pseudo near compactness of a fts.

In Chapter 6, Left fuzzy topological ring (left ftr), as introduced by Deb Ray, has been discussed in general and certain properties of the same from categorical point of view are interpreted.

We find that the collection of all left ftr-valued fuzzy continuous functions on a fuzzy topological space form a ring. The interplay between its ring structure and its topological and fuzzy topological behaviour are also observed.